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A Universal Formula for Network Functions

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Abstract—**A linear electrical network can be described in a convenient way by means of the node equations. This letter presents a universal formula which expresses any network function as the quotient of two determinants. The determinants belong to matrices derived from the indefinite nodal admittance matrix $\underline{\underline{Y}}$.**

I. Introduction

The main objective of an analysis of a linear circuit is to obtain some information on a network function. Analysis programs, such as ECAP [1], compute the modulus and argument of a network function frequency by frequency. Other programs, such as CORNAP [2], [3] and ANP3 [4], compute the poles, zeros, and gain factor of a network function. This approach is advantageous when a large number of points are wanted on a frequency response curve. The placing of the poles and zeros is also of great interest in many problems; for example, in filter design.

A characteristic for programs like CORNAP and ANP3 is that the poles and zeros are computed directly by means of eigenvalue techniques. The addition and subtraction of polynomials and computation of roots is thus avoided, and it *should* be avoided as it can give rise to serious numerical instability [5].

CORNAP and ANP3 are representative of the state-variable analysis formulation and the nodal analysis formulation, respectively. The state equations are difficult to establish but straightforward to solve, and conversely, the nodal equations are easily formulated but more difficult to solve.

A circuit containing no inductors has an indefinite admittance matrix [6] of the form $\underline{\underline{Y}} = \underline{\underline{G}} + s\underline{\underline{C}}$, where s is the complex frequency (inductors may be included by gyrator–capacitor simulation). Any network function $H(s)$ can be computed from the first and second cofactors of $\underline{\underline{Y}}$. The two-sets-of-eigenvalues technique [7] requires, however, that $H(s)$ can be expressed as the ratio of the determinants of just *two* matrices which are linear in s. This problem has been partly solved in [8] which gives formulas for the network functions Z_{21}, V_{21}, and Z_{11} $(= 1/Y_{11})$. The results from [8] have been utilized in the analysis program NAPPE [9], where it is possible to retain some of the parameters as symbols.

In the following, we describe a new formula, which is used in ANP3 for computation of network functions, including I_{21} and Y_{21}, by means of the two-sets-of-eigenvalues technique. The matrices, the determinants of which are computed, are, in general, different from the corresponding matrices used in [8].

II. Definitions and Notation

A network function is defined by a source/detector configuration. Fig. 1 shows the network connected to a voltage or current source S at nodes p and q, and connected to a voltage or current detector D at nodes r and s. The signs and arrows define the voltage polarities and current directions assumed in the formula.

In the following, the notation that will be used is

$\underline{\underline{Y}}^{abc\cdots}_{xyz\cdots}$	matrix $\underline{\underline{Y}}$ with rows $a,b,c,\cdots$ and columns $x,y,z,\cdots$ deleted
y_{rs}	element at position (r,s) in $\underline{\underline{Y}}$
$\underline{\underline{Y}}(R_a \rightarrow R_b)$	matrix $\underline{\underline{Y}}$ after the row operation row b: = row b + row a
$\underline{\underline{Y}}(C_a \rightarrow C_b)$	matrix $\underline{\underline{Y}}$ after the analogous column operation
Y	the determinant of $\underline{\underline{Y}}$
K_{rs}	the cofactor of element y_{rs} in $\underline{\underline{Y}}$.

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Fig. 1.

No renumbering is performed when rows or columns are deleted.

The operations previously given may appear in different combinations, e.g., $Y(R_p \rightarrow R_q, C_p \rightarrow C_q)^{sp}_{sr}$: perform the row operation $R_p \rightarrow R_q$, the column operation $C_p \rightarrow C_q$, delete rows s and p, and delete columns s and r. Compute the determinant.

III. The General Formula

Denominator

$\underline{\underline{Y}}$ (the indefinite admittance matrix) is modified as follows.

1) Perform the row operation $R_p \rightarrow R_q$ and the column operation $C_p \rightarrow C_q$.

2) Delete row s and column s.

3) For a voltage source, delete row p and column p. For a current detector, delete row r and column r.

The determinant of the final matrix is the denominator of the network function.

Numerator

$\underline{\underline{Y}}$ is modified as follows.

1) The same as 1) for the denominator.

2) The same as 2) for the denominator.

