SADLER UNIT 3 MATHEMATICS METHODS

WORKED SOLUTIONS

Chapter 9 Bernoulli and binomial distributions

Exercise 9A

Question 1

$\mathrm{E}(X) = p = 0.6$

$$\begin{aligned}\mathrm{Var}(X) &= p \times (1 - p)\\ &= 0.6 \times 0.4\\ &= 0.24\end{aligned}$$

Question 2

Refer to textbook answer.

Note: The skew of the histogram indicates the probability.

Question 3

Refer to textbook answer.

The skew of the histogram indicates the probability.

Question 4

$$\begin{aligned}\text{Binomial E}(X) &= np \\ &= 12 \times 0.25 \\ &= 3\end{aligned}$$

$$\begin{aligned}\text{Var}(X) &= np(1-p) \\ &= 12 \times 0.25 \times 0.75 \\ &= \frac{9}{4}\end{aligned}$$

$$\text{SD}(X) = \frac{3}{2} = 1.5$$

Question 5

$$np = 9.6$$

$$\begin{aligned}\sqrt{np(1-p)} &= 2.4 \\ np(1-p) &= 5.76 \\ 9.6(1-p) &= 5.76 \\ 1-p &= 0.6 \\ p &= 0.4\end{aligned}$$

$$\begin{aligned}n(0.4) &= 9.6 \\ n &= 24\end{aligned}$$

Question 6

a

$$a = \mathrm{P}(X = 3)$$
$$= \binom{8}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^5$$
$$= 0.2076$$

$$b = \mathrm{P}(X = 4)$$
$$= \binom{8}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^4$$
$$= 0.0865$$

b

$$\mu = np$$
$$= 8 \times \frac{1}{4}$$
$$= 2$$

$$\mathrm{SD}(X) = \sqrt{np(1-p)}$$
$$= \sqrt{8 \times \frac{1}{4} \times \frac{3}{4}}$$
$$= \sqrt{\frac{3}{2}}$$
$$= 0.5\sqrt{6} \quad \text{or} \quad 1.225$$

c

$$\mathrm{P}(\mu - \sigma \le X \le \mu + \sigma)$$
$$= \mathrm{P}(0.775 \le X \le 3.225)$$
$$= \mathrm{P}(X = 1,\ 2,\ 3)$$
$$= 0.267 + 0.3115 + 0.2076$$
$$= 0.786 \text{ (3 dp)}$$

Question 7

a $\mathrm{P}(X = 8) = \binom{9}{8}(0.6)^8(0.4)$

$= 0.0605$

b $\mathrm{P}(X = 9) = \binom{9}{9}(0.6)^9$

$= 0.0101$

c $\mathrm{P}(X \geq 8) = 0.0705$

d $\mathrm{P}(X < 8) = 1 - 0.0705$

$= 0.9295$

Question 8

a $\mathrm{P}(X = 5) = \binom{6}{5}(0.7)^5 \times 0.3$

$= 0.3025$

b $\mathrm{P}(X = 6) = \binom{6}{6}0.7^6$

$= 0.1176$

c $\mathrm{P}(X \geq 5) = 0.3025 + 0.1176$

$= 0.4202$

d $\mathrm{P}(X < 5) = 1 - 0.4202$

$= 0.5798$

Question 9

Let the random variable X denote the number of sixes rolled. $X \sim \text{Bin}(8, \frac{1}{6})$

a $\quad \text{P}(X = 2) = \binom{8}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^6$

$= 0.2605$

b $\quad n = 8,\ p = \frac{1}{6}$

$\text{P}(X = 6) = \binom{8}{6}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^6$

$= 0.0004$

Question 10

Let the random variable X denote the number of goals achieved. $X \sim \text{Bin}(9, 0.3)$

$\text{P}(X = 4) = \binom{9}{4}(0.3)^4(0.7)^5$

$= 0.1715$

Question 11

Let the random variable X denote the number of bulls scored. $X \sim \text{Bin}(10, 0.7)$

a $\quad \text{P}(X = 6) = \binom{10}{6}(0.7)^6(0.3)^4$

$= 0.2001$

b $\quad \text{P}(X = 8) = \binom{10}{8}(0.7)^8(0.3)^2$

$= 0.2335$

c $\quad \text{P}(X > 8) = \text{P}(X = 9) + \text{P}(X = 10)$

$= \binom{10}{9}(0.7)^9(0.3) + \binom{10}{10}(0.7)^{10}$

$= 0.1210608 + 0.028248$

$= 0.1493$

d $\quad \text{P}(X \geq 8) = \text{P}(X = 8) + \text{P}(X = 9) + \text{P}(X = 10)$

$= 0.2335 + 0.1493$

$= 0.3828$

Question 12

Let the random variable X denote the number of answers correctly guessed.

$X \sim \text{Bin}(20, 0.25)$

a $\quad \text{P}(X = 5) = \binom{20}{5}\left(\frac{1}{4}\right)^5\left(\frac{3}{4}\right)^{15}$

$= 0.202$

b $\quad P(X = 10) = \binom{20}{10}\left(\frac{1}{4}\right)^{10}\left(\frac{3}{4}\right)^{10}$

$= 0.010$

c $\quad \text{P}(8 < X \leq 10) = \text{P}(X = 9) + \text{P}(X = 10)$

$= \binom{20}{9}\left(\frac{1}{4}\right)^9\left(\frac{3}{4}\right)^{11} + \binom{20}{10}\left(\frac{1}{4}\right)^{10}\left(\frac{3}{4}\right)^{10}$

$= 0.037$

Question 13

Let the random variable X denote the number of people who are successfully treated.
$X \sim \text{Bin}(6, 0.4)$

$$\begin{aligned} \text{P}(X > 4) &= \binom{6}{4}(0.4)^4(0.6)^2 + \binom{6}{5}(0.4)^5(0.6) + \binom{6}{6}(0.4)^6 \\ &= 0.1792 \end{aligned}$$

