Complete Revision for Mathematics Methods Units 3 & 4

Topic by Topic – Exam Questions

Full Solutions and Marking Guide

Calculator and Non–Calculator Questions

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Preface for Students

No subject is more inherently Australian than mathematics!
“ Have a go”, make lots of mistakes, ask about and discuss those mistakes in light of the solutions given, and move on to the next challenge. Remember, learning mathematics comes from doing mathematics. Use this book to start your revision program from the day you finish your first topic, not the day you finish your last topic. In so doing, you will reduce the stress of exams to micro rather than mega.

Note to Teachers.

It is assumed that students have calculators, and that the use of those calculators may shorten the solution of many of the questions in this book. Our solutions, in general, are provided with limited use of calculator apps.

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Chapter 1: Functions – Exponential and Logarithmic

This chapter includes questions on

Limits of exponential functions
Equations with Relationship between Logs and Exponentials
Equations with Logs
Log–linear Relationship

Exponential Functions

1. [5 marks]

State the value of each correct to 2 decimal places.

(a) e [1]

(b) $\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n$ [1]

(c) $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}\ldots$ [1]

(d) $\lim_{p \to \infty}\left(1 + \frac{1}{p}\right)^p$ [2]

2. [4 marks]

State the exact value of

(a) $\lim_{n \to \infty}\left(1 + \frac{2}{n}\right)^n$ [2]

(b) $\lim_{n \to \infty}\left(1 + \frac{0.1}{n}\right)^n$ [2]

3. [2 marks]

Write the limiting value of each sequence.

(a) $(1.3)^1$, $(1.03)^{10}$, $(1.003)^{100}$, $(1.0003)^{1000}$, ……… [1]

(b) $(1.5)^1$, $(1.05)^{10}$, $(1.005)^{100}$, $(1.0005)^{1000}$, ………. [1]

4. [4 marks]

Solve the following exactly.

(a) $e^x.e^2 = \frac{1}{e^x}$ [2]

(b) $2e^{x+1} + 3e^{x-1} = xe^x$ [2]

Logarithmic Functions

1. (9 marks)

Express the following exponential equations as logarithmic equations.

(a) $y = 3^x$ [1]

(b) $6^{x+1} = g$ [1]

(c) $2M^t + 3 = h$ [2]

Express the following logarithmic equations as exponential equations.

(d) $\log_x 3 = y$ [1]

(e) $f = 1 + \log_4 p$ [2]

(f) $\frac{1}{2}\log x + y = 0$ [2]

2. [13 marks]

Write each equation in the form $y = f(x)$.

(a) $2^y = x^2$ [1]

(b) $\log y + 1 = x$ [2]

(c) $\ln 2y - \ln 3x = 2$ [2]

Solve each of the following, giving your answer in exact form.

(d) $2^x = 6$ [1]

(e) $\log_x 5 = 2$ [1]

(f) $3(2^{x-3}) = 5$ [3]

(g) $2 - 2\log_x 2 = \dfrac{2 - \log_x 4}{2}$ [3]

3. [6 marks]

Solve for x and/or y *exactly*.

(a) $\log_2 x = -2$ [1]

(b) $\log_3 x + 2 = \log_3 (x + 1)$ [2]

(c) $x = \log_8 y$ and $\log_2 (3xy) = 1$ [3]

4. [6 marks]

Given that $y = \log_a 2$ and $x = \log_a 5$, determine the following in terms of x and y.

(a) $\log_a 10$ [1]

(b) $\log_a 0{\cdot}4$ [1]

(c) $\log_a 50$ [2]

(d) $\log_a \sqrt{2a}$ [2]

5. [6 marks]

Write y in terms of x for each of the following.

(a) $\log_x y = 3$ [1]

(b) $\log_2 y + 2\log_2 x = 2$ [2]

(c) $\log x + 1 - 2\log y = 0$ [3]

6. [11 marks]

Solve for x and/or y ***exactly***,

(a) $5^x = 73$ [1]

(b) $3^{2x} = \dfrac{27^x}{81^{x+1}}$ [3]

(c) $2^{2x+2} - 2^{3.}(2^x) - 12 = 0$ Hint : Let $y = 2^x$ [4]

(d) $3^{5x+1} - 3^{5x} = \dfrac{2}{9}$ [3]

7. [13 marks]

Solve each of the following for x and/or y.

(a) $2\log_3 x = 1$ [2]

(b) $\log_4 x + 2 = \log_4 (2x + 7)$ [3]

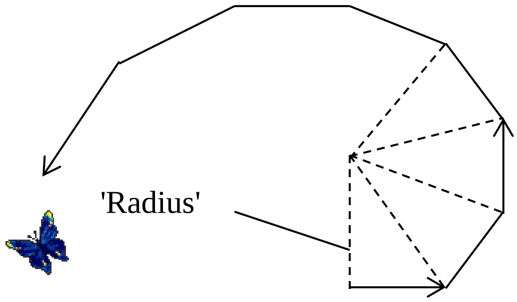
(c) $8^{x-y} = \frac{1}{16}$ and $3^{x+3y} = \sqrt{27^{2x+2}}$ [4]

(d) $2e^x(e^x + 1) - 3\left(2^2\right) = 0$ [Hint : Let $y = e^x$] [4]

8. [20 marks]

A Chemical firm installed perimeter lights for security purposes but found that many insects, especially moths, were attracted to the lights causing fouling of the light bulbs. A spray was developed to operate when the lights were switched on so that insects were repelled instead of being attracted.

Measurements were carried out and it was noticed that the insects spiralled away from the light when the spray took effect. After moving through 30° anticlockwise the insect turned again as is shown in the diagram.



The following table shows the measurements taken with r equalling the distance from the light after each turn.

number of turns (x)	1	2	3	4	5
distance from light (r cm)	1.2	1.44	1.73	2.07	2.49
y = ℓn r					

It is supposed that the relationship between x and r is of the form $r = Ae^{kx}$.

(a) Show by plotting ℓn r against x will prove this relationship. [2]
(b) Complete the above table, correct to 2 decimal places. [3]
(c) Using appropriate scales draw the graph of x against y. [4]
(d) Using your graph, or otherwise, find the values of A and k. [4]
(e) What measurement does A represent on the diagram? [2]
(f) How far from the light will the insect be after one complete revolution? [2]
(g) How many turns will the insect have made in order to be at least 30 cm from the light? [3]

Chapter 2: Growth & Decay

This chapter includes questions on

e as the banker's number

solution of $\frac{dx}{dt} = kt$ as exponential growth or decay

1. [2 marks]

If a quantity P earns 8% interest compounded yearly, after 1 year it becomes $P\,(1 + 0.08)^1$

If it is compounded more often, P becomes $P\left(1 + \frac{0.08}{n}\right)^n$ where n is the number of times that interest is calculated and added during the year.

If n approaches infinity, interest is being compounded continuously.

What does P become with continuous compounding? [2]

2. [3 marks]

When money \$$M$ is invested at 3% compounded continuously, the amount received after one year, A_1, is given by the expression

$$A_1 = M \lim_{n \to \infty} \left(1 + \frac{0.03}{n}\right)^n$$

When \$$M$ is invested at 6% compounded continuously, the amount received after one year, A_2, is given by the expression

$$A_2 = M \lim_{n \to \infty} \left(1 + \frac{0.06}{n}\right)^n$$

Determine the exact value of $\frac{A_1}{A_2}$. [3]

3. [9 marks]

The number of white rhinos in Africa has been decreasing at a rate proportional to the number present since 1993. At the beginning of 1993 there were 440 white rhinos in Africa.

That is $\frac{dW}{dt} = -kW$ where k is the constant of proportionality and t is the number of years since 1993.

(a) Show clearly that $W = W_0 e^{-kt}$ satisfies the above equation. [2]

By the beginning of 2000 there were only 356 white rhinos in Africa.

(b) Determine:

(i) the value of W_0 [1]

(ii) the value of the constant of proportionality, correct to three decimal places. [2]

Hence, or otherwise,

(c) determine the expected number of white rhinos at the beginning of 2010. [2]

(d) during which year the number of white rhinos will first fall below 300. [2]

4. [10 marks]

Biologists analysing the spread of feral animals in outback Western Australia note that the rate of *growth* of the *population of* a species of European Wild rabbit (*Oryctolagus cuniculus) is proportional to its population at any time. The approximate number of rabbits at the start of 1980 was 850 000 and the continuous percentage growth rate is 2.5% per year.*

(a) Determine the equation relating the population(P) of the rabbits to the time(t) years after the beginning of 1980. [2]

(b) Find the population of the rabbits at the beginning of 1996, to the nearest 100 rabbits. [2]

(c) What is the instantaneous rate of growth of the number of rabbits at the start of 1996? [2]

(d) In order to control the rapid growth of the feral rabbits, the *myoxama* virus was introduced at the beginning of 1996. The population decay after the virus was introduced is modelled by $\frac{dP}{dt} = -0.185P$, where P is the population after the beginning of 1996.

In what year will the rabbit population be reduced to less than 250 000? [4]

5. [8 marks]

A super volcano can potentially produce devastation on an enormous scale. One such volcano is located at Lake Toba in Sumatra.
The continual emission of ash by the volcano means that the radius of the base of the volcano increases at a continual rate of 4.5% per year.

i.e. $\frac{dr}{dt} = 0.045r$

where r is the radius of the volcano in km at any time t years after January 1st 2002.

It is known that the base circumference is 284km on January 1st 2005.

Determine:

(a) the base radius of the volcano on January 1st 2002. [3]

(b) the radius at any time t. [1]

(c) during which year the volcano will reach the National Park, 84km away from the centre of the volcano. [2]

(d) the rate of growth of the radius of the volcano in km per year, on January 1st 2011. [2]

6. [8 marks]

In a laboratory experiment on the growth of insects, there were 74 insects three days after the beginning of the experiment and 108 after an additional two days. Assume that the growth is exponential and is of the form $A_t = me^{kt}$ where A_t is the number of insects after t days, m is the initial insect population and k is a constant.

(a) Prove that $m = A_o$ [2]

(b) Use your graphics calculator to find the values of m and k correct to three significant figures. [2]

(c) Find the initial population and the population after 10 days. [2]

(d) How many days will it take for the population to be at least 500? [2]

7. [6 marks]

The pressure of a tractor tyre is P_t units after t hours and is given by the formula

$$P_t = Ae^{-kt}$$

(a) The tractor tyre is inflated to a pressure of 50 units and twenty–four hours later the pressure has dropped to 10 units. Use your graphics calculator to find the value of k, correct to three decimal places. [2]

(b) The tyre manufacturer advises that serious damage to the tyre will result if it is used when the pressure drops below 30 units. If a farmer inflates the tyre to 50 units and drives the tractor for 4 hours, can the tractor be driven further without inflating the tyre and without risking serious damage to the tyre?

If your answer is 'yes' calculate how much longer it is possible to drive without causing damage to the tyres. [4]

8. [5 marks]

An isotype of protactinium decays rapidly into a stable isotope of uranium. A particle detector set up to monitor such a decay recorded the levels of activity summarised in the table below.

Time after formation of Protactinium sample, t seconds	80	120	160	200	240
Number of particles detected per second, N	80	67	55	46	38

It is believed that N and t are related by an exponential relationship.

(a) Determine the exponential relation between N and t. [3]

(b) Calculate the initial number of particles detected and also the number of particles detected after 150 seconds. [2]

9. [6 marks]

A function y can be expressed as the infinite series $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$

where $a_0, a_1, a_2, a_3, ...$ are real constants.

The values of $a_0, a_1, a_2, a_3, ...$ are such that $\frac{dy}{dx} = y$

(a) By equating $\frac{dy}{dx}$ and *y*, find the values of a_1, a_2 and a_3 given that $a_0 = 1$ [4]

(b) Find *y* for *x* = 1. [2]

10. [10 marks]

The rate at which a certain drug is metabolised by the body is proportional to the concentration c units of the drug in the blood at any time t hours and is given by

$$\frac{dc}{dt} = -kc \text{ where } k > 0$$

The initial concentration of the drug was 54 units and one hour later the concentration was 39 units.

(a) Solve the above differential equation to express *c* as a function of *t*. [2]

(b) Determine the value of *k*. [3]

(c) How long does it take for the concentration of the drug to fall to one half of the initial concentration? [3]

(d) Find the concentration after 5 hours. [2]

11. [9 marks]

There are 1000 litres of brine (ie salt dissolved in water) in a tank and the brine contains 70kg of dissolved salt. Fresh water runs into the tank at a rate of 4 litres per minute, and the mixture, kept uniform by stirring, runs out at the same rate.

Let *x* kilograms be the mass of salt in the tank after t minutes.

(a) Show that $\frac{dx}{dt} = kx$ where *k* is a constant. [2]

(b) Find *k*. [2]

(c) Hence or otherwise, find out how many kilograms of salt will be in the tank at the end of one hour. [3]

(d) After how many hours will the tank contain less than 1 kg of salt? [2]

12. [6 marks]

A population, *y*, increases according to the differential equation:

$$\frac{dy}{dt} = 0.04y \quad \text{where } t \text{ is the time, in years, after the start of 2000}$$

The population at the start of 2000 has size 1 000.

(a) State the equation for population, *y*, in terms of *t*. [2]

(b) State the population size when *t* = 5. [2]

(c) Determine the doubling time for the population. [2]

Chapter 3: Differentiation

This chapter includes questions about

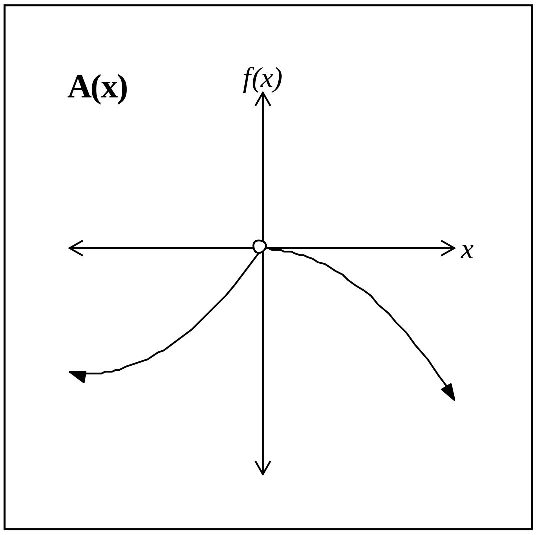
Rules of Differentiation
Leibnitz Notation and Chain Rule
Geometrical Interpretation of Derivative
The Second derivative

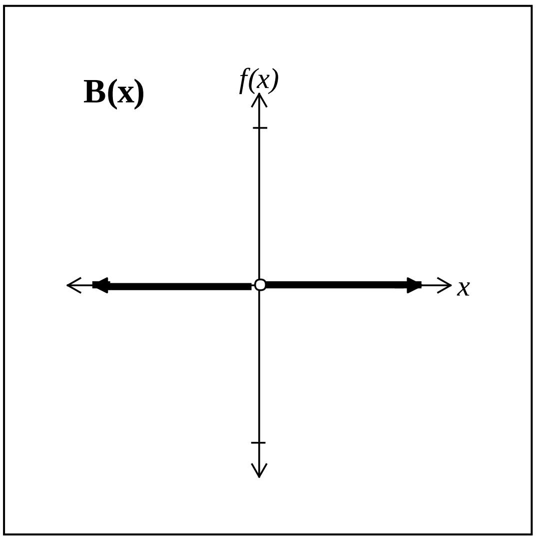
Derivatives of Polynomials

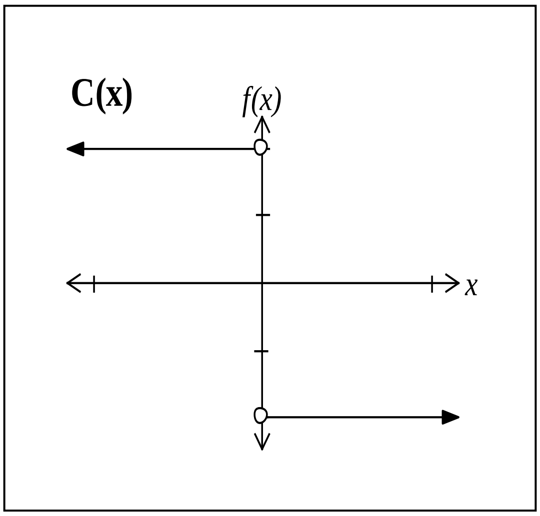
1. [4 marks]

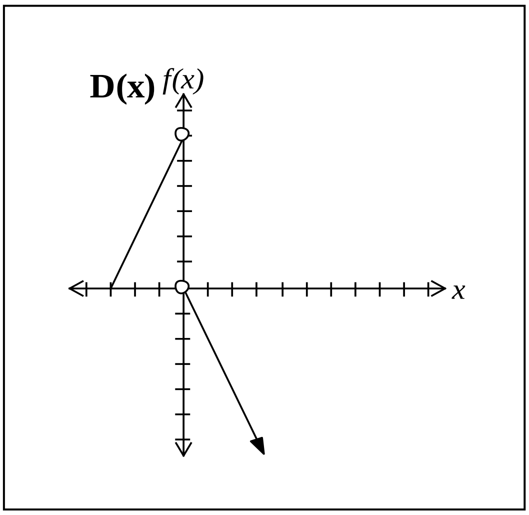
Here are the graphs of four functions A(x), B(x), C(x) and D(x) ; each defined for $x \neq 0$. The graph of the derivative of each function is also one of the graphs.
State which graph represents:

(a) $A'(x)$ (b) $B'(x)$ (c) $C'(x)$ (d) $D'(x)$









2. [5 marks]

$y = f(x) = 3x^2 + 2x.$

Use Calculus:

(a) to determine the coordinates of the turning point. [2]

(b) to verify that this function has no points of inflection. [1]

(c) to prove the conjecture that no quadratic function has a point of inflection. [2]

3. [8 marks]

Differentiate each of the following with respect to x;

(a) $y = 3x^2 + \dfrac{4}{x} - 3\sqrt{x^3}$ (Leave with positive indices.) [3]

(b) $y = \dfrac{2 - x}{2x + 1}$ (Do not simplify.) [2]

(c) $y = \dfrac{1}{2}x^2\,(\sqrt[3]{1 - 3x}\ )$ (Do not simplify.) [3]

4. [8 marks]

Sketch the graph of $f(x) = x(x - 1)(x + 2)$ on your graphic calculator.

(a) Determine, correct to two decimal places, the coordinates of: [4]

(i) the x – intercepts of $f(x)$.

(ii) the local extrema.

(iii) the point(s) of inflection.

(b) Determine the x – value of the point where the maximum gradient occurs. [1]

(c) Determine the x value(s), correct to two decimal places where necessary, where: [3]

(i) $f(x) > 0$

(ii) $f'(x) < 0$

(iii) $f''(x) > 0$

5. [5 marks]

Consider the functions given by $y = t^3 - 4t$ and $t = \sqrt{x + 1}$.

(a) Determine:

(i) $\dfrac{dy}{dt}$ [1]

(ii) $\dfrac{dt}{dx}$ [1]

(b) Show clearly how to use (a), (b) and the "Leibnitz Rule" to determine $\dfrac{dy}{dx}$ in terms of x. [3]

6. [11 marks]

Determine the derivative of each of the following. Express your answers with positive indices.

(a) $f(x) = \dfrac{1}{2}x^3 - \dfrac{2}{x^3} + \sqrt{2x} - \sqrt{3}$ [3]

(b) $g(y) = \dfrac{2x}{3 - x}$ [2]

(c) $m = v^2 \sqrt{v^3 - 4}$ [3]

(d) $x(t) = \sqrt[4]{(2t - 1)^3}$ [3]

7. [4 marks]

Find $\dfrac{dy}{dx}$ for $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$ (Do not simplify). [4]

8. [9 marks]

$$f(x) = -x^2 + 4x + 5$$

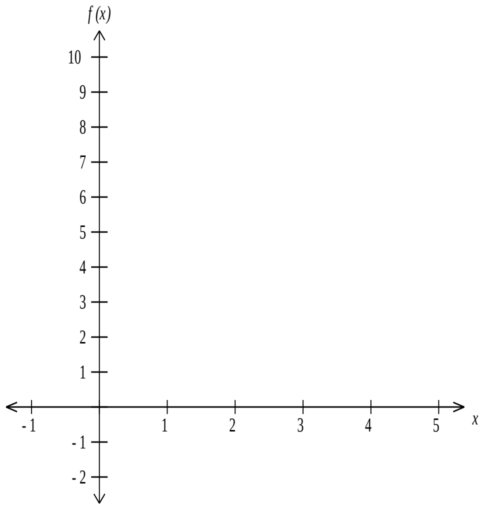
(a) Evaluate:

(i) $f(2)$ [1]

(ii) $f'(2)$ [1]

(iii) $f''(2)$ [1]

(b) (i) Sketch the graph of $f(x)$ over the domain $-1 \le x \le 5$ on the axes provided. [3]



(ii) With reference to your sketch, explain the significance of each answer from part (a). [3]

9. [6 marks]

(a) Simplify $\dfrac{4x+12}{x^2-9}$, stating any exclusions from the domain. [2]

Hence, make use of the chain rule with Leibniz notation, to determine:

(b) $\dfrac{dz}{dy}$, if $z = \dfrac{1}{3x}$ and $y = \dfrac{4x+12}{x^2-9}$ [4]

Derivatives of Exponential Functions

1. [4 marks]

(a) Differentiate $y = \dfrac{3x-1}{e^{3x}}$, without simplifying. [2]

(b) Differentiate $y = \sqrt{3e^{4x}}$ [2]

2. [4 marks]

Consider the function $f(x) = e^{-2x}x^3$.

(a) Show clearly that $f'(x) = x^2e^{-2x}(3-2x)$. [2]

Hence, or otherwise;

(b) determine the exact co–ordinates of the curve $f(x) = e^{-2x}x^3$ where the gradient is zero. [2]

3. [6 marks]

The gradient function of $y = f(x)$ is given as $f'(x) = e - 3x^2 + 4x$.

The point P$(2, 2e-1)$ lies on $f(x)$.

Determine, ***exactly***,

(a) $f(x)$ [2]

(b) the equation of the tangent to the curve $y = f(x)$ at the point P. [2]

(c) the co–ordinates of any points of inflection on $y = f(x)$ [2]

4. [7 marks]

(a) Determine the equation of the tangent to the curve $y = \dfrac{e^x}{x}$ at (1, e). [3]

(b) Determine the value(s) of k such that the equation $e^x = kx$ has:

(i) two roots. [2]

(ii) one root. [1]

(iii) no roots. [1]

5. [5 marks]

(a) Show that if $f(x) = x^2 e^x$ then $f'(x) = x e^x (x + 2)$ [1]

Hence, or otherwise,

(b) determine the equation(s) of the tangent(s) to the curve $f(x) = x^2 e^x$ at the point(s) where the gradient is zero. [4]

Derivatives of Logarithmic Functions

1. [10 marks]

Differentiate each of the following. Leave with positive indices.

(a) $y = \ln 2x + \ln(1 - x)$ [2]

(b) $f(x) = (\ln x)^2 + \ln x^2 - \ln 3e$ [2]

(c) $p = e^{x^2} - \ln(ex^2)$ [3]

(d) $h(t) = (t^3)(\ln t^3)$ [3]

2. [4 marks]

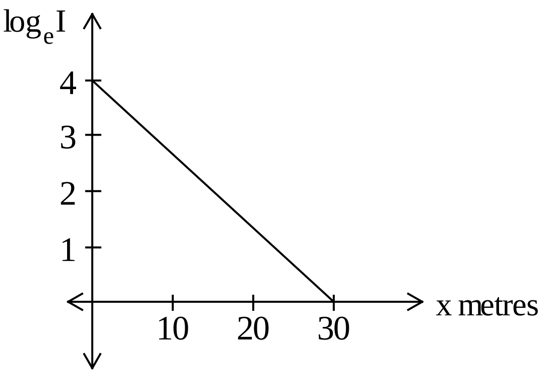
Find the equation of the tangent to the curve $f(x) = x^2 \ln x^2$ at the point where $x = 1$. [4]

3. [3 marks]

Find coordinates of the point on the curve $y = \ln(2x - 1)$ where the gradient is equal to 2. [3]

4. [8 marks]

The graph below shows a relationship between $\log_e I$ and x metres; where I units is the intensity of light at depth x metres below the surface of sea water.



(a) Express $\log_e I$ in terms of x. [2]

(b) Hence express I in terms of x. [1]

(c) Show that $\frac{dI}{dx} = kI$ and state the value of k. [3]

(d) At what depth is the light intensity 10% that of the surface? [2]

Derivatives of Trig Functions

1. [6 marks]

Find the derivative of each of the following.

(a) $y = \sin 2x + \cos\frac{x}{2}$ [2]

(b) $x = \sin^2 t + 3\cos 2t + \pi$ [2]

(c) $f(x) = (\sin x - \cos x)^3$ [2]

2. [4 marks]

Find the equation of the line tangential to the curve $y = \cos^2 3x$ at the point $\left(\frac{\pi}{4}, \frac{1}{2}\right)$. [4]

3. [9 marks]

Find each of the following. Do not simplify, but leave with positive indices.

(a) $\frac{d}{dx}\left(\sqrt[3]{(\cos^3 2x - \sin \pi x)}\right)$ [3]

(b) $\frac{d^2y}{dx^2}$ if $y = \cos^4(3x)$ [3]

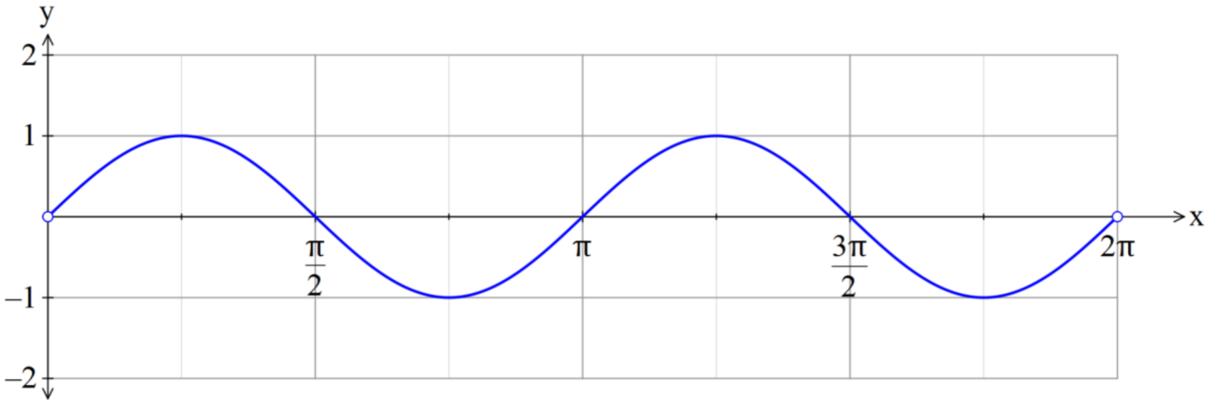
(c) $f\,'(x)$ if $f(x) = \frac{\sin x}{x - \cos x}$ [3]

4. [4 marks]

Find the equation of the tangent perpendicular to $y = 3 - \sin(1 - 2x)$ at the point when $x = \frac{1}{2}$. [4]

5. [4 marks]

For the function y = $f(x)$ shown, sketch the graph of $y = f\,'(x)$ over the domain $0 < x < 2\pi$. [4]



Derivatives of Combined Functions

1. [5 marks]

Determine the derivative of each of the following. Express your answers with positive indices, where appropriate.

(a) $y = (3x - 3)^3$ [1]

(b) $p(a) = 2\sqrt{a} + e^{3a+2}$ [2]

(c) $g(p) = \dfrac{2p^2 - 3}{p}$ [2]

2. [3 marks]

Given $y = 3\sin x$ and $x = \dfrac{\pi}{u}$ find $\dfrac{dy}{du}$ when $u = 2$. [3]

3. [10 marks]

Differentiate each of the following functions. Do not simplify.

(a) $e^{\sqrt{x}}$ [2]

(b) $(3x + 2)\cos(x^2)$ [3]

(c) $\dfrac{x^3 + 3}{x^4 + 4}$ [2]

(d) $\ln(2 - \sin^3 x)$ [3]

4. [3 marks]

Find the derivative of $h(x) = (4x^2 - x)^{-1}$. [3]

5. [6 marks]

Find the derivatives of

(a) $f(\theta) = e^{\cos 2\theta}$ [2]

(b) $g(t) = 3\cos^2(6\pi - 2t)$ [2]

(c) $L(t) = (1 - t)^2 \ln t$, $t > 0$ [2]

6. [4 marks]

The function f is defined by
$f(x) = x\,e^{-x}$ for all x.
Find exactly the global maximum of f.

7. [8 marks]

(a) Show that $5^x = e^{x \ln 5}$ [2]

(b) Using the result from part (a), find $\frac{dy}{dx}$ if $y = 5^x . \sin x$ [2]

(c) Hence find x for $x \,\varepsilon\, [0, \pi]$ such that y is a maximum. [4]

Chapter 4: Applications of Differentiation

This chapter includes questions about

Incremental Change
Optimisation
Curve Sketching
Marginal Cost

It includes the use of the second derivative.

It includes exponential, log and trig functions and their applications

1. [6 marks]

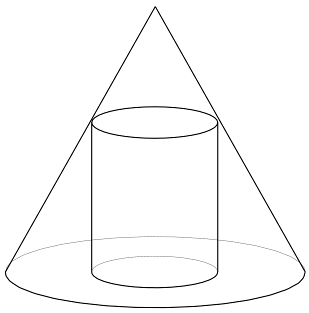
An explosion produces a sound wave which expands through the air as a sphere. The radius increases at a rate of 300 m/s.

(a) At what rate is the volume of the sphere increasing two seconds after the explosion? [3]

(b) Using the Incremental formula, determine the percentage increase in the surface area of the sphere when the radius increases by two percent? [3]

2. [9 marks]

A cylinder of radius *r* metres and height *h* metres is to be constructed inside a cone of height 6 metres and base radius 3 metres as shown.



(a) Show clearly why $h = 6 - 2r$. [2]

(b) Show clearly why the volume of the cylinder is given as:

$$V = 6\pi r^2 - 2\pi r^3$$ [2]

(c) Use *Calculus* to determine the dimensions of such a cylinder which gives maximum volume. [3]

(d) State the maximum volume. [2]

3. [5 marks]

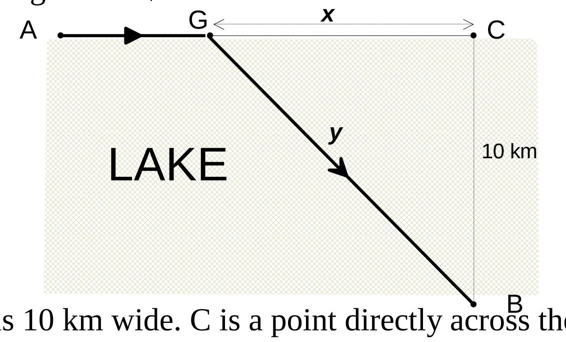
(a) Show that if $f(x) = x^2 e^x$ then $f'(x) = x e^x (x + 2)$ [1]

Hence, or otherwise,

(b) determine the equation(s) of the tangent(s) to the curve $f(x) = x^2 e^x$ at the point(s) where the gradient is zero. [4]

4. [8 marks]

Nuclear material must be transported from A to B, via G, on the diagram below. The train can transport the material alongside the lake at a cost of \$7 000 / km, while transportation costs across the lake are given as \$9 000 / km.



The lake is 10 km wide. C is a point directly across the river from B and 15 km from A.
A boat can take the material from any point along the rail line, AC.

If the distance GC is given as x, and the distance travelled by boat is y then:

(a) Show clearly that: $y = \sqrt{x^2 + 100}$ [1]

(b) Explain clearly why the total transportation costs, in \$ 000, are given as:

$$C = 105 - 7x + 9\sqrt{100 + x^2}$$ [2]

(c) How far from C, correct to the nearest metre, should the boat pick up the material to minimise costs? [2]

(d) What is the minimum costs of transportation? [1]

(e) What Calculus method could be used to prove that your answer to part (d) is the minimum? Explain how to use this method without calculation. [2]

5. [7 marks]

Water is flowing into a cylindrical tank of base radius 5 m and height h at a rate of 2 m^3/ min.

(a) Show that the volume of the water in the tank at any time t minutes is given by;
$V = 25\pi h$ [2]

(b) Determine the rate of change of the volume of water with respect to its height. [1]

Using the chain rule,

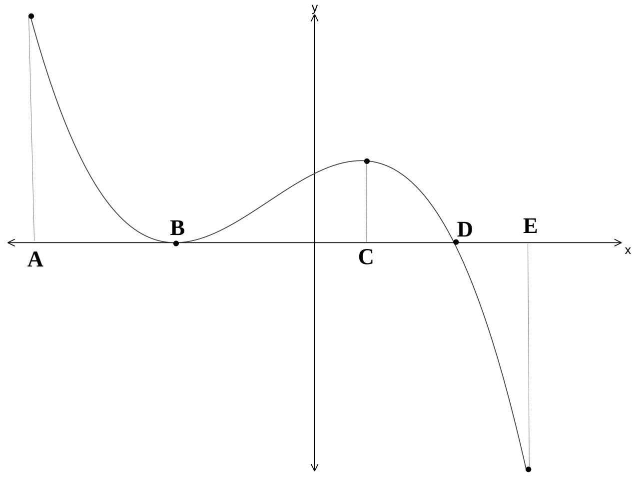
(c) find the rate of change of the height of the water with respect to time. [3]

Hence, or otherwise,

(d) determine the rate of change of the height of the water when the volume of the water is 10 m^3. [1]

6. [5 marks]

The graph of $y = f'(x)$ is given below.



Give *all* x values where:

(a) the function is stationary. [1]

(b) the function is decreasing. [1]

(c) $f''(x) > 0$. [1]

(d) the function has a horizontal point of inflection. Explain. [2]

7. [7 marks]

David owns a company and his marginal costs, in dollars, on x items are given as:

$$C'(x) = \frac{400}{\sqrt{x}}$$

(a) Determine $C'(100)$ and explain its meaning. [2]

It is known that the cost of producing 100 items is \$ 10 000.
The revenue on each item is \$ 200

(b) Show that David's profit on x items is given as:

$$P(x) = 200x - 800\sqrt{x} - 2\,000$$ [2]

(c) How many items would David expect to sell before he makes a profit? [1]

(d) What is the maximum loss that David would expect to make? [2]

8. [7 marks]

A balloon, spherical in shape, has air escaping at a constant rate of 100π cm^3/sec. Originally the balloon has a volume of 36000π cm^3.

Use the fact that the volume of a sphere is $\frac{4}{3}\pi r^3$, to determine:

(a) the rate of change of volume with respect to the radius, when the radius is 25 cm. i.e. $\frac{dV}{dr}$ [2]

(b) the rate of change of the radius with respect to time, correct to two decimal places, after 10 seconds, using Leibnitz rule $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$. [5]

9. [8 marks]

A function has ***all*** the following features. Sketch a possible function for each on the axes provided.

(a) [3]

- $f(-2) = f(0) = f(2) = 0$
- $f'(-1) = f''(0) = f'(1) = 0$
- $f(1) > 0$

(b) [5]

- $f(x) < 0$ for $x > -2$
- $f''(-2) = f'(-2) = f(-2) = 0$
- $f'(0) = 0$
- $f''(-1) = f''(1) = 0$

10. [11 marks]

A particle moves in rectilinear motion, such that its displacement (x) from the origin O, at any time t (seconds), is given as $x = t\left(2t^2 - 5t + b\right) + 1$ metres.

It is known that at $t = 3$, the particle is 4m to the right of the origin.

(a) Determine the value of b. [1]

(b) Determine the velocity of the particle when $t = 3$. [2]

(c) What was the initial speed of the particle? [2]

(d) Find the value(s) of t when the particle comes to rest, ***and*** the distance(s) from the origin at that time(s). [3]

(e) Calculate the acceleration of the particle when the velocity is 6m/s. [3]

11. [12 marks]

Harry and Sarah are organising their Yr 12 Leaver's jumpers for 2006. They feel that they can sell only 20 jumpers, if they set the price of each jumper at $100.
For every $2 decrease in the price, they expect to sell an extra 5 jumpers.
Let x = the number of reductions of $2 in the price of the jumper.

(a) Show clearly why the Revenue (R) is given by:

$$R(x) = 2\,000 + 460x - 10x^2$$ [2]

(b) If the cost to produce each jumper is $12, show clearly that the $P(x)$, is given by:

$$P(x) = 1760 + 400x - 10x^2$$ [3]

(c) Find the maximum profit they will make and the cost of each jumper when this occurs. [3]

(d) If there are 170 students in the year how many have not bought a jumper at the price determined in part (c)? [2]

(e) If all 170 students were to buy a jumper, at what price would they be sold? [2]

12. [6 marks]

Organisers of the 2007 *Slam–it Festival* know that if they sell tickets to their two day festival at $150 each they will sell 5000 tickets. For every 50c drop in ticket price, the number of tickets sold will increase by 50. It costs the organisers $250 000 per day to run the festival.

(a) (i) Write an expression that represents the price of the tickets, if the number of tickets sold is given by ($5000 + 50x$), where x is the number of 50c increases. [1]

(ii) Hence, or otherwise, show that $R(x) = 750000 + 5000x - 25x^2$. [2]

(b) Show that the total profit per day from the concert in terms of x is given as:

$$P(x) = 500\,000 + 5\,000x - 25x^2$$ [1]

(c) What is the maximum profit the organisers can expect, and at what price should the tickets be sold to achieve this profit? [2]

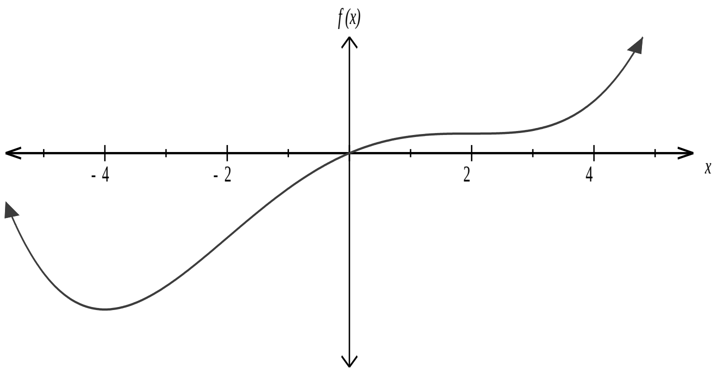
13. [5 marks]

Ben is designing an open rectangular toy box (i.e. no top) that is to have a volume of 562500cm^3. The length of the wooden box is to be double the height.

Using calculus methods, determine the dimensions of the box that meet the volume requirement **and** minimises the amount of wood used to construct it. [5]

14. [6 marks]

The function $f(x)$ is shown below. Sketch $f'(x)$ and $f''(x)$. [6]



15. [6 marks]

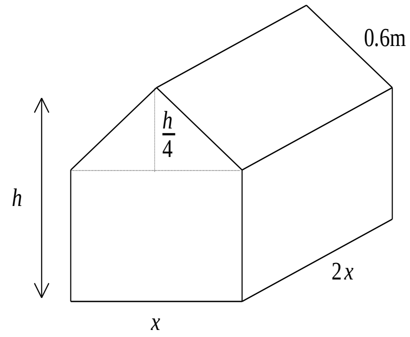
The curve $y = ax^4 + bx^3 - 12x + c$ has a horizontal point of inflection at (1, 7).
Find a, b and c.

[6]

16. [8 marks]

The Smiths are designing a new kennel for their dog. They have 5m^2 of plastic available to cover the kennel in the shape below. The length of the kennel must be twice that of the width (x), while the height of the triangular section is 25% of the total height, (h), of the kennel. Note: The front of the kennel is open and the base is not covered.

(a) Show that the volume, V, equals $\frac{7}{4}hx^2$. [3]



(b) Show that $h = \frac{8}{31x}(5 - 2.4x)$ [3]

(c) Calculate the dimensions of the kennel which will give the greatest space for their dog, whilst meeting their restrictions on plastic available. [2]

17. [7 marks]

Sketch each of the functions with all the properties.

(a) $y = f(x)$ such that:

$f(-1) = f(3) = f(5) = f'(1) = f'(4) = 0$

$f(x) \geq 0$ for $-1 \leq x \leq 3$ and $x \geq 5$

$f'(x) < 0$ for $1 < x < 4$ only. [3]

(b) $y = g(x) = -f(x)$; [2]

(c) $y = g'(x)$ [2]

18. [5 marks]

The Lucky Bay high school committee is organising the annual school ball. It is predicted that 100 people will come when the tickets cost $100 each. For every $2 decrease in ticket price, it is predicted that an extra two couples will come.

(a) If x = number of $2 decreases in the cost of each ticket show that revenue, $R(x)$ is given by: $R(x) = 10\,000 + 200x - 8x^2$ [2]

(b) The cost to the school to organise the ball is $10 000. Calculate the cost of the ticket and the number of people who would need to come for the school to "break even". Show working. [3]

19. [9 marks]

A company produces commercial dishwashers, at a marginal cost per dishwasher, in dollars, given by:

$C'(d) = 25d - 0.1d^2$ where d is the number of dishwashers produced.

(a) Find the marginal cost of producing the 5th dishwasher and interpret your answer. [2]

It is known that the initial costs are $55.

(b) Find $C(d)$. [2]

(c) Calculate the average cost per dishwasher of producing 10 dishwashers. [2]

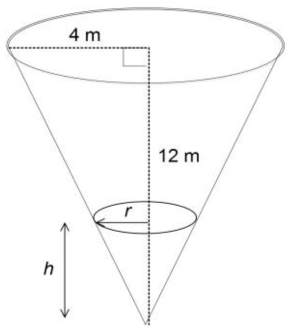
The revenue on each dishwasher is $320.

(d) Calculate the number of dishwashers that will result in maximum profit. [3]

20. [6 marks]

A cone shaped wheat silo is releasing its wheat at a rate of 3π m^3/min.
Use the fact that the volume of a cone is $V = \frac{1}{3}\pi r^2 h$ and that
the silo has a height of 12m and a top radius of 4m.