3) Delete row p and column r.

The determinant of the final matrix is the numerator of the network function.

Sign

The sign is determined by the numerator. Let p' and r' be the actual row number and column number of row p and column r after execution of (2). The sign of the network function is then the sign of

$$(-1)^{(p'+r')}.$$

The formula is valid also in the following degenerate cases.

1) node q = node s: three-terminal two-port function;

2) node q = node s where this node is isolated: one-port admittance; and

3) node p = node r and node q = node s: one-port impedance.

For some source-detector configurations, the universal formula leads to superflous row and column operations. These operations can be easily avoided, however, in a computer implementation.

IV. Derivation of the Formula

In the derivation, we need the well-known properties of the indefinite admittance matrix and the following lemma which is valid for a general square matrix.

Lemma

For the $n \times n$ matrix $\underline{\underline{Y}}$ where $q \neq s$ and $q \neq r$, we have

$$\text{a)}\quad K(C_q \rightarrow C_s)_{qq} = K_{qq} - K_{qs}$$

and analogously

$$\text{b)}\quad K(R_q \rightarrow R_r)_{qq} = K_{qq} - K_{rq}.$$

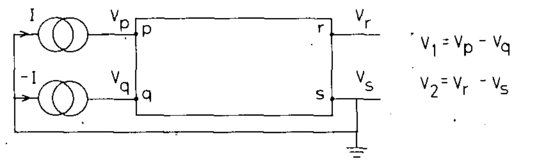

Fig. 2.

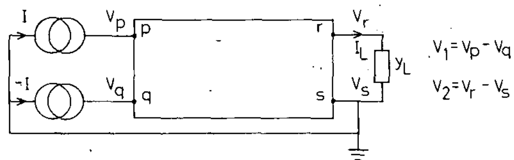

Fig. 3.

Proof: As the relations are quite analogous, only the first one is proved.

a) Assume $q < s$. $K(C_q \to C_s)_{qq}$ and K_{qq} are computed by decomposition after column s, and K_{qs} is computed by decomposition after column q, as seen by the equations

$$K(C_q \to C_s)_{qq} = \sum_{i=1}^{q-1} (-1)^{(s-1+i)} Y_{qs}^{qi}(y_{iq} + y_{is}) + \sum_{i=q+1}^{n} (-1)^{(s-1+i-1)} Y_{qs}^{qi}(y_{iq} + y_{is})$$

$$K_{qq} - K_{qs} = \left(\sum_{i=1}^{q-1} (-1)^{(s-1+i)} Y_{qs}^{qi} y_{is} + \sum_{i=q+1}^{n} (-1)^{(s-1+i-1)} Y_{qs}^{qi} y_{is}\right) - (-1)^{(q+s)} \left(\sum_{i=1}^{q-1} (-1)^{(i+q)} Y_{sq}^{qi} y_{iq} + \sum_{i=q+1}^{n} (-1)^{(i+q-1)} Y_{sq}^{qi} y_{iq}\right).$$

As

$$Y_{qs}^{qi} = Y_{sq}^{qi}$$

$$-(-1)^{(q+s)}(-1)^{(i+q)} = -(-1)^{(s+i)} = (-1)^{(s-1+i)}$$

and

$$-(-1)^{(q+s)}(-1)^{(i+q-1)} = -(-1)^{(i+s-1)} = (-1)^{(s-1+i-1)}$$

we have $K(C_q \to C_s)_{qq} = K_{qq} - K_{qs}$.

b) Assume $q > s$. In this case, the cofactors of the decomposition change signs, and the proof is unchanged.

Transfer Function Z_{21}

In Fig. 2, node p, q, r, and s are different (two-port function), or node q = node s (three-terminal two-port function). Note

$$\underline{Y}_s^s \underline{V} = (0,\cdots,I,\cdots,-I,\cdots,0)^T.$$

Perform the row operation $R_p \to R_q$ such that

$$\underline{\underline{Y}}(R_p \to R_q)_s^s \underline{V} = (0,\cdots, ,\cdots,0,\cdots,0)^T.$$

Cramer's rule gives

$$Z_{21} = \frac{V_2}{I} = (-1)^{(p'+r')} \frac{Y(R_p \to R_q)_{sr}^{sp}}{Y(R_p \to R_q)_s^s} = (-1)^{(p'+r')} \frac{Y(R_p \to R_q, C_p \to C_q)_{sr}^{sp}}{Y(R_p \to R_q, C_p \to C_q)_s^s} \tag{1}$$

where p' and r' are row number and column number of row p and column r after removal of row s and column s.

