SEQUENCES AND FINANCE

Arithmetic and Geometric Sequences Formulae

Type	Explicit	Recursive	Sum of Series	Sum to Infinity
Arithmetic (+ or −)	$T_n = a + (n-1)d$	$T_{n+1} = T_n + d$, $T_1 = a$	$S_n = \frac{n}{2}(2a + (n-1)d)$	$S_\infty = \infty$ or $-\infty$
Geometric (× or ÷)	$T_n = ar^{n-1}$	$T_{n+1} = T_n \times r$, $T_1 = a$	$S_n = \frac{a(1-r^n)}{1-r}$	$S_\infty = \frac{a}{1-r}$

T_n: n^{th} term in the sequence
a: first term in the sequence (i.e. T_1)
r: common ratio between terms
d: common difference between terms
S_n: sum of the first n terms in the sequence
S_∞: sum of all possible terms in the sequence

Growth or Decay Sequences Formulae

Type	Explicit	Recursive
Growth (+)	$P_t = a(1+r)^t$	$P_{t+1} = (1+r)P_t$, $P_1 = a$
Decay (−)	$P_t = a(1-r)^t$	$P_{t+1} = (1-r)P_t$, $P_1 = a$

r: rate of growth or decay (as a decimal)
P_t: population at time t
a: initial amount (i.e. P_1)
t: time in years

Arithmetic Sequence Examples

Some values of an arithmetic sequence are shown in the table below:

n	4	5	6	7
T_n	21.5	24.2	26.9	29.6

Find the explicit rule for the n^{th} term.

Need to determine a and d:
Calculating a: $a = 21.5 - (3 \times 2.7) = 13.4$
Calculating d: $d = 24.2 - 21.5 = 2.7$
Substitute values into $T_n = a + (n-1)d$
Hence, $T_n = 13.4 + (n-1) \times 2.7$

Find the recursive rule for the $(n+1)^{th}$ term.

From above, $a = 13.4$ and $r = 2.7$
Substitute values into $T_{n+1} = T_n + d,\ T_1 = a$
Hence, $T_{n+1} = T_n + 2.7,\ T_1 = 13.4$

Geometric Sequence Examples

Some values of a geometric sequence are shown in the table below:

n	3	4	5	6
T_n	0.5	2	8	32

Find the explicit rule for the n^{th} term.

$T_3 = ar^{3-1} = \frac{1}{2} \ldots$ *Equation* 1
$T_4 = ar^{4-1} = 2 \ldots$ *Equation* 2
Solve for a and r: $a = 0.03125$ and $r = 4$
Substitute into $T_n = ar^{n-1}$
Hence, $T_n = 0.03125 \times 4^{n-1}$

Find the recursive rule for the $(n+1)^{th}$ term.

From above, $a = 0.3125$ and $r = 4$
Substitute values into $T_{n+1} = T_n \times r,\ T_1 = a$
Hence, $T_{n+1} = 4T_n,\ \ T_1 = 0.03125$

Recurrence Relation Example

A recurrence relation is defined as:
$T_{n+1} = aT_n + b$ for some value of a and b.
Find the recurrence relation of a sequence where the first three terms are 3, 4 and 7.

3 → $\times a + b$ → 4 → $\times a + b$ → 7

From diagram above, create two equations that links T_1 with T_2 and T_2 with T_3.
$T_2 = aT_1 + b \to 4 = 3a + b \ldots$ *Equation* 1
$T_3 = aT_2 + b \to 7 = 4a + b \ldots$ *Equation* 2

Using ClassPad, solve *Equation* 1 and *Equation* 2 to find a and b: $a = 3$ and $b = -5$
Substitute into $T_{n+1} = aT_n + b,\ T_1 = 3$
Hence $T_{n+1} = 3T_n - 5,\ T_1 = 3$

Bouncing Ball Formulae

Below are shortcut formulae for the geometric sequence that models a ball dropped from an initial height a bouncing at r% efficiency.