(a) Show that $h = 3r$. [1]



Hence, or otherwise determine,

(b) the rate of change of volume with respect to the radius, when the radius of the wheat is 3m. i.e. $\frac{dV}{dr}$ [2]

(c) the rate of change of the radius with respect to time, correct to two decimal places, when the height of the wheat is 3m, using Leibnitz $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$. [3]

21. [3 marks]

Show that $x = Ae^{kt}$ is a solution of the equation

$$x.\frac{d^2x}{dt^2} = \left(\frac{dx}{dt}\right)^2$$ where A and k are constants. [3]

22. [4 marks]

As sand leaks out of a container it forms a conical pile whose altitude is always the same as the radius. If, at a certain instant, the radius is 10 cm, use differentiation to approximate the change in radius that will increase the volume of the pile by 2 cm^3. [4]

23. [4 marks]

The period, T seconds, of a simple pendulum varies directly as the square root of its length, ℓ

ie $T = k\sqrt{\ell}$ where k = real constant and ℓ = length of pendulum.

Determine the effect on the period corresponding to an increase of 1% in the length of the pendulum. [4]

24. [9 marks]

At 12 noon ship A is 50 km West of ship B. Ship A is sailing East at 30 km/hr and ship B is sailing South at 20 km/hr.

(a) Let d(t) be the distance in km between ships A and B at time t hours after 12 noon. Express d as a function of t. [2]

(b) Find the rate at which the distance between the ships is changing at 1.00 pm. [3]

(c) Investigate if the ships are on a collision course. If not, determine the closest distance the ships will come near each other. [4]

25. [9 marks]

A rectangle with fixed area, A, has perimeter, P, where $P = 2x + \frac{128}{x}$.
The length of one of the sides of the rectangle is x metres.

(a) If x increases from 2 metres to 2.01 metres, determine the associated change in P, by using the Incremental Formula (small change). [4]

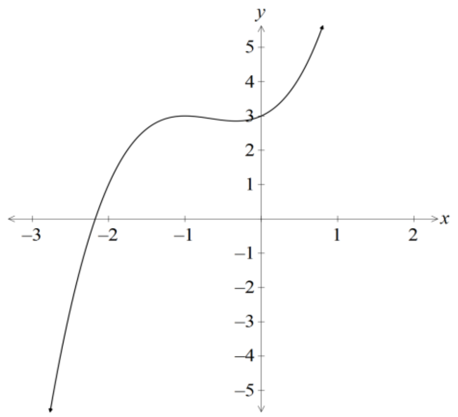
(b) Determine the value of x such that P is a minimum.
Show using Calculus, that P is the minimum. [3]

(c) Determine the dimensions of the rectangle from (b). [1]

(d) Determine the area of the rectangle from (b). [1]

26. [11 marks]

The graph of $y = x^3 + 2x^2 + x + 3$ is shown.



(a) Use the second derivative to show that a possible point of inflection exists at $x = -\frac{2}{3}$. [3]

(b) Use a sign test to verify that the point where $x = -\frac{2}{3}$ is, in fact, a point of inflection. [2]

(c) Calculate the equation of the tangent to the curve drawn at the point of inflection. [2]

(d) A conjecture is made that a tangent drawn through a point of inflection will go through at least one turning point.
Use a counter–example to show that this conjecture is false. [4]

27. [11 marks]

The number of bacteria in a certain biological culture at time *t* hours is N(t). At any time *t* the instantaneous rate of increase of N(t) is 4 per cent of N(t) per hour. Initially, when $t = 0$, there are 10,000 bacteria in the culture.

(a) Write down an expression for N(t). [2]

(b) Calculate the number of bacteria after 1 hour, and after 1 day. [3]

(c) Calculate, correct to the nearest hour, the time it takes for the number of bacteria to reach one million. [3]

Suppose $N^*(t)$ is the number of bacteria at time *t* hours in a second culture.
Suppose $N^*(0) = 5000$ and $\frac{dN^*}{dt} = 0{\cdot}08N^*$

(d) Find, correct to the nearest hour, the time *t* at which the cultures have the same number of bacteria. [3]

28. [6 marks]

A crystal of salt, in the shape of a cube, is growing in a solution. At time *t* minutes the length of one side of the cube is closely given by $x = \sqrt{ln(0.1t + 1)}$ where *x* is in millimetres.

(a) Find the rate of increase in its surface area at t = 60 minutes. [3]

(b) The formula above clearly matches the real growth of the crystal with an error of *x* within ±0.01 minutes. Calculate the possible error in calculating the surface area of the crystal at time = 60 minutes. [3]

29. [5 marks]

The cost function in dollars of a new product is given by

$C(x) = \frac{2}{x-3} + ln(x-3)$ for x > 3

where *x* represents the number of units produced.

Determine the number of units which will minimise the cost. [5]

30. [5 marks]

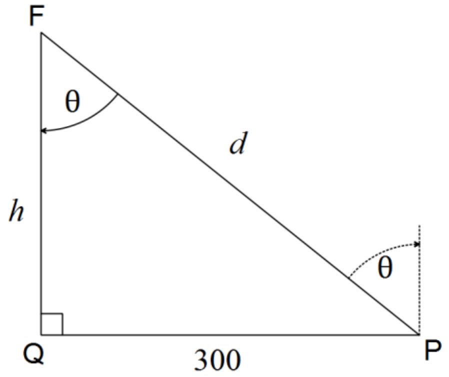
Consider the formula $L = 100\sqrt{1 - \frac{v^2}{a^2}}$ where $a = 3 \times 10^8$ ms^{-1}.

Use the method of small incremental change to calculate the approximate change in *L* when *v* changes from $v = 0.99a$ to $v = 0.999a$. [5]

31. [24 marks]

Light from a flare F shines on a small plate P which lies on a horizontal plane. The intensity I of the light is directly proportional to $\frac{\cos\theta}{d^2}$, where d is the distance from F to P, and θ is the angle of incidence as shown in the diagram.

At time t seconds the flare is h metres above the point Q on the plane, and P is 300 metres from Q.



(a) Show that $I = k \cos\theta \sin^2\theta$ for some constant k. [4]

(b) Evaluate k, given that I = 72 when h = 400. [3]

(c) Find $\frac{dI}{d\theta}$. [3]

(d) Find, to the nearest metre, the height h at which I is greatest. [4]

(e) What is the maximum value of I? [1]

The flare is falling vertically at a constant rate of 5 metres per second.

(f) Show that $\frac{d\theta}{dt} = \frac{1}{60}\sin^2\theta$. [5]

(g) Find $\frac{dI}{dt}$ when the flare is 125 metres above the point Q. [4]

32. [4 marks]

A guy rope, of length 20 metres, is stretched tight from the top of a flagpole and pegged firmly into the ground. The angle of elevation of the top of the pole is measured as 60^o, with a possible error of 0.25 degrees.

Use the incremental formula $\delta y \approx \left(\frac{dy}{dx}\right)\delta x$ to approximate the calculated height of the flagpole. [4]

33. [7 marks]

A one cubic metre block of ice is left out of the freezer and begins to melt. After 5 hours its volume has halved to 0.5 cubic metres. A student decides that this melting process can be modelled by the differential equation,

$\frac{dV}{dt} = kV$, where V is the volume of ice, in cubic metres, remaining after t hours.

(a) Use this relationship to find an equation for the volume, V, in terms of t, giving the value of k to 4 decimal places. [4]

(b) According to this equation what volume of ice will remain after 12 hours? [1]

(c) This equation does not model this situation well. Why not? [2]

Chapter 5: Anti–Differentiation

This chapter includes questions on

Theory of Integration
Primitives
Definite Integration
Fundamental Laws of Calculus
Exponential Functions
Trig Functions
Log Functions

Fundamental Theorem

1. [12 marks]

Find each of the following.

(a) $\frac{d}{dx}\int_0^x \sqrt{x^2-3}\ dx$ [1]

(b) $\frac{d}{dt}\int_{-1}^t \frac{1}{(x-3)^3}\,dx$ [1]

(c) $\frac{d}{dy}\int_0^{2y} (a^2+2)^3\ da$ [2]

(d) $\frac{d}{dq}\int_{q^2}^3 \sin q^3\ dq$ [3]

(e) $\int_0^x \left(\frac{d}{dx}(3x^3+1)\right)\ dx$ [2]

(f) $\int_t^{\frac{\pi}{2}} \left(\frac{d}{dy}(\sin^2 y)\right)\ dt$ [3]

2. [2 marks]

$F'(x)$ if $F(x) = x^3\int_0^x \sqrt{1+t^3}\,dt$

3. [6 marks]

Suppose that $f(x) = (1+x^2)^{\frac{1}{2}}$ and $g(x) = 1 + \frac{x^2}{2}$

(a) Evaluate the derivatives $f'(x)$ and $g'(x)$ and show that $f'(x) \le g'(x)$ for $x \ge 0$. [4]

(b) Using the fundamental theorem of calculus and the inequality in (a), or otherwise, deduce that $f(x) \le g(x)$ for $x \ge 0$ [2]

4. [3 marks]

Show that $G'(x) = \frac{1}{x}\left[\left(x^3 - 4\right)^{\frac{1}{2}} - \int_2^x \left(t^3 - 4\right)^{\frac{1}{2}} dt \right]$

if $x.G(x) = \int_2^x \left(t^3 - 4\right)^{\frac{1}{2}} dt$. [3]

5. Let $G(x) = \int_2^x g(u)\,du$

where $g(u) = \begin{cases} ku^2 - \frac{7}{2} & u \le 4 \\ \frac{1}{u - 2} & u > 4 \end{cases}$

Find the value of k, given that $G'(x) = g(x)$ for all $x \ge 2$. [4]

Integration

1. [9 marks]

(a) Evaluate $\displaystyle\int_1^2 \frac{4x^3 - 2x}{x}\, dx$ [3]

(b) Determine y, the primitive function, if $\dfrac{dy}{dx} = y' = \sqrt{x} + \dfrac{1}{\sqrt{2x}}$ [3]

Give your answer in the original format, with rational denominators.

(c) (i) Differentiate $(2x^7 - x^4)^4$. [1]

(ii) Hence, or otherwise, determine $\displaystyle\int 5(2x^7 - x^4)^3 (14x^6 - 4x^3)\, dx$ [2]

2. [7 marks]

Determine each of the following. Leave your answers with positive indices.

(a) $\displaystyle\int (4x^3 + 2\sqrt[3]{x} - \frac{4}{x^3})\, dx$ [3]

(b) $\displaystyle\int \frac{5 - x}{2x}\, dx$ [2]

(c) $\displaystyle\int (4e^{1-3x})\, dx$ [2]

3. [8 marks]

(a) Determine the value of each of the following ***exactly***.

(i) $\displaystyle\int_0^{-3} (x - 1)\, dx$ [2]

(ii) $\displaystyle\int_0^1 (e^{2x} + e^{x+1})\, dx$ [3]

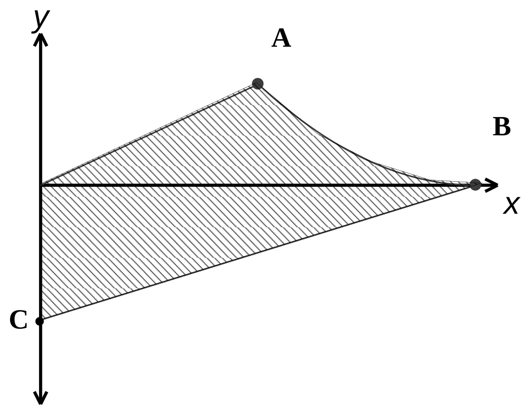
(b) Differentiate each of the following with respect to x.

(i) $\displaystyle\int_0^x \sqrt{t^2 - 4t}\, dt$ [1]

(ii) $\displaystyle\int_{-2}^{3x^2} \frac{5}{g^2 - 3}\, dg$ [2]

4. [8 marks]

Consider the following graphs of the functions $y = \frac{3x}{2}$, $y = \frac{3}{2}(x-2)^2$ and $y = x - 2$.



Determine:

(a) the coordinates A, B and C. [3]

(b) definite integrals which, when added, will give the area of the sail cloth (shaded). (Do not evaluate) [3]

(c) the area, correct to two decimal places, of the sail cloth. [2]

5. [6 marks]

Determine each of the following. Express your answers with positive indices.

(a) $\int x\left(4 - x^2\right) dx$ [2]

(b) $\int \frac{3}{(2t - 1)^2} dt$ [2]

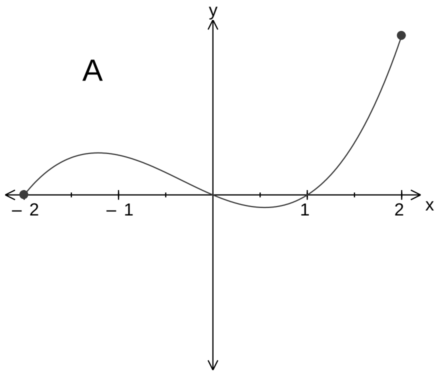
(c) $\int \frac{e^{2x} + e^{-3x}}{e^x} dx$ [2]

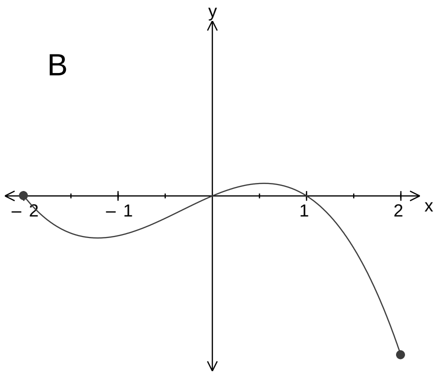
6. [6 marks]

Two functions; $f(x)$ and $g(x)$ exist such that:

$$\int_{-2}^{0} f(x)\,dx = 2 \quad \text{and} \quad \int_{1}^{0} g(x)\,dx = -1$$

(a) Determine which of the following functions are $f(x)$ and $g(x)$. [2]





A = ________ and B = __________

(b) Answer true or false for each of the following. [4]

(i) $\int_{-2}^{2} f(x)\,dx > \int_{-2}^{2} g(x)\,dx$

(ii) $\int_{0}^{2} f(x)\,dx > \int_{-2}^{2} f(x)\,dx$

(iii) $\int_{-2}^{2} g(x)\,dx > 0$

(iv) $\int_{-2}^{2} g(x)\,dx + \int_{-2}^{2} f(x)\,dx = 0$

7. [6 marks]

A function $y = f(x)$ passes through the point (1, –2). The tangent has a slope of –3 at that point. The associated second derivative is given by:

$$f''(x) = \frac{2}{\sqrt{3 - 2x}}$$

Determine the function $f(x)$. [6]

8. [6 marks]

$g(x)$ is a function such that $g(-1) = 4$ and $g'(-1) = 2$.
$f(x)$ is a function such that $f(-1) = f'(-1) = 3$.

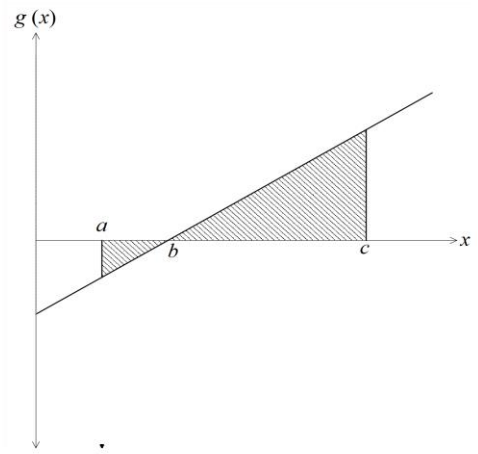
Determine ;

(a) $P'(-1)$ where $P(x) = \dfrac{f(x)}{g(x)}$ [3]

(b) $R'(-1)$ where $R(x) = f(x) \, . \, (g(x))^2$ [3]

9. [6 marks]

A function *g(x)* is sketched below.



(a) State the integral which represents the area trapped between the function *g(x)* and the *x*–axis. [2]

(b) Given that $\int_b^a g(x)\,dx = 4$ and $\int_b^c g(x)\,dx = 6$ determine ;

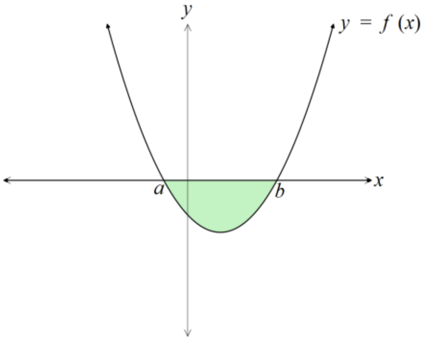
(i) $\int_a^b g(x)\,dx$ [1]

(ii) $\int_a^c g(x)\,dx$ [1]

(iii) $\int_a^c [g(x) - 2x]dx$ [2]

10. [6 marks]

(a) The shaded region below has an area of 5 square units.



Evaluate $\int_a^b 3f(x)\ dx$ [2]

(b) $\int_p^q 1\ dx = 7$ and $\int_1^p x\ dx = 4$

Determine the values of p and q. [4]

11. [6 marks]

The gradient function of $y = f(x)$ is given as $f'(x) = e - 3x^2 + 4x$.

The point P$(2, 2e - 1)$ lies on $f(x)$.

Determine, ***exactly***,

(a) $f(x)$ [2]

(b) the equation of the tangent to the curve $y = f(x)$ at the point P. [2]

(c) the co–ordinates of any points of inflection on $y = f(x)$ [2]

12. [7 marks]

Determine the value of each of the following ***exactly***.

(a) $\int_0^1 (x^2 - \sqrt{x} + e^{2x})\, dx$ [3]

(b) $\int_1^2 \left(\frac{3}{11 - 5x} + \frac{14x}{7x^2 + 2} \right) dx$ [4]

13. [9 marks]

Determine each of the following. Leave your answers with positive indices.

(a) $\int (4x^3 + 2\sqrt[3]{x} - \frac{4}{x^3})\, dx$ [3]

(b) $\int \frac{5 - x}{2x}\, dx$ [2]

(c) $\int \left(4e^{1 - 3x} - \frac{x^2}{x^3 + 1} \right) dx$ [4]

14. [7 marks]

Determine the value of each of the following ***exactly***.

(a) $\int_1^e \left(\frac{1}{2} - \frac{1}{2x} \right) dx$ [3]

(b) $\int_0^{\pi} \left(ex^{e-1} + \frac{25x^4}{15x^5 + e} \right) dx$ [4]

15. [3 marks]

(a) Determine $\frac{d}{dx}(\sin^4 ax)\, dx$. [1]

Hence determine:

(b) $\int \sin^3 ax \cos ax\, dx$ [2]

16. [8 marks]

Evaluate the following integrals:

(a) $\int_0^{\frac{\pi}{2}} (\sin x + \cos x)^2 \, dx$ [4]

(b) $\int_0^{\pi} (\sin x - \cos x)^2 \, dx$ [4]

17. [9 marks]

(a) Verify that $y = \frac{\sin x}{\cos x}$ is a solution of the equation

$$\frac{d^2 y}{dx^2} = 2y + 2y^3$$ [4]

(b) Find the function f, defined for all real x for which
$f'(x) = \sin x \cos x$ and $f(\pi) = -1$ [4]

(c) Consider the following statements

1. If $y = \sin^2 x$ then $\frac{dy}{dx} = 2\sin x \cos x = \sin 2x$

Therefore $\int \sin 2x \, dx = \sin^2 x + \text{constant}$

2. $\int \sin 2x \, dx = \frac{-\cos 2x}{2} + \text{constant}$

from first principles [1]

So from statements 1. and 2. we have

$$\sin^2 x = \frac{-\cos 2x}{2} + \text{constant}$$

Is this true? Comment

18. [5 marks]

(a) Differentiate $x\left(\cos x - e^{-x}\right)$ with respect to x. [2]

(b) By making use of your answer in part (i) find $\int x\left(\sin x - e^{-x}\right) dx$ [3]

19. [5 marks]

A heavy chain, suspended between 2 posts, a metres apart, has a curve shape defined by

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$$

(a) Find $\frac{dy}{dx}$. [3]

(b) Find the length of the chain where $length = \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ [2]

20. [3 marks]

Let $y = \int_1^{x^2} \frac{dt}{1 + t^2}$

Find

(a) $\frac{dy}{dx}$. [2]

(b) $y'(2)$. [1]

21. [6 marks]

(a) Determine the derivative of the following function.

$y = x^2 \log_e x$ [2]

Hence, or otherwise,

(b) determine the following $\int 2x \log_e x \; dx$. [4]

22. [7 marks]

(a) The function $g(x)$ passes through the point (4, 16).
If $g'(x) = \frac{2x + 5}{\sqrt{x}}$, determine $g(x)$. [3]

(b) The gradient function $\frac{dy}{dx}$ is inversely proportional to x for all values of (x, y)
If both the gradient function and $y(x)$ pass through (1, 2), determine $y(x)$. [4]

Chapter 6: Applications of Integrals

This chapter includes questions on

Approximation to Areas
Areas between curves
Rectilinear motion

1. [6 marks]

A particle moves in rectilinear motion with a velocity of 7 m/s as it passes through a fixed point O.

t is the number of seconds since passing through O. Acceleration a is defined as $a = mt - n$, where m and n are constants.

When $t = 1$, the velocity is 12 m/s, and when $t = 7$ the particle is instantaneously at rest.

(a) Calculate the values of m and n. [3]

(b) Determine the expression for the velocity as a function of time. [1]

(c) Determine when and where the maximum velocity is attained. [2]

2. [8 marks]

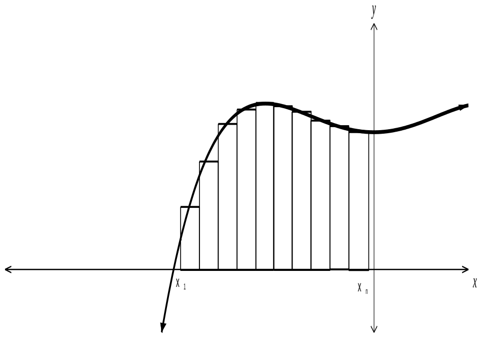
At any point (x, y) on a particular curve, $\dfrac{d^2y}{dx^2}$ is a quadratic function of x.

(a) From the information about $\dfrac{d^2y}{dx^2}$,

(i) what is the maximum number of stationary points for this original curve? [1]

(ii) what is the maximum number of points of inflection for this original curve? [1]

(b) Part of the curve looks like this. The rectangles can be used to approximate the area under the curve.



(i) What is represented by the expression $\lim\limits_{n \to \infty} \left(\sum_{i=1}^{n} (f(x_i) \times \Delta x) \right)$? [1]

(ii) Simplify the expression in (i), using Calculus symbols. [1]

At any point (x, y), a particular curve is defined by $\dfrac{d^2y}{dx^2} = 1 - x^2$.

A tangent drawn to the curve at (1, 1) has equation $y = 2 - x$.

(c) Determine the equation of the curve. [4]

3. [10 marks]

A particle is moving in rectilinear motion with acceleration a at any time t, in m s^{-2}, given as

$$a = 6t - 1$$

Initially the particle is at the origin with a velocity of -2 m/s.

Determine:

(a) the velocity of the particle at any time t. [2]

(b) when the particle is again at the origin. [3]

(c) the minimum velocity of the particle. [2]

(d) the total distance travelled by the particle in the first three seconds. [3]

4. [6 marks]

The air in a hot air balloon is being inflated such that the rate of change of its volume at any time t, minutes, is given as:

$$\frac{dV}{dt} = 3t^2 - 2t$$

If initially the balloon has 3 m^3 of air in it, determine:

(a) the rate of change in volume when $t = 1$. Explain the meaning of this. [2]

(b) for what values of t the volume is increasing. [2]

(c) the volume of the balloon after five seconds. [2]

5. [4 marks]

Lisa and Kym are sisters who sell computer software.

Lisa has structured her company such that the profit, in hundreds of dollars, on the sale of x items of software is given as $L(x) = 9x^2 - x^3$, while Kym's company has marginal profit, in hundreds of dollars, on the sale of x items of software, given as $K'(x) = 100 + 20x - 3x^2$.
Kym's initial costs are $1 000.

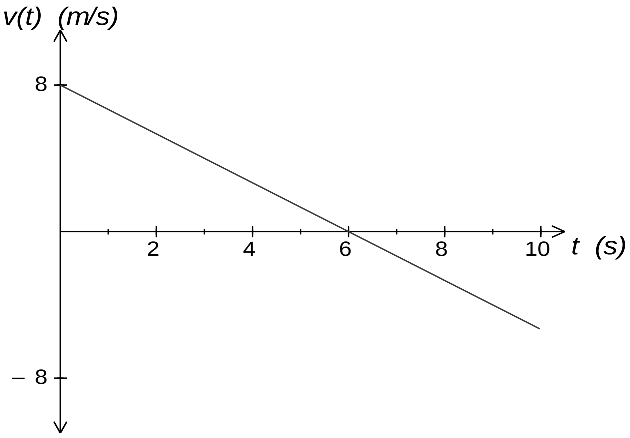
Determine:

(a) Lisa's maximum profit. [1]

(b) Kym's maximum profit. [3]

6. [7 marks]

The following graph is the velocity – time graph of a particle undergoing rectilinear motion.



Determine:

(a) the acceleration of the particle at $t = 6$. [1]

(b) the speed of the particle when $t = 10$. [2]

(c) the total change in displacement over the first ten seconds. [2]

(d) the total distance travelled by the particle in the first ten seconds. [2]

7. [6 marks]

Consider the following.

$$f'(x) = 2\sqrt{x} - \frac{3}{x} \quad \text{and } f(1) = 0$$

(a) Determine the coordinate(s) of any stationary points on the graph of $y = f(x)$. [3]

(b) Discuss the nature of the point(s) found in (a), using *Calculus* methods. [3]

8. [9 marks]

A particle is moving in rectilinear motion such that the velocity, m/ min at any time t, minutes, is given as:

$$v(t) = 4t^3 - 12t^2 + 6t + 4$$

If the particle begins 4 metres to the left of the origin, determine:

(a) when the particle stops. [2]

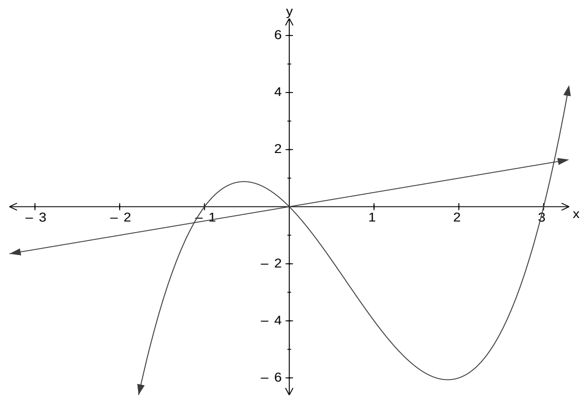
(b) the displacement at any time t. [2]

(c) the acceleration as the particle passes through the origin. [3]

(d) the total distance travelled by the particle in the first two minutes. [2]

9. [6 marks]

The functions $f(x) = x^3 - 2x^2 - 3x$ and $g(x) = \frac{1}{2}x$ are sketched below.



Determine, correct to two decimal places:

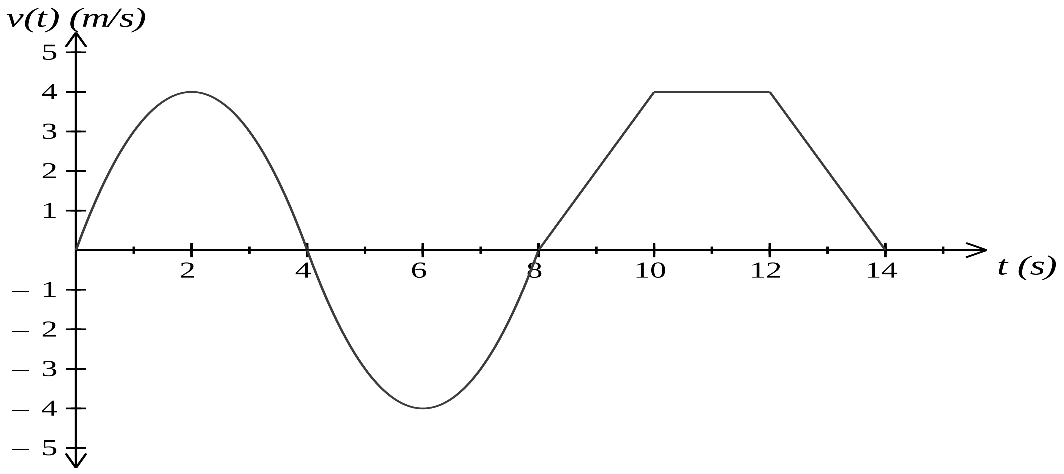
(a) $\int_{-1}^{0} [f(x) - g(x)]\, dx$ [2]

(b) $\int_{0}^{3} [f(x) - g(x)]\, dx$ [2]

(c) the area enclosed between $y = f(x)$, $y = g(x)$, and the lines $x = -1$ and $x = 3$ [2]

10. [8 marks]

A particle undergoes rectilinear motion such that the velocity– time graph is given below.



Determine:

(a) the acceleration of the particle at:

(i) $t = 2$. [1]

(ii) $t = 9$ [1]

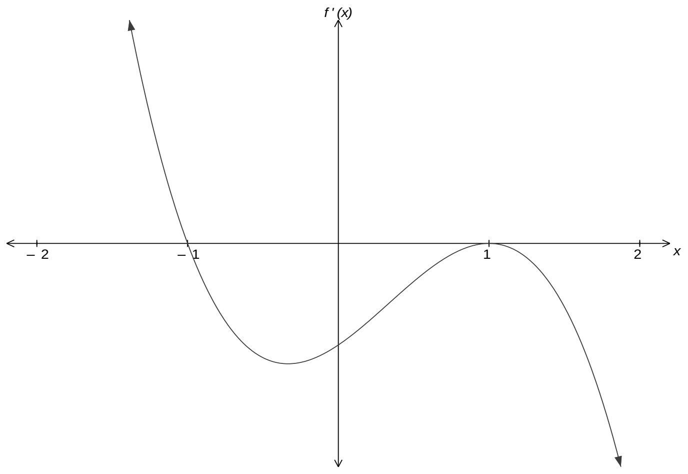
(b) the speed of the particle when $t = 6$. [2]

(c) the total change in displacement over the first ten seconds. [2]

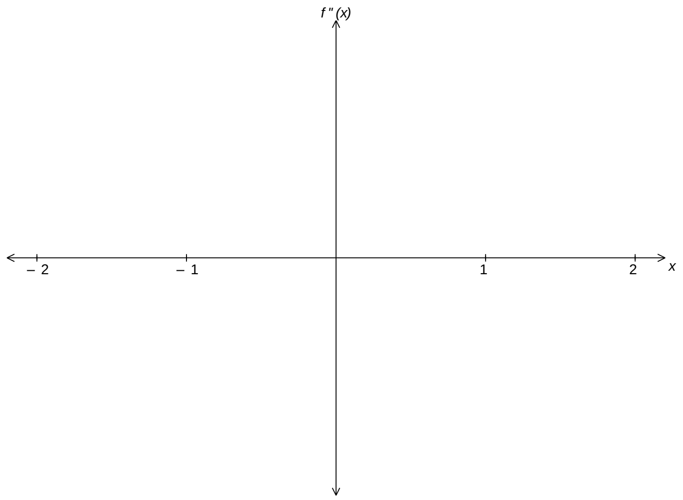
(d) the total distance travelled by the particle in the <u>last</u> six seconds. [2]

11. [5 marks]

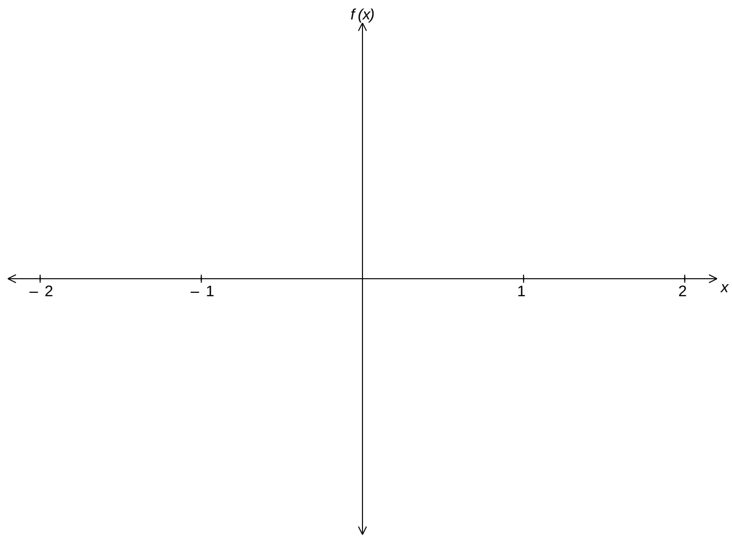
The graph of $y = f'(x)$ is drawn below.



(a) Sketch a possible graph of $y = f''(x)$ [2]

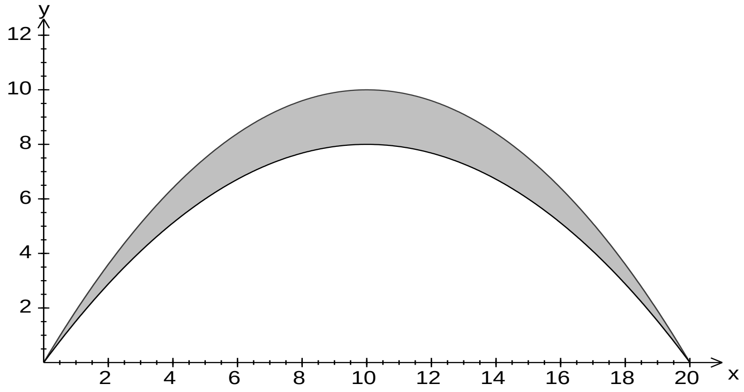


(b) Sketch a possible graph of $y = f(x)$. [3]



12. [10 marks]

The roof of a stage is constructed using the plans drawn below. The distances are in metres and the horizontal axis represents the floor of the stage. The two parabolas shown represent supports.



The equations of the supports are given by:

$$f(x) = -0.1x(x-20) \text{ and } g(x) = -0.08x(x-20)$$

Determine:

(a) the distance between the supports at the highest point above the stage. [2]

(b) the distance between the supports 4 metres from the right hand side of the stage. [2]

A tight wire is to be connected from the origin to the lower support at a point where $x = 8$.

(c) Determine the equation of this tight wire. [2]

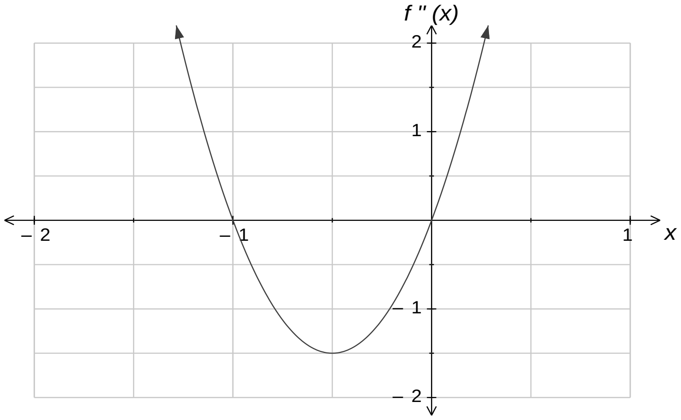
(d) State the integral which would be used to determine the area between the supports above the stage.(i.e. the shaded region) [2]

Hence, or otherwise;

(e) determine the area between the supports, correct to two decimal places. [2]

13. [10 marks]

The graph of the function $y = f''(x)$ is drawn below.



(a) Give the x value(s) of any points of inflection on $y = f(x)$. [2]

(b) Determine the function $f''(x)$. [2]

Stationary points of the function $y = f(x)$ occur at the point (0, 1) and where $x = -1.5$

(c) Are any of the points of inflection found in (a) horizontal? Explain. [2]

(d) Determine $y = f'(x)$. [2]

(e) Determine the function $y = f(x)$. [2]

14. [5 marks]

The rate of change of cost, in dollars, of producing x tonnes of fertiliser is such that:

$$\frac{dC}{dx} = 30x - 30\sqrt{x}$$

Determine:

(a) the rate at which cost is changing at the instant when 4 tonnes of fertiliser are being produced. [1]

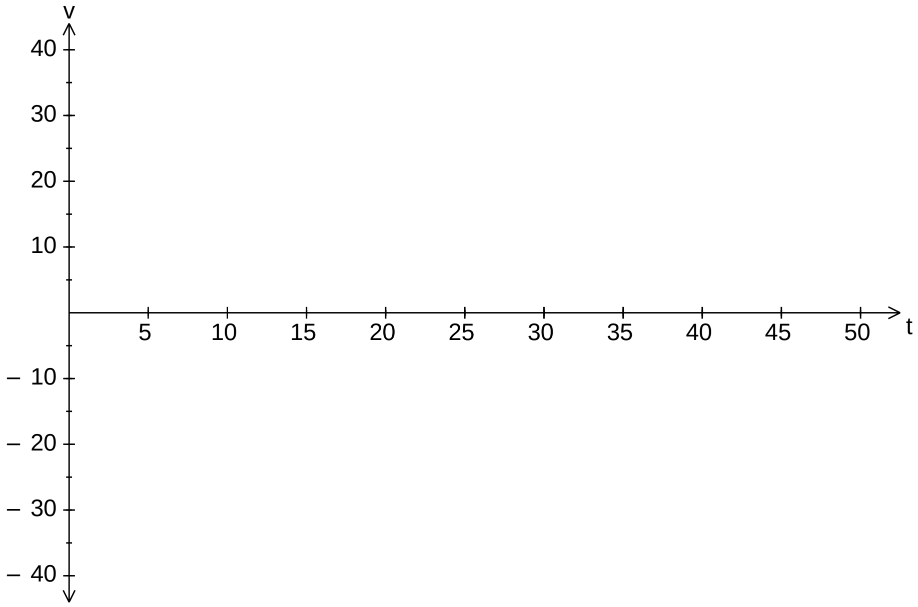
(b) the extra cost involved in producing 16 tonnes instead of 9 tonnes of fertiliser. [2]

(c) an integral which displays the actual cost of producing the 25^{th} tonne of fertiliser. (Do not evaluate.) [2]

15. [7 marks]

A particle accelerates from rest at 2 ms^{-2} for 10 seconds. It maintains this speed attained for a further 10 seconds. It then decelerates at 1 ms^{-2} until stopped.

(a) Sketch this motion of the vehicle on the axes below. [4]



Instantaneously, the particle turns and returns to its starting position in 10 seconds at a constant speed of ***k*** m/s.

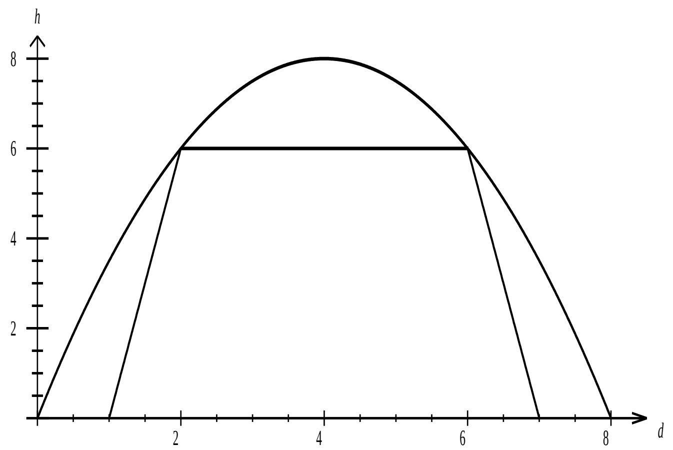
(b) Determine the value of ***k***. [2]

(c) Determine how far the particle travelled in total. [1]

16. [9 marks]

A new tunnel is being built. The outer frame of the tunnel is given by $h = -\frac{1}{2}d^2 + 4d$, where h is the height of the tunnel in metres above the road and d is the cross–sectional width of the tunnel in metres.

The frame of the tunnel, showing the internal supports is sketched below.



(a) State the equations and the domain that represent the three internal steel support beams. [3]

(b) Write an integral that represents the area trapped between the steel supports and the outer frame. [2]

(c) Calculate the area found in (b). [1]

(d) The area trapped between the three support beams and the outside of the tunnel is to be filled with cement costing \$9.80/m^3. If the tunnel is 1.5km long, calculate the cost of the cement to the nearest \$100. [3]

17. [7 marks]

A particle is projected upwards in rectilinear motion from the ground at 16ms^{-1}.
The acceleration due to gravity working **against** the particle is constant at 9.8ms^{-2}.
(i.e. $a = -9.8$).

Determine;

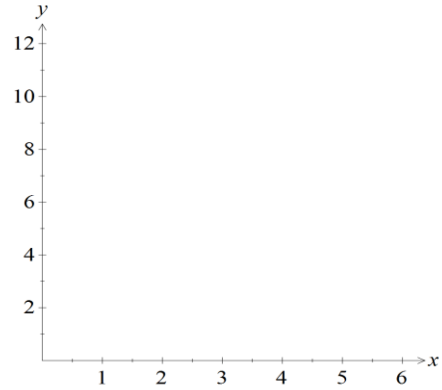
(a) the velocity equation of the particle. [2]

(b) the distance travelled by the particle in the first 2 seconds. [3]

(c) when the particle returns to the ground **and** the speed of its impact at this time. [2]

18. [8 marks]

(a) Sketch the function $y = 2x + \dfrac{4}{x}$ on the axes below in the domain $0 < x \le 5$ [2]



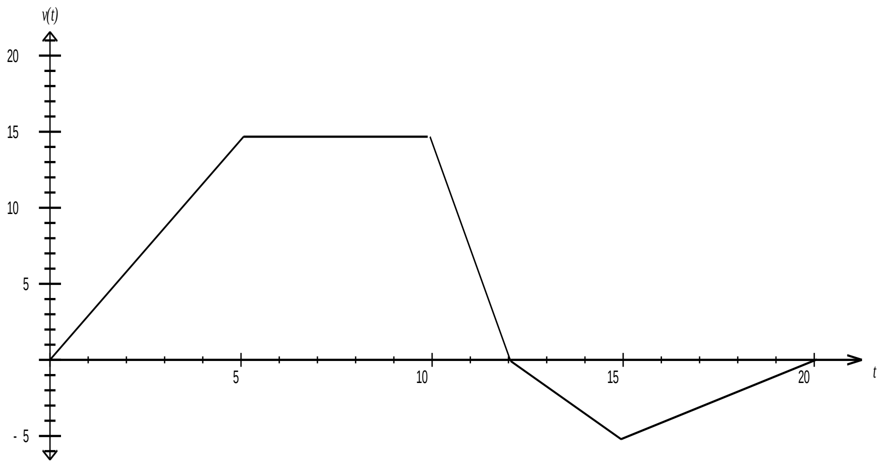
(b) State the integral that represents the area of the region bounded by this curve, the x – axis and the lines $x = 0.5$ and $x = 4$. [2]

(c) Hence, calculate the area of this region to 2 d.p. [2]

(d) If you were asked to find $\displaystyle\int_{0.5}^{4} \left(-2x - \frac{4}{x}\right) dx$, describe how you could utilise your answer from (c) in order to gain your solution. [2]

19. [3 marks]

The velocity–time graph shown below represents a particle, P, moving in a straight line from rest at the origin O. Initially P is moving to the right of the origin.