Exercise 9B

Question 1

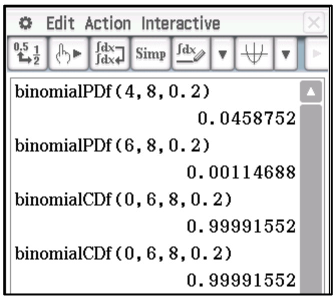
a $n = 8,\ p = 0.2$

$\text{P}(X = 4) = 0.0459$

b $\text{P}(X = 6) = 0.0011$

c $P(X \le 6) = 0.9999$

d $\text{P}(X < 7) = 0.9999$



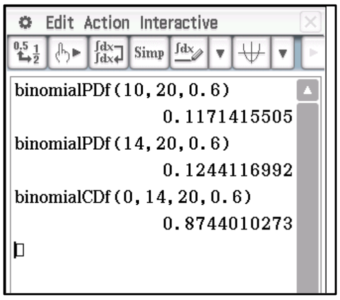
Question 2

a $\text{P}(X = 10) = 0.1171$

b $\text{P}(X = 14) = 0.1244$

c $\text{P}(X \le 14) = 0.8744$

d $\text{P}(X < 15) = 0.8744$



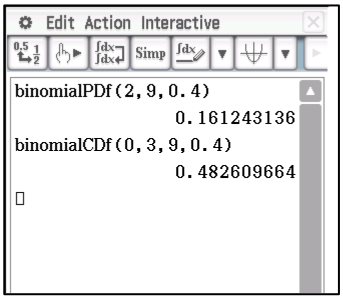
Question 3

a $\text{P}(X = 2) = 0.1612$

b $\text{P}(X \le 3) = 0.4826$

c $\text{P}(X = 2 \mid X \le 3) = \dfrac{0.1612}{0.4826}$

$= 0.334$



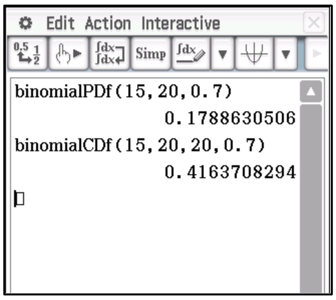
Question 4

a $\text{P}(X = 15) = 0.1789$

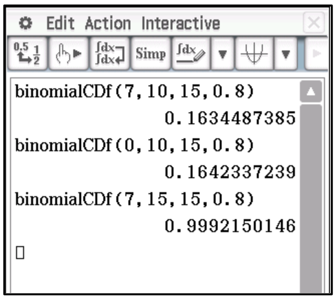
b $\text{P}(X \ge 15) = 0.4164$

c $\text{P}(X = 15 \mid X \ge 15) = \dfrac{0.1789}{0.4164}$

$= 0.430$



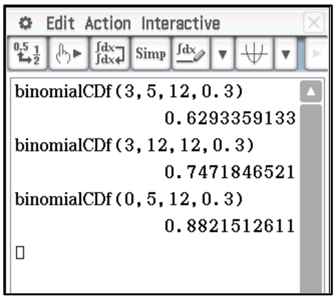
Question 5



a $$\mathrm{P}(X \geq 7 \mid X \leq 10) = \frac{\mathrm{P}(7 \leq X \leq 10)}{\mathrm{P}(X \leq 10)}$$
$$= \frac{0.163449}{0.164234}$$
$$= 0.9952$$

b $$\mathrm{P}(X \leq 10) \mid X \geq 7) = \frac{\mathrm{P}(7 \leq X \leq 10)}{\mathrm{P}(X \geq 7)}$$
$$= \frac{0.163449}{0.999215}$$
$$= 0.1636$$

Question 6



a $$\mathrm{P}(X \leq 5 \mid X \geq 3) = \frac{\mathrm{P}(3 \leq X \leq 5)}{\mathrm{P}(X \geq 3)}$$
$$= \frac{0.629336}{0.747185}$$
$$= 0.8423$$

b $$\mathrm{P}(X \geq 3 \mid X \leq 5) = \frac{\mathrm{P}(3 \leq X \leq 5)}{\mathrm{P}(X \leq 5)}$$
$$= \frac{0.629336}{0.882151}$$
$$= 0.7134$$

Question 7

$\mathrm{P}(X = 0) = 0.0080$

$\mathrm{P}(X = 1) = 0.0960$

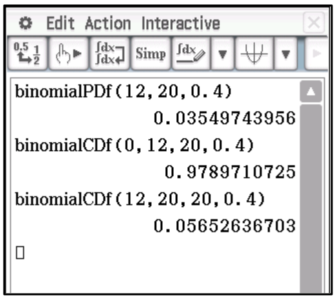
$\mathrm{P}(X = 2) = 0.3840$

$\mathrm{P}(X = 3) = 0.5120$

3 trials involved $\Rightarrow n = 3$

$p(\text{success}) = 0.8$

Question 8



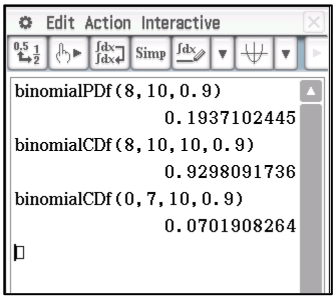
Let the random variable X denote the number of heads shown.
$X \sim \text{Bin}(20,\ 0.4)$

a $\quad \text{P}(X = 12) = 0.0355$

b $\quad \text{P}(X \leq 12) = 0.9790$

c $\quad \text{P}(X \geq 12) \ = 0.0565$

Question 9



Let the random variable X denote the seeds germinating.
$X \sim \text{Bin}(10,\ 0.9)$

a $\quad \text{P}(X = 8) = 0.1937$

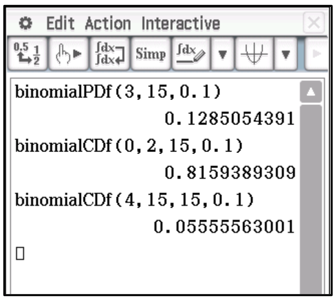
b $\quad \text{P}(X \geq 8) \ = 0.9298$

c $\quad \text{P}(X < 8) = 0.0702$

Question 10

Let the random variable X denote the number of fences incurring penalty points.