Ball height after n^{th} bounce:
$Height = ar^n$

Total vertical distance travelled (S_∞):
$Distance = a\left(\frac{1+r}{1-r}\right)$

Vertical distance travelled up to n^{th} bounce:
$Distance = a\left(\frac{1+r-2r^n}{1-r}\right)$

r: bounce common ratio (as a decimal)
a: drop height
n: number of bounces

Long Term Steady State Solution

Two methods to find steady state solution:
- Substitute T_{n+1} and T_n with T and solve for T.
- Using ClassPad Sequences App, find a term for a large value of n (e.g. T_{50}) and look for a consistency.

Find the long term steady state solution for the sequence $T_{n+1} = 0.8T_n + 24,\ T_1 = 196$
- Solving: $T = 0.8T + 24$ gives $T = 120$
- ClassPad Sequences App: $T_{30} = 120.0941$ and $T_{50} = 120.0011$ which approaches 120

Simple Interest Formulae

$I = PRT$ $A = I + P$

A: amount (principal plus interest)
P: principal (starting amount)
I: total amount of interest
R: interest rate (as a decimal)
T: time in years

Simple Interest Example

Noah purchased an iPhone worth $600 using his credit card that charges 19.8% p.a. simple interest on the 30th of March. He paid the account on the 11th of April.

What is that total interest that was charged?

$I = PRT = 600 \times 0.198 \times \frac{13}{365} = \$4.23$

What is the total amount Noah paid for the iPhone?

$A = I + P = 4.23 + 600 = \$604.23$

Effective Annual Rate

Effective annual rate of interest converts $i\%\ p.a.$ compounding n times per year to $i\%\ p.a.$ compounding annually.

$$i_{effective} = \left(1 + \frac{i}{n}\right)^n - 1$$

$i_{effective}$: effective annual rate of interest (as a decimal)
i: annual interest rate (as a decimal)
n: number of times per year that interest is compounded

Frequency of Compounding Interest

The more times interest compounds per year, the more interest is earned. The higher the value of n, the higher the effective annual rate of interest. There is diminishing returns on interest gained as n increases.

n	i	Converting to effective annual rate → $i_{effective}$
Yearly (1)	5%	5%
Half-Yearly (2)	5%	5.062%
Quarterly (4)	5%	5.095%
Monthly (12)	5%	5.116%
Fortnightly (26)	5%	5.122%
Weekly (52)	5%	5.125%
Daily (365)	5%	5.127%

Compound Interest Formulae

$A = P\left(1 + \frac{r}{n}\right)^{nt}$ $I = A - P$

A: amount (principal plus interest)
P: principal (starting amount)
I: total amount of interest
r: annual interest rate (as a decimal)
n: number of times interest is compounded per year
t: time in years

Compound Interest Recurrence Relation

$$A_{n+1} = \left(1 + \frac{i}{n}\right)A_n + r,\ A_0 = P$$

i: interest rate (as a decimal)
n: number of times interest is compounded per year
r: regular payments (for investments, r is positive and for loans and annuities, r is negative)
P: principal (initial amount)

Compound Interest Table Form

Investment: Lucas invests $1,000 into an account that pays 12% p.a. compounding monthly and makes monthly deposits of $200.

Month (n)	Amount @ Start (A_n)	Interest ($A_n \times \frac{i}{r}$)	Deposit ($+r$)	Amount @ End (A_{n+1})
1	$1,000	+ $10	+ $200	$1,210.00
2	$1,210.00	+ $12.10	+ $200	$1,422.10
3	$1,422.10	+ $14.22	+ $200	$1,636.32

Loan: Sophia borrows $25,000 at 4% p.a. compounding weekly and makes weekly payments of $3,000 to pay off the loan.

Week (n)	Amount @ Start (A_n)	Interest ($A_n \times \frac{i}{r}$)	Payment ($-r$)	Amount @ End (A_{n+1})
1	$25,000	+ $19.23	- $3,000	$22,019.23
2	$22,019.23	+ $16.94	- $3,000	$19,036.17
3	$19,036.17	+ $14.64	- $3,000	$16,050.81

Annuity: Charlotte invests $1,000 to buy an annuity that pays $200 per year at 7% p.a. compounding annually.