Transfer Function V_{21}

$$V_{21} = \frac{V_2}{V_1} = \frac{\frac{V_2}{I}}{\frac{V_1}{I}} = \frac{Z_{21}}{Z_0} \qquad \text{(see Fig. 2).}$$

Cramer's rule gives

$$Z_0 = \frac{K_{pp} - K_{pq}}{Y(R_p \to R_q)_s^s}$$

where K is a cofactor of the matrix $\underline{\underline{Y}}(R_p \to R_q)_s^s$. The lemma gives

$$Z_0 = \frac{K(C_p \to C_q)_{pp}}{Y(R_p \to R_q)_s^s} = \frac{Y(C_p \to C_q, R_p \to R_q)_{sp}^{sp}}{Y(R_p \to R_q)_s^s}$$

$$V_{21} = (-1)^{(p'+r')} \frac{Y(R_p \to R_q)_{sr}^{sp}}{Y(R_p \to R_q, C_p \to C_q)_{sp}^{sp}} = (-1)^{(p'+r')} \frac{Y(R_p \to R_q, C_p \to C_q)_{sr}^{sp}}{Y(R_p \to R_q, C_p \to C_q)_{sp}^{sp}}. \tag{2}$$

Transfer Function I_{21}

Add y_L to the original circuit, as seen in Fig. 3. Y_L modifies Y in four places, as seen by

$$\underline{\underline{Y}}' = \begin{bmatrix} y_{rr} + y_L & y_{rs} - y_L \\ y_{sr} - y_L & y_{ss} + y_L \end{bmatrix}$$

$$Z_{21}' = \frac{V_2}{I} = (-1)^{(p'+r')} \frac{Y(R_p \to R_q)_{sr}^{sp}}{Y(R_p \to R_q)_s^s + y_L Y(R_p \to R_q)_{sr}^{sr}}$$

$$I_{21}' = \frac{I_L}{I} = \frac{V_2}{I} y_L = Z_{21}' y_L$$

$$I_{21} = \lim_{y_L \to \infty} I_{21}' = (-1)^{(p'+r')} \frac{Y(R_p \to R_q)_{sr}^{sp}}{Y(R_p \to R_q)_{sr}^{sr}} = (-1)^{(p'+r')} \frac{Y(R_p \to R_q, C_p \to C_q)_{sr}^{sp}}{Y(R_p \to R_q, C_p \to C_q)_{sr}^{sr}}. \tag{3}$$

Transfer Function Y_{21}

See Fig. 3 and note

$$Y_{21} = \lim_{y_L \to \infty} \frac{I_L}{V_1} = \lim_{y_L \to \infty} \frac{\frac{I_L}{I}}{\frac{V_1}{I}} = \frac{I_{21}}{Z_s}$$

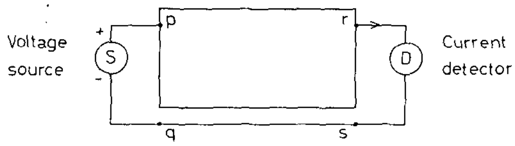

Fig. 4.

where

$$Z_s = \frac{Y(R_p \to R_q, C_p \to C_q)^{spr}_{spr}}{Y(R_p \to R_q)^{sr}_{sr}}.$$

The removal of row and column r means that node r is grounded and thus connected to node s.

$$Y_{21} = \frac{I_{21}}{Z_s} = (-1)^{(p'+r')} \frac{Y(R_p \to R_q, C_p \to C_q)^{sp}_{sr}}{Y(R_p \to R_q, C_p \to C_q)^{spr}_{spr}}. \quad (4)$$

One-Port Impedance Z_{11}

Cramer's rule gives immediately

$$Z_{11} = \frac{V_1}{I} = \frac{Y^{qp}_{qp}}{Y^q_q} \quad \text{(see Fig. 2)}. \quad (5)$$

This expression is equal to (1) when node p = node r and node q = node s.