$X \sim \text{Bin}(15, 0.1)$



a $\quad \text{P}(X = 3) = 0.1285$

b $\quad \text{P}(X < 3) = 0.8159$

c $\quad \text{P}(X > 3) = 0.0556$

Question 11

Let the random variable X denote the number of answers correctly guessed.

$X \sim \text{Bin}(20, 0.2)$

a $\quad \text{P}(X = 5) = 0.175$

b $\quad \text{P}(X = 10) = 0.002$

c $\quad \text{P}(X \geq 10) = 0.003$

d $\quad \text{P}(3 \leq X \leq 7) = 0.762$

Question 12

Let the random variable X denote the number of lambs inheriting the particular characteristic.

$X \sim \text{Bin}(6, 0.25)$

a $\text{P}(X = 0) = 0.1780$

b $\text{P}(X = 6) = 0.0002$

c $\text{P}(X = 3) = 0.1318$

d $\text{P}(X \geq 3) = 0.1694$

Question 13

Let the random variable X denote the number of faulty components.

$X \sim \text{Bin}(10, 0.01)$

$$\begin{aligned} &\text{P}(X = 0) = 0.9044 \\ &\therefore \text{P}(X \geq 1) = 1 - 0.9044 \\ &\qquad\qquad\quad = 0.0956 \end{aligned}$$

Question 14

a $\text{P(sum of 7)} = \dfrac{1}{6}$

Let the random variable X denote the number of times the uppermost faces have a sum of 7.

$X \sim \text{Bin}(10, \dfrac{1}{6})$

b $$\begin{aligned} \text{P}(X \geq 1) &= 1 - \text{P}(X = 0) \\ &= 1 - 0.1615 \\ &= 0.8385 \end{aligned}$$

c $\text{P}(X < 3) = 0.7752$

d $$\begin{aligned} \text{P}(X \geq 3) &= 1 - 0.7752 \\ &= 0.2248 \end{aligned}$$

Question 15

Let the random variable X denote the number correctly guessed answers.

$X \sim \text{Bin}(5, 0.25)$

$\text{P}(X \geq 3) = 0.1035$

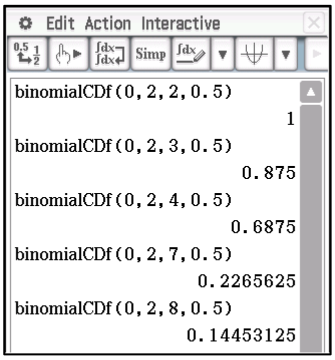
Question 16

Let the random variable M denote the number goals scored by Matt. $M \sim \text{Bin}(6, 0.2)$

Let the random variable J denote the number goals scored by Joel. $J \sim \text{Bin}(3, 0.4)$

$\text{P}(M \geq 1) = 0.7379$
$\text{P}(J \geq 1) = 0.7840$
Joel's probability of scoring at least one goal is higher.

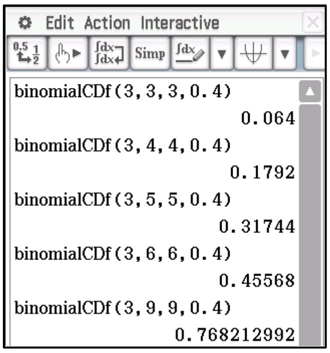
Question 17



$X \sim \text{Bin}(n, 0.5)$

n begins at 2 and increases
Use binomial CDf (0, 2, n, 0.5)
Change n until answer is less than 0.2.
$\text{P}(X \leq 2)$ $X \sim (7,\ 0.5) = 0.22656$
$\text{P}(X \leq 2)$ $X \sim (8,\ 0.5) = 0.1445$
$\therefore$ 7 tries.

Question 18



$X \sim \text{Bin}(n, 0.4)$

$\text{P}(X \geq 3) > 0.75$
n begins at 3 and increases
Using binomialCDf (3, n, n, 0.4)
When $n = 9$, $\text{P}(X \geq 3) = 0.768$
$\therefore$ 9 attempts

Question 19

Let us suppose the chance of a player improving on their second attempt by luck only to be 0.5.
The probability that 16 out of 20 would improve is $P(X \geq 16) = 5.9\%$ with $X \sim \text{Bin}(20, 0.5)$
$\therefore$ Depending on reliability of our assumptions it is possible the course did help as their is only a 5.9% of this sort of improvement happening without intervention.

Question 20

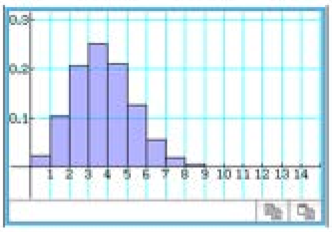
Let the random variable X denote the number correctly guessed answers for the 13 questions.

$X \sim \mathrm{Bin}(13, 0.25)$

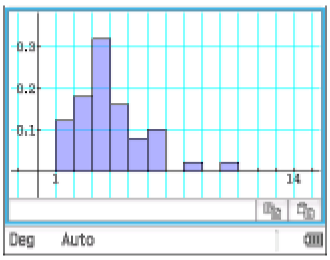
If we assume the students all got 2 correct, we can decrease the final results by 2 and compare them to an expected Binomial distribution for these parameters.

x	$P(X = x)$	0. Distribution
0	0.0238	0
1	0.1029	0.12
2	0.2059	0.18
3	0.2517	0.32
4	0.2097	0.16
5	0.1258	0.08
6	0.0559	0.1
7	0.0186	0
8	0.0047	0.02
9	0.0009	0
10	0.0012	0.02
11		
12		
13		

Binomial Distribution $X \sim \mathrm{Bin}(13, 0.25)$



Class Distribution



Assume they all got 2 correct, 13 random guesses.
Scores are now decreased by 2 to compare to a binomial.