Year (n)	Amount @ Start (A_n)	Interest ($A_n \times \frac{i}{r}$)	Withdraw ($-r$)	Amount @ End (A_{n+1})
1	$1,000	+ $70	- $200	$870.00
2	$870.00	+ $60.90	- $200	$730.90
3	$730.90	+ $51.16	- $200	$582.06

Compound Interest Example

Oliver borrowed $50,000 and makes monthly repayments of $1,120 to pay off the loan. Interest is 12% p.a. compounding monthly.

Find the recurrence relation that shows amount owing.

$A_{n+1} = \left(1 + \frac{0.12}{12}\right)A_n - 1120,\ A_0 = 50000$

How much does Oliver still owe after two years?

$A_{24} = \$33{,}276.45$

How much interest is charged during this period?

To calculate total interest, use formula: $I = A - P$
Total paid off loan $= 50000 - 33276.45 = \$16{,}723.55$
Total repayments $= 1120 \times 24 = \$26{,}880$
Total interest $= 26880 - 16723.55 = \$10{,}156.45$

Compound Interest Increasing Payments Example

Isaac deposits $300,000 into an account that earns interest at 8% p.a. compounded annually, withdrawing $37,500 at the end of the first year and the withdrawal amount increasing by 3% each year.

Find the recurrence relation that shows amount owing.

$A_{n+1} = 1.08A_n - 37500(1.03)^n,\ A_0 = 300000$

What is the final withdrawal amount?

Account reaches 0 in the 11^{th} year and final withdrawal is equal to $1.08A_{10}$ which is $1.08 \times 36421.04 = \$39{,}334.72$

ClassPad Compound Interest Variables

N	Number of time periods
I%	Annual interest rate (as a whole number)
PV	Present value
PMT	Regular payment amount
FV	Future value
P/Y	Number of payments per year
C/Y	Number of times interest is compounded per year

Compound vs. Simple Interest

Simple interest has a linear pattern (meaning that interest is constant overtime).

Compound interest has an exponential pattern (meaning that interest increases overtime).

Loans

Borrowing a sum of money that needs to be paid back in full.

PV	Positive Value
PMT	Negative Value
FV	0

Investments

Investments are a deposit that grows over time due to interest, making regular contributions.

PV	Negative Value
PMT	Negative Value
FV	Positive Value

Annuities

Investment that pays all of it out over time through regular intervals.

PV	Negative Value
PMT	Positive Value
FV	0

Perpetuities

Investing enough money to be able to "live off interest" and have the initial investment never deplete.

$$Q = PE$$

Q: annual withdrawal amount
P: principal (initial investment)
E: effective annual rate of interest (as a decimal)

ClassPad Compound Interest Examples

Jackson borrows $20,000 at 12% p.a. compounding monthly. He pays $350 every month to pay off the loan. How much would he still owe after 5 years of making payments?

N	60
I%	12
PV	20000
PMT	-350
FV	-7749.55
P/Y	12
C/Y	12

Emily borrows $25,000 at a rate of 12% p.a. compounding half-yearly. Her loan needs to be repaid in 4 years. What are Emily's half-yearly repayments?

N	8
I%	12
PV	25000
PMT	-4025.90
FV	0
P/Y	4
C/Y	4

James borrows $50,000 and is to be fully repaid in monthly repayments of $485.60 for 12 years. If interest is compounded monthly, determine the annual rate of interest.

N	144
I%	5.91%
PV	50000
PMT	-485.60
FV	0
P/Y	12
C/Y	12

Lily invests $10,000 at 7% p.a. compounding half-yearly. Lily wants her account to reach $50,000 in 10 years. How much does she need to deposit every six months?

N	20
I%	7
PV	-10000
PMT	-1064.44
FV	50000
P/Y	2
C/Y	2

Lachlan invests $2,000 and adds $200 to his account every quarter. Interest rate is 3.2% p.a. compounding quarterly. Determine how much is in his account in 5 years.

N	20
I%	3.2
PV	-2000
PMT	-200
FV	6664.63
P/Y	4
C/Y	4

Grace invests $700,000 to buy an annuity that pays $50,000 at 5.4% p.a. compounding annually. How many years will Grace be able to withdraw money?

N	26.82
I%	5.4
PV	-700000
PMT	50000
FV	0
P/Y	1
C/Y	1