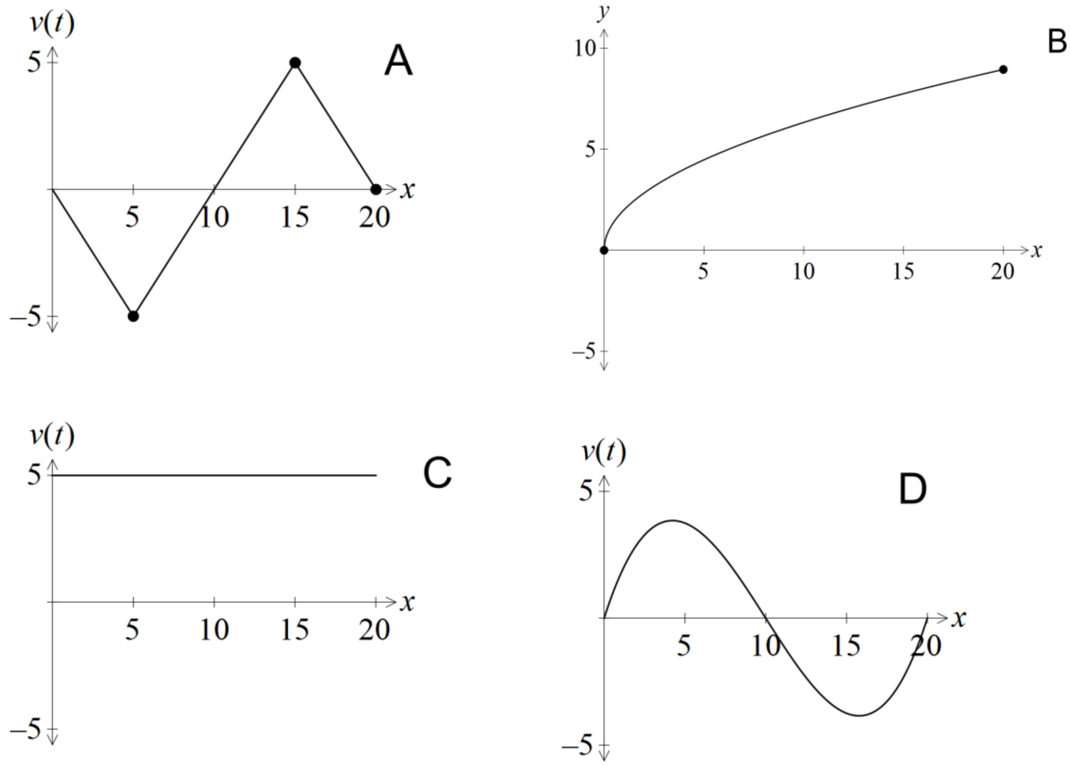


(a) At what time(s) does the particle change direction? [1]

(b) Find the position of P with relation to the origin at $t = 15$ seconds. [2]

20. [5 marks]

The graphs below display the velocities of 4 particles. Each particle moves along the x–axis for 20 seconds.



Determine which graph(s) shows a particle:

(a) having only a positive change of displacement. [2]

(b) where acceleration is zero throughout the journey. [1]

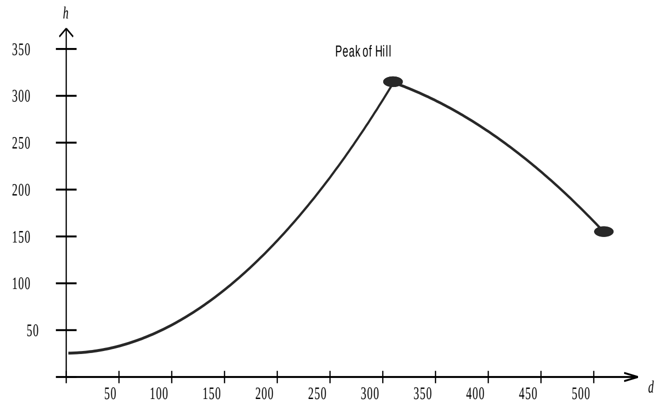
(c) which is the furthest away from its starting point. [1]

(d) which ends up back at its starting point. [1]

21. [12 marks]

The cross–section of the Mundabiddi Trail is being analysed by Outdoor Education students in preparation for their upcoming cycling camp. The trail to the top of the Mundabiddi hill is formed by two sections. These sections are represented in the graph below. h is the height above sea level and d is the distance from the base camp. All measurements are in metres.

$h = 0.003d^2 + 25$ and $h = -0.002d^2 + 0.84d + 246$. (Note : The track is 510m long.)



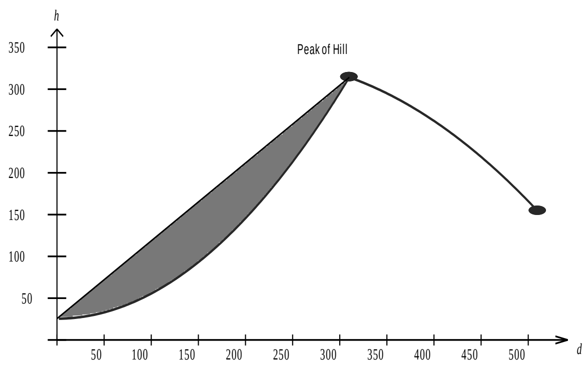
Determine:

(a) the height of the base camp above sea–level. [1]

(b) the co–ordinates of the peak of the hill. [2]

(c) the definite integral which when evaluated would give the cross–sectional area of the hill. (Do not evaluate.) [3]

The tourism branch wishes to make the trail more tourist friendly by filling the hill in with sand to make it easier to climb, as shown below.



(d) Determine the equation of the line that represents the new "user–friendly" path. [2]

(e) If the sand costs \$2 per 5m^3, how much will it cost the tourism branch to fill in the hill? (Note. The hill is 150 metres in width.) [4]

22. [13 marks]

Particle A is moving with uniform negative acceleration and has an initial velocity of $10.5\ ms^{-1}$. When $t = 5$ seconds the particle is travelling at $3\,ms^{-1}$ and is 10 m to the right of the origin.

(a) Show that the velocity of the particle at any time t is

$$v(t) = -1{\cdot}5t + 10{\cdot}5$$ [2]

Calculate ;

(b) the time(s) when the particle is at rest. [2]

(c) the initial position of the particle. [3]

(d) the time(s) when the particle is 8m from the origin. [4]

(e) the total distance travelled by the particle until it is at rest. [2]

23. [3 marks]

The curve, $y = f(x)$ is given by $f''(x) = 4(4x + 1)\, e^{2x^2 + x}$.

If $f'(0) = 1$, determine the exact gradient of the tangent to the curve where $x = 1$. [3]

24. [6 marks]

An object travelling in a straight line comes to *rest* in three stages: a 10 second period of constant deceleration, followed by a 10 second period of constant speed and, finally, a 5 second period of constant deceleration. If the total distance travelled during this 25 second period is 19.5 metres and the constant deceleration in the first 10 seconds is equal to that applied in the final 5 seconds, find the initial velocity of the object. [6]
Hint: a sketch of the velocity–time relationship would be helpful.

25. [10 marks]

The cost *per hour* of running a transport vehicle is given by the function,

$$C = \$(\frac{v^2}{64} + 81), \quad \text{where } v \text{ is the speed in km/h, and } 0 < v < 100.$$

(a) If the vehicle makes a 100 km journey, show that the *total cost*, of the journey is given by $T = \frac{25v}{16} + \frac{8100}{v}$

Hint: Let t = *total time of journey, and* $v = \frac{distance}{time}$. [2]

(b) Use Calculus techniques to find the speed at which the *total cost* of the journey is minimized. [4]

(c) Will this result be different for a 200km journey? Explain. [4]

26. [7 marks]

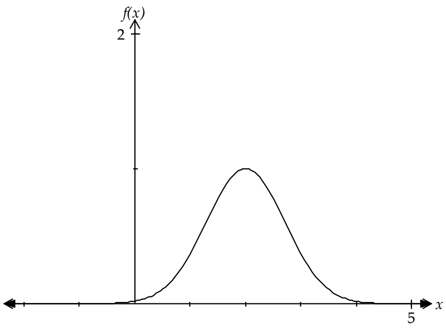
Becky is riding her motorcycle along a straight country road. Three signposts A, B and C are evenly spaced along the side of this road such that AB = BC = 90 metres. It takes her 6 seconds to travel between A and B and 4 seconds to travel between B and C.
If she is travelling with constant acceleration, use calculus techniques to find:

(a) the constant acceleration. [5]

(b) the speed at which she is travelling as she passes C. [2]

27. [7 marks]

The graph below shows the sketch of $y = e^{-(x-2)^2}$.



(a) Write an integral expression that will give the area of the region enclosed by the curve, the x–axis, the y–axis and the line $x = 2$, and find the approximate area of this region. [3]

(b) Find, correct to 4 decimal places, the area of the region between the curve and the x–axis which lies to the right of the line $x = 2$.
Show clearly the expressions you used to evaluate this area. [4]

28. [5 marks]

The variables P and x are related by the formula, $P = 2\pi\sqrt{\dfrac{x}{g}}$, where g is a constant.

Use the incremental formula to show that a 1% increase in x will result in a 0.5% increase in P.

29. [9 marks]

The volume of a meteorite decreases once it enters the Earth's atmosphere.
Its distance from the Earth's surface, s kilometres, is related to its volume, V cubic metres, via the equation, $V = 6 + 50\sqrt{s}$

(a) Find $\dfrac{dV}{ds}$ when the meteorite is 36 km above the Earth's surface. [2]

The meteorite is spherical in shape, and its approximate distance, s, from the Earth's surface, t minutes after entering the Earth's atmosphere is given by $s = 100 - 0.5t^{\frac{3}{2}}$.

(b) Find the rate of change of the volume of the meteorite with respect to time when $s = 36$. [4]

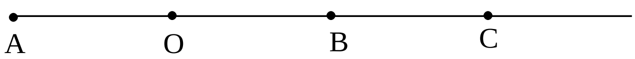
(c) Find the speed at which the meteorite hits the Earth's surface. [3]

30. [8 marks]

A curve has equation $4x^2 + y^2 = 4$. Find, exactly, the minimum and maximum distance from the point (1, 0) to this curve. Justify your answer. [8]

31. [14 marks]

A particle P travels along a straight line. It starts from a fixed point O, moves left, and then reverses direction at A. It then moves through O, passing through B and C in turn. B and C are points to the *right* of O such that OB = BC = AO.



It's velocity, after t seconds is given by $v = (t-8)^{1/3}$ cms^{-1}.

(a) Sketch the graphs of v against t and a against t, where a is the particle's acceleration after t seconds. Label your graphs clearly. [4]

You may use any feature available on your graphic calculator to answer the following questions. In each case, however, you need to show the mathematical expressions used and any calculus operations applied to these expressions.

(b) Show that the distance AO = 12 cm. [2]

(c) Show using calculus methods that P takes 21.45 seconds to travel from A to B. [4]

(d) Given that the particle is at C when $t = 23.91$ seconds, find the magnitude of the average acceleration between B and C. Justify your answer. [4]

32. [9 marks]

A particle is undergoing rectilinear motion. Its displacement from an origin, O, at time t seconds, is given by $y = t^3 - 9t^2 + 24t - 16$ where $0 \le t \le 6$ seconds.

(a) Determine the number of times the particle is at O. [1]

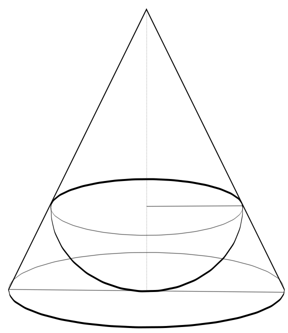
(b) Use calculus techniques to show that the particle changes its direction of motion twice. Your solution needs to include a sign diagram or a second derivative test. [4]

(c) The particle travels a total distance of 22 metres in the first q seconds.
Use a calculus technique to find q. [4]

33. [8 marks]

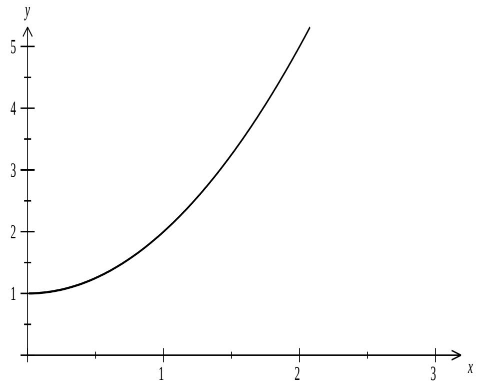
A solid hemisphere rests under a hollow cone on a horizontal table. The hemisphere has fixed radius R. The cone is regular in shape of height h and base radius r. The outer rim of the horizontal surface of the hemisphere is in complete contact with the inner surface of the cone. Disregard the thickness of the material used to make the hollow cone.

Use a calculus method to find the dimensions of the cone (in terms of R) if its capacity is to be maximised. [8]



34. [8 marks]

The curve drawn below is the function $y = x^2 + 1$.



Use the method of rectangles of width 1 unit to find:

(a) an over−estimate of the area enclosed between the x−axis and the curve within the domain $0 \le x \le 2$. Show this on a diagram. [3]

(b) an under−estimate of the area enclosed between the x−axis and the curve within the domain $0 \le x \le 2$. Show this on the same diagram as (a). [3]

Hence,

(c) give an accurate estimate of the area enclosed between the x−axis and the curve within the domain $0 \le x \le 2$. [1]

(d) How can you improve the accuracy of using this method to determine the area? [1]

35. [7 marks]

The acceleration of a particle t seconds after starting from rest is (2t – 1) m/sec^2.

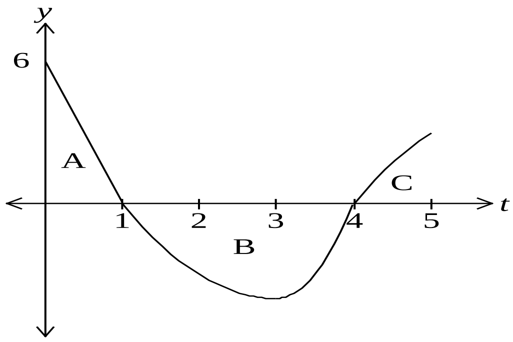
(a) Prove that the particle returns to the starting point after 1.5 seconds. [4]

(b) Find the distance of the particle from the starting point after a further 1.5 sec. [1]

(c) Find the time, after starting, that the particle reaches 20 m/sec. [2]

36. [5 marks]

An object moves along a straight line for a period of 5 seconds. The graph of its velocity, v metres per second, as a function of time, t seconds is shown below.

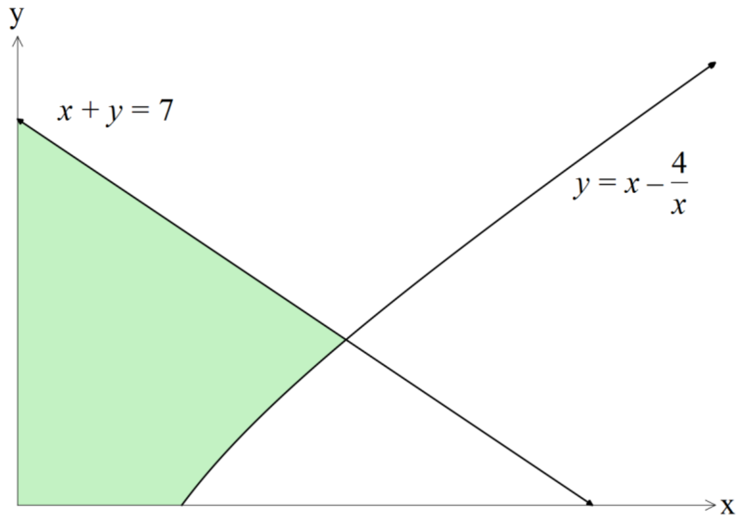


After 5 seconds the object returns to its starting point, after travelling a total of 18 metres. The acceleration is constant during the first second, and $v(0) = 6$.

(a) Find the areas of the regions A, B and C, as shown on the graph. [3]

(b) Use the graph to estimate the time t at which the acceleration is greatest. [2]

37. [7 marks]



The shaded area is bounded by the x and y axes, the line $x + y = 7$ and the curve with equation $y = x - \frac{4}{x}$.

(a) Determine the co–ordinates of A and B. [3]

(b) State an expression, which when evaluated will determine the area of the shaded region. Give the area correct to two decimal places. [4]

38. [3 marks]

Let the volume of a balloon at time t seconds be V $cm.^3$. If the rate of change of ln V (i.e. $\log_e V$) with respect to time is given by

$$\frac{d(\ln V)}{dt} = \frac{k}{\sqrt{t}} \quad , \; k \text{ a real constant}$$

Find V in terms of t.

39. [5 marks]

A particle moves in a straight line with acceleration (a) given by

$$a = \frac{8}{(t+3)^2} \quad m/sec^2$$

If the particle's initial displacement, from rest, is zero, find, correct to one decimal place, the displacement at t = 10 seconds.

40. [6 marks]

An experimental spacecraft starts at rest and travels in a straight line according to an acceleration of $\frac{2t}{t^2+3}$ metres/second^2

(a) Find the velocity of the spacecraft after t seconds. [3]

(b) At what time does the spacecraft reach a speed of 10 metres per second? [3]

41. [5 marks]

A projectile moves in a straight line with an acceleration $\frac{12}{(t+3)^2}$ metres/second^2, where t is the time in seconds. If it starts from rest at time $t = 0$, find the displacement after 9 seconds. [5]

42. [6 marks]

It is known that the temperature (T) at the external surface of insulation varies according to the thickness of the insulation (x) by the equation

$$\frac{dT}{dx} = \frac{k}{x} \quad \text{where k constant and } x \geq 1$$

Given the temperature of the surface of the insulation is -40^oC when x = 1 cm and -15^oC when x = 2 cm, find

(a) T in terms of x. [4]

(b) The thickness of insulation required (to the nearest mm), so that the insulation would be sufficient to prevent ice forming, i.e. at least 0^oC. [2]

43. [7 marks]

A body has initial velocity 20 metres/second and moves in a straight line with acceleration $4e^{-\frac{t}{20}}$ m/sec^2.

(a) Show that as t increases, the body approaches a limiting velocity. [3]

(b) Find the total distance travelled in the first 10 seconds. [4]

44. [4 marks]

The rate of change in temperature °C in a steel bar at any time *t* minutes after an initial reading is given as:

$$\frac{dT}{dt} = \frac{2}{t} + 50 \quad \text{where } t > 0$$

The temperature of the bar one minute after the initial reading is 80°C.

Determine:

(a) the change in the temperature of the bar during the second minute. [2]

(b) the temperature of the bar at any time *t*. [2]

45. [4 marks]

A particle is moving with acceleration k sin t, where k is a real constant, and is at rest at time $t = \frac{\pi}{2}$. Initially the particle was at the point x = k.

Where is the particle at time $t = \frac{3\pi}{2}$? [4]

46. [10 marks]

The velocity, V(t) metres per second, at time t seconds, of an object moving along a straight line for a period of 10 seconds, is given by

$$V(t)=t^2-6t$$

(a) Find the formula for X(t), the displacement at time t, given that X(0) = 0. [2]

(b) At what time t in the period $0 \le t \le 10$ does the object return to its starting point? [2]

(c) At what time t in the period $0 \le t \le 10$ is the object furthest away from its starting point? [2]

(d) At what time t in the period $0 \le t \le 10$ is the object moving towards its starting point? [2]

(e) How far does the object travel in the period $0 \le t \le 10$? [2]

47. [7 marks]

A lift takes 24 seconds to move vertically without stopping from rest at ground floor level to rest at top floor level. When the lift has been moving for t seconds, the speed in m/sec is given by

$$v = \frac{t(576t^{2})}{288}$$

(a) Show that the vertical distance between ground floor level and the top floor level is 288m. [3]

(b) Show that the magnitude of the final deceleration is twice the magnitude of the initial acceleration. [4]

48. [10 marks]

A particle, initially at rest at a displacement of 1 metre from the origin 0, moves in a straight line so that its velocity v metres/sec after t seconds is given by the equation

$$v = 3 \cos t - \sin t$$

(a) If after time t, the displacement is x metres and the acceleration is a metres/sec^2, show that $x + a = 0$. [5]

(b) Find the maximum acceleration of the particle. [3]

(c) Find the time in seconds, when the particle courses through the origin for the first time. [2]

49. [5 marks]

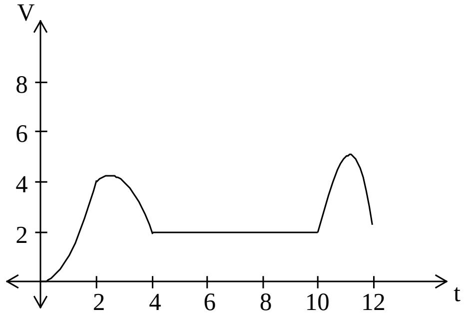
Water is flowing through a pipe at a linearly increasing rate, observed to be 10 m^3/sec initially and then 12 m^3/sec one minute later.
Let v be the volume of water in cubic metres.

(a) Show that $\frac{dv}{dt} = 120t + 600$ m^3/min. [3]

(b) Hence, or otherwise, find the amount of water which passed through the point of observation in this time interval. [2]

50. [12 marks]

The movement of a rock drill into an ore body can be represented by the velocity – time graph below.



The velocity v(t) in metres/min is given by

$$v(t) = \begin{cases} at^2 & 0 \le t < 2 \\ -t^2 + bt - c & 2 \le t < 4 \\ d & 4 \le t < 10 \\ (-t+e)^3 + 8 & 10 \le t \le 12 \end{cases}$$

(a) Determine the values of the constants a, b, c, d and e. [5]
(b) Determine the acceleration of the drill's progress when t = 11 minutes. [2]
(c) How far has the drill travelled from t = 0 and t = 12 minutes? [3]
(d) What is the mean velocity of the drill over the twelve minutes? [2]

51. [10 marks]

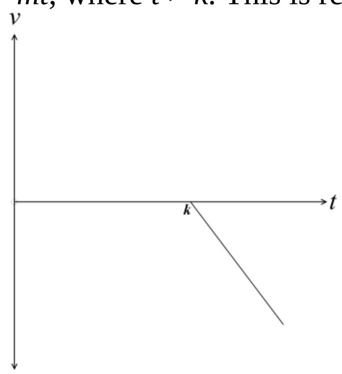
An object moves along a straight line, passing through point O with a velocity of 4 m/s. It has an acceleration, a ms^{-2}, given by $a = 16 - 2t$, t seconds after passing O. The object comes to rest when $t = k$ seconds.

(a) Determine the maximum velocity of the object. [2]

(b) Determine the *exact* value of k. [2]

(c) Sketch the v–t graph , showing and labelling all key points. [4]

After coming to rest, the object travels for a further three seconds with a velocity given by $v = n - mt$, where $t > k$. This is represented on the axes given below.



(d) State the expression, in terms of k, which represents the distance travelled in that three second period. [2]

Chapter 7: Discrete Random Variables

This chapter includes questions about

Discrete Random Variables
Change of Origin and Scale
Uniform Distribution
E(X) and Variance of a DRV

1. [3 marks]

Imagine a weighted die with 8 faces for which the probability of getting any particular number is directly proportional to that number. That is, for n = 1, 2, 3, 4, 5, 6, 7, 8, the probability of getting the number *n* is equal to *kn*, for some constant *k*.

Calculate the value of the constant *k*.

2. [7 marks]

A discrete random variable X takes integral values between 0 and 6 inclusive with probability given by the probability density function

$$P(X = x) = k - \frac{|3 - x|}{16}$$

(a) Find the value of k. [4]

(b) The expected value E(X) of the random variable X is given by

$$\sum_{x=0}^{6} \left(x . P(X = x)\right)$$

Calculate E(X). [3]

3. [6 marks]

A die with faces marked 1 to 6 is loaded in such a way that the probability of a face, marked n, turning up is given in the table below.

n	1	2	3	4	5	6
P(n)	$\frac{1}{20}$	$\frac{3}{20}$	$\frac{6}{20}$	$\frac{6}{20}$	$\frac{3}{20}$	$\frac{1}{20}$

Find the probability that the outcome of a throw of the die is

(a) an odd number. [2]

(b) a number greater than 2. [2]

(c) an even number, given that a number less than 5 is thrown. [2]

4. [10 marks]

Suppose that a die with faces marked 1 to 6 is loaded in such a way that for $n = 1, 2, \ldots 6$, the probability of a face marked n turning up when the die is thrown is kn.

(a) Show that k is $\frac{1}{21}$. [2]

Find the probability that the outcome of a throw of the die is

(b) a 3, [2]

(c) an odd number, [2]

(d) a number less than 5, [2]

(e) an odd number, given that a number less than 5 is thrown. [2]

5. [6 marks]

In a class of 20 students the probability of there being X girls is given by P(X = x).

Part of the probability distribution of the variable X is given in the table.

Use the table to find :

(a) P(X = 4) [2]

(b) P(X ≥ 16) [2]

(c) P(X ≥ 5) [2]

x	P(X = x)
0	0·0000
1	0·0000
2	0·0002
3	0·0011
4	0·0046
.	.
.	.
.	.
15	0·0148
16	0·0046
17	0·0011
18	0·0002
19	0·0000
20	0·0000

6. [6 marks]

Outcome	1	2	3	4	5	6
Probability	x	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

The table gives the probability distribution for the score obtained when a loaded die is thrown once.

(a) Find the value of x. [3]

The die is thrown twice and the total score is recorded.

(b) Find the probability of obtaining a total score of 6. [3]

7. [4 marks]

Outcome	1	2	3	4	5	6
Probability	$\frac{1}{5}$	$\frac{1}{6}x$	$\frac{1}{5}$	$\frac{1}{30}(8-5x)$	$\frac{1}{6}$	$\frac{1}{6}$

The table gives the probability distribution for the score obtained when a loaded die is thrown once.

Given that the die is thrown twice and that the total score is recorded, show that the probability of obtaining a total score of 6 is $\frac{48+40x-25x^2}{450}$ [4]

8. [5 marks]

The discrete random variable X has a probability density function given by

$$P(X = x) = \begin{cases} \left(\frac{1}{2}\right)^x, & x = 1, 2, 3, 4, 5 \\ k \quad, & x = 6 \\ 0 \quad, & \text{otherwise} \end{cases}$$

where k is a constant.

(a) Calculate the value of *k*. [3]

(b) Determine the mean and variance of X. [2]

9. [7 marks]

An eight sided die has the numbers 2, 2, 3, 3, 4, 5, 6, 6 painted on its triangular faces, one to a face. When the die is tossed each side is equally likely to be uppermost.

(a) Prepare a table to find the probability that any of the numbers 2, 3, 4, 5, 6 will be uppermost. [2]

(b) Confirm that this is a discrete probability density function. [2]

(c) What will be the probability of getting an even total when the die is tossed twice? [3]

10. [6 marks]

A sports journalist after studying the statistics for recent football games decides that in the space of any 5 minute period the ball is out of play either once, twice, three times or not at all. The journalist consults a statistician who decides that the probability of the ball being out of play t times in any 5 minute period is given by

$$P(t) = \frac{k}{3^t}, \quad t = 0,1,2,3 \qquad \text{where } k \text{ is a constant.}$$

(a) Find the value of k. [4]

(b) Find the probability that the ball is out of play exactly once in any 5 minute period. [2]

11. [9 marks]

A manufacturer makes pyjamas in three popular colours : red, white and blue. Of all the pyjamas he manufactures, 10% are red, 50% are white, and 40% are blue. Six pairs of pyjamas are chosen at random from the manufacturer's stock. Determine the probability that

(a) four of the chosen pairs are blue, [2]

(b) four of the chosen pairs are white, [2]

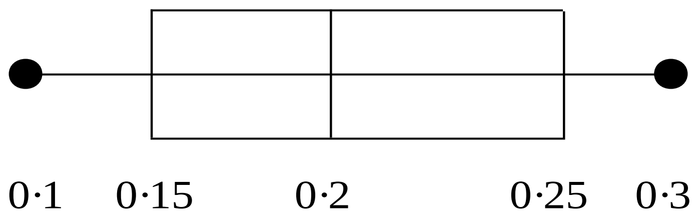
(c) at least two of the chosen pairs are red. [2]

Given that none of the pairs chosen is white,

(d) determine the probability that four of the six pairs are blue. [3]

12. [3 marks]

The median boxplot represents a distribution of probabilities from a discrete probability function.
State the possible values of the individual probabilities. [3]



13. [8 marks]

Luke fires at an archery target made up of three concentric circular regions. Luke never records a miss. He is given another shot if he misses. Points are scored for hitting various parts of the target.

The innermost region (Bullseye) scores 10 points. The next outer region scores 5 points. The outermost region scores 1 point.

The random variable X represents the number of points Luke scores. The probability distribution for X is given below.

x	1	5	10
P (X = x)	0.6	0.3	

(a) Calculate P (X = 10). [1]

Luke fires two arrows at the target.
Assume that each shot is independent of the other.

(b) Calculate the probabilities for the following events:

(i) The first arrow scores 5 and the second scores 1. [2]

(ii) The first arrow scores 5 or the second scores 1. [2]

(iii) The second arrow scores 5 given the first scored 1. [1]

(iv) Luke scores a total of 6 points. [2]

14. [9 marks]

Barry visits a local plant nursery, intending to purchase a camellia for his garden.
He is equally likely to choose any one of the following plants numbered 1 – 6.

1	2	3	4	5	6
Red sun– tolerant	White sun–tolerant	Pink sun–tolerant	Red needs shade	White needs shade	Pink needs shade

(a) (i) Draw the graph of this probability distribution of X, where X represents the type of camellia numbered above. [1]

(ii) State why k must be $\frac{1}{6}$. [1]

(b) (i) Determine P $(1 \le x \le 3)$. [1]

(ii) State the meaning of $1 \le x \le 3$. [1]

(c) State the probability of choosing a pink camellia. [1]

(d) The sales assistant suggests that Barry takes a red, a white and a pink – all sun tolerant, for his garden.

(i) In how many ways can Barry arrange them in a line in his garden? [1]

(ii) In how many ways can Barry arrange them in a line in his garden if the red and white are not to be together? [1]

Barry's wife wants at least one of the 6 types to give away as presents to her gardening friends.

(e) In how many unordered ways can she select at least one from the group of 6? [2]

15. [5 marks]

Consider this distribution for random variable X.

x	3	4	5	6
P(X=x)	0.25	0.25	0.25	0.25

(a) What name is given to such a distribution? [1]

(b) Determine E(X) and Var (X). [2]

If $Y = 2X - 1$

(c) determine E(Y) and Var (Y). [2]

16. [5 marks]

A game of chance uses a spinner with sections A, B and C, all equal in area.
The results of the spinner payout \$1, \$2 and \$3.

(a) Show that this is a fair game. (Hint: Calculate E(X)). [1]

The payouts are changed to \$0, \$2 and \$4.

(b) (i) Use Expected value to determine if this is still a fair game? [1]

(ii) Check your answer by determining a linear rule $Y = aX + b$ for the distribution, and then using change of origin and scale to determine E(Y). [2]

(c) If $Z = 0.5X - 0.5$, determine E(Z) and comment on the fairness of this game. [1]

17. [8 marks]

A standard fair die is thrown and the top number, X, is recorded.

(a) Sketch the graph of the probability function. [2]

(b) Calculate μ and Var(X). [2]

(c) Verify that Var (X) = $\frac{1}{12}(b-a)(b-a+2k)$ where a and b are the extremities of the domain of X and k is the size of the step between X values. [2]

(d) If X values were replaced by Y values, where $Y = 2X + 3$, determine E(Y) and Var(Y). [2]

18. [5 marks]

A formula which relates Var(X) to E(X) is given below.

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

If Var(X) = 4 and E(X) = 5, determine:

(a) $E(X^2)$ [2]

(b) $E(7X + 2)$ [2]

(c) $\text{Var}(3X - 1)$ [1]

Chapter 8: Bernoulli and Binomial Distributions

This chapter includes questions about

Bernoulli Distributions
Binomial Random Variables & their Distributions
Graphs of Binomial Distributions
Mean and Variance of these Distributions

Bernoulli Distribution

1. [14 marks]

A fair coin is tossed.
Let X be the number of heads obtained on the toss.

(a) (i) What is the probability distribution for this Bernoulli variable? [2]

(ii) Fill in the probability table. [2]

x	0	1
P (X = x)		

(b) State the value of the parameter , *p*. [1]

(c) What event represents " failure" in this experiment? [1]

(d) Determine E(X), Var (X) and standard deviation (X). [3]

(e) Complete these statements

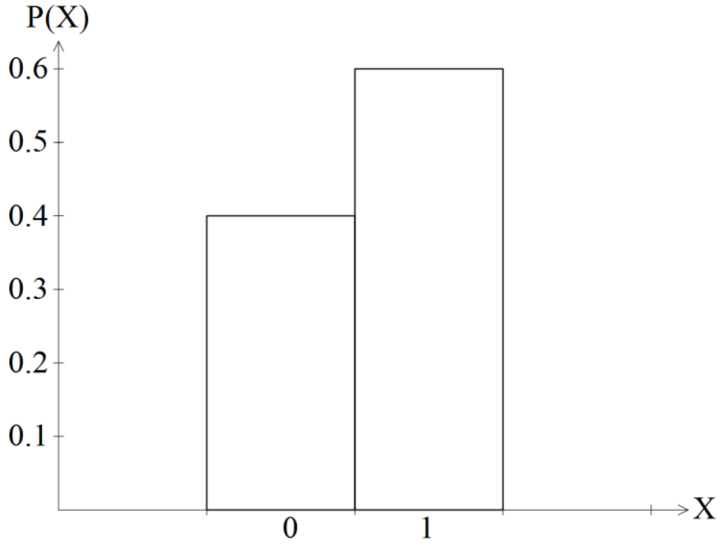
(i) For a variable to be a Bernoulli variable, and for trials of an experiment to be called Bernoulli trials, there must be__________outcomes for each trial, classified as ______________ and ____________. [3]

(ii) There must be a finite number of trials, all of which are _______________ of each other. [1]

(iii) The probability of success must be _____________for each trial. [1]

2. [2 marks]

Explain why this graph represents a Bernoulli distribution for the variable X. [2]



3. [9 marks]

A die is rolled.
Let X be the number of results less than 3 on the die obtained on any one roll.

(a) (i) What is the probability distribution for this Bernoulli variable? [2]

(ii) Complete the probability table. [2]

x	0	1
P (X = x)		

(b) State the value of the parameter , *p*. [1]

(c) What event represents " failure" in this experiment? [1]

(d) Determine E(X), Var (X) and the standard deviation (X). [3]

4. [2 marks]

At a particular college, 55% of the enrolment were males.
Define the Bernoulli distribution for X, where X is the random variable 'the student is a male" assuming a random selection of a student from the enrolment list. [2]

Binomial Distributions

1. [4 marks]

An unbiased die is thrown four times. What is the probability of getting

(a) two fives, [2]

(b) two or more fives? [2]

2. [6 marks]

A pack of 52 cards is shuffled, and the top card is turned up. This is done five times. What is the probability of the top card

(a) three times being a heart, [2]

(b) three times being a king, [2]

(c) three times being the queen of clubs? [2]

3. [2 marks]

A fair coin is tossed 5 times. Find the probability of obtaining exactly 2 tails.

4. [2 marks]

A fair coin is tossed 8 times. Find the probability of obtaining exactly 3 heads. [2]

5. [2 marks]

Two fair coins are tossed 5 times. Find the probability of obtaining two tails exactly twice. [2]

6. [2 marks]

Two fair coins are tossed 6 times. Find the probability of obtaining two heads exactly four times. [2]

7. [4 marks]

Of components from an automatic production process, 15% are defective. After thorough mixing, the components are packaged in fives. Find the probability that a package of 5 components contains

(a) 1 [2]

(b) 2 [2]

(c) 4 [2]

defective components.

8. [2 marks]

A throw of a die counts as a success if a 5 or 6 turns up. If four dice are thrown what is the probability of getting exactly 3 successes? [2]

9. [2 marks]

A fair coin is tossed 6 times. Find the probability of obtaining at least 4 heads. [2]

10. [3 marks]

One in twenty disks made by a computer company are faulty. What is the probability that there will be no more than one faulty disk in a set of 5 ? [2]

11. [3 marks]

From a packet containing sixty new and forty old $20 notes, a bank teller selected twenty notes at random. What is the probability that the twenty contained exactly twelve new, and eight old, notes?

12. [3 marks]

A fisherman has a $\frac{1}{40}$ chance of catching a fish every time the line is cast.
If the line is cast twenty times what is the probability that more than one fish is caught? [2]

13. [6 marks]

In a random examination of a supermarket's checkout slips it is found that 4% of all slips contain an error.

A supervisor selects 8 slips at random.

What is the probability that:

(a) exactly one slip contains an error? [2]

(b) at least one slip contains an error? [2]

One particular checkout develops a fault and 40% of all its slips contain an error.
If the examination is expanded to cover 500 slips selected at random from that checkout, what is the:

(c) μ and Variance of the number of errors to be expected? [2]

14. [13 marks]

The probability that a certain type of seed will germinate is 0·7. In field trials, seeds of this type are planted in rows of 10. What are the probabilities that, in a given row

(a) all the seeds germinate? [2]

(b) exactly 8 seeds germinate? [2]

(c) at least 8 seeds germinate? [2]

What is the probability that, in two adjacent rows, exactly

(d) 8 seeds germinate in each row? [2]

(e) 16 seeds germinate? [2]

If 20 rows are planted

(f) what number of seeds would be expected to germinate? [1]

(g) what is the variance of the expected number of seeds germinating? [2]

15. [16 marks]

Suppose that studies indicate that approximately 20% of 'Overnight Express' parcels sent by a firm through SPEEDY DELIVERY SERVICE are delivered late. If on one day 15 parcels are sent, what is the probability that

(a) 4 or fewer are late? [2]

(b) between 2 and 4 inclusive are late? [3]

(c) 4 are late given that at most 4 are late? [2]

On the next day 5 more parcels are sent through the same service. What is the probability that

(d) 2 of these parcels are late? [2]

(e) 2 parcels were late on each day? [2]

(f) a total of 4 parcels were late over the two days? [2]

In one month a total of 500 parcels are sent. Determine:

(g) E(X) and Var(X). [3]

16. [10 marks]

A delicate biological test has been found to be successful on just 40% of all subjects tested.

(a) If eight subjects are chosen at random and tested, what is the probability that:

(i) the test is successful on exactly 3 of the 8 subjects? [2]

(ii) the test is successful on at least 3 of the 8 subjects? [2]

(iii) the test is successful on the first 3 of the 8 subjects? [2]

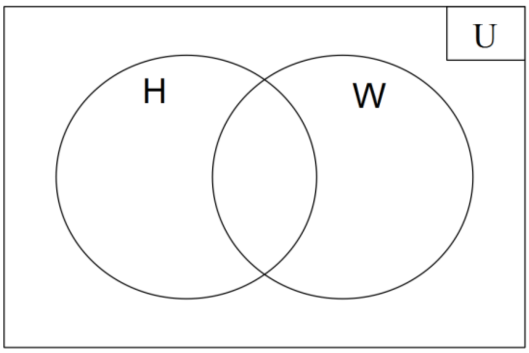
(b) How many subjects should be tested if the probability of having at least one successful test is to be at least 0·95? [4]

17. [12 marks]

Barry and Joy are members of a tennis club for married couples.
In their club, the probability of a husband having a tennis coaching certificate (TCC) is 0.6
If the wife has a TCC, then the probability that the husband has one is 0.2
The probability that a husband and wife have TCC's is 0.1

(a) Show that 50% of wives have a TCC. [2]

(b) Complete the Venn Diagram which represents the information, where H is husbands with a TCC and W is wives with a TCC. [2]



(c) The club has a rule about TCC's. Based on the Venn diagram, what do you think it is? [1]

(d) For a randomly selected married couple, what is the probability that:
(i) only one person has a TCC. [1]
(ii) the wife has a TCC, given the husband has a TCC. [2]

Two married couples are selected at random.

(e) What is the probability that both husbands have a TCC? [1]

Five married couples are chosen at random.

(f) What is the probability that in at least three of the couples both partners have a TCC? [3]

18. [4 marks]

John and Mary play a game. Mary's chance of winning is 0.25.
If the game is played 5 times, what is Mary's chance of winning:

(a) exactly 3 games? [2]

(b) at least 3 games? [2]

19. [9 marks]

Peter owns a service station.
He estimates that 15 % of cars entering his station need air in their tyres.

(a) Find the probability that of the next 100 cars entering the station

(i) at least 20 need air. [2]

(ii) exactly 20 need air. [2]

He also estimates that 20 % of cars need an oil change , quite independently of their need for air.
A car arrives at Peter's service station.

(b) Find the probability that it will need:

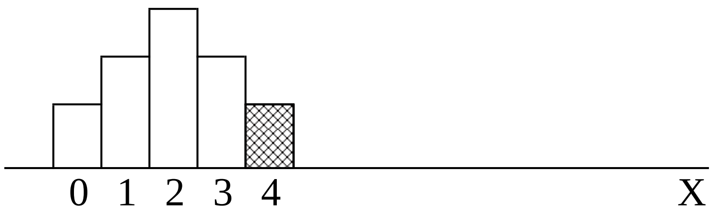
(i) air and an oil change. [2]

(ii) air or an oil change. [2]

(iii) air, given it needed an oil change. [1]

20. [4 marks]

This graph represents a binomial probability distribution.
The area of the shaded column is 0.053



(a) State the value of n. [1]

(b) State the value of p correct to 2 decimal places. [2]

(c) State the value of the total area under the five columns. [1]

21. [2 marks]

A wool classer finds that, on average, about 15 % of the fleeces that he sees have one or more defects.

In the next eight fleeces he sees, find the probability that exactly four of them will contain at least one defect. [2]

22. [5 marks]

One out of every eight people are left handed.

A cricket team has eleven batsmen.