The results are comparable to a binomial $X \sim \left(13, \dfrac{1}{4}\right)$

for the lower scores however in the binomial distribution,
$P(X > 9) = 0.00013$ while the class distrubition has 0.04 of its scores in this range.
No, I would not believe the students.

Miscellaneous exercise nine

Question 1

a $$T = 15 + 65e^{-0.004\times 300}$$
$$= 34.578$$
$$\therefore \approx 35°\text{ C}$$

b $$T = 15 + 65e^{-0.004\times 600}$$
$$= 20.897$$
$$\therefore \approx 21°\text{ C}$$

Question 2

$$y = x^2 + 5x + 6$$
$$\frac{dy}{dx} = 2x + 5$$

Question 3

$$\frac{dy}{dx} = 3(2x+1)^2 \times 2$$
$$= 6(2x+1)^2$$

Question 4

$$\frac{dy}{dx} = 2(3-2x)(-2)$$
$$= -4(3-2x)$$
$$= 4(2x-3)$$

Question 5

$$\frac{dy}{dx} = \frac{(x+1)\times 1 - x\times 1}{(x+1)^2}$$
$$= \frac{x+1-x}{(x+1)^2}$$
$$= \frac{1}{(x+1)^2}$$

Question 6

$$\frac{dy}{dx} = (2x+1)\times 18x^2 + (6x^3-5)\times 2$$
$$= 36x^3 + 18x^2 + 12x^3 - 10$$
$$= 48x^3 + 18x^2 - 10$$

Question 7

$$\frac{dy}{dx} = x \times 3(2-3x)^2(-3) + (2-3x)^3 \times 1$$
$$= (2-3x)^2(-9x+2-3x)$$
$$= (2-3x)^2(2-12x)$$
$$= 2(2-3x)^2(1-6x)$$

Question 8

$$\frac{dy}{dx} = (2x+1)\times 4(4+7x)^3 \times 7 + (4+7x)^4 \times 2$$
$$= 2(4+7x)^3\left(14(2x+1)+(4+7x)\right)$$
$$= 2(4+7x)^3(28x+14+4+7x)$$
$$= 2(4+7x)^2(35x+18)$$

Question 9

$$\frac{dy}{dx} = e^x$$

Question 10

$$\frac{dy}{dx} = 10x + e^x$$

Question 11

$$\frac{dy}{dx} = 4 \times e^{3x} \times 3 + 4x^3$$
$$= 12e^{3x} + 4x^3$$

Question 12

$$\frac{dy}{dx} = e^{2x-4} \times 2$$
$$= 2e^{2x-4}$$

Question 13

$$\frac{dy}{dx} = e^{3x+1} \times 3$$
$$= 3e^{3x+1}$$

Question 14

$$\frac{dy}{dx} = x^2 \times e^x + e^x \times 2x$$
$$= xe^x(x+2)$$

Question 15

$$\frac{dy}{dx} = 1 + x \times e^x + e^x \times 1$$
$$= xe^x + e^x + 1$$

Question 16

$$\frac{dy}{dx} = \cos x$$

Question 17

$$\frac{dy}{dx} = -\sin 3x \times 3$$
$$= -3\sin 3x$$

Question 18

$$\frac{dy}{dx} = \cos(3x-5) \times 3$$
$$= 3\cos(3x-5)$$

Question 19

$$\frac{dy}{dx} = e^{2x} \times \cos 4x \times 4 + \sin 4x \times e^{2x} \times 2$$
$$= 2e^{2x}(2\cos 4x + \sin 4x)$$

Question 20

$$\mathrm{P}(X = 12) = \binom{20}{12}(0.25)^{12}(0.75)^{8}$$
$$= 0.000752$$

Question 21

See answer in textbook.

Question 22

a $$\frac{d}{dx}(3x-1)^{-\frac{1}{2}}$$
$$= -\frac{1}{2}(3x-1)^{-\frac{3}{2}} \times 3$$
$$= -\frac{3}{2\sqrt{(3x-1)^3}}$$

b $$\frac{d}{dx}\left(5x + \frac{1}{x}\right)$$
$$= 5 - x^{-2}$$
$$= 5 - \frac{1}{x^2}$$

c $$\frac{d}{dx}\int_3^x \left(\frac{t-1}{t^3}\right) dt$$
$$= \frac{x-1}{x^3}$$

d $$\frac{d}{dx}\int_1^x \left(\frac{1+t^3}{\sqrt{t}}\right) dt$$
$$= \frac{1+x^3}{\sqrt{x}}$$