One-Port Admittance Y_{11}

Note

$$Y_{11} = \frac{1}{Z_{11}} = \frac{Y^r_r}{Y^{rp}_{rp}}.$$

Compute Y_{21}, according to (4), as the transfer function from the nodes p and q to the nodes r and s, as seen by

$$Y_{21} = (-1)^{(p'+r')} \frac{Y'(R_p \to R_q, C_p \to C_q)^{sp}_{sr}}{Y'(R_p \to R_q, C_p \to C_q)^{spr}_{spr}} \quad \text{(see Fig. 4)}. \quad (6)$$

$\underline{\underline{Y}}'$ is derived from $\underline{\underline{Y}}$ by adding a zero row and a zero column corresponding to node q (= node s). That is, $\underline{\underline{Y}}'^s_s = \underline{\underline{Y}}$, and furthermore, $(-1)^{p'+r'} Y^{sp}_{sr} = Y^r_r$ because all cofactors of $\underline{\underline{Y}}$ are equal. (This is a well-known property of the indefinite nodal admittance matrix.) Analogously, we have $Y'^{spr}_{spr} = Y^{rp}_{rp}$, and thus $Y_{11} = Y_{21}$ in the case of Fig. 4. This completes the proof of the universal formula stated in Section III.

V. Conclusion

The formula has lead to a very efficient organization of the two-sets-of-eigenvalues circuit analysis program ANP3 developed at the Technical University of Denmark.

Acknowledgment

This work is based on a less unified formula proposed by E. V. Sørensen during the development of the ANP3 program and proven by B. Guldbrandsen. Professor Sørensen's interest in this extension of his work is gratefully acknowledged.

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Tree Graphs and Tree Numbers

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***Abstract*—The concept of the tree graph of a given connected graph was first introduced and studied by Cummins [2]. Further properties of tree graphs were explored in [1], [6]–[10].**

In this correspondence, some additional properties of tree graphs are brought out. A related concept of tree numbers is introduced and explored.

Introduction

We shall consider only graphs that are nonnull, finite, undirected, connected, and simple (i.e., graphs without self loops and parallel edges), because what follows makes sense only for such graphs. Let $G = (V,E)$ denote a finite, undirected, connected, simple graph with vertex set V and edge set E. The tree graph $T(G) = (V^1,E^1)$ of G is defined as follows. There is a one-to-one correspondence between the spanning trees of G and the vertices of $T(G)$ such that two vertices in graph $T(G)$ are adjacent, if and only if the corresponding spanning trees in G are at a distance of one.[1] The properties of tree graph were studied by Kamae [6], Kishi and Kajitani [7]–[9], Amoia and Cottafava [1], and Malik [10]. The motivation behind these studies has been the relevance of tree graph to generation of all spanning trees of G and to central trees of G [3], [4].

It has been shown [2] that for any given connected graph G, there exists a tree graph $T(G)$, which is also connected. Furthermore, $T(G)$ always has at least one Hamiltonian circuit. A center of $T(G)$ corresponds to a central tree.

Using Harary's [5] terminology, we denote the complete graph of n vertices by K_n and the circuit of n vertices (i.e., the n-gon) by C_n. Let the usual symbol $\simeq$ denote the isomorphism between two graphs. Then we obtain the following results.

Properties of Tree Graphs

Lemma 1

The tree graph $T(G)$ of a graph G is isomorphic to G, if and only if G is K_3.

Proof: If G is K_3, it clearly has three spanning trees, each at a unit distance from the other two, and therefore, $T(K_3) \simeq K_3$. To prove the "only if" part let G be a graph such that $T(G) \simeq G$, and let n be the number of vertices in G as well as in $T(G)$. Since

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[1] It is well known that the distance between two spanning trees $d(t_i,t_j)$ is defined as the number of edges present in one spanning tree but not in the other. In other words,

$$d(t_i,t_j) = 1/2 \quad |t_i \oplus t_j|$$

where $\oplus$ denotes the symmetric difference of sets, and $|\ |$ denotes the cardinality. That this distance among spanning trees of a given graph satisfies the usual metric properties can be easily seen.