(a) Calculate the probability that nine of the eleven are left handed. [2]

There are several ways that a batsman can be given out . One way is to be out caught. Assume that three in every five dismissals will be "caught".

(b) Of the twenty dismissals that occurred in a match, what is the probability that nineteen of them were caught? [2]

(c) Assuming that being out "caught" is independent of being left handed, find the probability that nine of the eleven were left handed and of the twenty dismissals that occurred in a match, nineteen of them were "caught".
[NB Both these situations did occur, in the same match, WA against Queensland, November, 2006] [1]

23. [6 marks]

A machine produces spiggots. The probability that a spiggot is defective is 0.05.

(a) If 2 spiggots are selected, find the probability that:

(i) at least 1 is defective. [2]
(ii) both are defective. [1]

A sample of 50 spiggots is selected.

(b) Determine the probability (correct to 3 decimal places) that the sample will contain two or more defective items by using a Binomial distribution. [3]

24. [6 marks]

Consider this system of equations.

$$g + h + k = a$$
$$g - h - k = -0.875$$
$$g + h + 2k = 1.5625$$

(a) A probability distribution is defined as follows.

x	0	1	2
$P(X = x)$	g	h	k

If the restrictions on g, h and k are as in the system above, state the values of a, g, h and k. [4]

(b) Given that the variable X is Binomial, determine the parameters of the distribution in (a). [2]

25. [11 marks]

A survey at a Garden Nursery reported that 18% of customers purchased a wetting agent, 3% purchased liquid fertiliser, and that 20% purchased wetting agent or liquid fertiliser.

(a) Determine the probability that a randomly selected customer:

(i) purchased liquid fertiliser and a wetting agent. [2]

(ii) purchased liquid fertiliser, but not a wetting agent. [1]

(iii) purchased neither of those products. [1]

(iv) who had purchased at least one of the products, had purchased the wetting agent. [2]

(b) If four customers were sampled, determine the probability that:

(i) three purchased the liquid fertiliser only, and the other purchased neither of the products. [2]

(ii) three purchased the liquid fertiliser only. [3]

26. [4 marks]

A bag contains twenty casino chips used to represent cash. Five chips have a value of $10 each, and the remainder have a value of $5 each.

Four of the chips are drawn at random from the bag without replacement.

(a) Why is this not representative of a binomial situation? [1]

(b) Calculate the probability that at least one chip will be a $10 chip. [3]

27. [7 marks]

Of all the parrots captured and then relocated from the outback, only 1 in every 30 survives as a household pet.

If 60 parrots are captured in one week:

(a) how many could be expected to survive? [1]

(b) what is the probability that at most 2 will survive? [2]

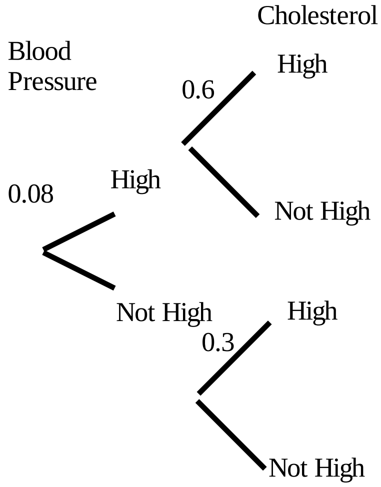
Over a four week period, 60 birds were captured each week.

(c) Find the probability that:
 (i) at most 2 will survive from each group. [2]

 (ii) a total of at most 8 will survive. [2]

28. [14 marks]

A population is examined for high blood pressure (B.P.) and cholesterol.
It is found that 8% have high blood pressure, and 60% of those with high blood pressure have high cholesterol as well.
Only 30% of the people who don't have high blood pressure have high cholesterol.
The information is shown on the probability tree below.



(a) Complete the probability tree above. [2]

(b) For a randomly selected member of the population, state the value of the following probabilities:

(i) P(high B.P. and high cholesterol.) [1]

(ii) P(high B.P. or high cholesterol.) [1]

(iii) P(high cholesterol.) [1]

(iv) P(high B.P. given that the person had high cholesterol.) [2]

Six people, known to have high blood pressure are tested for cholesterol.

(c) Based on information from the tree, calculate the probability that

(i) all six have high cholesterol. [1]

(ii) none have high cholesterol. [2]

(iii) exactly three have high cholesterol. [2]

A group of people, all known to have high B.P., were tested for cholesterol.
It was later calculated that within that group, the probability of 12 people having high cholesterol was 0.1797.

(d) How many people were tested? [2]

29. [3 marks]

If X is a binomial variable with $p = 0.31$, use a graphical method to determine the largest value of n such that: $P(X = 3) \leq 0.05$ [3]

30. [8 marks]

The ratio of boys to girls in Thailand is approximately 1.04 : 1

(a) What is the probability of a newborn Thai child being male? [2]

A sample of 2 000 babies were selected for a medical test.

(b) How many boys would you expect to be tested? [1]

(c) In a Thai family with 5 children, what is the probability;

(i) of there being at most three boys?
State any parameters and distributions used. [2]

(ii) that the fifth child is the fourth boy? Show all working. [3]

31. [8 marks]

"No−ache" claims to give relief from migraine headaches. It is claimed to be effective in 90% of cases.
Trials are conducted with four patients.

(a) If X is the number of patients obtaining relief, then assuming the claim is correct,

(i) why is X a binomial random variable? [3]

(ii) determine the mean and standard deviation of X. [2]

(iii) determine the probability that none of the four patients gains relief. [1]

(b) If none of the four gets relief, do you think that the 90% claim is reasonable?
Explain on the basis of the probabilities involved. [2]

32. [13 marks]

In 2009 US banks were caught with bad debts, and forced to renew their lending practices. As a result, 60% of loan applicants were refused a loan.

A sample of 50 such applications was analysed.

(a) State why the situation can be modelled by a Binomial distribution. [1]

(b) State the mean and standard deviation of refusals for the sample. [2]

40 of the 50 applications were accepted.

(c) Use relevant probabilities to determine if this result needs further investigation by the lending manager. [2]

(d) Given that another random sample of 50 applications had at least 25 refusals, find the probability that there were less than 30 refusals. [2]

Meanwhile, bank interest rates fell. Ashley's investment had a rate of 2% p.a., compounded daily for his principal of \$50 000.

His balance after one year is given by $A = 50\,000\left(1 + \dfrac{0.02}{365}\right)^{365}$.

(e) Calculate this value of *A*. [1]

Ashley dreamt that rates were 100% p.a., compounded continuously.
If this was the case, for his \$50 000 investment, $\text{A} = 50\,000\left[\lim_{n \to \infty}\left(1 + \dfrac{1}{n}\right)^{n}\right]$.

(f) Evaluate A:

(i) exactly. [1]

(ii) correct to the nearest dollar. [1]

(g) When interest is compounded continuously, the amount, A, increases at a rate proportional to its size at that time *t*.

(i) Write this as a differential equation. [1]

(ii) How many years would it take for an amount A to double at a rate of 2% p.a. compounded continuously? [2]

33. [10 marks]

When a set of random numbers is used to produce single digits, those digits are uniformly distributed, with probability function:

$f(x) = 0.1$, where $x = 0, 1, 2, 3, \ldots\ldots\ldots, 9$

(a) Determine the probability that a randomly selected digit is less than 7. [1]

(b) Find the probability that if ten digits are randomly selected:

(i) the first one is a 7. [1]

(ii) the first three digits are all 7's. [2]

(iii) exactly three out of the ten digits are 7's. [2]

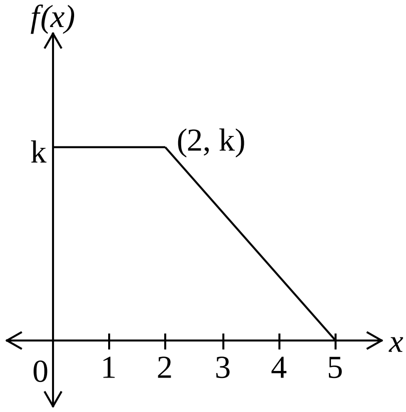
(iv) the first is a 7 or the second is a 7, but not both. [2]

(v) at least three out of the ten are 7's. [2]

Chapter 9: Continuous Random Variables

This chapter includes questions about
Continuous random variables and their distributions
Change of origin and scale

1. [6 marks]



The probability density function for a random variable, X, is given by the above graph.

(a) Find the value of k. [2]

(b) Find the probability that X is less than 2. [1]

(c) Given that X is less than 3, find the probability that X is less than 1. [3]

2. [3 marks]

A random variable X has probability density function given by

$$f(x) = \begin{cases} 1 - x & , \ 0 \le x \le 1 \\ x - p & , \ 1 < x \le 2 \\ 0 & , \ \text{otherwise} \end{cases}$$

Find the value of the constant p.

3. [7 marks]

The probability density that a car engine will start t seconds after the key has been turned in the ignition, is a triangular distribution starting at the origin, and rising until t = 4 seconds. If it hasn't started by then, it won't start.

(a) Find the equation of the probability density. [3]

(b) Calculate the probability that the car will start in the fourth second. [4]

4. [6 marks]

A continuous random variable X is uniformly distributed over the interval from 0 to 10 inclusive.

(a) Define the probability density function, f (x), over that interval and sketch its graph. [2]

Suppose a point is selected at random from the area defined in (a).

(b) Find the probability that its x co–ordinate is between 1.5 and 7.5 [2]

Suppose 5 points are selected at random from the area defined in (a).

(c) Find the probability that they all have x co–ordinates between 1.5 and 7.5 [2]

5. [8 marks]

A continuous random variable X has probability density function f where

$$f(x) = k(1 - ax), \quad 0 \le x \le 1$$
$$= 0 \quad , \quad \text{otherwise}$$

and k and a are positive constants.

(a) Show that $a \le 1$ (A diagram may be helpful) [4]

(b) Express k in terms of a. [4]

6. [4 marks]

A continuous variable X has probability density function f where

$$f(x) = kx^2, \quad 0 \le x \le 2$$
$$= 0 \quad , \text{otherwise}$$

and k is a positive constant.

Find the value of *k*.

7. [6 marks]

A continuous variable X has probability density function defined by

$$f(x) = \frac{x^2}{9} \quad \text{for } 0 \le x \le 3$$
$$= 0 \quad \text{otherwise.}$$

Calculate

(a) $P(1 \le x \le 2)$ [3]

(b) the value of k for which $P(X \ge k) = 0{\cdot}875$ [3]

8. [6 marks]

A continuous random variable X has probability density function defined by

$$f(x) = ke^{-0 \cdot 1x} + \frac{1}{x} \text{ for } x > 1$$
$$= 0 \quad \text{otherwise}$$

If $P(X < 2) = 0 \cdot 5$ find the value of *k*. [6]

9. [8 marks]

A continuous random variable X has probability density function defined by

$$f(x) = 0 \cdot 2e^{-kx} \text{ for } x \geq k$$
$$= 0 \quad \text{otherwise}$$

(a) State the value of k and Var(X). [2]

Calculate the probability that :

(b) $x > 5$. [3]

(c) $x > 10$, given that $x > 5$. [3]

10. [8 marks]

A continuous random variable X has probability density function defined by

$$f(x) = pe^{-qx} \text{ for } x \geq 0$$
$$= 0 \quad \text{otherwise}$$

(a) If $P(X < 1) = 0 \cdot 88$ find the values of p and q. [5]

(b) Evaluate $P(2 < x < 3)$. [3]

11. [8 marks]

The time, X seconds, between the counting of particles by a particle counter is a random variable defined by :

$$F(x) = \begin{cases} 0 \cdot 05e^{-0 \cdot 05x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Find :

(a) $P(X \geq 20)$ [4]

(b) $P(20 \leq X \leq 30)$ [4]

12. [8 marks]

$f(x) = \frac{x}{2}$, $0 < x < a$ is a continuous probability density function.

(a) Derive a quadratic equation involving *a*. [4]

(b) Hence solve for *a*. [2]

(c) One solution of the quadratic equation is not valid.
Give a brief reason why this is so. [2]

13. [4 marks]

If x is a random variable of the probability density function
$f(x) = k e^{-kx}$ ($x > 0$), find the probability that x is greater than the mean. [4]

14. [7 marks]

The number of hours, X, that an electric light bulb will burn from the moment it is first switched on is a continuous random variable with probability density function :

$f(x) = 0{\cdot}001e^{-0{\cdot}001x}$, $x \geq 0$

Find the maximum life of the bulb, in hours, such that the probability of lasting more than 100 hours is 0·53696. [7]
Answer to the nearest whole number.

15. [8 marks]

The time t in hours between 'system errors' on a home computer is thought to follow an exponential distribution with probability density function given by

$f(t) = 0{\cdot}5e^{-0{\cdot}5t}$, for $t \geq 0$

(a) What is the average time between system errors, and the variance of times? [2]

(b) Find the probability that a system error will occur within the first hour of operation of the computer. [2]

(c) Find the probability that the first system error will occur in the second hour after starting the computer. [2]

(d) State the cumulative density function for t. [2]

16. [10 marks]

A continuous probability density function is given by the formula

$P(x) = -kx(x - 20)$ where k is a positive quantity.

(a) For what values of x is the function defined? [3]

(b) Show that $k = \dfrac{3}{4000}$. [3]

A student solves the following equation to calculate *m*

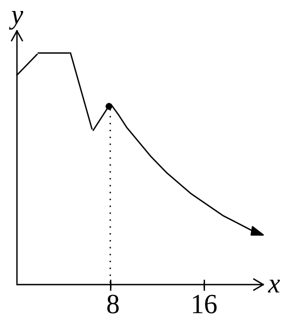
$$\int_0^m \left(-\frac{3}{4000}x^2 + 20\frac{3}{4000}x \right) dx = \frac{1}{2}$$

(c) What does *m* represent? [1]

(d) Calculate the value of *m*. [3]

17. [5 marks]

The graph drawn below represents a probability density function.
That portion of the graph to the right of x = 8 has equation $y = 0.1\,e^{-0.1x}$

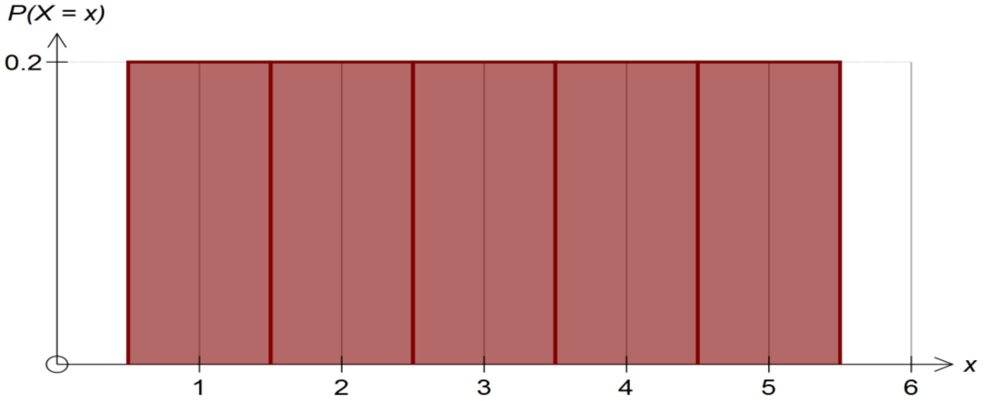


(a) Find the area to the left of x = 8. [3]

(b) Find the value of the upper quartile of this distribution. [2]

18. [12 marks]

(a) The following probability graph is of a discrete random variable, X.



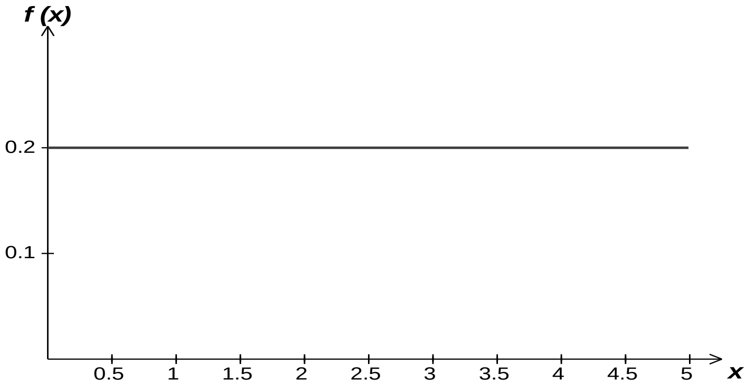
(i) Complete the rule: $P(X = x) =$ ______, for $x =$ ______ [2]

(ii) Calculate: $P(X = 4)$ [1]

(iii) Calculate: $P(X < 4)$ [1]

(iv) Calculate: $P(X < 4|X > 2)$ [2]

(b) The following graph is of a continuous random variable, X.



(i) Complete the description of the p.d.f. and the domain of x.

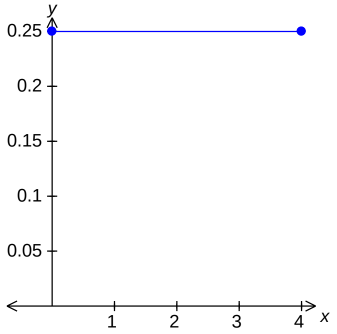
$f(x) =$ ________ for ________ [2]

(ii) Calculate: $P(X = 4)$ [1]

(iii) Calculate: $P(X < 4)$ [1]

(iv) Calculate: $P(X < 4|X > 2)$ [2]

19. [4 marks]



(a) State why the above function, f(x), could be a probability function. [2]

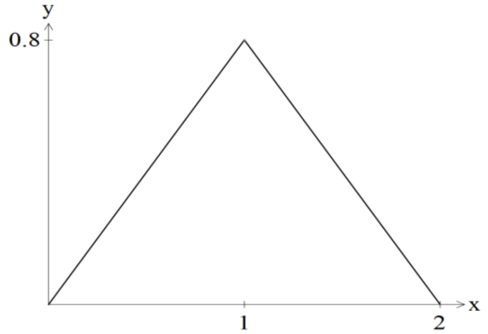
2f(x) – a is also a probability function for $0 \le x \le 4$.

(b) Determine the value of a. [2]

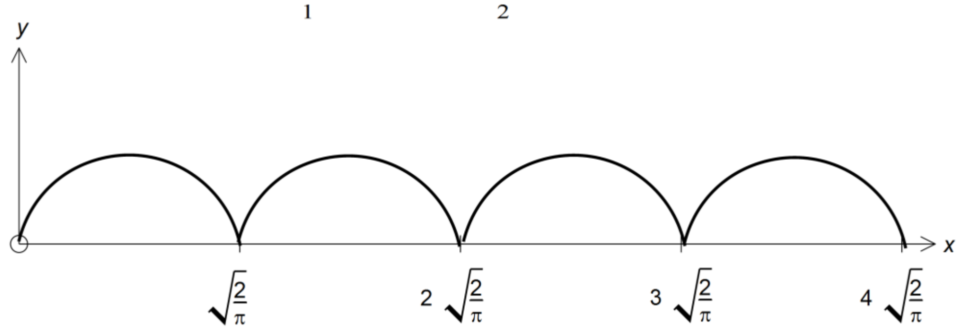
20. [9 marks]

Determine, with reasoning, whether each of the following represent probability distributions.

(a) [2]



(b) [3]



(c) $f(x) = \begin{cases} 0.2 & \textit{where } x = 1, 0, 7 \\ 0.4 & \textit{where } x = 5 \\ 0 & \textit{elsewhere} \end{cases}$ [2]

(d) $f(x) = \begin{cases} \frac{7}{9}x^2 - 2 & 0 \le x \le 3 \\ 0 & \textit{elsewhere} \end{cases}$ [2]

21. [8 marks]

State whether the following represent probability distributions of discrete or continuous random variables. State your answer as True or False. Give full reasoning for your answer.

(a) $f(x) = \begin{cases} {}^{10}C_x (0.3)^x (0.7)^{10-x} & x = 0, 1, 2, 3, 4, ...7 \\ 0 & \text{elsewhere} \end{cases}$ [2]

(b) $f(x) = \begin{cases} 0.3 & 0 \le x \le \frac{10}{3} \\ 0 & \text{elsewhere} \end{cases}$ [2]

(c) $f(x) = \begin{cases} x^2 & 0 \le x \le \sqrt[3]{3} \\ 0 & \text{elsewhere} \end{cases}$ [2]

(d) $f(x) = \begin{cases} 2e^{-2x} & x \ge 0 \\ 0 & \text{elsewhere} \end{cases}$ [2]

22. [6 marks]

(a) $f(x) = 0.2$ defines a probability function for a continuous random variable X for a fixed domain, between *a* and *b*.

(i) State the value of $b - a$. [1]

(ii) Calculate the median, in terms of *a* and *b*. [1]

(iii) Judging from the shape of the distribution, state an expression for the mean. [1]

(b) $f(x) = k(1 - x^2)$ defines a probability function for a continuous random variable X for a fixed domain $a \le x \le b$

If it is known that $\int_{-1}^{1} (1 - x^2)\,dx = \frac{4}{3}$ determine:

(i) k [1]

(ii) an obvious set of values for *a* and *b*. [1]

If it is also known that the mean = the median,

(iii) calculate the mathematical relationship between *a* and *b* . [1]

23. [9 marks]

The jumping length, l, of junior long jumpers is modelled by a continuous random variable, with probability density function given as:

$$f(l) = \begin{cases} \frac{3}{8}l^2 & 0 \le l < 1 \\ -\frac{1}{18}l + \frac{31}{72} & 1 \le l \le 4 \end{cases}$$

(a) Sketch the graph of $f(l)$. [2]

(b) Verify that this is a pdf. [3]

(c) Determine the median jumping length. [4]

24. [9 marks]

A shopping centre provides a child–minding facility for its customers. Parents are allowed a maximum of one hour of free child–minding time at any one visit to the shopping centre.

The actual times, X minutes, used may be modelled by the following probability distribution function, where k is a constant.

$$f(x) = \begin{cases} \frac{kx}{40} & 0 \le x \le 40 \\ k & 40 \le x \le 60 \\ 0 & \text{otherwise} \end{cases}$$

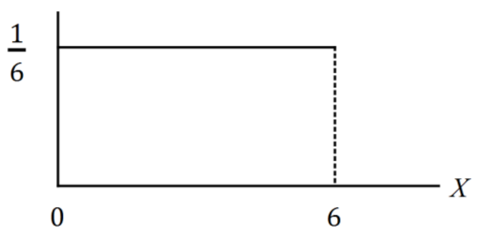
(a) Sketch the graph of f. [3]

(b) Show a clear method to calculate the value of k to be 0.025 [3]

(c) Hence, determine P ($X < 45$). [3]

25. [4 marks]

A probability distribution for random variable X is graphed below.



(a) Determine the mean and variance for X. [2]

The values of X are measurements made in cm.

(b) Determine E(Y) and Var (Y) if Y values are the X values converted to mm. [2]

26. [8 marks]

A uniform distribution is defined as $P(X = x) = \begin{cases} \frac{1}{2} & \text{when } 3 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$

(a) (i) State E(X). [1]

(ii) Use integration to determine Var(X). [3]

If $X = \begin{cases} \frac{1}{b-a} & \text{when } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$

(b) (i) State the cumulative probability distribution for X. [2]

(ii) Sketch the graph of the cdf. [2]

Chapter 10: Normal Distribution

This chapter includes questions about

Standard & Non Standard Normal Distribution
Normal Graphs
Theory & Applications
Binomial Inclusions

1. [16 marks]

Tuna are caught off the coast of Western Australia. Their weights are normally distributed with a mean of 22 kg.

The heaviest 4% of these tuna are processed as fertilizer.

The lightest 6% are exported fresh as Class A eating fish.

The remainder weigh between 17·33 kg and 27·25 kg, and are canned.

(a) Find the probability that a randomly selected tuna is canned. [1]

(b) Find the probability that a randomly selected tuna is canned, given that it is not exported. [2]

(c) Calculate the standard deviation of these tuna. [2]

Kingfish are also caught off the WA coast. Their weights are also normally distributed, with a mean of 20 kg and a standard deviation of 4 kg.

(d) A fish weighing approximately 22 kg is caught.
Is it more likely to be a tuna or a kingfish? Explain. [2]

Three kingfish are caught.

(e) Calculate the probability that two weigh less than 22 kg, and the other weighs more than 22 kg. [4]

At a particular time of the year, the ratio of numbers of tuna to numbers of kingfish is 20:1.

(f) Find the probability, correct to three decimal places, that a particular fish, known to be either a tuna or a kingfish, weighs less than 22 kg.
(Hint: a probability tree is useful here.) [5]

2. [3 marks]

A population is normally distributed with mean = μ and standard deviation = σ .

Determine the percentage of marks which will lie between $\mu - \sigma$ and $\mu + 2\sigma$.
Give your answer correct to one decimal place. [3]

3. [4 marks]

X and Y are normal random variables with means μ_1 and μ_2 and variances δ_1^2 and δ_2^2 respectively.

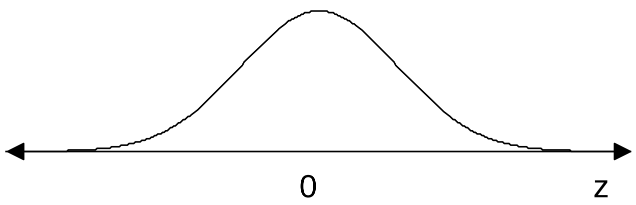
Sketch and label the two normal curves given that $\mu_1 = \mu_2$ and $\delta_1^2 > \delta_2^2$. [4]

4. [5 marks]

A standard normal variable, z, has $\mu = 0$ and $\delta^2 = 1$.
The defining rule for that probability distribution is f (z)

where $f(z) = \dfrac{1}{\sqrt{2\pi}} e^{-0.5z^2}$

It has a bell shaped graph as shown.



(a) Calculate f(0). [1]

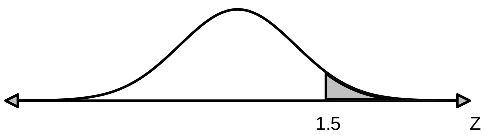
(b) Determine the area between the curve and the z axis to the left of the line z = 1. [1]

(c) Solve for z: f(z) > g(z), where $g(z) = z^2$. [3]

5. [6 marks]

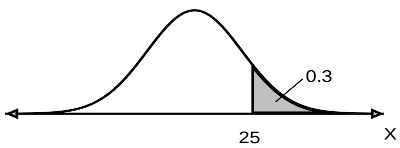
The following question relates to a Standard Normal distribution.

(a) Calculate the shaded area. [1]

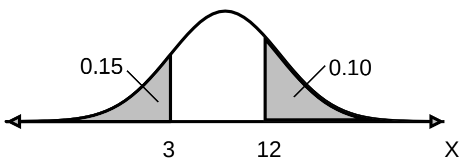


The following questions relate to non standard Normal distributions.

(b) If $\mu = 18$, calculate σ. [2]



(c) Calculate μ and σ. [3]



6. [12 marks]

A special spark plug has an average life of 5 000 kilometres, with a standard deviation of 500 kilometres. Assume such lives are normally distributed.

(a) Calculate the probability that a randomly selected plug will last

(i) more than 6 000 kilometres. [1]

(ii) between 4 000 and 6 000 kilometres. [1]

In my coupe, there are four such spark plugs, arranged in a line.

(b) Calculate the probability that:

(i) all four will last more than 4000 kilometres. [2]

(ii) the middle two will last more than 6000 kilometres , and the other two will last less than 6000 kilometres. [2]

(iii) at least two will last more than 6000 kilometres. [2]

(iv) exactly one will last more than 6000 kilometres (and the others less) or exactly one will last less than 4000 kilometres (and the others more). Give your answer correct to four decimal places. [4]

7. [6 marks]

The number of customers entering Deworths supermarket per day is modelled by a normal random variable with a mean of 350 and standard deviation of 18.

(a) Determine the probability that tomorrow there will be:

(i) less than 340 customers. [2]

(ii) less than 370 given that there is more than 340. [2]

(b) Determine the probability that over the next 5 days there will be between 340 and 370 on exactly 3 of those days.
Show all distributions and relevant parameters that you use. [2]

8. [7 marks]

"Chebychev's Inequality" is true for populations and samples.
The inequality states that at least $1-\dfrac{1}{k^2}$ of a set of scores lie within *k* standard deviations either side of the mean.

eg If $k = 2$, then the fraction $1-\dfrac{1}{2^2}$. ie $\dfrac{3}{4}$ or at least 75% of the scores in the distribution lie within 2 standard deviations of the mean.

(a) State the percentage of scores that will lie within 2.5 standard deviations. [2]

(b) Show that for $k = 2.5$, and for a normally distributed population, the inequality is true. [2]

A sample of scores is given below:
8, 8, 8, 8, 8, 8, 100

(c) Show that if $k = 2.5$, the inequality holds for the sample. [3]

9. [9 marks]

The cholesterol level in males under 21 years of age is normally distributed with a mean of 160 and a standard deviation of 10.

(a) Find the probability that a randomly selected male under 21 years of age has a cholesterol level of:

(i) less than 160. [1]

(ii) less than 170. [1]

(iii) less than 160, if it is known that the level was less than 170. [1]

(b) From a sample of four males aged less than 21 years of age, find the probability that at least two have cholesterol levels of less than 170. [2]

The cholesterol level of females under 21 years of age is also normally distributed with a mean of 160.

(c) Calculate the probability that a man and a woman, both under 21, have cholesterol levels higher than 160. [1]

It is known that 40% of females aged less than 21 years of age have levels higher than 165.

(d) Find the standard deviation of cholesterol levels for this group. [3]

10. [5 marks]

z	x
–3	22
–2	34.5
1	72
3	97

The following questions relate to the tabled data above.

(a) Calculate the correlation coefficient between z and x. [1]

(b) Calculate the equation used to predict x if z is known. [2]

x represents values of Normal random variable X, with parameters μ and σ^2 .
z represents corresponding values of Normal random variable Z, with parameters 0 and 1.

(c) Determine the values of μ and σ. [2]

11. [8 marks]

A manufacturer of MP3 players finds their "lives" are normally distributed with a mean life of 3 000 hours and a standard deviation of 350 hours.

(a) Calculate the probability that the next player sold lasts between 3 300 and 3 500 hours. [2]

(b) The WA distributor imports 5 000 of these players. How many could be expected to have life–times of less than 2 400 hours? [2]

The distributor guarantees to replace 5% of the players, for failing to have a reasonable life, free of charge.

(c) What minimum lifespan is guaranteed by the distributor? [2]

(d) If a player fails before 2 650 hours, what is the probability that it will be replaced for free? [2]

12. [10 marks]

Ewes lends money for home purchase.
The following represents a sample of the loan applications' amounts.

Amount	Number of these loans
$60\ 000 \le x < 80\ 000$	58
$80\ 000 \le x < 100\ 000$	75
$100\ 000 \le x < 120\ 000$	100
$120\ 000 \le x < 140\ 000$	81
$140\ 000 \le x < 160\ 000$	65

(a) Based on the sample given, find the probability that a randomly selected person applied for a loan less than $ 100 000. [2]

(b) Calculate to the nearest dollar;

(i) the mean. [1]

(ii) the standard deviation. [1]

(c) If this sample was normally distributed, what would be true of the mean and the median? [1]

For the questions that follow, we will assume that:

(i) The above sample is part of a population with the same mean and standard deviation as that of the sample, both written to the nearest $ 1 000.

(ii) The population is normally distributed.

(d) Calculate the probability that the next loan application will be for less than $ 110 000. [1]

(e) Calculate the probability that the next loan application will be for less than $ 110 000 given that it is more than $ 80 000. [2]

(f) Calculate the probability that of the next seven applications, four of them will be for amounts between $ 80 000 and $ 110 000. [2]

13. [9 marks]

The new train service from Perth to Mandurah opened on December 23, 2007. It averages 48 minutes for the 72 km journey.

(a) What is the average speed for the trip, in km/hr? [1]

The completion times for the journey have since been analysed, and seem to be normally distributed. The range of completion times (Max − Min) is approximately 30 minutes. The standard deviation of completion times is estimated to be 5 minutes.

(b) Explain how this estimate can be obtained. [1]

(c) What % of trips would take more than 58 minutes? [2]

(d) Determine the IQR of times. Use a labelled diagram of a normal curve to help explain your answer. [3]

John catches the train from Mandurah to Perth, then walks the two minutes to his workplace.

(e) State the mean and standard deviation of his total travel times. [2]

14. [6 marks]

An ice–cream factory produces ice–cream in cylindrical tubs. The volume, V litres, of ice–cream in a tub is normally distributed with a mean of 12 and a variance of σ^2 .

(a) Determine $P(V < 12 + \sigma)$ [2]

(b) Calculate the value of σ such that 98% of tubs contain more than 11 litres of ice–cream. [2]

(c) If a given tub contains more than 11 litres, what is the probability it will contain less than 11.5 litres? Assume σ is 1 litre. [2]

15. [4 marks]

A standard normal score, z, has mean 0 and standard deviation 1. Calculate

(a) $P(0 < z < 2)$ [2]

(b) $P(-1{\cdot}72 < z < 1{\cdot}72)$ [2]

16. [6 marks]

A variable W has a normal distribution with a mean of 60 and a standard deviation of 10.

Determine

(a) $P(W < 70)$ [2]

(b) $P(W < 40)$ [2]

(c) $P(40 < W < 70)$. [2]

17. [7 marks]

For a standard normal distribution find

(a) $P(-1 < z < 1{\cdot}5)$ [2]

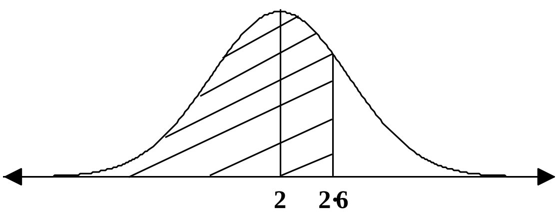
(b) $P(z > -0{\cdot}6)$ [2]

(c) u when $P(-u < z < u) = 0{\cdot}8164$ [3]

18. [5 marks]

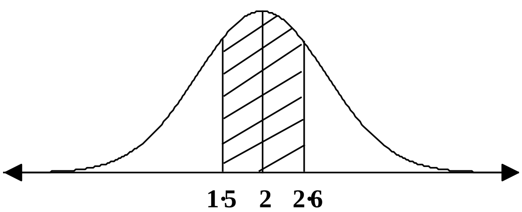
The following diagrams show a normal curve with mean 2 and standard deviation 0·4. For each diagram, calculate the area of the shaded region.

(a)



[2]

(b)



[2]

19. [5 marks]

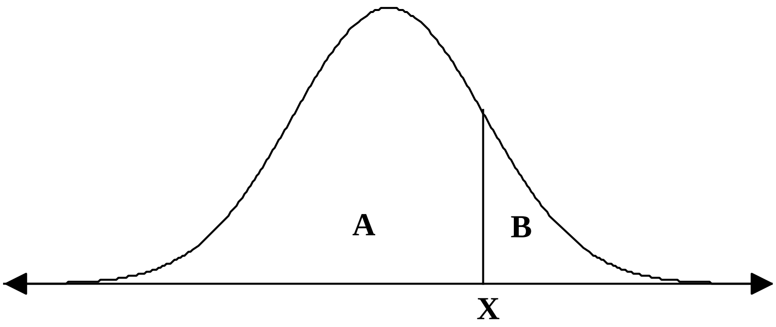
If a variable z has a normal distribution with mean 0 and standard deviation 1, determine

(a) $P(z > -1{\cdot}28)$ (2)

(b) the value of u when $P(z > u) = 0{\cdot}33$ (3)

20. [3 marks]

A normal distribution of mean 50 units and standard deviation 10 units is represented in the diagram below



The area under the curve is divided into two parts A and B.
If B has area 0·33 square units

(a) what is the area of A, [1]

(b) what is the value of X? [2]

21. [5 marks]

A continuous variate h is distributed normally with a mean of 60 and a standard deviation of 5.

(a) Above what value of h does 80% of the population lie? [2]

(b) Below what value of h does 50% of the population lie? [1]

(c) Below what value of h does 10% of the population lie? [2]

22. [9 marks]

The life of a certain brand of toaster is normally distributed with mean 1200 days and standard deviation 250 days. Assume that there are 365 days in one year and 30 days in one month. Calculate the probabilities that a toaster will last for

(a) more than four years, [2]

(b) less than three years, [2]

(c) between two and three years. [2]

The manufacturer guarantees to replace any toaster which fails to last for more than M months.

(d) Find the smallest integer M, such that he will, on average, replace no more than one toaster in fifty under the guarantee scheme. [3]

23. [6 marks]

The intensity I of 'clear–air turbulence' encountered by airlines on scheduled flights in a certain area of the world is said to be normally distributed, with a mean of 103 units and a standard deviation of 12 units.

(a) If there is a danger of structural damage when $I > 135$, find the percentage of encounters with clear–air turbulence which are likely to be dangerous. [3]

(b) On non–scheduled military flights, normally distributed as for airline scheduled flights, 99% of the encounters with clear–air turbulence are calculated to be safe. What will be the maximum value of I for safety? [2]

(c) If there are 15 000 non–scheduled military flights in one year, how many of these will be dangerous? [1]

24. [10 marks]

Coffee is packed in jars and the nominal weight of the contents is 1 kg. The actual quantity of coffee delivered to a jar by the automatic filling machine is normally distributed with a mean of 1 kg and a standard deviation of 12g.

(a) What proportion of the packed jars of coffee will contain

(i) less than 982g, [2]

(ii) more than 1030g, [2]

(iii) between 982g and 1030g? [2]

(b) How should the average weight of the contents be adjusted if not more than 1% of the packed jars are to contain less than 1kg of coffee?
Assume that the standard deviation of the quantities supplied to the jars remains constant at 12g. [4]

25. [7 marks]

The size of peas in a certain crop is normally distributed, the mean diameter being 6mm and the standard deviation 1·5mm. Peas with diameters equal to or less than 4mm are said to have a sweeter flavour than the larger peas.

(a) What percentage of the crop will have the sweeter flavour? [2]

(b) What percentage of the crop will have a diameter of more than 1cm? [2]

(c) If a national firm selects those peas with diameters less than 4mm for freezing, what is the probability of selecting a pea with greater than 3mm diameter from a packet of frozen peas? [3]

26. [12 marks]

The extra refills for a propelling pencil are supposed to be 0·5mm in diameter. Refills below 0·485mm will fall out, while those above 0·520mm are too big to fit in the pencil.

A firm makes refills with diameters which are normally distributed with mean 0·5mm and standard deviation 0·01mm.

(a) Find the probability that a randomly chosen refill will fit. [2]

(b) If a refill is not suitable what is the probability that it is too big? [4]

The firm decides that refills which fall out of the pencil are more acceptable than those which are too big and consequently attempt to set their machine so that no more than 1% of refills are too big and no more than 4% fall out.

(c) What adjustment should be made to the mean and the standard deviation to allow this to happen? [6]

27. [11 marks]

A machine is designed to drill holes in plastic plates. The diameters of holes drilled are approximately normally distributed with mean 1·02cm and standard deviation 0·01cm. Holes with diameters less than 1cm are too narrow, while holes with diameters greater than 1·05cm are too wide.

(a) What proportion of holes drilled by the machine have diameters less than 1cm? [2]

(b) What proportion of holes drilled are either too narrow or too wide? [3]

(c) What proportion of holes that are not too narrow are too wide? [3]

(d) Suppose that each plate contains two holes drilled independently. What is the probability that a plate will be rejected because at least one of the two holes will be too narrow or too wide? [3]

28. [11 marks]

A machine fills packets with powdered laundry detergent. Assume that each packet is filled independently of all other packets and that the weight of the contents of each packet is normally distributed with a mean of 500g and a standard deviation of 2·5g.

Each packet is weighed before sealing and if it does not contain at least 495g of detergent it is topped up.

(a) Calculate the probability that a randomly selected packet requires topping up. [2]

(b) For the day in which 1000 packets are produced, calculate

(i) the expected number of packets which require topping up,

(ii) the expected total cost of topping up if it costs 40 cents for each packet topped up. [2]

(c) Given that a certain packet was randomly selected from those which require topping up, calculate the probability that this packet contained more than 494g. [3]

(d) After overhauling the machine it was claimed that the weights of the contents of the packets had a normal distribution with a mean of 500g and that 98·9 percent of packets contained at least 495g. What then was the standard deviation of this distribution? (Give your answer correct to one decimal place.) [4]

29. [15 marks]

A new, but not always reliable, test has been devised to assist in the diagnosis of a certain disease in cats. The test is administered to a cat suspected of having the disease, and the animal's score is recorded. For cats which are later found to have the disease, scores on the test are normally distributed with mean 20 and standard deviation 2.

What is the probability that a cat with the disease will score

(a) at least 19 on the test? [2]

(b) between 16 and 21 on the test? [2]

In using the test veterinarians set a score x_o, so that if the cat's score exceeds x_o, the animal is diagnosed as having the disease, while if it is less than x_o, it is diagnosed as not having the disease.

(c) What is the value of x_o if veterinarians want to be able to correctly diagnose 90% of diseased cats as having the disease? [2]

(d) What is the probability of making a wrong diagnosis for cats with the disease? [1]

When the same test is applied to cats which do not have the disease it is found that scores on the test are also normally distributed but with mean 17 and standard deviation 1.

(e) What is the probability that the owner of a cat is told that the animal has the disease when in fact it does not? [3]

(f) If the test is administered to 200 cats, half of which have the disease, and half of which do not, how many wrong diagnoses would be expected to be made? [2]

(g) Do you think this is a useful test? Give your reasoning. [2]

30. [16 marks]

A research team investigated the reaction times of male drivers aged between 18 and 25 years. The drivers were classified into two groups depending on their blood alcohol content : Group A if their blood alcohol content was 0·05% or less and Group B if it was more than 0·05%. Thirty percent of drivers were classified as Group B.

Drivers in Group A were found to have reaction times which were approximately normally distributed with mean 0·32 seconds and standard deviation 0·05 seconds under normal driving conditions. Those in Group B had reaction times which were approximately normally distributed with mean 0·45 seconds and standard deviation 0·10 seconds under the same conditions.

(a) What is the likelihood that a driver's reaction time will be less than 0·45 seconds if he is in:

(i) Group A? [2]

(ii) Group B? [2]

It is believed that a driver whose reaction time exceeds 0·40 seconds will be unable to avoid an impending accident.