Question 23

a $f(2) = 2\left(2(2)-1\right)^3$

$= 2 \times 3^3$

$= 54$

b $f(0.5) = 2\left(2(0.5)-1\right)^3$

$= 0$

c $f'(x) = 2 \times 3(2x-1)^2 \times 2$

$= 12(2x-1)^2$

d $f'(3) = 12(6-1)^2$

$= 12 \times 25$

$= 300$

Question 24

$$\frac{dy}{dx} = 6x + 2e^{2x}$$

When $x = 1$,

$$\frac{dy}{dx} = 6(1) + 2e^2$$

$$= 2e^2 + 6$$

Question 25

a $\int 15x^4 dx$

$$= \frac{15x^5}{5} + c$$

$$= 3x^5 + c$$

b $\int (6x^2 - 4x + 6)\, dx$

$$= \frac{6x^3}{3} - \frac{4x^2}{2} + 6x + c$$

$$= 2x^3 - 2x^2 + 6x + c$$

c $\int \left(\frac{x+3}{\sqrt{x}} \right) dx$

$= \int \left(x^{\frac{1}{2}} + 3x^{-\frac{1}{2}} \right) dx$

$= \frac{2}{3} x^{\frac{3}{2}} + 3x^{\frac{1}{2}} \times 2 + c$

$= \frac{2}{3} \sqrt{x^3} + 6\sqrt{x} + c$

d $\int (2x+3)^5 dx$

$= \frac{1}{2} \int 2(2x+3)^5 dx$

$= \frac{1}{2} \times \frac{(2x+3)^6}{6} + c$

$= \frac{(2x+3)^6}{12} + c$

e $\int (5x-2)^3 dx$

$= \frac{1}{5} \int 5(5x-2)^3 dx$

$= \frac{1}{5} \times \frac{(5x-2)^4}{4} + c$

$= \frac{(5x-2)^4}{20} + c$

f $\int \sin x \, dx$

$= -\cos x + c$

g $\int \cos 2x \ dx$

$= \frac{1}{2} \int 2 \cos 2x \ dx$

$= \frac{1}{2} \sin 2x + c$

h $\int \sin(2x-1) \, dx$

$= \frac{1}{2} \int 2 \sin(2x-1) \ dx$

$= \frac{1}{2} \left(-\cos(2x-1) \right) + c$

$= -\frac{1}{2} \cos(2x-1) + c$

i $\int 4\sin 3x\ dx$

$$= \frac{4}{3}\int 3\sin 3x\ dx$$

$$= \frac{4}{3}(-\cos 3x) + c$$

$$= -\frac{4}{3}\cos 3x + c$$

j $\int 4x(x^2+3)^4 dx$

$$= 2\int 2x(x^2+3)^4 dx$$

$$= \frac{2(x^2+3)^5}{5} + c$$

k $\int \frac{d}{dx}(x^5 - 7x)$

$$= x^5 - 7x + c$$

l $\int \frac{d}{dx}(e^x\sqrt{x} - 7x)$

$$= e^x\sqrt{x} - 7x + c$$

Question 26

$$\frac{dA}{dx} = \frac{(x-1)\times 6 - (6x+3)\times 1}{(x-1)^2}$$

$$= \frac{6x - 6 - 6x - 3}{(x-1)^2}$$

$$= -\frac{9}{(x-1)^2}$$

Question 27

$$\frac{dT}{dp} = 3 \times \frac{1}{2} p^{-\frac{1}{2}}$$

$$= \frac{3}{2\sqrt{p}}$$

a When $p = 16$,

$$\frac{dT}{dp} = \frac{3}{2\sqrt{16}}$$

$$= \frac{3}{8}$$

b When $p = 25$,

$$\frac{dT}{dp} = \frac{3}{2\sqrt{25}}$$

$$= 0.3$$

c When $p = 36$,

$$\frac{dT}{dp} = \frac{3}{2\sqrt{36}}$$

$$= \frac{3}{12}$$

$$= \frac{1}{4}$$

Question 28

$$\frac{dy}{dx} = (2x+3) \times 2x + (x^2+3) \times 2$$

$$= 4x^2 + 6x + 2x^2 + 6$$

$$= 6x^2 + 6x + 6$$

$$= 6(x^2 + x + 1)$$

When $x = -1$,

$$\frac{dy}{dx} = 6(1-1+1)$$

$$= 6$$

Question 29

$$y = x^2 - 5x + c$$
$$7 = 1^2 - 5(1) + c$$
$$7 = -4 + c$$
$$c = 11$$
$$\therefore y = x^2 - 5x + 11$$

Question 30

$$y = 5x^2 - 6x + c$$
$$9 = 5(4) - 6(2) + c$$
$$9 = 8 + c$$
$$c = 1$$
$$\therefore y = 5x^2 - 6x + 1$$

Question 31

$$\begin{aligned} y &= \int 12(8-2x)^2 dx \\ &= (-6)\int (-2)(8-2x)^2 dx \\ &= -6 \times \frac{(8-2x)^3}{3} + c \\ &= -2(8-2x)^3 + c \end{aligned}$$

When $x = 4$, $f(x) = 6$

$$6 = -2\left(8 - 2(4)\right)^3 + c$$
$$c = 6$$

$$\begin{aligned} f(x) &= -2(8-2x)^3 + 6 \\ &= 6 - 2(8-2x)^3 \end{aligned}$$

Question 32

$$y = \int -18(3t+1)^{-2}\,dt$$
$$= -6\int 3(3t+1)^{-2}\,dt$$
$$= -6 \times \frac{(3t+1)^{-1}}{-1} + c$$
$$= \frac{6}{(3t+1)} + c$$

When $t = 1, x = 4.5$

$$4.5 = \frac{6}{4} + c$$
$$c = 3$$
$$\therefore\ x = \frac{6}{(3t+1)} + 3$$

Question 33

a $y = \int 5(2x+1)^4 dx$

$$= \frac{5}{2}\int 2(2x+1)^4 dx$$

$$= \frac{5}{2} \times \frac{(2x+1)^5}{5} + c$$

$$= \frac{(2x+1)^5}{2} + c$$

When $x = 1, y = 125$

$$125 = \frac{(2+1)^5}{2} + c$$

$$= 121.5 + c$$

$$c = 3.5$$

$$\therefore y = \frac{(2x+1)^5}{2} + 3.5$$

b When $x = 0$,

$$y = \frac{1^5}{2} + 3.5$$

$$= 4$$

c When $y = 19.5$,

$$19.5 = \frac{(2x+1)^5}{2} + 3.5$$

$$16 = \frac{(2x+1)^5}{2}$$

$$32 = (2x+1)^5$$

$$2x+1 = 2$$

$$2x = 1$$

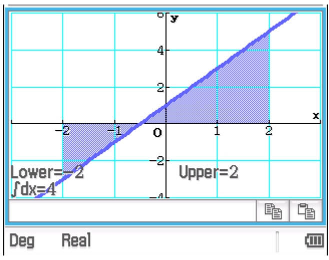
$$x = \frac{1}{2}$$

Question 34

x-intercept:

$$2x+1=0$$

$$x=-\frac{1}{2}$$



$$\int_{-\frac{1}{2}}^{2}(2x+1)\,dx$$

$$=\left[x^2+x\right]_{-\frac{1}{2}}^{2}$$

$$=(4+2)-\left(\frac{1}{4}-\frac{1}{2}\right)$$

$$=6-\left(-\frac{1}{4}\right)$$

$$=6\frac{1}{4}$$

$$\int_{-2}^{-\frac{1}{2}}(2x+1)\,dx$$

$$=\left[x^2+x\right]_{-2}^{-\frac{1}{2}}$$

$$=\left(\frac{1}{4}-\frac{1}{2}\right)-(4-2)$$

$$=-\frac{1}{4}-2$$

$$=-2\frac{1}{4}$$

$\therefore$ Area $=6\frac{1}{4}+2\frac{1}{4}=8\frac{1}{2}$ square units

Using area formula,

$$A_1=\frac{1}{2}\times\frac{5}{2}\times 5 \qquad A_2=\frac{1}{2}\times\frac{3}{2}\times 3$$

$$=\frac{25}{4} \qquad =\frac{9}{4}$$

$$=6\frac{1}{4} \qquad =2\frac{1}{4}$$

$\therefore$ Total area $=6\frac{1}{4}+2\frac{1}{4}$

$$=8\frac{1}{2}\text{ units}^2$$

Question 35

$$\frac{dy}{dx} = \int (30x - 14)\, dx$$
$$= 15x^2 - 14x + c$$

$$y = \int (15x^2 - 14x + c)\, dx$$
$$= \frac{15x^3}{3} - 7x^2 + cx + d$$
$$= 5x^3 - 7x^2 + cx + d$$

Using $x = 1,\ y = 1,$

$5 - 7 + c + d = 1$

$c + d = 3 \quad \rightarrow \quad \text{Equation 1}$

Using $x = -1,\ y = -9,$

$-5 - 7 - c + d = -9$

$-c + d = 3 \quad \rightarrow \quad \text{Equation 2}$

$\text{Equation 1} + \text{Equation 2}$

$2d = 6$

$d = 3$

$c = 0$

$\therefore\ y = 5x^3 - 7x^2 + 3$

Question 36

$$\frac{dP}{dt} = 500 \times 0.12e^{0.12t}$$
$$= 60e^{0.12t}$$

a When $t = 1$,

$$\frac{dP}{dt} = 60e^{0.12}$$
$$= 67.65$$
$\therefore$ \$67.65 per year

b When $t = 5$,

$$\frac{dP}{dt} = 60e^{0.12\times 5}$$
$$= 109.33$$
$\therefore$ \$109.33 per year

c When $t = 10$,

$$\frac{dP}{dt} = 60e^{0.12\times 10}$$
$$= 199.21$$
$\therefore$ \$199.21 per year

d When $t = 25$,

$$\frac{dP}{dt} = 60e^{0.12\times 25}$$
$$= 1205.13$$
$\therefore$ \$1205.13 per year

Question 37

Let X denote the number of times treble 20 is scored

$\mathrm{P}(X \geq 1)$

$= 1 - \mathrm{P}(X = 0)$

$= 1 - \binom{10}{0}(0.1)^0(0.9)^{10}$

$= 0.6513$

$\therefore 0.65$

Question 38

$$f(x) = 3x^2 + x$$
$$f'(x) = 6x + 1$$
$$\frac{\delta f(x)}{\delta x} \approx f'(x)$$
$$\begin{aligned}\delta f(x) &\approx f'(x) \times \delta x \\ &\approx (6x+1) \times \delta x \\ &\approx (6 \times 5 + 1) \times 0.04 \\ &\approx 1.24\end{aligned}$$

$$\begin{aligned}f(5.04) &= 3 \times 5.04^2 + 5.04 \\ &= 81.2448\end{aligned}$$

$$\begin{aligned}f(5) &= 3 \times 5^2 + 5 \\ &= 80\end{aligned}$$

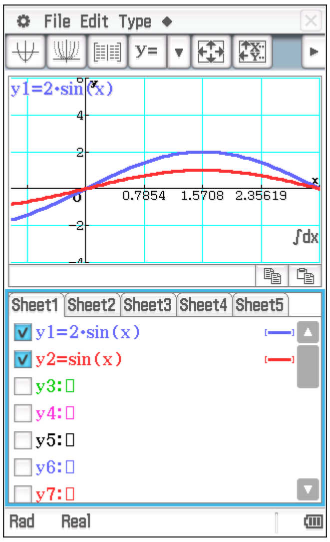
$$\begin{aligned}&f(5.04) - f(5) \\ &= 81.2448 - 80 \\ &= 1.2448\end{aligned}$$

Question 39

$$V = 5x^3$$
$$\frac{dV}{dx} = 15x^2 \qquad \frac{\delta x}{x} = 0.03$$
$$\frac{\delta V}{\delta x} \approx \frac{dV}{dx}$$
$$\begin{aligned}\frac{\delta V}{V} &\approx \frac{dV}{dx} \times \frac{\delta x}{V} \\ &\approx \frac{15x^2 \times \delta x}{5x^3} \\ &\approx \frac{15x^2}{5x^2} \times \frac{\delta x}{x} \\ &\approx 3 \times 0.03 \\ &\approx 0.09\end{aligned}$$

$\therefore$ Approximately a 9% increase in V.

Question 40



$$y = 2\sin x$$
$$y = \sin x$$

$$\int_0^{\frac{\pi}{2}} (2\sin x - \sin x)\, dx$$
$$= \int_0^{\frac{\pi}{2}} \sin x \, dx$$
$$= \left[-\cos x\right]_0^{\frac{\pi}{2}}$$
$$= \left(-\cos\frac{\pi}{2}\right) - (-\cos 0)$$
$$= 0 - (-1)$$
$$= 1$$

Area = 1 unit^2

Question 41

$$f'(x) = \frac{(2x+a)\times 2 - (2x+3)\times 2}{(2x+a)^2}$$
$$= \frac{4x + 2a - 4a - 6}{(2x+a)^2}$$
$$= \frac{2a-6}{(2x+a)^2}$$

$$f'(3) = \frac{2a-6}{(6+a)^2} \qquad = -16$$

By classpad $x = -7.125, -5$

Question 42

$$\frac{dA}{dt} = 1.2e^{0.01t}$$
$$A = 120 \times 0.01 e^{0.01t}$$
$$= 120e^{0.1t}$$

When $x = 180$,

$$A = 120e^{0.01\times 180}$$
$$= 725.96$$

$\therefore 725.96 - 120 = 605.96$

$\therefore \$606$

Question 43

$$x = e^{\cos t}$$

$$\begin{aligned} v &= \frac{dx}{dt} \\ &= e^{\cos t} \times (-\sin t) \\ &= -\sin t \times e^{\cos t} \end{aligned}$$

When $t = \frac{\pi}{2}$,

$$\begin{aligned} V &= -\sin\frac{\pi}{2} \times e^{\cos\frac{\pi}{2}} \\ &= -1 \text{ m/s} \end{aligned}$$

Question 44

a Let random variable X represent the total of two uppermost faces.

$$\mathrm{P}(X = 6) = \frac{9}{36} = \frac{1}{4}$$

b

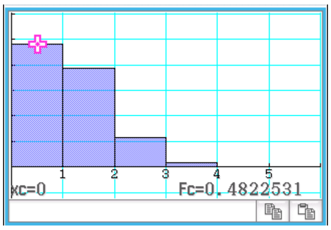
$$\begin{aligned} \mathrm{E}(X) &= \sum p_i x_i \\ &= 2 \times \frac{4}{36} + 4 \times \frac{12}{36} + 7 \times \frac{4}{36} + 6 \times \frac{9}{36} + 9 \times \frac{6}{36} + 12 \times \frac{1}{36} \\ &= 5\frac{2}{3} \end{aligned}$$

Question 45

$$\begin{aligned} \frac{dy}{dx} &= \frac{(1-\sin x)(\cos x) - (1+\sin x)(-\cos x)}{(1-\sin x)^2} \\ &= \frac{\cos x - \sin x \cos x + \cos x + \sin x \cos x}{(1-\sin x)^2} \\ &= \frac{2\cos x}{(1-\sin x)^2} \end{aligned}$$

Question 46

Let the random variable X denote the number of sixes rolled.



a $\mathrm{P}(0 \text{ sixes}) = 0.4823$

$$\binom{4}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^4$$

The most likely number of sixes is zero.

b $1 - 0.4823 = 0.5177$

$\therefore$ It is more likely zero sixes will not occur.

c Let X represent the number of sixes.

x	0	1	2	3	4
$\mathrm{P}(X = x)$	0.4823	0.3858	0.1157	0.0154	0.0008

$$\begin{aligned} E(X) &= \sum x_i p_i \\ &= 0.6666 \end{aligned}$$

$\therefore$ Long term average number of sixes $\approx \dfrac{2}{3}$

Question 47

$$\frac{dy}{dx}=\frac{x(-\sin x)-\cos x\times 1}{x^2}$$
$$=\frac{-x\sin x-\cos x}{x^2}$$

When $x=\frac{\pi}{3}$,

$$\frac{dy}{dx}=\frac{-\frac{\pi}{3}\times\left(\sin\frac{\pi}{3}\right)-\cos\left(\frac{\pi}{3}\right)}{\left(\frac{\pi}{3}\right)^2}$$
$$=\left(-\frac{\pi}{3}\times\frac{\sqrt{3}}{2}-\frac{1}{2}\right)\div\frac{\pi^2}{9}$$
$$=-\left(\frac{\sqrt{3}\pi}{6}-\frac{1}{2}\right)\times\frac{9}{\pi^2}$$
$$=\frac{-\sqrt{3}\pi-3}{6}\times\frac{9}{\pi^2}$$
$$=\frac{-3(\sqrt{3}\pi+3)}{2\pi^2}$$

Question 48

$$y = \sqrt{3}\sin x + \cos x$$

$$\frac{dy}{dx} = \sqrt{3}\cos x - \sin x = 0$$

$$\sqrt{3}\cos x = \sin x$$

$$\tan x = \sqrt{3}$$

$$x = \frac{-5\pi}{3}, \frac{-2\pi}{3}, \frac{\pi}{3}, \frac{4\pi}{3}$$

When $x = \frac{-5\pi}{3}, y = 2$ $\quad (\frac{-5\pi}{3}, 2)$

When $x = \frac{-2\pi}{3}, y = -2$ $\quad (\frac{-2\pi}{3}, -2)$

When $x = \frac{\pi}{3}, y = 2$ $\quad (\frac{\pi}{3}, 2)$

When $x = \frac{4\pi}{3}, y = -2$ $\quad (\frac{4\pi}{3}, -2)$

$$\frac{d^2y}{dx^2} = \sqrt{3}(-\sin x) - \cos x$$

$$= -\sqrt{3}\sin x - \cos x$$

By ClassPad

When $x = \frac{-5\pi}{3}, \frac{d^2y}{dx^2} = -2 < 0$

When $x = \frac{-2\pi}{3}, \frac{d^2y}{dx^2} = 2 > 0$

When $x = \frac{\pi}{3}, \frac{d^2y}{dx^2} = -2 < 0$

When $x = \frac{4\pi}{3}, \frac{d^2y}{dx^2} = 2 > 0$

$\therefore \left(\frac{-5\pi}{3}, 2\right)$ and $\left(\frac{\pi}{3}, 2\right)$ are maximum turning points.

$\therefore \left(\frac{-2\pi}{3}, -2\right)$ and $\left(\frac{4\pi}{3}, -2\right)$ are minimum turning points.