(b) What is the probability that a driver in Group B can avoid an impending accident, given that his reaction time is at least 0·32 seconds? [3]

(c) What is the maximum reaction time of the quickest reacting 20% of drivers in Group A? [2]

(d) What is the likelihood that a randomly chosen driver has a reaction time of less than 0·40 seconds? [4]

(e) If two drivers are chosen at random from Group A, what is the probability that one has a reaction time of more than 0·40 seconds and the other has a reaction time of less than 0·40 seconds? [3]

31. [12 marks]

On a certain section of a freeway the legal speed limit is 90 km/h. Vehicles are also required to maintain a minimum of 70 km/h wherever the traffic conditions permit.

Assume that under typical driving conditions the speed of cars along this section of freeway is normally distributed with mean 85 km/h and a standard deviation of 8 km/h.

(a) What is the probability that a randomly chosen car exceeds the speed limit? [2]

(b) What proportion of cars drive within the legal speed limits? [2]

(c) What is the probability that out of three cars on the freeway exactly one exceeds the speed limit? [2]

A police patrol stops all cars travelling more than 10 km/h above the legal speed limit.

(d) What proportion of cars is stopped for speeding? [2]

When the maximum permitted speed is raised to 100 km/h, it is found that the speeds of vehicles still fit a normal distribution, with mean 94 km/h and standard deviation 12 km/h.

(e) Of 100 cars exceeding the new speed limit, how many would be stopped by the police for driving at more than 10 km/h over the allowed limit? [4]

32. [7 marks]

During sleep, the reduction of a person's oxygen consumption has a normal distribution with mean 38·4 millilitres per minute (ml/min) and standard deviation 4·6 ml/min.

(a) Determine the probability that during sleep a person's oxygen consumption will be reduced by

(i) more than 43·5 ml/min. [2]

(ii) at most 36·4 ml/min. [2]

(b) Of people whose oxygen consumption is reduced by less than 43·5 ml/min, what proportion have reduction below 36·4 ml/min? [3]

33. [4 marks]

A normal distribution has parameters $(\mu, \sigma^2) = (0, 1)$.

(a) If $Y = 2X + 7$, determine (μ, σ^2) for Y. [2]

(b) Comment on the shape of Y if Y is also normally distributed. [2]

34. [8 marks]

A Binomial probability distribution has n trials and probability of success of p.
A Normal distribution with mean of μ can be used to approximate Binomial distributions.
The Binomial probability $P(X = \mu)$ is approximated by the Normal probability
$P(a < X < b) = 0.3169$

(a) Show that the standard deviation of that distribution is 1.225, correct to three decimal places. [4]

(b) If it is known that the mean of the Normal distribution used is 2, that is $\mu = 2$, determine the value of n and the value of p, correct to three decimal places. [3]

(c) Hence, calculate the Binomial probability that $X = \mu$, i.e. $P(X = \mu)$ [1]

Chapter 11: Sampling & Inference

This chapter includes questions about

Random Samples
Confidence Intervals based on sample proportion

1. [7 marks]

Roses – 2 – Go promise fast delivery of orders received for flowers.

Their delivery times, *t*, are uniformly distributed, and the range is between 100 and 180 minutes.

(a) Sketch, and define, the density function $T(t)$. [2]

$T(t) =$

(b) What is the probability that the next order will take between 2 and 2.5 hours? [1]

(c) What is the probability that three of the next five orders will each take between 2 and 2.5 hours? [2]

100 delivery times were recorded and their mean was calculated to be 135 minutes. This process was repeated many times. The group of means was graphed.

(d) (i) Sketch the shape of the graph. [1]

(ii) State the mean of those means. [1]

2. [4 marks]

On average, one in every 25 customers wins a prize at a promotion for Jenny's ice–cream. On each day of the promotion, 100 customers tried for a prize.

(a) Calculate the mean and standard deviation of the number of prize winners each day. [2]

(b) Estimate the probability that for 40 randomly selected promotion days, the mean was more than 4.5 prize winners. [2]

3. [2 marks]

Seaside Senior High School has 3 mathematics classes in Year 12, containing 25, 20, and 16 students respectively. A sample of 6 students is required and so 2 students are taken from each class.

(a) How many such samples of 6 can be selected? [1]

(b) Are these samples of 6 "random samples"? [1]

4. [2 marks]

Consider 200 samples each of size 50 drawn from a population with mean = μ .

The 200 samples are used to calculate 200 separate 90% confidence intervals for μ .

How many of these intervals would you expect to NOT include μ ? [2]

5. [3 marks]

A single die is tossed, and the result, X, is recorded.
The density function associated with the random variable X, has mean 3.5 and variance 2.916.

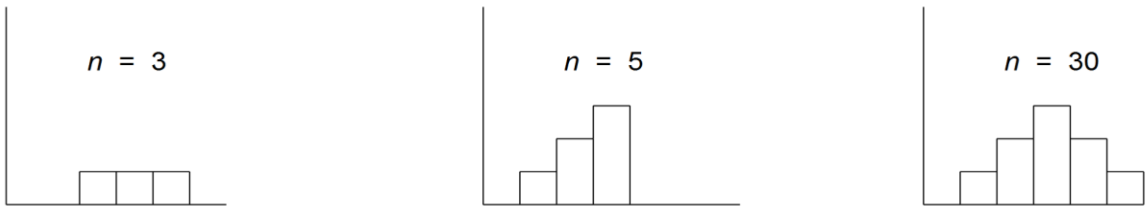
(a) Describe the distribution of X. [1]

The die is rolled 100 times and the mean recorded. This experiment is repeated many times.

(b) Complete: "These means should fluctuate about the value ________, and have a variance of ______." [2]

6. [1 mark]

Samples of size *n* are taken from a population. The distribution of these samples are graphed below for increasing values of *n*.



What is the effect of increasing the value of *n*? [1]

7. [6 marks]

(a) Investigate the truth of this statement:
"A 90% confidence interval for μ is 2.47 times as wide as a 50% confidence interval." [3]

(b) Complete the following statement"
"A 99% confidence interval is _________times as wide as a 90% confidence interval." [3]

8. [7 marks]

A sample of 57 Dockers supporters were asked whether they took their own food to the game. Seventeen of the group said they did.

(a) Is the sample size sufficiently large to use for inference? Explain. [2]

(b) Calculate the sample proportion. [1]

(c) Determine a 90% confidence interval for p , the population proportion. [2]

The ground capacity at the new Perth stadium will be 60 000.

(d) Use your answer to (c) to estimate the number of people who will be expected to bring their own food to a Dockers game. [1]

(e) How would that information be used by the management team at the new stadium? [1]

9. [3 marks]

A treatment for liverworm in a species of dog is effective in 45% of cases. A young vet believes he has a more effective treatment. He reported in the Dog Daily that he had successfully treated 25 dogs from a sample of 50. Use a 95% confidence interval to determine if his treatment is better. [3]

10. [3 marks]

A sample of 400 tomato seeds were genetically modified to produce a crop resistant to a strain of tomatoitis. Twenty of the seeds produced plants resistant to the disease. Calculate a 95% confidence interval for the proportion of plants which would react positively to the genetic modification. [3]

11. [3 marks]

A sample of 59 Perth footballers showed that 21 were above their playing weight when pre–season training began.
Find a 95% confidence interval for this proportion and write a brief interpretation for the coach with a "positive spin". [3]

12. [2 marks]

What is mathematically wrong with making a statement of this type, based on a normal distribution of sample proportions, with population proportion p?

$$P(2 < p < 3) = 0.95$$ [2]

13. [6 marks]

A die is rolled 100 times. On 15 of those rolls, the uppermost number was 4.

(a) What is the sample proportion? [1]

(b) Determine the mean and standard deviation of the distribution representing sample sizes of size 100. [2]

(c) If the die is rolled 1000 times, resulting in 150 fours, state a 90% confidence interval for the value of p, the population mean for the proportion of 4's. [3]

14. [2 marks]

A survey samples a population. The sample proportion is 0.8
To be 95% confident that the population proportion will be between 0.7 and 0.9, what would be the appropriate size of the survey? [2]

15. [3 marks]

For a sample of size 6000 and a sample proportion of 0.5, what is the margin of error at the 95% confidence level? [3]

16. [12 marks]

Two normal cubical dice are rolled and the sum of their uppermost faces is recorded.

(a) Complete the probability distribution for the discrete random variable X, where X is the sum of the 2 uppermost numbers. [4]

x	2	3	4	5	6	7	8	9	10	11	12
P (X = x)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	a	b	c	d	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

The dice are rolled 500 times in a simulation, and 80 times, the result was 10 or more.

(b) Calculate $\hat{p}$, the sample proportion of results of 10 or more in 500 trials. [1]

(c) (i) Determine a 95% confidence interval for p, the population proportion. [3]

(ii) Is p = Ex (X) found in that interval? [1]

(d) State the number of rolls required such that 99% of intervals of width 6 standard deviations, placed symmetrically in the distribution of those sample means, would contain p. [3]

17. [5 marks]

A radio poll of 980 people suggested that 42% of people were happy with the government's proposal to remove negative gearing as a tax deduction for property investment.

(a) Determine a 95% confidence interval for p, the proportion of people in the general population who are happy with the proposal. [3]

(b) Is it fair to say that a majority of Australians are not in favour of the government proposal? Explain. [2]

18. [3 marks]

State the circumstances where the formula for calculating sample size, n, required for a confidence interval for p could be written as

$$n = 0.25 \left(\frac{Z}{E}\right)^2$$

where Z is the z value corresponding to the confidence level, and E is the margin of error. Show using calculus or quadratic theory that those circumstances lead to that formula. [3]

19. [3 marks]

A travel survey question was : "Have you been to Bali?"
The survey was given to 400 people and 10% said they had been to Bali.
A report said that, based on the sample, between 7% and 13 % of Perth people had been to Bali.
How confident could the author of that report be with the accuracy of his statement? [3]

20. [5 marks]

The time, T minutes, that Erica spends at her local coffee shop has a mean of 30 and a standard deviation of 20.

(a) It is *not* likely that the distribution of T is normal. Explain why. [1]

$\bar{T}$ represents the mean time of a random sample of 40 of Erica's visits.

(b) Even though T is not normally distributed, the distribution of $\bar{T}$ will be normally distributed. Explain why. [1]

A sample of 40 such visits showed 65% lasted longer than 20 minutes.

(c) Calculate the probability, that another sample of 40 would show that more than 70% lasted more than 20 minutes. [3]

21. [14 marks]

A trout farm is selling all its current stock of adult trout, kept in a huge tank. It is estimated that there are 10 000 trout in the tank. Weights of trout in the tank are thought to be normally distributed with a mean of 1 kg, and a standard deviation of 0.2 kg.

(a) Determine the percentage of trout weighing between 0.9 kg and 1.2 kg. [2]

(b) Six trout are selected at random from the tank. What is the probability that exactly four of them will weigh between 0.9 kg and 1.2 kg? [3]

Grade A trout weigh more than one standard deviation above the mean.

(c) How many grade A fish are in the tank? [2]

The smallest 10% (by weight) of trout are used for fertiliser.

(d) What is the largest weight for such a trout? [2]

A sample of 25 carp are randomly selected. 16 of these weigh more than 1kg.

(e) Determine a 99% confidence interval for the proportion of carp which, based on this sample, would weigh more than 1 kg. [3]

(f) How large a sample should be taken to ensure, with 95% confidence, that the population proportion of carp is between 0.54 and 0.74? [2]

22. [5 marks]

An election had two candidates, A and B.
A random sample of 100 voters had 43% voting for A. Everybody votes.

There is a 95% chance that the true proportion of votes for candidate A will be between x and y.

(a) Determine x and y. [4]

Based on that sample, a polling company predicted the outcome with a margin of error of k%.

(b) Determine k. [1]

Solutions

Chapter 1: Functions

Exponential Functions

1. (a) 2.72✓
   (b) 2.72✓
   (c) 2.72✓
   (d) 2.72✓✓

2. (a) e^2 ✓✓
   (b) $e^{0.1}$ ✓✓

3. (a) $e^{0.3}$✓
   (b) $e^{0.5}$✓

4 (a) $e^{x+1} = e^{-x}$ ✓
   $\therefore 2x = 2 \rightarrow x = 1$ ✓
   (b) $2e^x.e + 3e^x\left(\frac{1}{3}\right) = xe^x$ ✓
   $\therefore 2ee^x + e^x = xe^x$
   $\therefore x = 2e + 1$ ✓

Logarithmic Functions

1. (a) $\log_3 y = x$ ✓
   (b) $\log_6 g = x + 1$ ✓
   (c) $M^t = \frac{h-3}{2}$ ✓
   $\therefore \log_M \left(\frac{h-3}{2}\right) = t$
   (d) $x^y = 3$ ✓
   (e) $\log_4 p = f - 1$ ✓
   $\therefore p = 4^{f-1}$ ✓
   (f) $\log x = -2y$ ✓
   $\therefore x = 10^{-2y}$ ✓

2. (a) $y\ln 2 = 2\ln x \rightarrow y = \frac{2\ln x}{\ln 2}$ ✓
   (b) $\log y = x - 1$ ✓
   $\therefore y = 10^{x-1}$ ✓
   (c) $\ln\left(\frac{2y}{3x}\right) = 2 \rightarrow \frac{2y}{3x} = e^2$ ✓
   $\therefore y = \frac{3xe^2}{2}$ ✓
   (d) $x = \frac{\ln 6}{\ln 2}$ ✓
   (e) $x^2 = 5 \rightarrow x = \sqrt{5}$ as $x > 0$ ✓
   (f) $2^{x-3} = \frac{5}{3}$ ✓
   $\therefore (x-3)\ln 2 = \ln 5 - \ln 3$ ✓
   $\therefore x = \frac{\ln 5 - \ln 3}{\ln 2} + 3$ ✓
   (g) $4 - \log_x 16 = 2 - \log_x 4$ ✓
   $\log_x 16 - \log_x 4 = 2$ ✓
   $\therefore \log_x 4 = 2 \rightarrow x = 2$ ✓

3. (a) $\frac{1}{4}$ ✓
   (b) $\log_3 (9x) = \log_3 (x+1)$ ✓
   $\therefore x = \frac{1}{8}$ ✓
   (c) $y = 8^x$ *and* $xy = \frac{2}{3}$ ✓
   $\therefore x = \frac{1}{3}$ *and* $y = 2$ ✓✓

4. (a) $\log_a 10 = \log_a 2 + \log_a 5$
   $= y + x$ ✓
   (b) $\log_a 0.4 = \log_a \frac{2}{5}$
   $= y - x$ ✓
   (c) $\log_a 50 = \log_a (25 \times 2)$
   $= \log_a 5^2 + \log_a 2$ ✓
   $= 2x + y$ ✓
   (d) $\log_a \sqrt{2a} = \frac{1}{2}[\log_a 2 + \log_a a]$ ✓
   $= \frac{1}{2}y + \frac{1}{2}$ ✓

5. (a) $y = x^3$✓
   (b) $\log_2 y + \log_2 x^2 = 2$ ✓
   $\therefore \log_2 (yx^2) = 2$
   $\therefore y = \frac{4}{x^2}$ ✓
   (c) $\log x + \log 10 - \log y^2 = 0$
   $\therefore \log \frac{10x}{y^2} = 0$ ✓
   $\therefore \frac{10x}{y^2} = 1 \rightarrow y^2 = 10x$ ✓
   $\therefore y = \sqrt{10x} \quad x > 0$ ✓

6. (a) $x = \dfrac{\log 73}{\log 5}$ ✓

(b) $3^{2x} = \dfrac{3^{3x}}{3^{4x+4}} \rightarrow 3^{2x} = 3^{-x-4}$ ✓

$\therefore 3x = -4 \rightarrow x = -\frac{4}{3}$ ✓✓

(c) $4y^2 - 8y - 12 = 0$

$\rightarrow 4(y-3)(y+1) = 0$ ✓✓

$\therefore 2^x = 3 \text{ or } 2^x = -1 \rightarrow x = \dfrac{\log 3}{\log 2}$ ✓✓

(d) $3^{5x}\left(3^1 - 1\right) = 2\left(3^{-2}\right)$ ✓

$3^{5x} = 3^{-2} \rightarrow 5x = -2 \therefore x = -\frac{2}{5}$ ✓✓

7. (a) $2\log_3 x = 1 \Rightarrow \log_3 x = \dfrac{1}{2}$ ✓✓

$\therefore x = \sqrt{3}$

(b) $\log_4 x + 2\log_4 4 = \log_4 (2x + 7)$

$\Rightarrow \log_4 16x = \log_4 (2x + 7)$ ✓

$\Rightarrow 14x = 7 \therefore x = \frac{1}{2}$ ✓✓

(c) $2^{3(x-y)} = 2^{-4}$ and $3^{x+3y} = 3^{\frac{3}{2}(2x+2)}$ ✓

$\Rightarrow 3x - 3y = -4$ and $3y - 2x = 3$ ✓

$\therefore x = -1,\ y = \frac{1}{3}$ ✓✓

(d) $2e^{2x} + 2e^{x} - 12 = 0$ ✓

$\Rightarrow 2y^2 + 2y - 12 = 0$

$\Rightarrow 2(y-2)(y+3) = 0$ ✓✓

$\therefore y = 2,\ -3$

$\Rightarrow e^x = 2 \Rightarrow x = \ln 2$ ✓

8. (a) $r = Ae^{kx}$

$\ell nr = \ell n(Ae^{kx})$ ✓

$\ell nr = \ell nA + kx$ ✓

ℓnA and k are constants

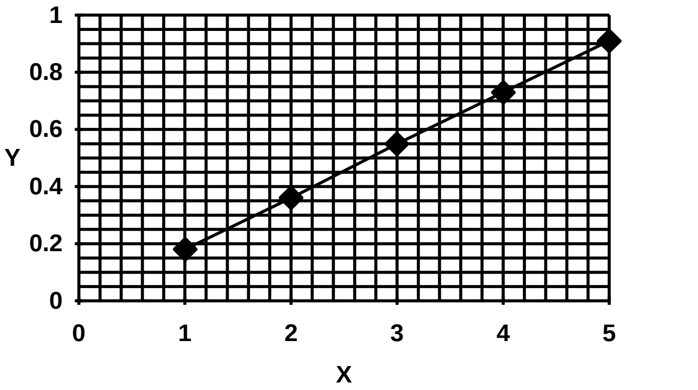
$\Rightarrow$ if relationship is true then ℓnr against x will graph as a straight line.

(b)

$y = \ell nr$	0·18	0·36	0·55	0·73	0·91

✓✓✓

(c)



✓✓✓✓

(d) $x = 1 \Rightarrow y = 0{\cdot}18$ etc. using calculator ✓

$\ell nr = -0{\cdot}003 + 0{\cdot}183x$ ✓

$k = 0{\cdot}183$ and $\ell n\,A = -0{\cdot}003$ ✓

$\therefore A = 0{\cdot}997$ ✓

(e) $A = 0{\cdot}997$ corresponds to $x = 0$ ✓

$\Rightarrow$ 0·997 cm is initial distance from light. ✓

(f) 1 complete revolution $\Rightarrow 360^\circ \Rightarrow 12 \times 30^\circ$

$\Rightarrow x = 12$ ✓

$\Rightarrow \ell nr = 2{\cdot}193$

$r = 8{\cdot}96$ cm ✓

after 1 revolution the insect will be 8·96 cm from light.

(g) $r = 30 \Rightarrow \ell nr = 3{\cdot}401$ ✓

$\Rightarrow x = 18{\cdot}6$ ✓

$\Rightarrow$ after 19 turns. ✓

Chapter 2: Growth & Decay

1. $Pe^{0.08}$ ✓✓

2. $\frac{A_1}{A_2} = \frac{e^{0.03}}{e^{0.06}}$ ✓✓
$= e^{-0.03}$ ✓

3. (a) If $W = W_0 e^{-kt}$ then $\frac{dW}{dt} = -kW_0 e^{-kt}$ ✓✓
$\therefore \frac{dW}{dt} = -kW$
(b) (i) $W_0 = 440$ ✓
(ii) $356 = 440e^{-7k}$ ✓
$\therefore$ $k = 0.030$ ✓
(c) $W = 440\, e^{-0.30(17)} = 264$ ✓✓
(d) $300 = 440\, e^{-0.30t}$ ✓
$\therefore$ $t = 12.8$ ie. 2005 ✓

4. (a) $P = 850\,000\, e^{0.025t}$
or $P\,(1000's) = 850\, e^{0.025t}$ ✓✓
(b) $P(16) = 850\, e^{0.025(16)}$ ✓
$= 1\,268\,100$ ✓
(c) $\frac{dP}{dt} = 0.025P = 0.025 \times 1268100$ ✓
$= 31703$ ✓
(d) $250\,000 = 1\,268\,100\, e^{-0.185t}$ ✓
$\therefore$ $t = 8.78$ ✓
$\therefore$ During 2004 the population of rabbits will fall to below 250 000. ✓✓

5. (a) On January 1st 2005
$\Rightarrow 284 = 2\pi r \Rightarrow r = \frac{284}{2\pi}$ ✓
As $r = r_0 e^{kt} \Rightarrow \frac{284}{2\pi} = r_0\, e^{0.045 \times 3}$ ✓
$\therefore$ $r_0 = 39.49$ ✓
(b) $r = 39.49e^{0.045t}$ ✓
(c) $84 = 39.49\, e^{0.045t} \Rightarrow t = 16.77$
During 2018 ✓✓
(d) $\frac{dr}{dt} = 1.777e^{0.045(9)} = 2.66$ ✓✓

6. (a) $A_t = me^{kt}$
$t = 0 \Rightarrow A_0 = me^0$ ✓
$A_0 = m$ ✓
(b) m = 42.0 and k = 0.189 correct to 3 sf ✓✓
(c) From calculator t = 0 $\Rightarrow$ $A_0 = 41.97$ ✓
From calculator t = 10 $\Rightarrow$ $A_{10} = 277.909$
Initial population = 42 insects ✓
Population after 10 days = 278 insects ✓
(d) $A_t = 500 \Rightarrow t = 13.107$ ✓
Population at least 500 after 14 days. ✓

7. (a) $k = 0{\cdot}067$ ✓✓
(b) $P_4 = 38.236 \Rightarrow$ Yes, tractor can be ✓
driven further (P > 30) ✓
$P_t = 30 \Rightarrow t = 7.617$ ✓
3.62 hours further before damage to tyre. ✓

8. (a) Let $N_t = N_0 e^{-kt}$
$N_0 = 110.17$ and $k = 0.0047$ ✓✓
$N_t = 110.17e^{0.00047t}$ ✓
(b) Initial number of particles detected = 117
t = 150 $\Rightarrow$ $N_{150} = 60.37$ ✓
60 particles detected after 150 seconds ✓

9. (a) $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$
$\therefore \frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$ ✓
$\therefore \frac{dy}{dx} = y$ ✓
$\rightarrow a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$
$a_0 = a_1 = 1\,;\, 2a_2 = a_1 \rightarrow a_2 = \frac{1}{2}$ ✓
and $3a_3 = a_2 \rightarrow a_3 = \frac{1}{6}$ ✓
(b)
$y = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \ldots\ldots$
$y = e$ when $x = 1$ ✓✓
Note: $\frac{dy}{dx} = y$ *gives a clue to this.*

10. (a) $\frac{dc}{dt} = -kc$ ✓

$c = Ae^{-kt}$ ✓

(b) $t = 0$, $c = 54$ $t = 1$, $c = 39$✓

$A = 54 \qquad 39 = 54e^{-k}$

$\ln 39 = \ln 54 - k$

$-k = -0.325$ ✓✓

$c = 54e^{-0.325t}$

(c) $c = 27 \Rightarrow 27 = 54e^{-0.325t}$ ✓

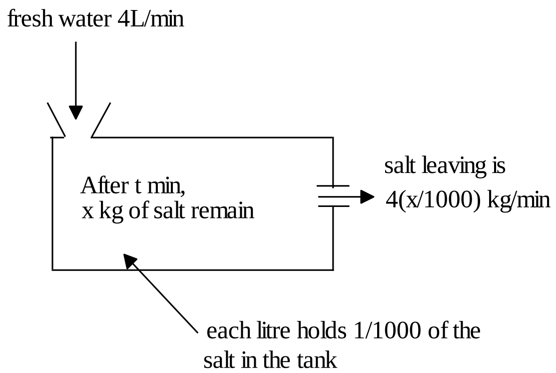
t = 2.13 hours from GC✓✓

one half concentration after 2.13 hours

(d) $t = 5 \Rightarrow c = 54e^{-0.325 \times 5}$ ✓

c = 10.63.✓

11.



(a) The rate of change of salt in the tank is

$\frac{dx}{dt} = -\frac{4}{1000}x$ ✓

(b) Hence k = - 0.004✓

(c) Hence $x = Ae^{-0.004t}$

At t = 0, x = 70

i.e. $70 = Ae^{-0.004(0)}$ ✓

i.e. A = 70

$x = 70e^{-0.004t}$

at t = 60, x = 55.06 kg✓✓

(d) $1 = 70e^{-0.004t}$

i.e. $\frac{1}{70} = e^{-0.004t}$

$\therefore -0.004t = \ln\left(\frac{1}{70}\right)$ ✓✓✓

∴ t = 1062.12 min

i.e. t must be greater than 17.7 hours

12. (a) $y = 1\,000e^{0\cdot 04t}$ ✓✓

(b) $y = 1\,000e^{0\cdot 04\times 5} = 1\,221$ ✓✓

(c) $2 = e^{0\cdot 04t}$ ✓

$\therefore \quad t = 17.33$✓

Chapter 3: Differentiation

Derivatives of Polynomials

1. (a) A′(x) is represented by D(x) ✓
   (b) B′(x) is represented by B(x) ✓
   (c) C′(x) is represented by B(x) ✓
   (d) D′(x) is represented by C(x) ✓

2. (a) $f'(x) = 6x + 2$
   $f'(x) = 0$ when $x = -\frac{1}{3}$ ✓
   ∴ TP $\left(-\frac{1}{3}, -\frac{1}{3}\right)$✓
   (b) $f''(x) = 6$
   Since $f''(x) \neq 0$, then no POI exists.✓
   (c) $f(x) = ax^2 + bx + c$
   ∴ $f'(x) = 2ax + b$ and $f''(x) = 2a$ ✓
   Since a is a non-zero constant $2a \neq 0$
   ∴ No POI exists ✓

3. (a) $\frac{dy}{dx} = 6x \quad \frac{4}{x^2} \quad \frac{9}{2}x^{\frac{1}{2}}$ ✓✓✓
   (b)
   $$\frac{dy}{dx} = \frac{(2x+1)(-1) - (2-x)(2)}{(2x+1)^2}$$ ✓✓
   (c)
   $$\frac{dy}{dx} = x\sqrt[3]{1-3x} + \frac{1}{2}x^2\frac{1}{3}(1-3x)^{-\frac{2}{3}}(-3)$$ ✓✓✓

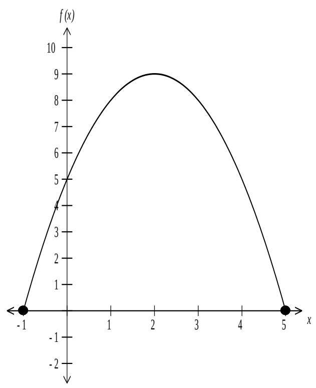
4. (a) (i) (–2, 0), (0, 0), (1, 0) ✓
   (ii) (–1.22, 2.11), (0.55, –0.63) ✓✓
   (iii) (–0.33, 0.74) ✓
   (b) –0.33✓
   (c) (i) $-2 < x < 0$ or $x > 1$✓
   (ii) $-1.22 < x < 0.55$✓
   (iii) $x > -0.33$✓

5. (a) (i) $\frac{dy}{dt} = 3t^2 - 4$ ✓
   (ii) $\frac{dt}{dx} = \frac{1}{2\sqrt{x+1}}$ ✓
   (b) $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$ ✓
   ∴ $\frac{dy}{dx} = (3t^2 - 4) \cdot \frac{1}{2\sqrt{x+1}}$ ✓
   ∴ $\frac{dy}{dx} = \frac{3(x+1) - 4}{2\sqrt{x+1}} = \frac{3x-1}{2\sqrt{x+1}}$ ✓

6. (a)
   $$f'(x) = \frac{3x^2}{2} + 6x^{-4} + \sqrt{\frac{2}{2}}\,x^{-\frac{1}{2}}$$ ✓✓
   $$= \frac{3x^2}{2} + \frac{6}{x^4} + \sqrt{\frac{2}{2\sqrt{x}}}$$ ✓
   (b)
   $$g'(x) = \frac{(3-x)(2) - (2x)(-1)}{(3-x)^2}$$ ✓✓
   (c)
   $$\frac{dm}{dv} = v^2 \times \frac{1}{2}\left(v^3 - 4\right)^{-\frac{1}{2}} \times \left(3v^2\right) + 2v \times \sqrt{v^3 - 4}$$ ✓✓
   $$= \frac{3v^4}{2\sqrt{v^3 - 4}} + 2v \times \sqrt{v^3 - 4}$$ ✓
   (d)
   $$x'(t) = \frac{3}{4}(2t-1)^{-\frac{1}{4}}(2)$$ ✓✓
   $$= \frac{3}{2\sqrt[4]{2t-1}}$$ ✓

7.
   $$y = \left(x + \left(x + (x)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}$$
   $$\frac{dy}{dx} = \frac{1}{2}\left(x + \left(x + x^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{-\frac{1}{2}}\left(1 + \frac{1}{2}\left(x + x^{\frac{1}{2}}\right)^{-\frac{1}{2}}\right).\left(1 + \frac{1}{2}x^{-\frac{1}{2}}\right)$$
   ✓✓✓✓

8. (a) (i) 9 ✓
   (ii) 0 ✓
   (iii) –2 ✓
   (b) (i)



✓✓✓

   (ii) $f(2)$ is the y value of the turning point. ✓
   (ii) $f'(2)$ suggests the function has a stationary point at $x = 2$.
   i.e. the gradient of the tangent at (2, 9) is zero. ✓
   (iii) $f''(2)$ suggests that the function is concave down at $x = 2$. ✓

9. (a) $\dfrac{4(x+3)}{(x-3)(x+3)} = \dfrac{4}{x-3}$ ✓

Restrictions are $x \neq \pm 3$ ✓

(b) $\dfrac{dz}{dx} = -\dfrac{1}{3x^2}$ and $\dfrac{dy}{dx} = \dfrac{-4}{(x-3)^2}$ ✓✓

$$\frac{dz}{dy} = \frac{dz}{dx} \times \frac{dx}{dy}$$

$\therefore \quad \dfrac{dz}{dy} = \dfrac{-1}{3x^2} \times \dfrac{(x-3)^2}{-4}$ ✓

$\therefore \quad \dfrac{dz}{dy} = \dfrac{(x-3)^2}{12x^2}$ ✓

Derivatives of Exponential Functions

1. (a) $\dfrac{dy}{dx} = \dfrac{e^{3x}(3) - (3x-1)e^{3x}(3)}{e^{6x}}$ ✓✓

(b) $y = \left(3e^{4x}\right)^{\frac{1}{2}} = \sqrt{3}\ e^{2x}$

$\therefore \quad \dfrac{dy}{dx} = \sqrt{3}\ e^{2x}(2) = 2\sqrt{3}\ e^{2x}$ ✓✓

2. (a) $f'(x) = -2e^{-2x}x^3 + 3x^2e^{-2x} = x^2e^{-2x}(3-2x)$ ✓✓

(b) $f'(x) = 0 \Rightarrow x = 0, \dfrac{3}{2}$ ✓

$\therefore$ Co–ordinates $(0,0)$, $\left(\frac{3}{2}, \frac{27}{8}e^{-3}\right)$ ✓

3. (a) $f(x) = ex - x^3 + 2x^2 + c$ ✓

$\therefore (2, 2e-1) \rightarrow f(x) = ex - x^3 + 2x^2 - 1$ ✓

(b) $f'(2) = e - 4$ ✓

$\therefore \quad y = (e-4)x + c \rightarrow (2, 2e-1)$

$\rightarrow y = (e-4)x + 7$ ✓

(c) $f''(x) = -6x + 4$ ✓

$\therefore f''(x) = 0 \rightarrow \left(\dfrac{2}{3}, \dfrac{2}{3}e - \dfrac{11}{27}\right)$ ✓

4. (a) $\dfrac{dy}{dx} = \dfrac{xe^x - e^x}{x^2}$ ✓

when $x = 1$ $\dfrac{dy}{dx} = 0$ ✓

$\rightarrow \quad y = e$ ✓

(b) (i) $k > e$ ✓✓

(ii) $k = e$ ✓

(iii) $k < e$ ✓

5. (a) $f'(x) = x^2e^x + e^x(2x)$ ✓

$= e^x x(x+2)$

(b) $e^x x(x+2) = 0$ ✓

$\therefore \quad x = 0$ *or* $x = -2$ ✓

$\therefore (0, 0)$ *and* $\left(-2, \dfrac{4}{e^2}\right)$ ✓

Equations: $y = 0$ *or* $y = \dfrac{4}{e^2}$ ✓

Derivatives of Logarithmic Functions

1. (a) $\dfrac{dy}{dx} = \dfrac{2}{2x} + \dfrac{-1}{1-x} = \dfrac{1}{x} - \dfrac{1}{1-x}$ ✓✓

(b) $f'(x) = 2(\ln x)\left(\dfrac{1}{x}\right) + \dfrac{2}{x}$ ✓✓

(c) $\dfrac{dp}{dx} = 2xe^{x^2} - \dfrac{2ex}{ex^2} = 2xe^{x^2} - \dfrac{2}{x}$ ✓✓✓

(d) $h'(t) = 3t^2(\ln t^3) + t^3\left(\dfrac{3}{t}\right) = 3t^2(\ln t^3) + 3t^2$ ✓✓✓

2. $f'(x) = x^2 \times \dfrac{2}{x} + 2x(\ln x^2) = 2x + 4x\ln x$ ✓

$m = 2$ when $x = 1$ ✓

$\therefore y = 2x + c$ ✓

$(1, 0) \rightarrow c = -2$

$\therefore y = 2x - 2$ ✓

3. $\dfrac{dy}{dx} = \dfrac{2}{2x-1}$ ✓

$\therefore \dfrac{2}{2x-1} = 2 \rightarrow 2x - 1 = 1$ ✓

$\therefore x = 1 \rightarrow (1, 0)$ ✓

4. (a) Equation joining (0, 4) and (30, 0) is

$\ln I = \frac{-2}{15}x + 4$ ✓✓

(b) $I = e^{-\frac{2}{15}x+4}$ ✓

(b) $I = e^{\left(\frac{-2}{15}x + 4\right)}$

$\frac{dI}{dx} = \frac{-2}{15}I$ ✓✓✓

$\Rightarrow k = \frac{-2}{15}$

(d) At surface ln I = 4

$I = e^4$

$0.1e^4 = e^{\left(\frac{-2}{15}x + 4\right)}$

$\ln 0.1 + 4 = \frac{-2}{15}x + 4$ ✓✓

$\frac{-2}{15}x = \ln 0.1$

$\Rightarrow x = \frac{-15}{2}\ln 0.1$

$\Rightarrow x = 17.27$ *metres*

Derivatives of Trig Functions

1. (a) $\frac{dy}{dx} = 2\cos x - \frac{1}{2}\sin\frac{x}{2}$ ✓✓

(b) $\frac{dx}{dt} = 2\sin t\cos t - 6\sin 2t$ ✓✓

(c)

$f'(x) = 3(\sin x - \cos x)^2(\cos x + \sin x)$ ✓✓

2. $\frac{dy}{dx} = (2\cos 3x)(-3\sin 3x) = -6\cos 3x\sin 3x$ ✓

$\therefore \frac{dy}{dx} = 3$ when $x = \frac{\pi}{4}$ ✓

$\therefore y = 3x + c$

$\left(\frac{\pi}{4}, \frac{1}{2}\right) \rightarrow c = \frac{1}{2} - \frac{3\pi}{4}$ ✓

$\therefore y = 3x + \frac{1}{2} - \frac{3\pi}{4}$ ✓

3. (a) $\frac{d}{dx}\left((\cos^3 2x - \sin \pi x)^{\frac{1}{3}}\right)$ ✓

$= \frac{1}{3}(\cos^3 2x - \sin \pi x)^{\frac{2}{3}}((3\cos^2 2x)(-2\sin 2x) - \pi\cos \pi x)$

✓✓

(b)

$\frac{dy}{dx} = 4(\cos^3 3x)(-3\sin 3x) = (-12\cos^3 3x)(\sin 3x)$ ✓

$\frac{d^2y}{dx^2} = (-36\cos^2 3x)(-3\sin 3x)(\sin 3x) + (3\cos 3x)(-12\cos^3 3x)$

✓✓

(c)

$f'(x) = \frac{(x - \cos x)(\cos x) - (\sin x)(1 + \sin x)}{(x - \cos x)^2}$ ✓✓✓

4. $\frac{dy}{dx} = \cos(1 - 2x) \times (-2) = -2\cos(1 - 2x)$ ✓

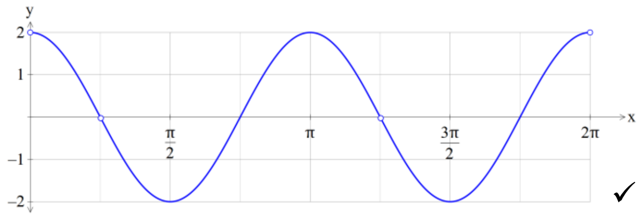
Since $m = -2$ when $x = \frac{1}{2}$, then $\perp m = \frac{1}{2}$ ✓

$\therefore y = \frac{1}{2}x + c$

$\left(\frac{1}{2}, 3\right) \rightarrow c = \frac{11}{4}$

$\therefore y = \frac{1}{2}x + \frac{11}{4}$ ✓✓

5.



✓✓✓✓

Derivatives of Combined Functions

1. (a) $\frac{dy}{dx} = 9(3x-3)^2$ ✓

(b) $p'(a) = \frac{1}{\sqrt{a}} + 3e^{3a+2}$ ✓✓

(c) $g(p) = 2p - \frac{3}{p} \rightarrow g'(p) = 2 + \frac{3}{p^2}$ ✓✓

2.

$y = 3\sin x \qquad x = \frac{\pi}{u} = \pi u^{-1}$

$\frac{dy}{dx} = 3\cos x \qquad \frac{dx}{du} = -\frac{\pi}{u^2}$

$\frac{dy}{du} = \frac{dy}{dx}.\frac{dx}{du}$ ✓

$\frac{dy}{du} = 3\cos x.-\frac{\pi}{u^2}$

$\frac{dy}{du} = \frac{-3\pi\cos x}{u^2}$ ✓

$\text{when } u = 2, \ \frac{dy}{du} = \frac{-3\pi\cos\frac{\pi}{2}}{4}$

$= 0$ ✓

3. (a)

$y = e^{\sqrt{x}} = e^{x^{\frac{1}{2}}}$

$y' = \frac{1}{2}x^{-\frac{1}{2}}.e^{x^{\frac{1}{2}}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$ ✓ ✓

(b)

$= 3cos(x^2) + (3x+2).2x\,sin(x^2).(-1)$ ✓✓✓

$= 3cos(x^2) - 2x(3x+2)sin(x^2)$

(c)

$y' = \frac{(x^4+4)(3x^2) - (x^3+3).4x^3}{(x^4+4)^2}$ ✓

$= \frac{-x^6 - 12x^3 + 12x^2}{(x^4+4)^2}$ ✓

(d)

$y' = \frac{1}{2-\sin^3 x}.(-3)\sin^2 x.\cos x$ ✓ ✓ ✓

$= \frac{3\sin^2 x\cos x}{2-\sin^3 x}$

4.

$h(x) = (4x^2 - x)^{-3}$

$h'(x) = -3(4x^2 - x)^{-4}.(8g - 1)$ ✓✓

$= \frac{-3(8x-1)}{(4x^2-x)^4}$ ✓

5. (a) $f'(\theta) = -2\sin 2\theta\, e^{\cos 2\theta}$ ✓✓

(b)

$g'(t) = 3.2\cos(6\pi - 2t).-(-2)\sin(6\pi - 2t)$

$= 12\sin 2(6\pi - 2t)$

✓✓

(c) $L'(t) = 2(1-t).(-1)\ln t + \frac{1}{t}.(1-t)^2$

$= -2(1-t)\ln t + \frac{(1-t)^2}{t}$ ✓ ✓

6. $f(x) = xe^x \quad \textit{for all } x$

$f'(x) = 1.e^{-x} + (-1)xe^{-x} = e^{-x}(1-x)$ ✓

$\textit{Let } f'(x) = 0 \Rightarrow e^{-x}(1-x) = 0$

$\textit{and } f''(x) = -e^{-x}(1-x) + e^{-x}(-1)$

$= -e^{-x}(2-x)$ ✓

and $f''(1) < 0 \Rightarrow$ relative minimum at $x = 1$

As $f'(x) > 0$ *for* $x < 1$

$f'(x) < 0$ *for* $x > 1$ ✓

then the global maximum of

f *is at* $x = 1$ *and* $f(1) = \frac{1}{e}$ ✓

7. (a) $e^{x\ln 5} = (e^{\ln 5})^x$ ✓

$= (5)^x$ ✓

(b) If $y = 5^x\sin x$

then $\frac{dy}{dx} = \ln 5.e^{x\ln 5}.\sin x + 5^x.\cos x$ ✓

i.e. $y' = 5^x(\ln 5\sin x + \cos x)$

(c) For stationary points let $y' = 0$

$\therefore\ 5^x\left(\ln 5.\sin x + \cos x\right) = 0$

i.e. $\ln 5.\sin x = -\cos x$ ✓

i.e. $\tan x = -\dfrac{1}{\ln 5}$

i.e. $\tan x = -0.6213$ ✓

i.e. $x\ -0.556\ ,\ 2.586$ ✓

$y'(2.5) = 9.05$ *and* $y'(2.6) = -1.78$

$\therefore$ maximum at $x \approx 2.586$ ✓

Chapter 4: Applications of Differentiation

1. (a) $\dfrac{dV}{dt} = \dfrac{dV}{dr} \times \dfrac{dr}{dt} = 4\pi r^2 \times 300 = 1200\pi r^2$ ✓

When $t = 2,\ r = 600$ ✓

$\therefore \dfrac{dV}{dt} = 1200\pi \times 360\,000 = 432\,000\,000\pi$

$\therefore \dfrac{dV}{dt} \approx 1\,357\,168\,026\ m^3/\text{s}$ ✓

(b) $\delta A = \dfrac{dA}{dr} \times \delta r = 8\pi r \times 0.02r = 0.16\pi r^2$ ✓

$\therefore \dfrac{\delta A}{A} = \dfrac{0.16\pi r^2}{4\pi r^2} = 0.04$ ✓

$\therefore$ Increase of 4% ✓

2. (a) Since similar triangles, then $\dfrac{r}{3} = \dfrac{6-h}{6}$ ✓

$\therefore\ 6r = 18 - 3h \rightarrow h = 6 - 2r$ ✓

(b) $V = \pi r^2 h$ ✓

$\therefore\ V = \pi r^2(6 - 2r)$ ✓

$\therefore\ V = 6\pi r^2 - 2\pi r^3$

(c) $V'(r) = 12\pi r - 6\pi r^2 = 0$ ✓

$\therefore\ 6\pi r(2 - r) = 0$

$\therefore\ r = 2$ m ✓

Since $V''(r) = 12\pi - 12\pi r$

then $V''(2) < 0\ \therefore$ maximum ✓

(d) $V = \pi(4)(2) = 8\pi\ \text{m}^3$ ✓✓

3. (a) $f'(x) = x^2 e^x + e^x(2x)$ ✓

$= e^x x(x + 2)$

(b) $e^x x(x + 2) = 0$ ✓

$\therefore\ x = 0$ *or* $x = -2$ ✓

$\therefore\ (0, 0)$ *and* $\left(-2, \dfrac{4}{e^2}\right)$ ✓

Equations: $y = 0$ *or* $y = \dfrac{4}{e^2}$ ✓

4. (a) $y^2 = 100 + x^2$ ✓

$\therefore\ y = \sqrt{100 + x^2}$

(b) $Cost = 7(15 - x) + 9y$ ✓

$\therefore\ = 105 - 7x + 9\sqrt{100 + x^2}$ ✓

(c) 12.374 km = 12374 m ✓✓

(d) $161 569 ✓

(e) First derivative test ✓

- find the first derivative and then take values each side of 12.37, and test to make sure that it is negative on the left and positive on the right. ✓

5. (a) $V = \pi r^2 h = \pi(25)\,h$ ✓✓

$\therefore\ V = 25\pi\, h$

(b) $\dfrac{dV}{dh} = 25\pi$ m³/min ✓

(c) $\dfrac{dV}{dt} = \dfrac{dV}{dh} \times \dfrac{dh}{dt}$ ✓

$\therefore\ 2 = 25\pi \times \dfrac{dh}{dt}$ ✓

$\therefore\ \dfrac{dh}{dt} = \dfrac{2}{25\pi}$ m/min ✓

(d) $\dfrac{dh}{dt}$ *is constant* $\therefore\ \dfrac{2}{25\pi}$ m/min ✓

6. (a) B, D ✓

(b) D < x ≤ E ✓

(c) B < x < C ✓

(d) B, since $f'(x) = 0$ and $f''(x) = 0$ ✓✓

7. (a) $C'(100) = \$40$ ✓

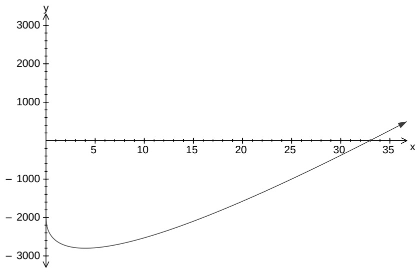
The approximate cost of producing the 101st item. ✓

(b) $C(x) = 800\sqrt{x} + 2\,000$ ✓

$P(x) = 200x - (800\sqrt{x} + 2\,000)$ ✓

$\therefore\ P(x) = 200x - 800\sqrt{x} - 2\,000$

(c)



33 items ✓

(d) \$2 800 ✓✓

8. (a) $\dfrac{dV}{dr} = 4\pi r^2 = 4 \times \pi \times 25^2$ ✓✓

$= 2\,500\pi = 7\,853.98$

(b) $V_0 = 36\,000\pi$ ✓

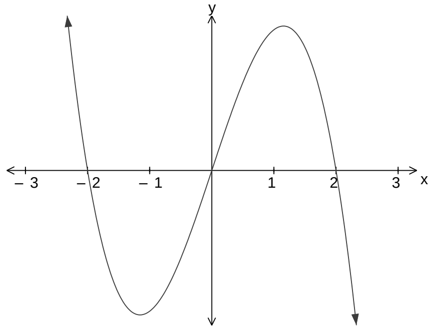
$\therefore$ r = 29.72 ✓

$\therefore \dfrac{dV}{dt} = 4\pi r^2 \times$ dr/dt ✓

$\rightarrow\ 4\pi(29.72)^2 \times$ dr/dt = -100π

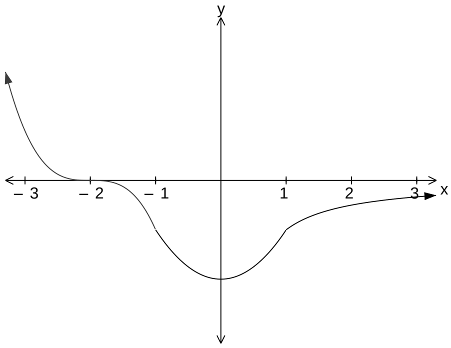
$\dfrac{dr}{dt} = -\ 0.03$ cm/sec ✓✓

9. (a)



✓✓✓

(b)



✓✓✓✓✓

10. (a) $4 = 2(3)^3 - 5(3)^2 + 3b + 1 \therefore b = -2$ ✓

(b) $\dfrac{dx}{dt} = v = 6t^2 - 10t - 2$ ✓

$v(3) = 22$ m/s ✓

(c) $v(0) = -2$ m/s $\therefore$ Speed = 2 m/s ✓✓

(d) $0 = 6t^2 - 10t - 2$

$\rightarrow\ t = -0.18s$ or $1.85s$ ✓✓

$\therefore$ Distance from origin when particle stops is 7.15 m at 1.85s. ✓

(e) $6 = 6t^2 - 10t - 2 \therefore t = -0.59$ *or* 2.26 seconds. ✓

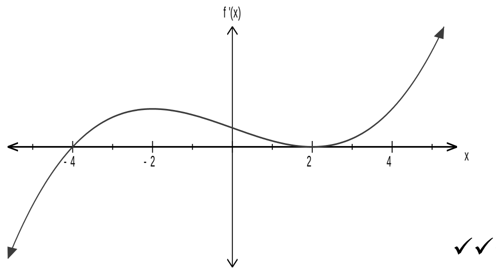
$\dfrac{d^2x}{dt^2} = a = 12t - 10$ ✓

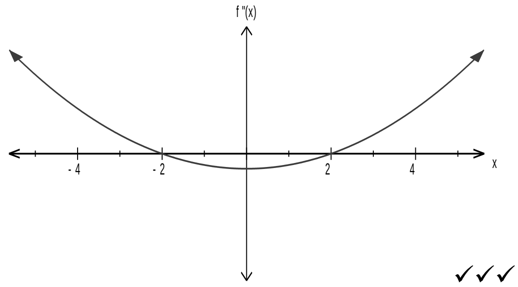
$\therefore\ a(2.25) = 17\text{ms}^{-2}$ ✓

11. (a) Number = $20 + 5x$, Cost = $100 - 2x$
Revenue =
$(20 + 5x)(100 - 2x) = 2000 + 460x - 10x^2$ ✓✓
(b) $P(x) = R(x) - C(x)$
$R(x) = 2000 + 460x - 10x^2$ and $C(x) = 12(20 + 5x)$ ✓✓
$\therefore P(x) = 1760 + 400x - 10x^2$ ✓
(c) $P'(x) = 400 - 20x = 0 \rightarrow x = 20$ ✓
Maximum profit of \$5760 when jumpers cost \$60. ✓✓
(d) Number = $20 + 5(20) = 120$ students. ✓
∴ 50 students have not brought a jumper. ✓
(e) $170 = 20 + 5x \rightarrow x = 30$ ✓
∴ The jumpers should be sold at \$(100 – 2(30)) = \$40. ✓

12. (a) (i) Price = $(150 - 0.5x)$ ✓
(ii) $R(x) = (150 - 0.5x)(5000 + 50x)$
$= 750000 + 7500x - 2500x - 25x^2$
$= 750000 + 5000x - 25x^2$ ✓✓
(b) $P(x) = R(x) - C(x)$
$P(x) = 750000 + 5000x - 25x^2 - 250000$
$\therefore P(x) = 500000 + 5000x - 25x^2$ ✓
(c) Maximum when $\frac{dP}{dx} = 0 \Rightarrow 5000 - 50x = 0$
$\therefore x = 100$ ✓
$P(100) = 750000$ and
Price = $(150 - (0.5 \times 100)) = 100$
Maximum Profit of \$750,000 when tickets are sold for \$100. ✓

13. Volume = 562500cm^3
$\therefore 562500 = 2x^2y \Rightarrow y = \frac{562500}{2x^2}$ ✓
$\text{SA} = (2xy) + 2(2x^2) + 2(xy)$
$\therefore \text{SA} = 4x^2 + 4xy$ ✓
$= 4x^2 + 4x\left(\frac{562500}{2x^2}\right) \Rightarrow 4x^2 + \frac{1125000}{x}$ ✓
$\frac{dSA}{dx} = 0$
$\therefore 8x - \frac{1125000}{x^2} = 0 \therefore x = 52.002$ ✓
∴ The dimensions of the box that minimize the SA, are Length 104.004 cm, Height 52.002 cm and Width 104.004 cm. ✓

14.
(a)


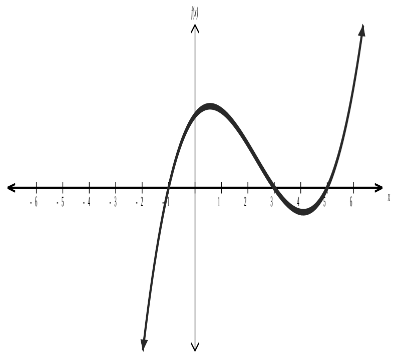
✓✓✓
(b)


✓✓✓

15. When $x = 1$, $f'(x)$ and $f''(x) = 0$.
$\therefore f'(1) = 0 \Rightarrow 4a + 3b = 12$ ✓
$\therefore f''(1) = 0 \Rightarrow 12a + 6b = 0$ ✓
$(1, 7) \Rightarrow a + b - 12 + c = 7$ ✓
Solve $a = -6$, $b = 12$, $c = 13$ ✓✓✓

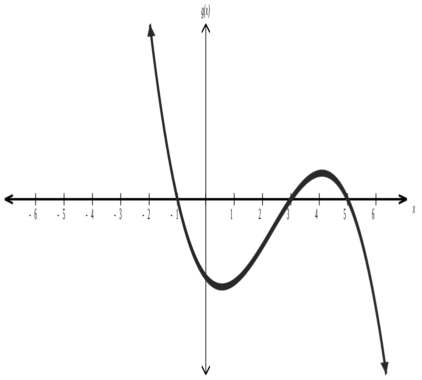
16. (a) Area Triangle= $\frac{1}{2}x \cdot \frac{1}{4}h = \frac{1}{8}xh$,
Area Rectangle = $x\left(\frac{3}{4}h\right)$ ✓
$\therefore$ Total Area = $\frac{7}{8}xh$
$V = \frac{7}{8}\text{xh} \cdot 2x = \frac{7}{4}x^2h$ ✓✓
(b) SA = 5 m^2
$5 = 2\left(\frac{3}{4}h \times 2x\right) + 2(0.6 \times 2x) + \frac{7}{8}xh$ ✓
$h = (5 - 2.4x) \div \frac{31x}{8} = \frac{8}{31x}(5 - 2.4x)$ ✓✓
(c)
$V = \frac{7}{4}\left(\frac{8}{31x}(5 - 2.4x)\right)x^2 = \frac{14}{31}\left(5x - 2.4x^2\right)$
Calculator gives ✓
Width = 1.04 , Length = 2.08,
Height= 0.619 m. ✓

17. (a)



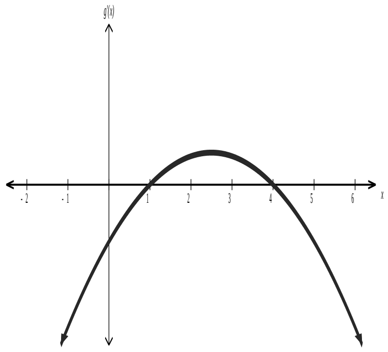
✓✓✓

(b)



✓✓

(c)



✓✓

18. (a) $R = N \times C$

$\rightarrow$ $N = (100 + 4x)$, $C = (100 - 2x)$

$R = (100 + 4x)(100 - 2x) = 10\,000 + 400x - 200x - 8x^2$ ✓✓

$= 10\,000 + 200x - 8x^2$

(b) Break even when $P = 0$.

$\therefore$ $10\,000 = (100 + 4x)(100 - 2x)$

Using GC $x = 0$ *or* $x = 25$ ✓

Tickets cost \$50, school will break even. ✓

If tickets cost \$50, 200 people will attend the ball. ✓

19. (a) $C'(5) = \$122.50$ ✓

This is the change in the cost in producing the 5th washing machine from the 4th machine.✓

(b) $C(d) = \frac{25}{2}d^2 - \frac{1}{30}d^3 + 55$ ✓✓

(c) $\frac{C(10)}{10} = \frac{1271\frac{2}{3}}{10} = \$127.17$ ✓✓

(d)

$P = R - C \rightarrow 320d - \left(\frac{25}{2}d^2 - \frac{1}{30d^3} + 55\right)$

$= 320d - \frac{25}{2}d^2 + \frac{1}{30}d^3 - 55$

✓

Local Maximum (13.53,2068.9). ✓

Maximum profit occurs when 14 dishwashers produced. ✓

20. (a) $\frac{r}{4} = \frac{h}{12} \therefore h = 3r$ ✓

(b) $V = \pi r^3 \therefore \frac{dV}{dr} = 3\pi r^2$ ✓✓

$\rightarrow \frac{dV}{dr}(3) = 27\pi$

(c) $\frac{dV}{dt} = 3\pi r^2 \times \frac{dr}{dt} = 3\pi$ ✓

$\therefore \frac{dr}{dt} = \frac{3\pi}{3\pi(1)^2} = 1$ m/min ✓✓

21. $x = Ae^{kt}$

$\therefore \frac{dx}{dt} = k\,Ae^{kt}$

$\therefore \frac{dx}{dt} = kx$ ✓

$\therefore \frac{d^2x}{dt^2} = \frac{d}{dt}(kAe^{kt}) = k^2\,Ae^{kt}$ ✓

$\therefore \frac{d^2x}{dt^2} = k^2 x$ ✓

$\therefore x\frac{d^2x}{dt^2} = k^2x^2 = \left(\frac{dx}{dt}\right)^2$ ✓

$\Rightarrow$ $x = Ae^{kt}$ is a solution.

22. $V = \frac{1}{3}\pi r^3 \quad (r = h)$

$$\frac{dV}{dr} = \pi r^2 \quad ✓$$

$$\left.\frac{dV}{dr}\right|_{r=10} = 100\pi \quad ✓$$

$$\delta V = \left(\frac{dV}{dr}\right)\delta r$$

$$2 = 100\pi\delta r$$

$$\delta r = \frac{2}{100\pi} \quad ✓$$

Change in radius required is 0.0063661 cm ✓
Note: Without using calculus result is 0.006362 cm

23. $T = k\sqrt{\ell}$

$$\frac{dT}{d\ell} = \frac{1}{2}k\frac{1}{\sqrt{\ell}} \quad ✓$$

$$\delta T = \left(\frac{dT}{d\ell}\right)\delta\ell$$

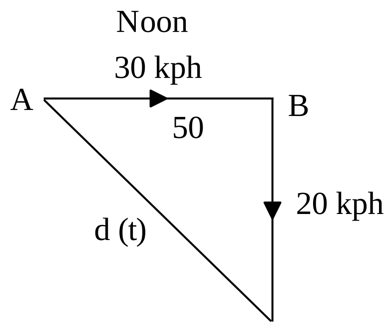
$$\delta T = \frac{1}{2}k\frac{1}{\sqrt{\ell}}0.01\ell \quad ✓$$

$$\delta T = 0.01.\frac{1}{2}k\sqrt{\ell}$$

$$\delta T = 0.005T \quad ✓$$

i.e. an increase of $\frac{1}{2}\%$ ✓

24. (a)



$$d(0) = 50$$

$$d(t) = \sqrt{(50 - 30t)^2 + (20t)^2} \quad ✓$$

$$= \sqrt{2500 - 3000t + 900t^2 + 400t^2}$$

$$= \sqrt{2500 - 3000t + 1300t^2} \quad ✓$$

(b) $d'(t) = \dfrac{2600t - 3000}{2\sqrt{2500 - 3000t + 1300t^2}}$ ✓

$$d'(1) = \frac{-400}{2\sqrt{800}} \quad ✓$$

$$= \frac{-40}{2\sqrt{8}} = -5\sqrt{2}$$

(c) collision if d(t) = 0

$$2500 - 3000t + 1300t^2 = 0 \quad ✓$$

$$25 - 30t + 13t^2 = 0$$

$$\Delta = 900 - 4x25x13 < 0 \quad ✓$$

$\Rightarrow$ no solution

$\Rightarrow$ no collision ✓

minimum $d(t)$ when $d'(t) = 0$

$$2600t - 3000 = 0$$

$$t = \frac{15}{13} \quad ✓$$

closest at 1.09 pm when 27.73 km apart ✓

25. (a) $\delta x = 0{\cdot}01$ and $\delta P = \frac{dP}{dx} \times \delta x$ ✓

$\therefore\ \delta P = \left(2 - \frac{128}{2^2}\right)(0{\cdot}01) = -0.3$ ✓✓

$\therefore$ 30 cm decrease in P ✓

(b) $P = 2x + \frac{128}{x}$

$P' = 2 - \frac{128}{x^2}$ ✓

$P'(x) = 0$ when $2 - \frac{128}{x^2} = 0$

i.e. $x = 8$ ✓

$P''(x) = \frac{256}{x^3}$

Since $P''(8) > 0 \therefore$ a minimum ✓

(c) $P = 32$ m $\therefore$ Dimensions are 8 x 8 m ✓

(d) $A = 64\ m^2$ ✓

26. (a) $\frac{dy}{dx} = 3x^2 + 4x + 1$ and $\frac{d^2y}{dx^2} = 6x + 4$ ✓

Point of Inflection occurs when $\frac{d^2y}{dx^2} = 0$ ✓

$\therefore$ $6x + 4 = 0 \rightarrow x = -\frac{2}{3}$ ✓

(b)

x	$-\frac{2}{3}^-$	$-\frac{2}{3}$	$-\frac{2}{3}^+$
$\frac{d^2y}{dx^2}$	–	0	+
$\frac{dy}{dx}$	–	–	–

$\therefore$ Point of inflection ✓✓

(c) When $x = -\frac{2}{3}$ then $m = \frac{dy}{dx} = -\frac{1}{3}$ ✓

$\therefore$ $y = -\frac{1}{3}x + \frac{73}{27}$ ✓

(d) Turning points occur when

$3x^2 + 4x + 1 = 0$ ✓

$\therefore$ $(-1, 3)$ and $\left(-\frac{1}{3}, \frac{77}{27}\right)$ ✓✓

Neither of these two points lie on

$y = -\frac{1}{3}x + \frac{73}{27}$ ✓

$\therefore$ Not true.

27. (a)

$\frac{dN(t)}{dt} = 0.04N(t)$

$\Rightarrow N(t) = Ae^{0.04t}$

$t = 0$ when $N(t) = 10{,}000$ ✓

$\Rightarrow N(t) = 10{,}000e^{0.04t}$ ✓

(b) After 1 hour $\Rightarrow$ t = 1

$N(1) = 10{,}000.\ e^{0.04}$ ✓

N(1) = 10,408

After 1 day $\Rightarrow$ t = 24

$N(24) = 10{,}000.\ e^{0.04x24}$ ✓

N(24) = 26,117 ✓

(c) $10{,}000.\ e^{0.04t} = 1{,}000{,}000$ ✓

$e^{0.04t} = 100$

$0.04t = \ln 100$ ✓

$t = \frac{\ln 100}{0.04}$

t = 115 hours ✓

(d) $N^* = 5{,}000e^{0.08t}$

Number of bacteria are equal when

$5{,}000\,e^{0.08t} = 10{,}000.\ e^{0.04}$ ✓

$e^{0.08t} = 2\,e^{0.04t}$ ✓

$0.08t = \ln 2 + 0.04t$

$0.04t = \ln 2$

$t = \frac{\ln 2}{0.04}$

t = 17 hours to the nearest hour ✓

28. (a) $S.A._{(cube)} = 6x^2$

$= 6\left(\sqrt{\ln(0.1t+1)}\right)^2$ ✓

$A = 6\ln(0.1t+1)$

$\frac{dA}{dt} = 6.\frac{1}{0.1t+1}.01$ ✓

$= \frac{0.6}{0.1t+1}$ ✓

(b)

$\left.\frac{dA}{dt}\right|_{t=60} = \frac{0.6}{0.1x60+1} = 0.0857$ ✓

$\delta A = \left(\frac{dA}{dt}\right)\delta t$

$\delta A = (0.0857).(\pm 0.01)$ ✓

$\delta A = \pm 0.0008567$ ✓

Possible error in surface area = ± 0.000857 cm^2

29. $C(x) = \frac{2}{x-3} + \ln(x-3)$

$C'(x) = \frac{-2}{(x-3)^2} + \frac{1}{x-3}$ ✓

For max. or min. $C'(x) = 0$

$\frac{-2}{(x-3)^2} + \frac{1}{x-3} = 0$ ✓

$-2 + x - 3 = 0$

$x = 5$ ✓

$C''(x) = \frac{4}{(x-3)^3} - \frac{1}{(x-3)^2}$

$C''(5) = \frac{4}{8} - \frac{1}{4}$

> 0 ✓

$\Rightarrow$ minimum cost when x = 5 ✓

30. $\frac{dL}{dv} = 100 \times \frac{1}{2}\left(1-\frac{v^2}{a^2}\right)^{-\frac{1}{2}} \times \frac{-2v}{a^2}$ ✓✓

$\delta L = -\frac{100v}{a^2}\left(1-\frac{v^2}{a^2}\right)^{-\frac{1}{2}}\delta v$ ✓

$\delta v = 0.009a$ when $v = 0.99a$ $\Rightarrow$ $\delta L = 6.32$ m✓✓

31. (a)

$I \propto \frac{\cos\theta}{d^2}$

$\Rightarrow I = \frac{K\cos\theta}{d^2}$ ✓

$\frac{300}{d} = \sin\theta$

$d = \frac{300}{\sin\theta}$

$d^2 = \frac{90000}{\sin^2\theta}$ ✓

$I = \frac{K\cos\theta\sin^2\theta}{90000}$

$I = k\cos\theta\sin^2\theta$ ✓

where $k = \frac{K}{90000}$ ✓

(b)

$h = 400 \Rightarrow \tan\theta = \frac{3}{4}$ ✓

$\Rightarrow \sin\theta = \frac{3}{5}$

$\Rightarrow \cos\theta = \frac{4}{5}$

$72 = k.\frac{4}{5}.\frac{9}{25}$ ✓

$k = 250$ ✓

(c)

$I = 250\cos\theta\sin^2\theta$

$\frac{dI}{d\theta} = 250\cos\theta.2\sin\theta\cos\theta + 250.-\sin\theta\sin^2\theta$ ✓

$\frac{dI}{d\theta} = 250\left(2\sin\theta\cos^2\theta - \sin^3\theta\right)$ ✓

$\frac{dI}{d\theta} = 250\left(2\sin\theta - 3\sin^3\theta\right)$ ✓

(d)

$$\frac{dI}{d\theta} = 0 \Rightarrow \sin\theta\left(2-3\sin^2\theta\right) = 0$$

$$\Rightarrow \sin\theta = 0,\ \sin\theta = \pm\sqrt{\frac{2}{3}} \quad ✓$$

$$\text{but } 0^0 < \theta < 90^0 \Rightarrow \sin\theta = \sqrt{\frac{2}{3}} \quad ✓$$

$$\Rightarrow \theta = 54.74^0 \quad ✓$$

$$\frac{h}{300} = \frac{1}{\tan 54.74^0}$$

h = 212 m ✓

(e)

$$I = 250\cos 54.74^0 \sin^2 54.74^0$$

I = 96.23 ✓

(f)

$$\frac{dh}{dt} = -5$$

$$\frac{h}{300} = \frac{1}{\tan\theta} \quad ✓$$

$$h = \frac{300}{\tan\theta}$$

$$h = \frac{300\cos\theta}{\sin\theta} \quad ✓$$

$$\frac{dh}{d\theta} = \frac{\sin\theta.-300\sin\theta - 300\cos\theta.\cos\theta}{\sin^2\theta}$$

$$= \frac{-300\left(\sin^2\theta + \cos^2\theta\right)}{\sin^2\theta} \quad ✓$$

$$= \frac{-300}{\sin^2\theta} \quad ✓$$

$$\frac{d\theta}{dt} = \frac{d\theta}{dt}.\frac{dh}{dt}$$

$$= \frac{-\sin^2\theta}{300}.-5 \quad ✓$$

$$= \frac{1}{60}\sin^2\theta$$

(g) When $h = 125$, $d = 325 \rightarrow \sin\theta = \frac{12}{13}$ ✓

$$\frac{dI}{dt} = \frac{dI}{d\theta} \times \frac{d\theta}{dt}$$

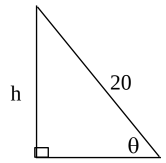
$$= 250\,(2\sin\theta - 3\sin^3\theta)\,\frac{1}{60}\sin^2\theta \quad ✓$$

$$\frac{dI}{dt} = 250 \times 2\left(\frac{12}{13} - 3\left(\frac{12}{13}\right)^3\right) \times \frac{1}{60}\left(\frac{12}{13}\right)^2 \quad ✓$$

$$= -1.823 \quad ✓$$

NB. Intensity is decreasing.

32.



h = 20 sin θ ✓

$$\frac{dh}{d\theta} = 20\cos\theta \quad ✓$$

$$\frac{dh}{d\theta}_{\theta = 60^0}$$

$$\delta\theta\left(\frac{dh}{d\theta}\right) = \delta h$$

$$\delta h = \pm\ 0.25 \text{x} \frac{\pi}{180} \text{x} 10 \approx \pm\ 0.0436 \quad ✓$$

Height lies between 17.364 and 17.276 m ✓

Note: trigonometry result is 17.277 to 17.364 m

33. (a) $V = V_0 e^{kt}$, $V_0 = 1$ ✓

$0.5 = e^{5k}$ ✓ $\Rightarrow$ $k = -0.1386$ ✓

Hence, $V = e^{-0.1386t}$ ✓

(b) $V(12) = 0.1895$ m^3. ✓

(c)
- Model does not allow for $V = 0$ ✓
- Model assumes rate of melting is constant which is not true in real life. ✓

Chapter 5: Anti-Differentiation

Fundamental Theorem

1. (a) $\sqrt{x^2 - 3}$ ✓

(b) $\dfrac{1}{(t-3)^3}$ ✓

(c) $((2y)^2 + 2)^3(2) = 2(4y^2 + 2)^3$ ✓✓

(d) $-\dfrac{d}{dq}\displaystyle\int_3^{q^2} \sin q^3 \, dq$ ✓

$= -\sin(q^2)^3 \times (2q) = -2q\sin q^6$ ✓✓

(e) $[3x^3 + 1]_0^x = 3x^3 + 1 - 1 = 3x^3$ ✓✓

(f) $-\displaystyle\int_{\frac{\pi}{2}}^{t} \frac{d}{dy}(\sin^2 y)\, dy$ ✓

$= [\sin^2 y]_{\frac{\pi}{2}}^{t}$ ✓ $= \sin^2 t - 1$ ✓

2. $F'(x) = x^3.\sqrt{1+x^3} + 3x^2\displaystyle\int_0^x \sqrt{1+t^3 dt}$ ✓✓

3. (a) $f'(x) = \dfrac{1}{2}(1 + x^2)^{-\frac{1}{2}\cdot 2} x$ and $g'(x) = x$ ✓✓

$= \dfrac{x}{(1 + x^2)^{\frac{1}{2}}}$

$f'(x) = \dfrac{g'(x)}{(1 + x^2)^{\frac{1}{2}}}$ ✓✓

$\leq g'(x) \quad for \quad x \geq 0$

(b) $f'(t) \leq g'(t)$ for $t \geq 0$ ✓

$\therefore \displaystyle\int_0^x f'(t)\ \mathrm{dt} \leq \int_0^x g'(t)\ \mathrm{dt}$ ✓

$\therefore f(x) \leq g(x)$ for $x \geq 0$

4. $x.G(x) = \displaystyle\int_2^x (t^3 - 4)^{\frac{1}{2}}\ \mathrm{dt}$

$x.G'(x) + G(x) = (x^3 - 4)^{\frac{1}{2}}$ ✓

$x.G'(x) = (x^3 - 4)^{\frac{1}{2}} - G(x)$ ✓

$G'(x) = \dfrac{1}{x}\left[(x^3 - 4)^{\frac{1}{2}} - \displaystyle\int_2^x (t^3 - 4)^{\frac{1}{2}}\, dt\right]$ ✓

5. g must be continuous

$\displaystyle\lim_{u\to 4^+} g(u) = \frac{1}{2}$

and $\displaystyle\lim_{x \to 4^-} = k \times 4^2 - 3{\cdot}5$ ✓

$\therefore g(4) = 16k - 3{\cdot}5$ ✓

$\therefore \dfrac{1}{2} = 16k - 3{\cdot}5$ ✓

$\therefore k = \dfrac{1}{4}$ ✓

ie $k = \dfrac{1}{4}$ for g to be continuous.

Integration

1. (a) $\displaystyle\int_1^2 (4x^2 - 2)\, dx = \left[\frac{4x^3}{3} - 2x\right]_{,2}^{1}$ ✓✓

$= \left(\dfrac{32}{3} - 4\right) - \left(\dfrac{4}{3} - 2\right) = 7\dfrac{1}{3}$ ✓

(b) $y = \displaystyle\int \left(\sqrt{x} + \frac{1}{\sqrt{2}} x^{-\frac{1}{2}}\right) dx$ ✓

$\therefore y = \dfrac{2x^{\frac{3}{2}}}{3} + \dfrac{2}{\sqrt{2}}\sqrt{x} + c$ ✓✓

(c) (i) $\dfrac{d}{dx}\left[(2x^7 - x^4)^4\right] = 4(2x^7 - x^4)^3(14x^6 - 4x^3)$ ✓

(ii) $\therefore \displaystyle\int (2x^7 - x^4)^3(14x^6 - 4x^3)\, dx = \frac{1}{4}(2x^7 - x^4)^4 + c$ ✓

$\therefore \displaystyle\int 5(2x^7 - x^4)^3(14x^6 - 4x^3)\, dx = \frac{5}{4}(2x^7 - x^4)^4 + c$ ✓

2. (a) $\displaystyle\int 4x^3 + 2x^{\frac{1}{3}} - 4x^{-3}\ dx$

$= x^4 + \dfrac{3}{2}x^{\frac{4}{3}} + \dfrac{2}{x^2} + c$ ✓✓✓

(b) $\displaystyle\int \left(\frac{5}{2}x^{-1} - \frac{1}{2}\right) dx$

$\dfrac{5}{2}\ln|x| - \dfrac{1}{2}x + c$ ✓✓

(c) $-\dfrac{4}{3}\displaystyle\int -3e^{1-3x}\ dx$

$= -\dfrac{4}{3}e^{1-3x} + c$ ✓✓

3. (a) (i) $= \left[\frac{x^2}{2} - x\right]_0^{-3}$ ✓

$= 7.5$✓

(ii) $= \left[\frac{1}{2}e^{2x} + e^{x+1}\right]_0^1$ ✓✓

$= \frac{3}{2}e^2 - e - \frac{1}{2}$ ✓

(b) (i) $\sqrt{x^2 - 4x}$ ✓

(ii) $\frac{30x}{9x^4 - 3}$ ✓✓

4. (a) A(1, 1.5), B(2, 0), C(0, –2) ✓✓✓

(b)

$$Area = \int_0^1 \frac{3}{2}x\,dx + \int_1^2 \frac{3}{2}(x-2)^2 dx + \left|\int_0^2 (x-2)dx\right|$$ ✓✓✓

(c) $Area = 0.75 + 0.5 + 2$ ✓

$= 3.25$ ✓

5. (a) $2x^2 - \frac{1}{4}x^4 + c$ ✓✓

(b) $-\frac{3}{2(2t-1)} + c$ ✓✓

(c) $\int \left(e^x + e^{-4x}\right)dx$

$= e^x - \frac{1}{4e^{4x}} + c$ ✓✓

6. (a) $A = f(x)$ and $B = g(x)$ ✓✓

(b) (i) T✓

(ii) F✓

(iii) F✓

(iv) T✓

7. $f'(x) = \int 2(3-2x)^{-\frac{1}{2}}\,dx$ ✓

$= -2(3-2x)^{\frac{1}{2}} + c$ ✓

$f'(1) = -3 = -2 + c \rightarrow c = -1$ ✓

$\therefore\ f(x) = \frac{2}{3}(3-2x)^{\frac{3}{2}} - x + k$ ✓

$f(1) = \frac{2}{3} - 1 + k = -2 \rightarrow k = -\frac{5}{3}$✓

$\therefore\ f(x) = \frac{2}{3}(3-2x)^{\frac{3}{2}} - x - \frac{5}{3}$ ✓

8. (a) $P'(x) = \frac{f'(x)\,g(x) - g'(x)\,f(x)}{[g(x)]^2}$ ✓

$\Rightarrow P'(-1) = \frac{(3)(4) - (2)(3)}{16} = \frac{3}{8}$ ✓✓

(b) $R'(x) = f'(x)(g(x))^2 + 2g'(x)f(x)g(x)$✓

$\Rightarrow R'(-1) = 96$ ✓✓

9. (a) $\int_b^a g(x)dx + \int_b^c g(x)dx$ ✓✓

(b) (i) $\int_a^b g(x)\,dx = -4$ ✓

(ii) $\int_a^c g(x)\,dx = 6 - 4 = 2$ ✓

(iii)

$\int_a^c [g(x) - 2x]dx = \left[\int g(x)dx - x^2\right]_a^c$

$= \left[\int g(x)\right]_a^c - \left[x^2\right]_a^c$ ✓

$= 2 - c^2 + a^2$ ✓

10. (a) $3\int_a^b f(x)\,dx = 3\times(-5) = -15$ ✓✓

(b) $q - p = 7$ and $\left[\frac{x^2}{2}\right]_1^p = 4$ ✓✓

$\therefore\ q - p = 7$ and $\frac{p^2}{2} - \frac{1}{2} = 4$

$\therefore\ q - p = 7$ and $p^2 = 9$ ✓

$\therefore$ $p = -3$ and $q = 4$ or $p = 3$ and $q = 10$✓

11. (a) $f(x) = ex - x^3 + 2x^2 + c$✓

$\therefore$ $(2, 2e - 1) \rightarrow f(x) = ex - x^3 + 2x^2 - 1$ ✓

(b) $f'(2) = e - 4$ ✓

$\therefore\ y = (e-4)x + c \rightarrow (2, 2e - 1)$ ✓

$\rightarrow y = (e-4)x + 7$

(c) $f''(x) = -6x + 4$ ✓

$\therefore\ f''(x) = 0 \rightarrow \left(\frac{2}{3}, \frac{2}{3}e - \frac{11}{27}\right)$ ✓

12. (a) $\left[\frac{x^3}{3} - \frac{2x^{\frac{3}{2}}}{3} + \frac{e^{2x}}{2}\right]_0^1 = \frac{e^2}{2} - \frac{5}{6}$ ✓✓✓

(b) $-\frac{3}{5}\int_1^2 \frac{-5}{11-5x}\,dx + \int_1^2 \frac{14x}{7x^2+2}\,dx$ ✓

$= \left[-\frac{3}{5}\ln|11-5x| + \ln|7x^2+2|\right]_1^2$ ✓

$= -\frac{3}{5}\{\ln 1 - \ln 6\} + \ln 30 - \ln 9$ ✓

$= \frac{3}{5}\ln 6 + \ln 30 - \ln 9$ ✓

13. (a) $\int (4x^3 + 2x^{\frac{1}{3}} - 4x^{-3})\,dx$

$= x^4 + \frac{3}{2}x^{\frac{4}{3}} + \frac{2}{x^2} + c$ ✓✓✓

(b) $\int \left(\frac{5}{2}x^{-1} - \frac{1}{2}\right)dx$

$= \frac{5}{2}\ln|x| - \frac{1}{2}x + c$ ✓ ✓

(c) $\frac{4}{3}\int (3e^{1-3x})\,dx - \frac{1}{3}\int\left(\frac{3x^2}{x^3+1}\right)dx$

$= -\frac{4}{3}e^{1-3x} - \frac{1}{3}\ln|x^3+1| + c$ ✓✓✓✓

14. (a) $\left[\frac{x}{2} - \frac{1}{2}\ln|x|\right]_1^e = \frac{e}{2} - 1$ ✓✓✓

(b) $\int_0^\pi (ex^{e-1})\,dx + \frac{1}{3}\int_0^\pi \frac{75x^2}{15x^5+e}\,dx$ ✓

$= \left[x^e + \frac{1}{3}\ln|15x^5+e|\right]_0^\pi$ ✓

$= \pi^e + \frac{1}{3}\{\ln|15\pi^5 + e| - \ln e\}$ ✓

$= \pi^e + \frac{1}{3}\ln(15\pi^5 + e) - \frac{1}{3}$ ✓

15. (a) $\frac{d}{dx}(\sin^4 ax) = 4(\sin^3 ax)(a\cos ax)$ ✓

(b) $\int \cos ax \sin^3 ax\,dx$

$= \frac{1}{4a}\sin^4 ax + c$ ✓✓

16. (a)

$\int_0^{\frac{\pi}{2}} (\sin x + \cos x)^2 dx$

$= \int_0^{\frac{\pi}{2}} (\sin^2 x + \cos^2 x + 2\sin x\cos x)dx$ ✓

$= \int_0^{\frac{\pi}{2}} (1 + \sin 2x)dx$ ✓

$= \left[x + \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$ ✓

$= \frac{\pi}{2} + \frac{1}{2} - \left(0 - \frac{1}{2}\right)$

$= \frac{\pi}{2} + 1$ ✓

(b)

$\int_0^{\pi} (\sin x - \cos x)^2\,dx$

$= \int_0^{\pi} (\sin^2 x + \cos^2 x - 2\sin x\cos x)\,dx$

$= \int_0^{\pi} (1 - \sin 2x)\,dx$ ✓

$= \left[x - \frac{1}{2}.-\cos 2x\right]_0^{\pi}$ ✓

$= \left[x + \frac{1}{2}\cos 2x\right]_0^{\pi}$ ✓

$= \pi + \frac{1}{2} - 0 - \frac{1}{2}$ ✓

$= \pi$

17. (a)

$$y = \frac{\sin x}{\cos x}$$

$$\frac{dy}{dx} = \frac{1}{\cos^2 x} = (\cos x)^{-2} \quad ✓$$

$$\frac{d^2y}{dx^2} = -2(\cos x)^{-3}.-\sin x$$

$$= +\frac{2\sin x}{\cos^3 x} \quad ✓$$

$$2y + 2y^3 = \frac{2\sin x}{\cos x} + \frac{2\sin^3 x}{\cos^3 x} \quad ✓$$

$$2y + 2y^3 = \frac{2\sin x\cos^2 x + 2\sin^3 x}{\cos^3 x}$$

$$2y + 2y^3 = \frac{\sin x\left(2\cos^2 x + 2\sin^2 x\right)}{\cos^3 x} \quad ✓$$

$$2y + 2y^3 = \frac{2\sin x}{\cos^3 x} \quad ✓$$

$$\Rightarrow y = \frac{\sin x}{\cos x} \text{ satisfies } \frac{d^2y}{dx^2} = 2y + 2y^3$$

(b)

$$f'(x) = \sin x\cos x$$

$$f(x) = \int \sin x\cos x\, dx$$

$$f(x) = \frac{1}{2}\int \sin 2x\, dx \quad ✓$$

$$f(x) = \frac{1}{2}.\frac{1}{2}.-\cos 2x + C \quad ✓$$

$$f(x) = -\frac{1}{4}\cos 2x + C \quad \text{and} \quad f(\pi) = -1$$

$$-1 = -\frac{1}{4}\cos 2\pi + C \quad ✓$$

$$C = -\frac{3}{4}$$

$$f(x) = -\frac{1}{4}\cos 2x - \frac{3}{4} \quad ✓$$

(c) Not true, as the constant of integration is different in each case. ✓✓

18. (a)

$$y = x\left(\cos x - e^{-x}\right)$$

$$\frac{dy}{dx} = 1\left(\cos x - e^{-x}\right) + x\left(-\sin x + e^{-x}\right) \quad ✓$$

$$\frac{dy}{dx} = \cos x - e^{-x} - x\left(\sin x - e^{-x}\right) \quad ✓$$

(b)

$$\frac{dy}{dx} = \cos x - e^{-x} - x\left(\sin x - e^{-x}\right)$$

$$x\left(\sin x - e^{-x}\right) = \cos x - e^{-x} - \frac{dy}{dx} \quad ✓$$

$$\int x\left(\sin x - e^{-x}\right)dx = \int\left(\cos x - e^{-x} - \frac{dy}{dx}\right)dx \quad ✓$$

$$\int x\left(\sin x - e^{-x}\right)dx = \sin x + e^{-x} - y \quad ✓$$

$$\int x\left(\sin x - e^{-x}\right)dx = \sin x + e^{-x} - x\cos x + xe^{-x} + C$$

19. (a)

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$$

$$\frac{dy}{dx} = \frac{a}{2}\left(\frac{1}{a}e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right) \quad ✓$$

$$= \frac{1}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right) \quad ✓$$