Question 49

x	1	2	3	$\dots$	n
$P(X=x)$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$		$\frac{1}{n}$

$$\begin{aligned} \mathrm{E}(X) &= 1\times\frac{1}{n}+2\times\frac{1}{n}+3\times\frac{1}{n}+\ldots+n\times\frac{1}{n} \\ &= \frac{1}{n}(1+2+3+\ldots+n) \\ &= \frac{1}{n}\times\frac{n}{2}\left[2(1)+(n-1)\times 1\right] \\ &= \frac{1}{2}\left[2+n-1\right] \\ &= \frac{1}{2}\left[n+1\right] \end{aligned}$$

Question 50

a When $x = -3$,

$$y = \frac{5(-3)-7}{2(-3)+10}$$

$$= \frac{-22}{4}$$

$$= -5\frac{1}{2} = a$$

$$\frac{dy}{dx} = \frac{(2x+10)\times 5-(5x-7)\times 2}{(2x+10)^2}$$

$$= \frac{10x+50-10x+14}{(2x+10)^2}$$

$$= \frac{64}{(2x+10)^2}$$

When $x = -3$,

$$\frac{dy}{dx} = \frac{64}{(-6+10)^2}$$

$$= 4$$

$$\therefore b = 4$$

The tangent is of the form $y = 4x + c$ and passes through $(-3, -5.5)$.

$$-5.5 = 4(-3) + c$$

$$c = 6.5$$

b $$\frac{dy}{dx} = \frac{64}{(2x+10)^2} = 4$$

$$(2x+10)^2 = 16$$

$$2x+10 = \pm 4$$

$2x+10 = 4$ or $2x+10 = -4$

$2x = -6$ $\quad$ $2x = -14$

$x = -3$ $\quad$ $x = -7$

When $x = -7$,

$$y = \frac{5(-7)-7}{2(-7)+10}$$

$$= \frac{-42}{-4}$$

$$= 10.5$$

$\therefore$ At the point $(-7, 10.5)$

Question 51

$$-\sin(\pi x) = 0$$

$$\pi x = 0 \quad \text{or} \quad \pi x = \pi$$

$$x = 0 \qquad\qquad x = 1$$

$$\int_0^1 -\sin(\pi x)\,dx$$

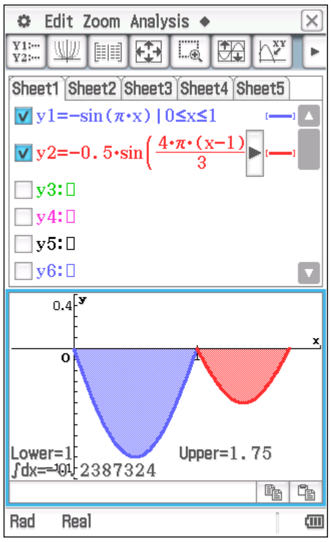
$$= -\frac{1}{\pi}\int_0^1 \pi\sin(\pi x)\,dx$$

$$= -\frac{1}{\pi}\left[-\cos(\pi x)\right]_0^1$$

$$= \frac{1}{\pi}\left[\cos(\pi x)\right]_0^1$$

$$= \frac{1}{\pi}(\cos\pi - \cos 0)$$

$$= -\frac{2}{\pi}$$



$$-\frac{1}{2}\sin\left(\frac{4\pi}{3}(x-1)\right) = 0$$

$$\frac{4\pi}{3}(x-1) = 0 \quad \text{or} \quad \frac{4\pi}{3}(x-1) = \pi$$

$$x - 1 = 0 \qquad\qquad x - 1 = \pi \times \frac{3}{4\pi}$$

$$x = 1 \qquad\qquad x = \frac{3}{4}$$

$$x = 1.75$$

$$\int_1^{1.75}\left(-\frac{1}{2}\sin\left(\frac{4\pi}{3}(x-1)\right)\right)dx$$

$$= -\frac{3}{8\pi}\int_1^{1.75}\frac{4\pi}{3}\sin\left(\frac{4\pi}{3}(x-1)\right)dx$$

$$= -\frac{3}{8\pi}\left[\left(-\cos\left(\frac{4\pi}{3}(x-1)\right)\right)\right]_1^{1.75}$$

$$= \frac{3}{8\pi}\left[\cos\left(\frac{4\pi}{3}(x-1)\right)\right]_1^{1.75}$$

$$= \frac{3}{8\pi}(\cos\pi - \cos 0)$$

$$= \frac{3}{8\pi}(-1-1)$$

$$= -\frac{3}{4\pi}$$

$$\text{Area} = -\left(-\frac{2}{\pi} - \frac{3}{4\pi}\right)$$

$$= -\left(\frac{-11}{4\pi}\right)$$

$$= \frac{11}{4\pi}\ \text{unit}^2$$

Question 52

a $$\frac{dx}{d\theta} = 10\left(\sin\theta(-\sin\theta) + \cos\theta\cos\theta\right)$$
$$= 10(\cos^2\theta - \sin^2\theta)$$

b $$10(\cos^2\theta - \sin^2\theta) = 0$$
$$\cos^2\theta - \sin^2\theta = 0$$

$$1 - \sin^2\theta - \sin\theta = 0$$
$$\sin^2\theta = \frac{1}{2}$$
$$\sin\theta = \pm\frac{1}{\sqrt{2}}$$
$$\theta = \pm 45°$$
$$\theta > 0 \therefore \theta = 45°$$

$$\frac{d^2x}{d\theta^2} = 10\left(2\cos\theta(-\sin\theta) - 2\sin\theta\cos\theta\right)$$
$$= 10(-4\sin\theta\cos\theta)$$
$$= -40\sin\theta\cos\theta$$
When $\theta = 45°$
$$\frac{d^2x}{d\theta^2} = -40 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}$$
$$= -20$$
$$\frac{d^2x}{d\theta^2} < 0 \therefore \theta = 45° \text{ is a maximum point}$$

$$x_{\max} = 10 \sin 45° \cos 45° = 10 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = 5$$

c $$x = 10\sin\theta\cos\theta$$
$$= 5 \times 2\sin\theta\cos\theta$$
$$= 5\sin 2\theta$$

$5\sin 2\theta$ has an amplitude of 5 when $\sin 2\theta = 1$
$$\sin 2\theta = 1$$
$$2\theta = 90°$$
$$\theta = 45°$$