(b)

$$length = \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^a \sqrt{1 + \frac{1}{4}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)^2}\, dx \quad ✓$$

$$= \int_0^a \frac{1}{2}\sqrt{\left(e^{\frac{2x}{a}} + 2 + e^{-\frac{2x}{a}}\right)}\, dx$$

$$= \int_0^a \frac{1}{2}\left(e^{\frac{2x}{a}} + e^{-\frac{2x}{a}}\right) dx \quad ✓$$

$$= \frac{1}{2}\left[\frac{e^{\frac{x}{a}}}{\frac{1}{a}} + \frac{e^{-\frac{x}{a}}}{-\frac{1}{a}}\right]_0^a \quad ✓$$

$$= \frac{1}{2}\left\{a\left(e - e^{-1}\right) - a(1 - 1)\right\}$$

$$= \frac{a}{2}\left(e - e^{-1}\right) \text{ units} \quad ✓$$

20. (a) $y = \int_1^{x^2} \frac{dt}{1+t^2}$

$\frac{dy}{dx} = \frac{1}{1+(x^2)^2} \times 2x$ ✓✓

$\therefore \frac{dy}{dx} = \frac{2x}{1+x^4}$ ✓

(b) $y'(2) = \frac{4}{17}$ ✓

21. (a) $\frac{dy}{dx} = x^2 . \frac{1}{x} + 2x \ln x$ ✓

$= x + 2x \ln x$ ✓

(b) $\therefore 2x \ln x = \frac{dy}{dx} - x$ ✓

$\int (2x\ln x)\, dx = \int \frac{dy}{dx}\, dx - \int x\, dx$ ✓

$= y - \frac{x^2}{2} + c$ ✓

$= x^2 \ln x - \frac{x^2}{2} + c$ ✓

22. (a) $g(x) = \int \left(2\sqrt{x} + \frac{5}{\sqrt{x}}\right) dx$

$= \frac{4}{3}x^{\frac{3}{2}} + 10\sqrt{x} + c$ ✓✓✓

(b) $\frac{dy}{dx} = \frac{k}{x} = kx^{-1} \rightarrow y = k\ln x + c$ ✓✓

$(1, 2) \rightarrow 2 = c$ *and* $k = 2$ ✓✓

$\therefore y = 2\ln x + 2$

Chapter 6: Applications of Integration

1. (a) $v = \frac{mt^2}{2} - nt + c$ ✓

$v(0) = 7 \rightarrow c = 7$

$v(1) = 12$ and $v(7) = 0 \rightarrow m = -2$ and $n = -6$ ✓✓

(b) $v(t) = -t^2 + 6t + 7$ ✓

(c) Max v occurs when $a = 0$

$\therefore -2t + 6 = 0 \rightarrow t = 3$ ✓

$\therefore x(t) = -\frac{1}{3}t^3 + 3t^2 + 7t$

$\therefore x(3) = 39$ ✓

2. (a) (i) 3 ✓

(ii) 2 ✓

(b) (i) As the rectangles increase in number, but decrease in width, the limit represents the area under the curve ✓

(ii) $\int_{x_1}^{x_n} f(x)\, dx$ ✓

(c) $\frac{dy}{dx} = x - \frac{x^3}{3} + c_1$ ✓

$\frac{dy}{dx} = -1$ when $x = 1 \rightarrow c_1 = -1\frac{2}{3}$ ✓

$y = \frac{x^2}{2} - \frac{x^4}{12} - \frac{5x}{3} + c_2$ ✓

$(1, 1) \rightarrow c_2 = \frac{27}{12}$

$\therefore y = \frac{x^2}{2} - \frac{x^4}{12} - \frac{5x}{3} + \frac{27}{12}$ ✓

3. (a) $v = 3t^2 - t + c$ ✓

$v(0) = -2 \rightarrow c = -2$

$\therefore v(t) = 3t^2 - t - 2$ ✓

(b) $x = t^3 - \frac{1}{2}t^2 - 2t + k$ ✓

$x(0) = 0 \rightarrow k = 0$

$\therefore x(t) = t^3 - \frac{1}{2}t^2 - 2t$ ✓

$\therefore x = 0$ when $t(t^2 - 0.5t - 2) = 0$

$\therefore t = 1.69$ s ✓

(c) $a = 0$ when $t = \frac{1}{6}$ ✓

$\therefore \quad v\left(\frac{1}{6}\right) = -2{\cdot}08$ m/s✓

(d) Total distance =

$\int_0^3 \left| 3t^2 - t - 2 \right| dt$ ✓✓

$= 1.5 + 16.5 + 1.5 = 19.5$ m✓

4. (a) $\frac{dV}{dt} = 3(1) - 2(1) = 1$ ✓

∴ The balloon is instantaneously increasing in volume at 1 m^3/min✓

(b) $3t^2 - 2t > 0$ ✓

$\therefore \quad t > \frac{2}{3}$ mins ✓

(c)

$V = \int_0^5 \left(3t^2 - 2t\right) dt + 3$ ✓

$V = 103\ m^3$ ✓

5. (a) $10 800✓

(b) $K = \int (100 + 20x - 3x^2)\, dx$ ✓

$\therefore \ = 100x + 10x^2 - x^3 + c$

Initial costs of $1 000 → $c = -10$

$\therefore \quad K = 100x + 10x^2 - x^3 - 10$ ✓

∴ Max profit of $99 000 ✓

6. (a)

$a(6) = -\frac{4}{3}$ ✓

(b) $v = -\frac{4}{3}t + 8$ ✓

$v(10) = -5\frac{1}{3}$

∴ speed = $5\frac{1}{3}$ m/s✓

(c) Change in displacement =

$24 + \left(-10\frac{2}{3}\right) = 13\frac{1}{3}$ ✓✓

(d) Distance =

$24 + 10\frac{2}{3} = 34\frac{2}{3}$ ✓✓

7. (a) $2\sqrt{x} - \frac{3}{x} = 0 \rightarrow x = 1.31$✓

$\therefore \quad f(x) = \frac{4}{3}x^{\frac{3}{2}} - 3\ln x + c$ ✓

$(1, 0) \rightarrow c = -\frac{4}{3}$ ✓

∴ Coordinate (1.31, – 0.14) ✓

(b) $f''(x) = \frac{1}{\sqrt{x}} + \frac{3}{x^2}$ ✓

$f''(1{\cdot}31) > 0$

∴ local minimum ✓

8. (a) $4t^3 - 12t^2 + 6t + 4 = 0$ ✓

∴ 1.37 min and 2 min ✓

(b) $x = t^4 - 4t^3 + 3t^2 + 4t - 4$ ✓✓

(c) $a = 12t^2 - 24t + 6$ ✓

$x = 0$ when $t = 1, 2$ ✓

$\therefore \ a(1) = -6$ and $a(2) = 6$ ✓

(d) Total $= \int_0^2 \left| \left(4t^3 - 12t^2 + 6t + 4\right) \right| dt$ ✓

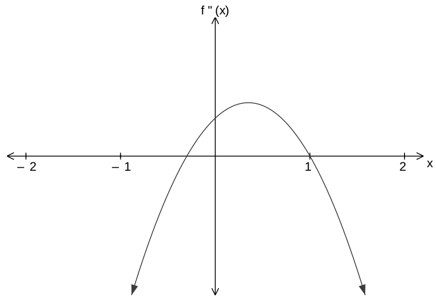
$= 4.35 + 0.35$

$= 4.7$ metres. ✓

9. (a) 0.83 ✓✓
(b) -13.50 ✓✓
(c) 14.33 ✓✓

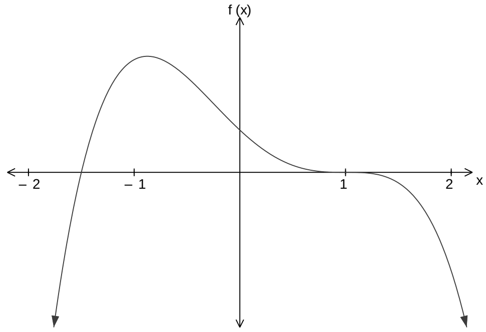
10. (a) (i) 0 ✓
(ii) 2 ms^{-2} ✓
(b) 4 m/s ✓✓
(c) Area of triangle = 4 m ✓✓
(d) Area of trapezium = 16 m ✓✓

11. (a)



✓✓

(b)



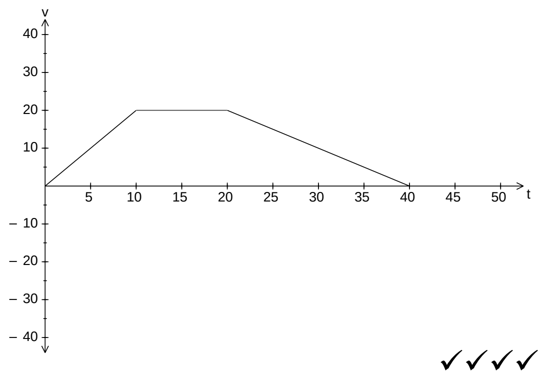
✓✓✓

12. (a) $f(10) = 10$ and $g(10) = 8$ ✓
$\therefore$ 2 metres. ✓
(b) $f(16) = 6.4$ and $g(16) = 5.12$ ✓
$\therefore$ 1.28 metres. ✓
(c) $g(8) = 7.68$
$\therefore \quad y = \frac{7.68}{8}x = 0.96x$ ✓✓
(d) $A = \int_0^{20} [-0.1x(x-20) + 0.08x(x-20)]\,dx$ ✓✓
$= \int_0^{20} -0.02x(x-20)\,dx$
(e) 26.67 ✓✓

13. (a) $x = -1$ or $x = 0$ ✓✓
(b) $f''(x) = ax(x+1)$
$f''(-0.5) = -1.5 \rightarrow a = 6$ ✓
$\therefore f''(x) = 6x(x+1) = 6x^2 + 6x$ ✓
(c) Since $f''(0) = f'(0) = 0$, then at $x = 0$ ✓ exists a horizontal point of inflection.✓
(d)
$f'(x) = \int \left(6x^2 + 6x\right) dx = 2x^3 + 3x^2 + c$ ✓
$f'(0) = 0 \rightarrow c = 0$
$\therefore f'(x) = 2x^3 + 3x^2$ ✓
(e)
$f(x) = \int \left(2x^3 + 3x^2\right) dx = \frac{1}{2}x^4 + x^3 + k$ ✓
$\therefore f(0) = 1 \rightarrow k = 1$
$\therefore f(x) = \frac{1}{2}x^4 + x^3 + 1$ ✓

14. (a) $\frac{dC}{dx} = \$60$ ✓
(b) Extra Cost $= \int_9^{16} (30x - 30\sqrt{x})\,dx = \$1\ 885$ ✓✓
(c) Cost $= \int_{24}^{25} (30x - 30\sqrt{x})\,dx$ ✓✓

15. (a)



✓✓✓✓

(b) $k = \frac{500}{10} = 50$ m/s ✓✓
(c) 1 000 m ✓

16. (a) $h = 6d - 6;\ 1 \le d \le 2$ ✓
$h = -6d + 42;\ 6 \le d \le 7$ ✓
$h = 6;\ 2 \le d \le 6$ ✓

(b) $\left[\int_0^8 \left(-\frac{1}{2}d^2 + 4d\right) dd\right] - \left[\frac{1}{2}(4 + 6)\,6\right]$ ✓✓

(c) $\therefore$ Area $= 42\frac{2}{3} - 30 = 12\frac{2}{3}\,m^2$ ✓

(d) Volume $= 12\frac{2}{3} \times 1500 = 19000m^3$

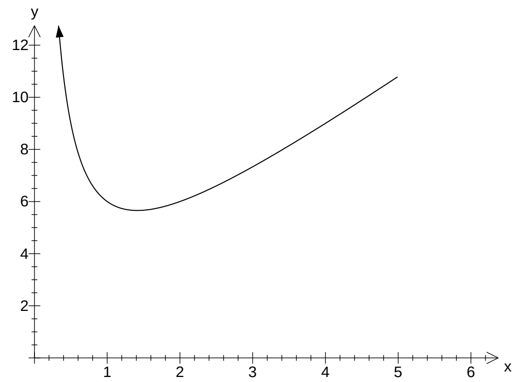
$\Rightarrow 19000 \times 9.8 = \$186200$ ✓✓✓

17. (a) $v(t) = \int -9.8 \Rightarrow -9.8t + c$ ✓
$v(0) = 16 \therefore v(t) = -9.8t + 16$ ✓

(b) $\int_0^2 |-9.8t + 16|dt = 13.72$ ✓✓
$\therefore$ Total distance travelled in the first 2 seconds is 13.72m.✓

(c)
$s(t) = 0 \Rightarrow -4.9t^2 + 16t = 0 \Rightarrow t = 0,\ 3.27$
$\Rightarrow v(3.27) = -16.05\,ms^{-1}$ ✓
$\therefore$ Particle returns to the ground when $t = 3.27s$, at a speed of $16.05ms^{-1}$. ✓

18. (a)



✓✓✓✓

(b) $\int_{0.5}^{4} \left[2x + \frac{4}{x}\right] dx$ ✓✓

(c) $\left[x^2 + 4\ln|x|\right]_{0.5}^{4} = 24.07$✓✓

(d) As
$\int_{0.5}^{4} \left(-2x - \frac{4}{x}\right) dx = -\int_{0.5}^{4} \left(2x + \frac{4}{x}\right) dx$
$\therefore$ Integral $= -24.07$✓✓

19. (a) $t = 12$ s ✓
(b) 127.5 – 7.5 = 120 m to the right of O.✓✓

20. (a) C and B ✓✓
(b) C ✓
(c) B ✓
(d) A and D ✓

21. (a) $25m$ ✓
(b) (310.4, 314.0) ✓✓
(c)
$\int_0^{310.4} \left(0.003d^2 + 25\right)dd + \int_{310.4}^{510} \left(-0.002d^2 + 0.84d + 246\right)dd$
✓✓✓
(d) $h = 0.931d + 25$ ✓✓
(e)
$V = 150 \times \int_0^{310.4} (0.931d + 25) - \left(0.003d^2 + 25\right) dd = 14\,943.6$ ✓✓

Cost $= \frac{14934.6}{5} \times 2 = \$5\,977.44$ ✓✓

22. (a) $a(t) = c \rightarrow v(t) = ct + 10.5$ ✓
$\therefore$ v (5)=5c+10.5=3
$\therefore c = -1.5 \therefore v(t) = -1.5t + 10.5$✓

(b) When $v = 0$, $t = \frac{10.5}{1.5} = 7s$ ✓✓

(c)
$x = \int (-1.5t + 10.5)dt = -0.75t^2 + 10.5t + c$ ✓
$x(5) = 33.75 + c = 10 \therefore c = -23.75m$
$\therefore x(0) = 23.75$ m to left of origin. ✓✓

(d) $-0.75t^2 + 10.5t - 23.75 = 8$, ✓
$-0.75t^2 + 10.5t - 23.75 = -8$ ✓
$t = 4.42s$ or $t = 1.71s$ or $t = 9.58s$ or $t = 12.29s$
✓✓

(e) $\int_0^7 (-1.5t + 10.5)dt = 36.75m$ ✓✓

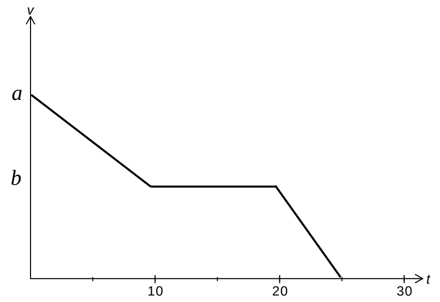
23. $y' = 4e^{2x^2 + x} + c$ ✓
$\rightarrow y'(0) = 1 \therefore 1 = 4e^0 + c \rightarrow c = -3$
$\therefore y' = 4e^{2x^2 + x} - 3$ ✓

Sub $y'(1) = 4e^3 - 3 \therefore m = 4e^3 - 3$ ✓

24. Constant deceleration in first 10 seconds
= Constant deceleration in final 5 seconds

$\Rightarrow \quad \frac{a-b}{10} = \frac{b}{5}$ ✓

$\Rightarrow \quad a = 3b$ ✓



Total distance travelled = 19.5

$\Rightarrow \quad \frac{1}{2}(a+b)\times 10 + 10b + 2.5b = 19.5$ ✓✓

$\therefore \quad 5a + 17.5b = 19.5$ ✓

Since $a = 3b$,

$32.5b = 19.5$

Hence, $b = 0.6 \Rightarrow a = 1.8$ ✓

25. (a) Journey time = $\frac{100}{v}$

Total Cost $T = \frac{100}{v} \times \left(\frac{v^2}{64}+81\right)$ ✓✓

$= \frac{25v}{16} + \frac{8100}{v}$

(b) $\frac{dT}{dv} = \frac{25}{16} - \frac{8100}{v^2}$ ✓

T is min when $\frac{25}{16} - \frac{8100}{v^2} = 0$ ✓

$\Rightarrow \quad v = 72$

$\frac{d^2T}{dv^2} = \frac{16\,200}{v^3} > 0$ for $v = 72$. ✓

Hence, T is min when $v = 72$ km/h. ✓

(c) New Cost $N = 2 \times \left[\frac{100}{v}\times\left(\frac{v^2}{64}+81\right)\right]$ ✓

As $N = 2 \times T$, a vertical scaling of factor 2 has been applied. ✓

Hence, minimum point of N and the minimum point of T share the same v-coordinate. ✓

$\therefore$ the cost is still minimised at $v = 72$ km/h ✓

OR

$N = 2 \times \left[\frac{100}{v}\times\left(\frac{v^2}{64}+81\right)\right]$ ✓

$= \frac{25v}{8} + \frac{16\,200}{v}$

$\frac{dN}{dv} = \frac{25}{8} - \frac{16\,200}{v^2}$ ✓

N is min when $\frac{25}{8} - \frac{16\,200}{v^2} = 0$ ✓

$\Rightarrow \quad v = 72$ ✓

26. (a) Let constant acceleration be represented by a.

Then, velocity $v = \int a\ dt = at + C$ ✓

Also, displacement $x = \int at + C\ dt$

$= \frac{at^2}{2} + Ct + K$ ✓

When $t = 0$, $x = 0 \quad \Rightarrow \quad K = 0$

Also, when $t = 6$, $x = 90 \quad \Rightarrow \quad 18a + 6C = 90$ ✓

and, when $t = 10$, $x = 180 \quad \Rightarrow \quad 50a + 10C = 180$ ✓

Hence, $a = 1.5$ and $C = 10.5$

Therefore, constant acceleration is 1.5 ms^{-2} . ✓

(b) When $t = 10$, $v = 1.5 \times 10 + 10.5$

= 25.5 ms^{-1} ✓✓

= 91.8 kph

27. (a) Area = $\int_0^2 e^{-(x-2)^2}\ dx$ ✓✓ = 0.8821 ✓

(b) Area between x =2 and x = 50

$= \int_2^{50} e^{-(x-2)^2}\ dx$

= 0.8862

Area between x =2 and x = 100

$= \int_2^{100} e^{-(x-2)^2}\ dx$

= 0.8862 ✓✓✓

Hence, the area to the right of $x = 2$ approaches 0.8862 (limiting area) ✓

28. Rewrite P = $\dfrac{2\pi}{\sqrt{g}}x^{\frac{1}{2}} \Rightarrow \dfrac{dP}{dx} = \dfrac{2\pi}{\sqrt{g}}\times\dfrac{1}{2}x^{-\frac{1}{2}}$ ✓

Hence, $\delta P \approx \dfrac{dP}{dx}\times\delta x$

$= \dfrac{2\pi}{\sqrt{g}}\times\dfrac{1}{2}x^{-\frac{1}{2}}\times\delta x = \dfrac{\pi x^{-\frac{1}{2}}}{\sqrt{g}}\times\delta x$ ✓

If x increases by 1%, then, $\delta x = 0.01\times x$;

$\delta P \approx \dfrac{\pi x^{-\frac{1}{2}}}{\sqrt{g}}\times 0.01x$ ✓

$= 0.01\times\dfrac{\pi x^{\frac{1}{2}}}{\sqrt{g}} = 0.01\times\dfrac{P}{2} = 0.005P$ ✓✓

29. (a) $\dfrac{dV}{ds} = \dfrac{25}{\sqrt{s}}$. ✓

Hence, when $s = 36$, $\dfrac{dV}{ds} = \dfrac{25}{6}$ ✓

(b) $\dfrac{dV}{dt} = \dfrac{25}{\sqrt{s}}\times -\dfrac{3}{4}t^{\frac{1}{2}}$ ✓

When $s = 36, 100 - 0.5t^{\frac{3}{2}} = 36$

$\Rightarrow t = 25.3984.$ ✓✓

Hence, when $s = 36$,

$\dfrac{dV}{dt} = -15.75 \text{ km}^3 \text{ min}^{-1}$ ✓

(c) When it hits the earth, $s = 0$.

Speed = $\left|\dfrac{dS}{dt}\right| = \left|-\dfrac{3}{4}t^{\frac{1}{2}}\right| = \dfrac{3}{4}t^{\frac{1}{2}}$ ✓

When $s = 0$,

$100 - 0.5t^{\frac{3}{2}} = 0 \Rightarrow t = 34.1995$ ✓

Hence, when $s = 0$,

speed = 4.39 km min^{-1} ✓

30. Since (1, 0) lies on the curve $4x^2 + y^2 = 4$, minimum distance from (1, 0) to this curve is 0. ✓

Distance from any point (x, y) on this curve to the point (1, 0) is

$s = \sqrt{(x-1)^2+(y-0)^2} = \sqrt{x^2-2x+1+y^2}$ ✓

But $y^2 = 4 - 4x^2 \Rightarrow s = \sqrt{x^2-2x+1+(4-4x^2)}$

$= \sqrt{-3x^2-2x+5}$ ✓

$\dfrac{ds}{dx} = \dfrac{1}{2}\times\dfrac{(-6x-2)}{\sqrt{-3x^2-2x+5}}$ ✓

For optimum values,

$\dfrac{ds}{dx} = \dfrac{1}{2}\times\dfrac{(-6x-2)}{\sqrt{-3x^2-2x+5}} = 0$ ✓

Hence, $x = -\dfrac{1}{3}$ ✓

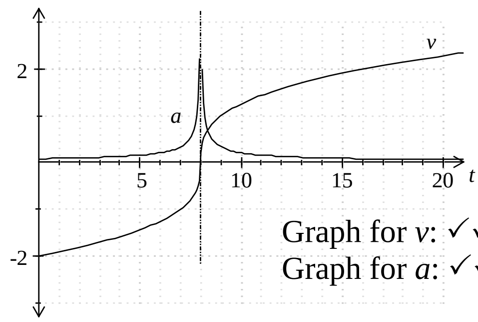
Sign Test: ✓

x	$-\frac{1}{3}^-$	$-\frac{1}{3}$	$-\frac{1}{3}^+$
$\frac{ds}{dx}$	+	0	–

Hence, s is maximum when $x = -\dfrac{1}{3}$,

$\Rightarrow$ Maximum s = $\dfrac{4\sqrt{3}}{3} = 2.3094$ ✓

31. (a)



Graph for v: ✓✓
Graph for a: ✓✓

(b) At A, $v = 0 \Rightarrow t = 8$ seconds ✓

$\Rightarrow$ AO = $\displaystyle\int_0^8 \left|(t-8)^{\frac{1}{3}}\right| dt = 12$ cm ✓

(c) At A, $t = 8$; At B, let $t = k$.

$\Rightarrow$ Distance AB = $\int_8^k (t-8)^{1/3}\, dt$ ✓

= 24 ✓

$\dfrac{3(k-8)^{4/3}}{4} - 0 = 24 \Rightarrow k = 21.45$ seconds✓

(d) Time taken to travel between B and C
= 23.91 – 21.45 = 2.46 seconds.

$\Rightarrow$ Average acceleration = $\dfrac{1}{2.46}\int_{21.45}^{23.91}\left|\dfrac{dv}{dt}\right| dt$ ✓✓

$= \dfrac{1}{2.46}\int_{21.45}^{23.91}\left|\dfrac{(t-8)^{-2/3}}{3}\right| dt$ ✓

$= 0.06$ cms^{-2} ✓

Or Average acceleration
$= \dfrac{1}{2.46}\left[v(23.91) - v(21.45)\right] = 0.06$ cms^{-2}

32. (a) $t^3 - 9t^2 + 24t - 16 = 0 \Rightarrow t = 1, 4$ (repeated)
Hence, twice. ✓

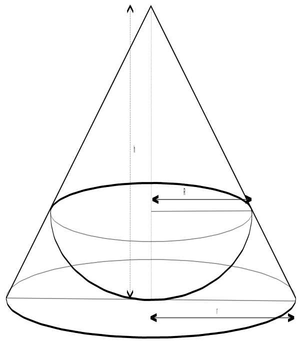
(b) $dy/dt = 3t^2 - 18t + 24$ ✓
$dy/dt = 0 \Rightarrow 3t^2 - 18t + 24 = 0$
$\Rightarrow t = 2, 4$ ✓
$d^2y/dt^2 = 6t - 18$: $t = 2$, $d^2y/dt^2 = -6 \neq 0$
$t = 4$, $d^2y/dt^2 = 6 \neq 0$
Hence, P changes direction twice. ✓

(c) $\int_0^2 3t^2 - 18t + 24\, dt = 20$ ✓

$\left|\int_2^q 3t^2 - 18t + 24\, dt\right| = 2$ ✓

Hence, $(q^3 - 9q^2 + 24q - 16) - 4 = -2$✓
$q = 3$ ✓

33.



$\dfrac{h}{r} = \dfrac{h-R}{R} \Rightarrow r = \dfrac{hR}{h-R}$ ✓

Volume V = $\dfrac{1}{3}\pi r^2 h$

$= \dfrac{1}{3}\pi R^2 \dfrac{h^3}{(h-R)^2}$ ✓

$\dfrac{dV}{dh} = \dfrac{1}{3}\pi R^2\left[\dfrac{(h-R)^2 3h^2 - 2h^3(h-R)}{(h-R)^4}\right]$ ✓

$\dfrac{dV}{dh} = 0 \Rightarrow (h-R)^2 3h^2 - 2h^3(h-R) = 0$

$\therefore h^2(h-R)[3(h-R) - 2h] = 0$ ✓

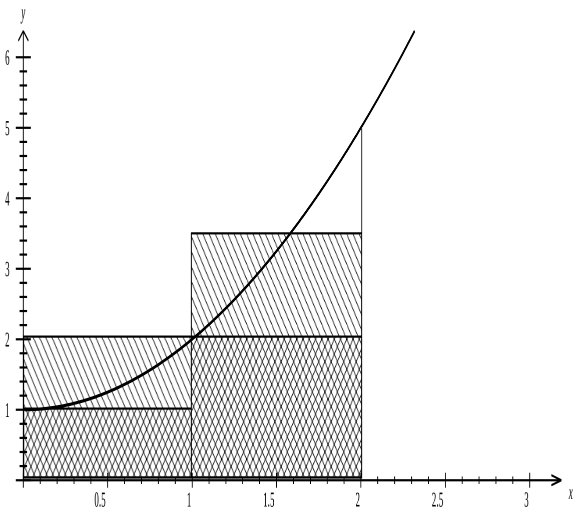
But $h \neq 0$ and $h \neq R \Rightarrow h = 3R$ ✓

When $h = 3R$: ✓

	$3R^-$	$3R$	$3R^+$
$\dfrac{dV}{dh}$	+	0	–

Hence, V is max when $h = 3R$ and $r = 3R/2$.✓✓

34. (a) Area = 1 x [$(1^2 + 1) + (2^2 + 1)$]
= 1 x (2 + 5) = 7 units^2 ✓✓✓



(b) Area = 1 x [$(0^2 + 1) + (1^2 + 1)$]
= 1 x (1 + 2)
= 3 units^2 ✓✓✓

(c) Area = $\frac{3 + 7}{2}$ = 5 units^2 ✓

(d) By making the rectangles of smaller width.✓

35. (a)

$$\text{Acceleration} = \frac{dv}{dt} = 2t - 1 \quad ✓$$

$$v = \int (2t-1)dt$$

$$v = t^2 - t + C$$

$$v = t^2 - t \quad ✓$$

$$\Rightarrow \frac{ds}{dt} = t^2 - t$$

$$\Rightarrow s = \frac{1}{3}t^3 - \frac{1}{2}t^2 + K$$

$$\Rightarrow s = \frac{1}{3}t^3 - \frac{1}{2}t^2 \Rightarrow \frac{1}{3}t^3 - \frac{1}{2}t^2 = 0 \quad ✓$$

$$\Rightarrow t = 0 \; or \; t = 1.5 \quad ✓$$

(b) t = 3 ⇒ ~~s~~ = 4.5 sec ✓

(c) v = 20 ⇒ 20 = t^2 - t
(t - 5)(t + 4) = 0 ✓
t = 5 or t = ~~-4~~ ✓

36. (a) Area(A) = ½.6.1
A = 3 units^2✓
A + B + C = 18
A - B + C = 0
Subtracting 2B = 18
⇒ B = 9 units^2✓
⇒ C = 6 units^2✓

(b) Acceleration is greatest when $\frac{dv}{dt}$ is greatest, ✓
when the slope of the graph is greatest
i.e. at t = 4 seconds✓

37. (a) A (4, 3) and B (2, 0) ✓✓✓

(b) Area $= \int_0^4 (7 - x)\, dx - \int_2^4 \left(x - \frac{4}{x}\right) dx$ ✓✓
$= 20 - [-4\ln 2 + 6] = 14 + \ln 16$ ✓✓

38. $$\frac{d(\ln V)}{dt} = \frac{k}{\sqrt{t}}$$

$\therefore \ln V = \int \frac{k}{\sqrt{t}}\, dt$ ✓

$= k \int t^{\frac{1}{2}}\, dt$

$= k \dfrac{t^{\frac{1}{2}}}{\frac{1}{2}} + c$ ✓

ie. $\ln V = 2k\sqrt{t} + c$

$\therefore V = e^{2k\sqrt{t} + c}$ ✓

or $V = ae^{2k\sqrt{t}}$ where $a = e^{c}$

39. $a = \frac{8}{(t+3)^2}$

$v = \int \frac{8}{(t+3)^2}\,dt$ ✓

$v = -\frac{8}{t+3} + c$

$c = \frac{8}{3}$ ✓

$s = \int \left(\frac{-8}{(t+3)} + \frac{8}{3} \right) dt$

$s = -8\ln(t+3) + \frac{8}{3}t + k$ ✓

$s = 0,\ t = 0 \Rightarrow 0 = -8\ln 3 + k$

$k = 8\ln 3$ ✓

$s = -8\ln(t+3) + \frac{8}{3}t + 8\ln 3$

$t = 10 \Rightarrow s = -8\ln 13 + \frac{80}{3} + 8\ln 3$

displacement = 14.9 *metres* ✓

40. (a) $a = \frac{2t}{t^2+3}$

$v = \int \frac{2t}{t^2+3}\,dt$ ✓

$v = \ln(t^2+3) + c$

$0 = \ln(0+3) + c$

$c = -\ln 3$

$v = \ln(t^2+3) - \ln 3$ ✓

$v = \ln\left(\frac{t^2+3}{3}\right)$ ✓

(b) $v = 10 \Rightarrow 10 = \ln(t^2+3) - \ln 3$ ✓

$\ln(t^2+3) = 10 + \ln 3$

$\ln(t^2+3) = \ln e^{10} + \ln 3$

$\ln(t^2+3) = \ln(3e^{10})$ ✓

$t^2 + 3 = 3e^{10}$

$t = \sqrt{3e^{10} - 3}$

$t = 257$ *seconds* ✓

41. $a = \frac{12}{(t+3)^2}$

$\therefore$ velocity $= \int \frac{12}{(t+3)^2}\,dt$ ✓

$= -12\left(\frac{1}{t+3}\right) + c_1$

at $t = 0,\ v = 0 \therefore c_1 = 4$ ✓

i.e. $v = -\frac{12}{(t+3)} + 4$

displacement from t = 0 to t = 9

$= \int_0^9 v\,dt$

$= \int_0^9 \left(-\frac{12}{(t+3)} + 4\right) dt$ ✓✓

$= \left[-12\ln|t+3| + 4t\right]_0^9$

= - 12 (ln 12 + 36) – (- 12 ln 3)

$= 36 - 12 \ln \frac{12}{3}$

i. e. displacement ≈ 19.4 m✓

42. (a) $\frac{dT}{dx} = \frac{k}{x}$

$T = \int \frac{k}{x}\,dx$

$T = k\ln x + c$ ✓

$T = -40^\circ,\ x = 1$ and $T = -15^\circ,\ x = 2$ ✓

$\Rightarrow -40 = k\ln 1 + c$ and $-15 = k\ln 2 + c$

$\Rightarrow c = -40$ and $k = 36.067$

$\Rightarrow T = 36.067\ln x - 40$ ✓ ✓

(b) $T = 0 \Rightarrow 0 = 36.067 \ln x - 40$

$\ln x = \frac{40}{36.067} \Rightarrow x = 3.03$ ✓✓

Thickness needed is 30 mm (to nearest mm)

43. (a) $a = 4e^{-\frac{1}{20}}$

$v = \int 4e^{-\frac{1}{20}}\,dt$

$v = -20.4e^{-\frac{1}{20}} + c$

$20 = -20.4e^{0} + c\ , t = 0\ , \ v = 20$ ✓✓✓

$\Rightarrow c = 100$

$\Rightarrow v = -80e^{-\frac{t}{20}} + 100$

as $t \to +\infty \ v \to 100$

(b) $s = \int v\,dt$

$s = \int \left(-80e^{-\frac{t}{20}} + 100 \right) dt$ ✓

$s = -80.-20e^{-\frac{t}{20}} + 100t + k$

$0 = -80.-20e^{0} + 0 + k$

$k = -1600$

$s = 1600e^{-\frac{t}{20}} + 100t - 1600$ ✓

$s = \int_0^{10} v\,dt = 370$ ✓

$s(10) = 1600e^{-.5} + 1000 - 1600$

$s = 370.4\ m$

Total distance travelled= 370 m ✓

44. (a) $\int_1^2 \left(\frac{2}{t} + 50\right) dt = 51.39$ °C✓✓

(b) $T(t) = 2\ln t + 50t + c$✓

$80 = 50 + c$

$\therefore\ c = 30$

$\therefore\ T(t) = 2\ln t + 50t + 30$ ✓

45. $\ddot{x} = k\sin t$

$\dot{x} = \int \ddot{x}\,dt = k\cos t + c_1$ ✓

at $t = \frac{\pi}{2}$, $\dot{x} = 0$ i.e. $0 = c_1$

$\therefore x = \int k\cos t\,dt$ ✓

i.e. $x = k\sin t + k$

at $t = 0$, $x = k$ i.e. $k = k\sin(0) + c_2$

$\therefore k = c_2$

$\therefore x = k\sin t + k$ ✓

$\therefore$ at $t = \frac{3\pi}{2}$, $x = k\sin\frac{3\pi}{2} + k$

$= 2k$ ✓

46. (a) $V(t) = t^2 - 6t$

$X(t) = \frac{1}{3}t^3 - 3t^2 + C$ ✓

$X(0) = 0 \rightarrow C = 0$

$\therefore X(t) = \frac{1}{3}t^3 - 3t^2$ ✓

(b) Starting point when X(0) = 0

$\therefore 0 = \frac{1}{3}t^3 - 3t^2$✓

$t = 0$ or 9 ✓

$\therefore$ returns when t = 9 seconds

(c) Furthest away when V(t) = 0

$t^2 - 6t = 0$ ✓

~~t = 0~~ or t = 6 seconds ✓

(d) Moving towards starting point when V(t) < 0

$t^2 - 6t < 0$ ✓

$t(t - 6) < 0$

$\Rightarrow t > 6$ seconds and $t < 9$ ✓

(e)

$$\text{distance} = \left|\int_0^6 \left(t^2 - 6t\right)dt\right| + \left|\int_6^{10}\left(t^2 - 6t\right)dt\right|$$ ✓

$$= |-36| + \left|69\frac{1}{3}\right|$$

$$= 105\frac{1}{3}$$ ✓

47. $v = \dfrac{t(576 - t^2)}{288}$

(a) distance = ∫ velocity dt

i.e. $d = \displaystyle\int_0^{24} \frac{t(576 - t^2)}{288}\,dt$ ✓

i.e. $d = \dfrac{1}{288}\displaystyle\int_0^{24}\left(576t - t^3\right)dt$ ✓

$= \dfrac{1}{288}\left[\dfrac{576t^2}{2} - \dfrac{t^4}{4}\right]_0^{24}$

$= \dfrac{1}{288}\left\{\left(288(24)^2 - \dfrac{(24^4)}{4}\right) - (0)\right\}$

$= \dfrac{1}{288}(82944) = 288\ \text{m}$ ✓

(b) $\text{acc} = \dfrac{dv}{dt} = \dfrac{d}{dt}\left(\dfrac{1}{288}\left(576t - t^3\right)\right)$ ✓

$= \dfrac{1}{288}\left(576 - 3t^2\right)$ ✓

i.e. $a = 2 - \dfrac{3}{288}t^2$ ✓

final acceleration:

i. e. $a(24) = 2 - \dfrac{3}{288}(24)^2 = -4\ \text{m/sec}^2$

initial: a(0) = 2 – 0 = 2

i. e. | a(24) | = 2 | a(0) | ✓

48. (a) v = 3 cos t – sin t

acceleration $a = \dfrac{dv}{dt}$

i.e. $a = -3\sin t - \cos t$

$= -\left(3\sin t + \cos t\right)$ ✓✓

and displacement $x = \displaystyle\int v\,dt$

i.e. $x = 3\sin t + \cos t + c$ ✓

when $t = 0,\ x = 1$

i.e. $1 = 3\sin 0 + \cos 0 + c$

i.e $c = 0$

$\therefore\ x = 3\sin t + \cos t$ ✓

$\therefore\ x + a = 0$ ✓

(b) $a = -\left(3\sin t + \cos t\right)$

$\dfrac{da}{dt} = -3\cos t + \sin t$ ✓

let $\dfrac{da}{dt} = 0 \therefore 3\cos t = \sin t$ i.e. $3 = \tan t$

$t = \tan^{-1}(3)$ ✓

i.e. $t = 1.249,\ 4.391,\ ....$

and a(1.249) = 3.162 a(4.391) = - 3.162 ✓

Hence maximum acceleration = 3.162 m/sec^2

or:

$a = -\left(3\sin t + \cos t\right)$

$= -\sqrt{10}\left(\dfrac{3}{\sqrt{10}}\sin t + \dfrac{1}{\sqrt{10}}\cos t\right)$

$= -\sqrt{10}\left(\sin\left(t + a\right)\right)$

where $a = \cos^{-1}\left(\dfrac{3}{\sqrt{10}}\right)$

$\therefore$ max acc $= \sqrt{10} \approx 3.162\ \text{m/sec}^2$

(c) $x = 0 \therefore 3\sin t + \cos t = 0$

i.e. $3\sin t = -\cos t$ ✓

$\tan t = -\dfrac{1}{3}$

$t = \tan^{-1}\left(-\dfrac{1}{3}\right)$ ✓

i. e. t ≈ 2.82 seconds

49. (a) $\frac{dv}{dt} = at + b$ ✓

$t = 0,\ \frac{dv}{dt} = 10 \times 60 \Rightarrow b = 600$

$t = 1,\ \frac{dv}{dt} = 12 \times 60 \Rightarrow a = 120$ ✓✓

$\frac{dv}{dt} = 120t + 600$

(b) Amount of water

$= \int_0^1 (120t + 600)\,dt$ ✓

$= 660 \text{ m}^3$ ✓

50. (a) $0 \le t < 2,\ v(t) = at^2$

$4 = 4a \Rightarrow a = 1$ ✓

$2 \le t < 4,\ v(t) = -t^2 + bt - c$

$4 = -4 + 2b - c$

$2 = -16 + 4b - c$

$b = 5,\ c = 2$ ✓✓

$4 \le t < 10,\ v(t) = d \quad d = 2$ ✓

$10 \le t \le 12,\ v(t) = (8 - t)(t - 10)(t - 12) + e$

$2 = 0 + e$

$e = 2$ ✓

(b) $v(t) = -t^3 + 30t^2 - 296t + 962$

$v'(t) = -3t^2 + 60t - 296$ ✓✓

$v'(11) = 1$

(c) $A = \int_0^{12} v(t)\,dt$

$= \int_0^2 t^2\,dt + \int_2^4 (-t^2 + 5t - 2)\,dt$

$+ \int_4^{10} 2\,dt + \int_{10}^{12} (-t^3 + 30t^2 - 296t + 962)\,dt$

✓✓

$= 30$ metres ✓

(d) Mean velocity $= \frac{30}{12}$ m/min ✓

$= 2.5$ m/min ✓

51. (a) $v(t) = 16t - t^2 + c$

Since $v(0) = 4$ then $c = 4$ then

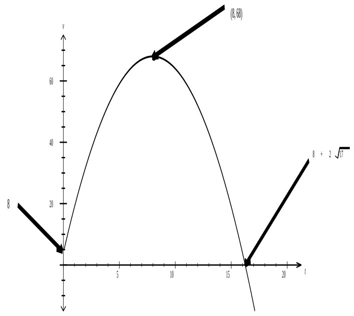
$v(t) = 16t - t^2 + 4$ ✓

$\therefore$ Max velocity $= v(8) = 68$ m/s ✓

(b) $v(k) = 0 \rightarrow 16k - k^2 + 4 = 0$ ✓

$\therefore \quad k = 8 + 2\sqrt{17}$ ✓

(c)



✓✓✓✓

(d) $\left| \int_k^{k+3} (n - mt)\,dt \right|$ ✓✓

Chapter 6: Discrete Random Variables

1. $k + 2k + 3k + 4k + 5k + 6k + 7k + 8k = 1$✓

   $36k = 1$✓

   $k = \frac{1}{36}$✓

2. (a) $k - \frac{3}{16} + k - \frac{2}{16} + k - \frac{1}{16} + k$

   $+ k - \frac{1}{16} + k - \frac{2}{16} + k - \frac{3}{16} = 1$✓✓

   $7k - \frac{12}{16} = 1$✓

   $7k = \frac{28}{16}$

   $k = \frac{1}{4}$✓

   (b) $0 + 1\left(\frac{1}{4} - \frac{2}{16}\right) + 2\left(\frac{1}{4} - \frac{1}{16}\right) + 3\left(\frac{1}{4}\right)$

   $+ 4\left(\frac{1}{4} - \frac{1}{16}\right) + 5\left(\frac{1}{4} - \frac{2}{16}\right) + 6\left(\frac{1}{4} - \frac{3}{16}\right)$✓✓

   $= \frac{1}{8} + \frac{3}{8} + \frac{3}{4} + \frac{3}{4} + \frac{5}{8} + \frac{3}{8}$

   $= 3$✓

3. (a) $\frac{1}{20} + \frac{6}{20} + \frac{3}{20} = \frac{1}{2}$✓✓

   (b) $\frac{6}{20} + \frac{6}{20} + \frac{3}{20} + \frac{1}{20} = \frac{4}{5}$✓✓

   (c) $\dfrac{\frac{3}{20} + \frac{6}{20}}{\frac{1}{20} + \frac{3}{20} + \frac{6}{20} + \frac{6}{20}} = \frac{9}{16}$✓✓

4. (a) $k + 2k + 3k + 4k + 5k + 6k = 1$✓

   $21k = 1$

   $k = \frac{1}{21}$✓

   (b) $\frac{3}{21} = \frac{1}{7}$✓✓

   (c) $\frac{1}{21} + \frac{3}{21} + \frac{5}{21} = \frac{3}{7}$✓✓

   (d) $\frac{1}{21} + \frac{2}{21} + \frac{3}{21} + \frac{4}{21} = \frac{10}{21}$✓✓

   (e) $\dfrac{\frac{1}{21} + \frac{3}{21}}{\frac{1}{21} + \frac{2}{21} + \frac{3}{21} + \frac{4}{21}} = \frac{4}{10} = \frac{2}{5}$✓✓

5. (a) $0{\cdot}0046$✓✓

   (b) $0{\cdot}0046 + 0{\cdot}0011 + 0{\cdot}0002 = 0{\cdot}0059$✓✓

   (c) $1 - P(x < 5)$

   $= 1 - (0{\cdot}0002 + 0{\cdot}0011 + 0{\cdot}0046)$ ✓

   $= 0{\cdot}9941$✓

6. (a) $x + \frac{1}{5} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$✓

   $x + \frac{26}{30} = 1 \quad \therefore \quad x = \frac{2}{15}$✓✓

   (b) P(score of 6) = P(1,5) + P(2,4) + P(3,3) + P(4,2) + P(5,1)

   $= 2 \times \frac{2}{15} \times \frac{1}{6} + 2 \times \frac{1}{5} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6}$✓

   $= \frac{2}{45} + \frac{1}{15} + \frac{1}{36}$✓

   $= \frac{5}{36}$✓

7. P(score of 6) = P(1,5) + P(2,4) + P(3,3) + P(4,2) + P5,1)

   $= 2\times\frac{1}{5}\times\frac{1}{6}+2\times\frac{1}{6}x\times\frac{1}{30}(8-5x)+\frac{1}{5}\times\frac{1}{5}$ ✓✓

   $= \frac{1}{15} + \frac{1}{90}x(8 - 5x) + \frac{1}{25}$

   $= \frac{30 + 40x - 25x^2 + 18}{450}$✓

   $= \frac{48 + 40x - 25x^2}{450}$✓

8. (a) $\left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4$ ✓✓

   $+ \left(\frac{1}{2}\right)^5 + k = 1$

   $\therefore \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + k = 1$ ✓

   $\therefore k = \frac{1}{32}$ ✓

   (b) $E(X) = 1\frac{31}{32}$ ✓

   $Var(X) = 1\frac{671}{1024}$ ✓

9. (a)

x	P(X = x)
2	$\frac{1}{4}$
3	$\frac{1}{4}$
4	$\frac{1}{8}$
5	$\frac{1}{8}$
6	$\frac{1}{4}$

✓✓

(b) Not continuous, sum of probabilities is 1 and each is positive. ✓

$\Rightarrow$ discrete probability density function. ✓

(c) P (even total)

= P (even + even) + P (odd + odd)

$= \frac{5}{8} \times \frac{5}{8} + \frac{3}{8} \times \frac{3}{8}$ ✓✓

$= \frac{34}{64} = \frac{17}{32}$ ✓

10. (a) $P(t) = \frac{k}{3^t}$

$k + \frac{k}{3} + \frac{k}{9} + \frac{k}{27} = 1$ ✓✓

27k + 9k + 3k + k = 27

40k = 27✓

$\therefore \quad k = \frac{27}{40}$ ✓

(b) $P(t) = \frac{27}{40 \cdot 3^t}$

$P(1) = \frac{27}{40 \cdot 3} = \frac{9}{40}$ ✓✓

11. (a) P (4 blue) = 0·13824✓✓
(b) P (4 white) = 0·23437✓✓
(c) P (at least 2 red) = 0·11426✓✓
(d) P (4 are blue | none white)

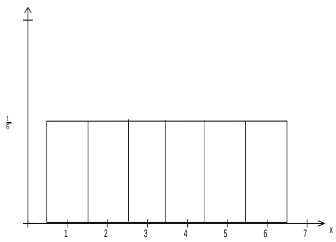
P (blue) = $\frac{0 \cdot 4}{0 \cdot 5} = 0 \cdot 8$✓✓

= 0·24576✓

12. Min = 0.1 Max = 0.3Middle score = 0.2
Σ x = 1 (prob.distribution) n must be 5.
Lower Quartile = 0.15 = mean of 0.1 and x_2.
So x_2 = 0.2
Upper Quartile = 0.25 = mean of 0.3 and x_3.
So x_3 = 0.2
Scores are 0.1, 0.2, 0.2, 0.2, 0.3 ✓✓✓

13. (a) 0.1 ✓
(b) (i) (0.3)(0.6) = 0.18✓✓
(ii) 0.3 + 0.6 – 0.18 = 0.72✓✓
(iii) 0.3✓
(iv) 5 and 1or 1 and 5= 0.18✓
0.18 = 0.36✓

14. (a) (i)



✓

(ii) $6k = 1 \quad \therefore \quad k = \frac{1}{6}$ ✓

(b) (i) $\frac{3}{6}$ ✓

(ii) Red, white or pink sun tolerant✓

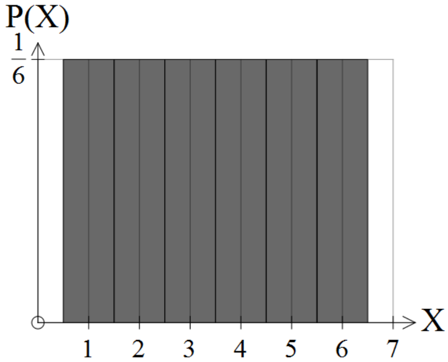
(c) $\frac{2}{6}$ ✓

(d) (i) 3! = 6 ✓
(ii) 2 ✓
(e) $2^6 - 1 = 63$ ✓✓

15. (a) Uniform✓
(b) E(X) = 4.5, Var(X) = 1.25✓✓
(c) E(Y) = 8, Var(Y) = 5✓✓

16. (a) E(X) = 2 = cost, so fair. ✓
(b) (i) E(X) = 2 = cost, so fair. ✓
(ii) Y = 2x – 2 ✓
E(Y) = 2 = cost, so fair. ✓
(c) E(Z) = 0.5 Not fair–you lose $1.50✓

17. (a)



✓ ✓

(b) $E(X) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} \times \; .... + 6 \times \frac{1}{6}$

$= \frac{7}{2}$ ✓

$\text{Var} = \Sigma(X-\mu)^2 \times P(X=x)$

$= \left(1-\frac{7}{2}\right)^2 \times \frac{1}{6} + \left(2-\frac{7}{2}\right)^2 \times \frac{1}{6}$ ✓

$+ \left(3-\frac{7}{2}\right)^2 \times \frac{1}{6} + ... + \left(6-\frac{7}{2}\right)^2 \times \frac{1}{6}$ ✓

$= \frac{35}{12}$

(c) $Var(X) = \frac{1}{12}(6-1)(6-1+2) = \frac{35}{12}$ ✓

(d) E(Y) = 7, Var(Y) = $\frac{35}{3}$ ✓✓

18. (a) $E(X^2) = 29$✓
(b) E(7X + 2) = 37✓✓
(c) Var(3X – 1) = 36✓✓

Chapter 7: Bernoulli & Binomial Distributions

Bernoulli Distribution

1. (a) (i) $P(X=x) = \begin{cases} 0{\cdot}5 & \text{if } x = 0 \\ 0{\cdot}5 & \text{if } x = 1 \end{cases}$ ✓✓

(ii)

x	0	1
P ($X = x$)	0.5✓	0.5✓

(b) 0.5✓
(c) Getting a Tail✓
(d) E(x) = 0.5✓
Var(x) = p(1–p) = 0.25✓
SD = $\sqrt{p(1-p)} = 0{\cdot}5$ ✓
(e) (i) two✓, success✓ and failure✓
(ii) independent✓
(iii) constant✓

2. Two outcomes; 0 and 1. ✓
Probabilities are 0.6 for success and 0.4 for failure. ✓

3. (a) (i) $P(X = x) = \begin{cases} \frac{4}{6} & \text{when } x = 0 \\ \frac{2}{6} & \text{when } x = 1 \end{cases}$ ✓✓

(ii)

x	0	1
P ($X = x$)	$\frac{4}{6}$ ✓	$\frac{2}{6}$✓

(b) $\frac{2}{6}$ ✓
(c) Getting a result greater than or equal to 3 on any one roll. ✓
(d) $E(X) = p = \frac{2}{6}$ ✓

$\text{Var}(X) = p(1-p) = \frac{2}{6} \times \frac{4}{6} = \frac{2}{9}$ ✓

$\text{SD} = \sqrt{\frac{2}{9}} = \frac{\sqrt{2}}{3}$ ✓

4. $X = \begin{cases} 0 & \text{when } x = 0.45 \\ 1 & \text{when } x = 0.55 \end{cases}$ ✓ ✓

Binomial Distribution

1. (a) P(2 fives) = 0.11574 = B(5 , $\frac{1}{6}$)✓✓

   (b) P(2 or more fives) = 1 – 0.86805
   = 0.13195✓✓

2. (a) P(heart 3 times) = 0.08789 = B(5 , $\frac{1}{4}$)✓✓

   (b) P(king 3 times) = 0.0038783✓✓

   (c) P(queen of clubs 3 times) = 0.00006841✓✓

3. P(2 tails) = 0.3125✓✓

4. P(3 heads) = 0.21875✓✓

5. P(2 tails) = 0·25✓
   P(2 tails, twice) = 0·26367✓

6. P(2 heads) = 0·25✓
   P(2 heads, 4 times) = 0·032958✓

7. (a) P(1 defective) = 0.3915✓

   (b) P(2 defective) = 0.13817✓

   (c) P(4 defective) = 0.0021515✓✓

8. P(5 or 6 in 1 throw) = $\frac{1}{3}$✓
   P(3 successes) = 0.098765✓

9. P(head) = 0·5
   P(at least 4 heads).= 0·34375✓✓

10. P(1 faulty) = 0·05✓
    P(no more than 1 faulty) = 0·9774✓✓

11. P(new) = 0·6✓
    P (12 new and 8 old) = 0·1797✓✓

12. P(1 fish) = 0·025✓
    P (more than 1 fish) = 0.088241✓✓

13. (a) P(1 error slip) = 0·24046✓

    (b) P(at least 1 error slip) = 0·27861✓✓

    (c) $\mu = 500 \times 0{\cdot}4$ ✓
    $\sigma = \sqrt{500 \text{x} 0.4 \text{x} 0.6}$ ✓
    = 10.95445

14. (a) P(all) = 0·028247✓✓

    (b) P(8) = 0.23347✓✓

    (c) P(at least 8) = 0·38279✓✓

    (d) P(8 each row) = P(8).P(8)
    = 0·054508✓✓

    (e) P(16) = 0·13042✓✓

    (f) Expected = 200 × 0·7 = 140✓

    (g) $\sigma = \sqrt{200 \text{ x } 0{\cdot}7 \text{ x } 0{\cdot}3}$ ✓
    = 6·4807✓

15. (a) P(4 or fewer late) = 0·83576✓✓

    (b) P(2 ≤ x ≤ 4) = P(2) + P(3) + P(4)
    = 0·66862✓✓✓

    (c) $P(x = 4 \mid x \le 4) = \frac{0.1876}{0.83576}$ ✓✓
    = 0·22447✓

    (d) P(2) = 0.2048✓✓

    (e) P(2 each day) = 0·047286✓

    (f) P(total of 4) = 0·21819✓✓

    (g) $\mu = 500 \times 0{\cdot}2 = 100$ ✓
    $\sigma = \sqrt{500 \text{ x } 0{\cdot}2 \text{ x } 0{\cdot}8}$ ✓
    = 8·94427✓

16. (a) P (3) = 0·27869✓✓

    (b) P (x ≥ 3) = 0.6846✓✓

    (c) P (1st 3) = $(0{\cdot}4)^3$✓
    = 0·064✓

    (d) P (at least 1) = 1 – P(all fail)
    = $1 - (0{\cdot}6)^n$ ✓
    ∴ $1 - (0{\cdot}6)^n \ge 0{\cdot}95$
    $(0{\cdot}6)^n \le 0{\cdot}05$✓
    $n \ge \frac{\log 0{\cdot}05}{\log 0{\cdot}6}$
    $n \ge 5{\cdot}86$✓
    at least 6 subjects required. ✓

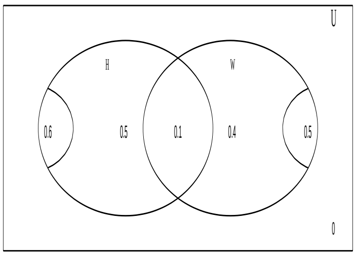
Note : trial and error will give answer.

17. (a) P(H) = 0.6, P(H|W) = 0.2 and P(H ∩ W) = 0.1

$$P(H|W) = \frac{P(H \cap W)}{P(W)}$$

$\therefore \quad 0.2 = \frac{0.1}{P(W)} \rightarrow P(W) = 0.5$ ✓✓

(b)



✓✓

(c) At least one must have a TCC ✓

(d) (i) 0.9 ✓

(ii) $\frac{0.1}{0.6} = \frac{1}{6}$ ✓✓

(e) $P(H \cap H) = 0.6^2 = 0.36$ ✓

(f) X ~ B(5, 0.1) ✓

18. (a) P(X = 3) = 0.08789 ✓✓
(b) P(X ≥ 3) = 0.10351 ✓✓

19. (a) (i) P (X ≥ 20) = 0.1065 ✓✓
(ii) P (X = 20) = 0.0402 ✓✓
(b) (i) P (Air and Oil) = 0.03 ✓✓
(ii) P (Air or Oil) = 0.32 ✓✓
(iii) P (Air | Oil) = 0.15 (independent)✓

20. (a) n = 4 ✓
(b) $p^4 = 0.053 \therefore p = 0.48$ (2 d.p.) ✓✓
(c) 1 ✓

21. B (8, 0.15,4) = 0.0185 ✓✓

22. (a) $X \sim B(11, \frac{1}{8})$

$P(X = 9) = 3 \cdot 1373 \times 10^{-7}$ ✓✓

(b) $Y \sim B(20, 0.6)$
$P(Y = 19) = 0.00049$ ✓✓
(c) $P(X = 9 \cap Y = 19) = 1.5 \text{ x } 10^{-10}$ ✓

23. (a) (i) 1 – P (Both are non–defective)
$= 1 - (0.95)^2 = 0.0975$ ✓✓
(ii) $(0.05)^2 = 0.0025$ ✓
(b) X ≈ B(50, 0.05) ✓
P(X ≥ 2) = 0.721 ✓✓

24. (a) Using calculator $a = 1$, $g = 0.0625$, $h = 0.375$ & $k = 0.5625$✓✓✓✓
(b) Parameters are $n = 2$ & $p = 0.75$ ✓✓

25. (a) (i) P(W ∩ L) = 0.01 ✓✓
(ii) P(L ∩ W′) = 0.02 ✓
(iii) P(W′ ∩ L′) = 0.8 ✓
(iv) $P(W \mid W \cup L) = \frac{0.18}{0.2} = 0.9$ ✓✓

(b) (i) $\binom{4}{3}(0.02)^3(0.8) = 0.0000256$ ✓✓

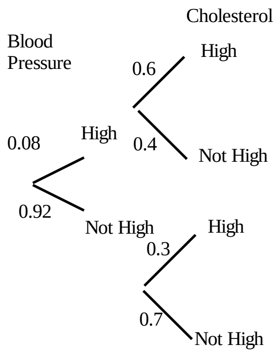
(ii) $\binom{4}{3}(0.02)^3(0.98) = 0.00003136$ ✓✓✓

26. (a) Not constant probability. ✓

(b) $P(x \geq 1) = 1 - \frac{\binom{15}{4}}{\binom{20}{4}} = 0 \cdot 718$ ✓✓✓

27. (a) 2 ✓
(b) P(X ≤ 2) = 0.6767 ✓✓
(c) (i) $(0.6767)^4 = 0.2097$ ✓✓
(ii) P(X ≤ 8) = 0.5925 ✓✓

28. (a) ✓✓



(b) (i) 0.048 ✓
(ii) 1–(0.92)(0.7) = 0.356 ✓
(iii) 0.048 + 0.276= 0.324 ✓
(iv) $\frac{0.048}{0.324} = 0.148$ ✓✓

(c) (i) $(0.6)^6 = 0.0467$ ✓
(ii) $(0.4)^6 = 0.0041$ ✓✓
(iii) ${}^6C_3(0.6)^3(0.4)^3 = 0.2765$ ✓✓

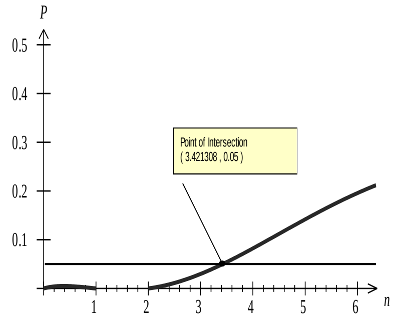
(d) By checking in the Binomial tables, n = 20. ✓✓

29. $X \sim B(n, 0.31)$

$P(X = 3) = \binom{n}{3}(0.31)^3(0.69)^{n-3}$ ✓

$\therefore \frac{n(n-1)(n-2)}{6} \times (0.31)^3(0.69)^{n-3} \leq 0.05$ ✓

$n \leq 3.42 \therefore n = 3$ ✓



30. (a) $\frac{104}{204} = \frac{26}{51}$ ✓✓

(b) $\frac{104}{204} \times 2\,000 = 1\,020$ ✓

(c) (i) $X \sim B(5, \frac{26}{51})$

$\therefore P(X \leq 3) = 0.8$ ✓✓

(ii)

$\left(\frac{26}{51}\right)^3 \times \left(\frac{25}{51}\right) \times \binom{4}{1} \times \frac{26}{51} = 0{\cdot}1324$ ✓✓✓

31. (a) (i) Independent trials, finite number of trials ✓✓
and two outcomes; success or failure. ✓

(ii) $\mu = 3{\cdot}6$ and $\sigma = 0{\cdot}6$ ✓✓

(iii) P(0) = 0.0001 ✓

(b) P(0) = 0.01%

∴ Unlikely to have occurred by chance alone. ✓

∴ Claim not reasonable ✓

32. (a) Independent trials, success or failure, constant *p*. ✓

(b) $\bar{x} = 30$ and $\sigma_x = \sqrt{12}$ ✓✓

X ~ B(50, 0.4)

(c) P(X = 10) = 0.14 ✓

∴ Not likely investigate. ✓

(d) $P(X < 30 | X \geq 25) = \frac{P(25 \leq X \leq 29)}{P(X \geq 25)}$ ✓

$= \frac{0.382}{0.943} = 0.405$ ✓

(e) $51 010.04 ✓

(f) (i) $50 000e ✓

(ii) $135 914 ✓

(g) (i) $\frac{dA}{dt} = kA$ ✓

(ii) $e^{0.02t} = 2 \rightarrow t = 34.66$ ✓✓

33. (a) $P(X < 7) = 0.7$ ✓

(b) (i) 0.1 ✓

(ii) $0.1^3 = 0.001$ ✓✓

(iii) $\binom{10}{3}(0.1)^3(0.9)^7 = 0.057$ ✓✓

(iv) (0.1 x 0.9) + (0.9 x 0.1)
= 0.18 ✓✓

(v) $P(X \geq 3) = 1 - P(X \leq 2)$ ✓
= 1 – (0.387 + 0.194 + 0.348)
= 0.070 ✓

or B(10, 0.1) = 0.070

Continuous Random Variables

1. (a) $2k + \frac{1}{2}.3.k = 1$ ✓

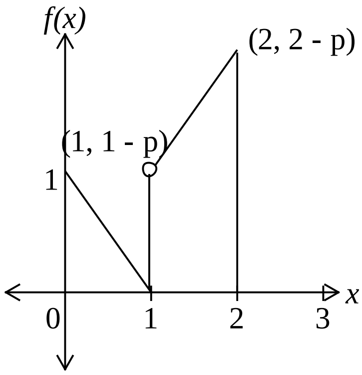
$k = \frac{2}{7}$ ✓

(b) $P(X < 2) = 2.\frac{2}{7} = \frac{4}{7}$ ✓

(c) $P(X < 1 \mid X < 3)$

$= \dfrac{\frac{2}{7}}{1 - \frac{1}{2}.2.\frac{4}{21}} = \frac{6}{17}$ ✓✓✓

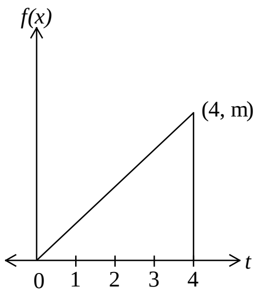
2.



✓

$\frac{1}{2} + \frac{1}{2}.1.(3 - 2p) = 1 \Rightarrow p = 1$ ✓✓

3. (a)



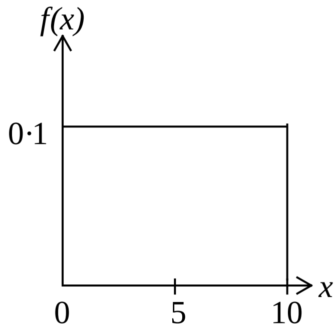
$\frac{1}{2}.4.m = 1 \Rightarrow m = \frac{1}{2}$ ✓

Equation of the probability distribution is $y = \frac{1}{8}x$ ✓✓

(b) $P(3 < t < 4) = \frac{1}{2}.\left(\frac{3}{8} + \frac{1}{2}\right).1$ ✓✓

$= \frac{7}{16}$ ✓

4. (a) $f(x) = 0.1 \quad 0 \le x \le 10$ ✓✓

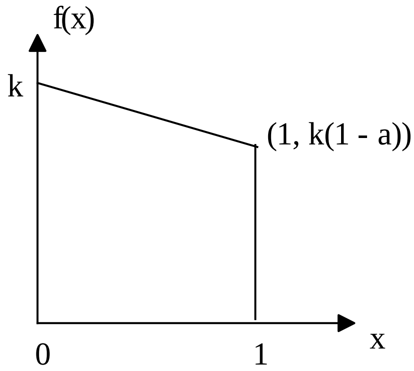


✓✓

(b) $P(1.5 \le X \le 7.5) = \frac{6}{10} = 0.6$ ✓✓

(c) $P(\text{all}) = (0.6)^5 = 0.07776$ ✓✓

5. (a)



✓✓

$k(1 - a) \ge 0$

$\Rightarrow 1 - a \ge 0$ ✓

$\Rightarrow a \le 1$ ✓

(b) Area = 1

$\Rightarrow \frac{1}{2}(k + k(1 - a)) = 1$ ✓✓

$k + k - ak = 2$ ✓

$k(2 - a) = 2$

$k = \frac{2}{2 - a}$ ✓

6. $\int_0^2 kx^2 dx = 1$✓

$\left[\frac{1}{3}kx^3\right]_0^2 = 1$✓

$\frac{1}{3}.k.8 = 1$✓

$k = \frac{3}{8}$✓

7. (a) $P(1 \le x \le 2) = \int_1^2 \frac{x^2}{9} dx$✓

$= \left[\frac{x^3}{27}\right]_1^2$

$= \frac{8}{27} - \frac{1}{27}$✓

$= \frac{7}{27}$✓

(b) $P(x \ge k) = 0{\cdot}875$

$\int_k^3 \frac{x^2}{9} dx = 0\cdot875$✓

$\left[\frac{x^3}{27}\right]_k^3 = 0{\cdot}875$

$1 - \frac{k^3}{27} = 0{\cdot}875$✓

$k^3 = 0{\cdot}125 \times 27$

$k = 1{\cdot}5$✓

8. $P(x < 2) = 0{\cdot}5$

$\int_1^2 (ke^{-0\cdot1x} + \frac{1}{x}) dx = 0\cdot5$✓

$\left[\frac{k}{-0\cdot1}e^{-0\cdot1x} + \ell n\, x\right]_1^2 = 0\cdot5$✓

$\frac{k}{-0\cdot1}e^{-0\cdot2} + \ell n2 - \left(\frac{k}{-0\cdot1}e^{-0\cdot1} + \ell n1\right) = 0\cdot5$✓

$-8{\cdot}187k + 9{\cdot}048k = 0{\cdot}5 - \ln 2 + \ln 1$✓

$0{\cdot}861k = -0{\cdot}1931$✓

$k = -0{\cdot}2246$✓

9. (a) $E(X) = 0{\cdot}2$✓ and $Var(X) = 25$✓

(b) $P(x > 5) = \int_5^\infty 0\cdot2e^{-0\cdot2x} dx$✓

$= \left[-e^{-0\cdot2x}\right]_5^\infty$✓

$= 0 + e^{-1}$

$= 0{\cdot}368$✓

(c) $P(x > 10|x > 5) = \frac{e^{-2}}{0\cdot368}$✓

(Note : $\frac{e^{-2}}{e^{-1}} = e^{-1}$)

$= \frac{0\cdot135}{0\cdot368} = 0{\cdot}368 = P(x > 5)$ ✓✓

10. (a) $\int_0^\infty pe^{-qx} dx = 1$

$\Rightarrow p = q$

$\int_0^1 pe^{-qx} dx = 0\cdot88$✓

$\left[-\frac{p}{q}e^{-qx}\right]_0^1 = 0{\cdot}88$✓

$-e^{-q} + e^0 = 0{\cdot}88$✓

$-e^{-q} = -0{\cdot}12$

$e^{-q} = 0{\cdot}12$

$q = -\ln 0{\cdot}12$✓

$q = 2{\cdot}120 = p$✓

(b) $P(2 < x < 3) = \int_2^3 2\cdot12\, e^{-2\cdot12x} dx$✓

$= \left[-e^{-2\cdot12x}\right]_2^3$

$= -e^{-6\cdot36} + e^{-4\cdot24}$

$= 0{\cdot}012678225$✓✓

11. (a) $P(X \ge 20) = \int_{20}^\infty 0\cdot05e^{-0\cdot05x} dx$✓

$= \left[-e^{-0\cdot05x}\right]_{20}^\infty$✓

$= 0 - \left(-e^{-0\cdot05 \times 20}\right)$✓

$= 0{\cdot}36788$ ✓

(b) $P(20 \le X \le 30) = \int_{20}^{30} 0\cdot05e^{-0\cdot05x} dx$✓

$= \left[-e^{-0\cdot05x}\right]_{20}^{30}$✓

$= -e^{-0\cdot05 \times 30} + e^{-0\cdot05 \times 20}$✓

$= 0{\cdot}14475$✓

12. (a) $\int_0^a \frac{x}{2}dx = 1$ ✓

$\left[\frac{x^2}{4}\right]_0^a = 1$ ✓

$\frac{a^2}{4} = 1$ ✓

$a^2 = 4$ ✓

(b) $a = \pm 2$ ✓✓

(c) $a = -2$ is not valid as outside domain and would also give negative values of f. ✓✓

13. $f(x) = ke^{-kx}$ mean $= \frac{1}{k}$ ✓

$P(x > \frac{1}{k}) = \int_{\frac{1}{k}}^{\infty} ke^{-kx}\,dx$ ✓

$= \left[-e^{-kx}\right]_{\frac{1}{k}}^{\infty} = 0 - (-e^{-1})$ ✓

$= e^{-1} = 0{\cdot}368$ ✓

14. $P(100 \le x \le a) = 0{\cdot}53696$ ✓

$\int_{100}^{a} 0\cdot001e^{-0\cdot001x}dx = 0{\cdot}53696$ ✓

$\left[-e^{-0\cdot001x}\right]_{100}^{a} = 0{\cdot}53696$ ✓

$-e^{-0\cdot001a} + e^{-0\cdot1} = 0{\cdot}53696$

a = 1000.005499 ✓

$-e^{-0\cdot001a} = -0{\cdot}36788$ ✓

$-0{\cdot}001a = \ln 0{\cdot}36788$

$-0{\cdot}001a = -0{\cdot}999998$

$a = 999{\cdot}998$ ✓

Maximum life of bulb is 1000 hours. ✓

15. (a) E(X) = 2 hours✓ and Var(X) = 4✓

(b)

$P(\text{error in } 1^{st} \text{ hour}) = \int_0^1 0.5e^{-0.5t}\,dt$

$= \left[-e^{-0\cdot5t}\right]_0^1$ ✓✓

$= -e^{-0\cdot5} + e^0$

$= 0\cdot393$

✓✓

(c) P(error in 2^{nd} hour)

$= P(\text{error in } 2^{nd} \text{ hour} | \text{no error in } 1^{st} \text{ hour})$

$\frac{[-e^{-0\cdot5t}]_{,2}^{1}}{1 - 0\cdot393} = 0\cdot393$ ✓✓

(d) $\text{cdf} = \begin{cases} 1 - e^{-0\cdot5t} & \text{when } x \ge 0 \\ 0 & \text{when } x < 0 \end{cases}$ ✓✓

16. (a) $P(x) = -kx^2 + 20kx$

Since the graph is quadratic and probabilities must be ≥ 0.

∴ Defined for $0 \le x \le 20$. ✓✓✓

(b) $\int_0^{20} \left(-kx^2 + 20kx\right) dx = 1$ ✓

$\left[\frac{-kx^3}{3} + \frac{20kx^2}{2}\right]_0^{20} = 1$

$\frac{-8000}{3}k + \frac{8000k}{2} = 1$ ✓

$\frac{8000k}{6} = 1$

$k = \frac{6}{8000} = \frac{3}{4000}$ ✓

(c) m is the median

$(P(X<\text{median}) = \frac{1}{2})$ ✓

(d) $\left[\frac{-3}{4000}\cdot\frac{x^3}{3} + \frac{20.3}{4000}\cdot\frac{x^2}{2}\right]_0^m = \frac{1}{2}$ ✓

$-\frac{m^3}{4000} + \frac{3m^2}{400} - \frac{1}{2} = 0$ ✓

∴ m = 10 ✓

17. (a) Area to the left of 8
= 1 – area to the right of 8
= 1 – 0.4493= 0.5507 ✓✓✓
(b) Area to the right of upper quartile = 0.25
UQ = 13.86 ✓✓

18. (a) (i) $P(X = x) = 0.2$, for $x = 0, 1, 2, 3, 4$ ✓✓
(ii) 0.2 ✓
(iii) 0.8 ✓
(iv) $\frac{0.2}{0.4} = 0.5$ ✓✓
(b) (i) $f(x) = 0.2, \quad 0 \le x \le 5$ ✓✓
(ii) 0 ✓
(iii) 0.8 ✓
(iv) $\frac{0.4}{0.6} = \frac{2}{3}$ ✓✓

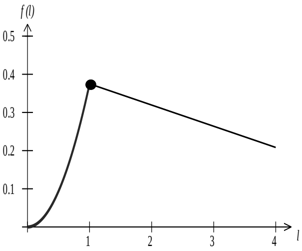
19. (a) Area = 4 x 0.25 = 1
All functional values are non negative.✓✓
(b) Functional values become $0.5 - a$,
so Area becomes
$4 \times (0.5 - a) = 2 - 4a = 1$
$\therefore a = 0.25$ ✓✓

20. (a) $A = \frac{1}{2} \times 2 \times 0{\cdot}8 = 0{\cdot}8$ ✓
∴ No ✓
(b) $A = 2\pi \times \left(0.5 \times \sqrt{\frac{2}{\pi}}\right)^2 = 1$ ✓✓
∴ Yes ✓
(c) $A = 3 \times 0{\cdot}2 + 0{\cdot}4 = 1$ ✓
∴ Yes ✓
(d) $f(1) < 0$ ✓
∴ No ✓

21. (a) False. X needs to go to 10. ✓✓
(b) True. Area = 1 ✓✓
(c) True. Area = 1 ✓✓
(d) True. Area = 1. ✓✓

22. (a) (i) 5 ✓
(ii) $\frac{b + a}{2}$ ✓
(iii) $\frac{b + a}{2}$ ✓
(b) (i) $\frac{3}{4}$ ✓
(ii) $a = -1$ and $b = 1$ ✓
(iii) $a = -b$ ✓

23. (a)



✓✓

(b) $\int_0^1 \left(\frac{3}{8} l^2\right) dl = 0.125$

$\int_1^4 \left(-\frac{1}{18} l + \frac{31}{72}\right) dl = 0.875$ ✓

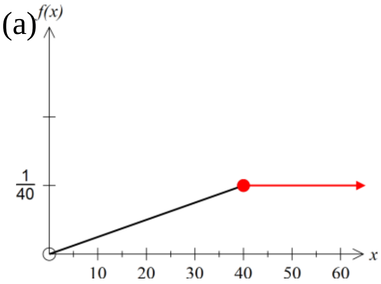
and since all $f(l) \ge 0$ ∴ pdf. ✓✓

(c) $\int_1^m \left(-\frac{1}{18} l + \frac{31}{72}\right) dl = 0.375$ ✓

∴ $\left[-\frac{1}{36} l^2 + \frac{31}{72} l\right]_1^m = 0.375$ ✓✓

∴ $m = 2.09$ or by area of trapezium.✓

24. (a)



✓✓✓

(b) $20k + 20k = 1$ ✓✓
∴ $k = \frac{1}{40}$ ✓

(c)
$P(X < 45) = P(X < 40) + P(40 < X < 45)$

$= \frac{20}{40} + \frac{5}{40}$ ✓✓

$= \frac{25}{40} = \frac{5}{8} = 0.625$ ✓

25. (a) E(X) = 3✓, Var(X) = 3✓
(b) E(Y) = 30 mm, Var(X) = 300 mm^2 ✓✓

26. (a) $E(X) = 4$ ✓

(b) $\int_3^5 (f(x) \times (x-\mu)^2 \; dx$

$= \int_3^5 \left(\frac{1}{2}(x-4)^2\right) dx$ ✓

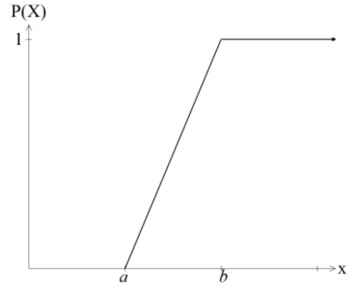
$= \frac{1}{2}\left[\frac{(x-4)^3}{3}\right]_3^5$ ✓

$= \frac{1}{2}\left(\frac{1}{3} - \left(-\frac{1}{3}\right)\right) = \frac{1}{3}$ ✓

(c)

$$P(X = x) = \begin{cases} 0 & \text{when } x < 0 \\ \dfrac{x-a}{b-a} & \text{when } a \le x \le b \\ 1 & \text{when } x > b \end{cases}$$ ✓✓

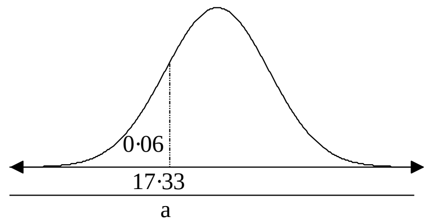
(d)



✓✓

Chapter 8: Normal Distribution

1.



(a) 0.9 ✓

(b) $\frac{0.9}{0.94} = 0.957$ ✓✓

(c) P (z < a) = 0.06 ✓
a = –1.5548 (from calc)
17.33 = 22 – 1.5548 s
∴ s = 3 ✓

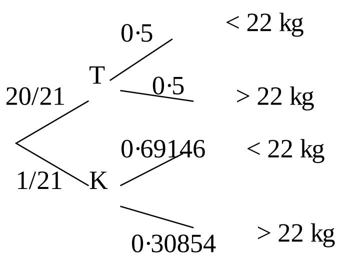
(d) Tuna. ✓Probability becomes less the further we go from the mean. ✓

(e) Binomial: n = 3, p = 0.69146 (calculator)

$P(X = 2) = \binom{3}{2} (0.69146)^2 (0.30854)$ ✓

$= 0.44255$ ✓

(f)



$P(T \cap <22) = \frac{10}{21}$ ✓✓

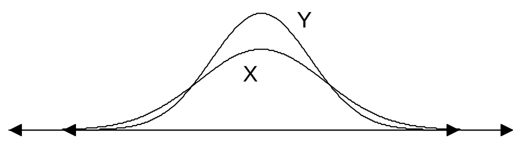
$P(K \cap <22) = \frac{0.69146}{21}$ ✓✓

$P(T \cap <22) + P(K \cap <22) = \frac{10}{21} + \frac{0.69146}{21}$

$= 0.509$ ✓

2. $P(\mu - \sigma < X < \mu + 2\sigma) = P(-1 < Z < 2)$ ✓
$= 0.8186$ ✓
$= 81.9\%$ ✓

3. ✓Normal shape✓ equal means ✓
Y higher than X✓
X more spread out than Y



4. (a) $f(0) = \frac{1}{\sqrt{2\pi}} = 0.3989$ ✓
(b) 0.8413 ✓
(c) $-0.58058 < z < 0.58058$ ✓✓✓

5. (a) 0.06681 ✓
(b) P (Z > a) = 0.3
$a = 0.5244$ ✓
$25 = 18 + 0.5244\sigma$
$\sigma = 13.349$ ✓
(c) $3 = \mu - 1.0364\sigma$ ✓
$12 = \mu + 1.2816\sigma$ ✓
Solve simultaneously to give $\mu = 7.024$
and $\sigma = 3.8827$ ✓

6. (a) (i) P (X > 6 000) = 0.0228 ✓
(ii) P (4 000 < X < 6 000) = 0.9545 ✓
(b) (i) $(0.9772)^4 = 0.9119$ ✓✓
(ii) $(0.9772)^2(0.0228)^2 = 0.000496$ ✓✓
(iii) P ($X \geq 2$) where X is Binomial (4, 0.0228)= 0.0030 ✓✓
(iv) $P(A \cup B)$
$= P(A) + P(B) - P(A \cap B)$ ✓
${}^4C_1\,(0.0228)^1(0.9772)^3 +$
${}^4C_1\,(0.0228)^1(0.9772)^3$
$- [{}^4C_1\,(0.0228)^1][{}^4C_1\,(0.0228)^1] \times$
$[\,P(4\,000 < X < 6\,000)]^2$ ✓✓
$= 0.1702 - 0.0076$
$= 0.1626$ ✓

7. (a) (i) 0.28926 ✓
(ii) $\frac{0.57748}{0.71074} = 0.8125$ ✓✓
(b) Bin (5, 0.57748) ✓
P (X = 3) = 0.3438 ✓✓

8. (a) $1 - \frac{1}{6.25} = 0.84 = 84\%$ ✓✓
(b) Consider Normal distribution with $\mu = 0$ and $\sigma = 1$
P (–2.5 < X < 2.5) = 0.9876
which is at least 84%. ✓✓
(c) For the sample given, $\bar{x} = 21.14$
and $s_x = 32.19$
so the interval is 21.14 – 80.48
up to 21.14 + 80.48
i.e. – 59.34 up to 101.62
and so 100% of scores lie in that interval. ✓✓✓

9. (a) (i) 0.5 ✓
(ii) 0.8413 ✓
(iii) $\frac{0{\cdot}5}{0{\cdot}8413} = 0{\cdot}501$ ✓
(b) $P(X \geq 2) = 0.9859$ ✓✓
(c) $(0.5)^2 = 0.25$ ✓
(d) $0{\cdot}2534 = \frac{165 - 160}{s}$ ✓✓
$\therefore \quad s = 19.73$ ✓

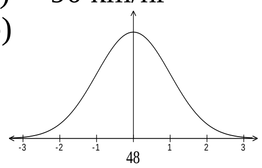
10. (a) r = 1 ✓
(b) x = 12.5 z + 59.5 ✓✓
(c) μ = 59.5, σ = 12.5 ✓✓

11. $X \sim N(3\,000, 350^2)$
(a) P(3 300 < X < 3 500) = 0.119 ✓✓
(b) P(X < 2 400) = 0.0432 ✓
∴ 5 000 x 0.0432 = 216 ✓
(c) P(X < k) = 0.05 ⇒ k = 2 424 ✓✓
(d) P(X < 2 424 | X < 2 650) =
$\frac{0.05}{0.1587} = 0.315$ ✓✓

12. (a) $\frac{133}{379} = 0.35$ ✓✓
(b) (i) 111 055 ✓
(ii) 26 130 ✓
(c) mean = median ✓
(d) μ = 111 000, σ = 26 000
P (X < 110 000) = 0.4847 ✓
(e) P (X < 110 000 | X > 80 000)
$= \frac{P(80\,000 < X < 110\,000)}{P(X > 80\,000)}$
$= \frac{0.3681}{0.8834} = 0.4169$ ✓✓
(f) B (7, 0.3681) ✓
P (X = 4) = 0.1621 ✓

13. (a) 90 km/hr ✓

(b)

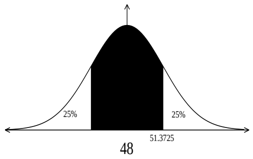


$\frac{30}{6}$ minutes ✓

(c) $P(X > 58) = 0.0228$✓

$\therefore$ 2.28%✓

(d)



$P(x < k) = 0.75 \rightarrow k = 51.3725$ ✓✓

$\therefore$ IQR = 2 x 3.3725 = 6.745 ✓

(e) $\bar{x} = 50$ min and $s_x = 5$ min ✓✓

14. (a) $P(V < 12 + \sigma) = P(z < 1) = 0{\cdot}8413$ ✓✓

(b) $P(z > k) = 0{\cdot}98$ ✓

$k = -2{\cdot}05$

$-2{\cdot}05 = \frac{11 - 12}{\sigma} \rightarrow \sigma = 0{\cdot}4869$ ✓

(c) $P\left(V < 11{\cdot}5 \mid V > 11\right) = \frac{P(11 < V < 11{\cdot}5)}{P(V > 11)}$ ✓

$P\left(V < 11.5 \mid V > 11\right) = \frac{0.1499}{0.8413} = 0.1782$ ✓

15 (a) P(0 < z < 2) = 0·47725✓✓

(b) P(– 1·72 < z < 1·72) = 0·91456✓✓

16. (a) P(W < 70) = 0·84134✓✓

(b) P(W < 40) = 0·02275✓✓

(c) P(40 < W < 70) = 0·81859✓✓

17. (a) P(– 1 < z < 1·5) = 0·77453✓✓

(b) P(z > – 0·6) = 0·72575✓✓

(c) P(– u < z< u) = 0·8164✓

P(z <u) = 0·9082✓

u = 1·3297✓

18. (a) Area = 0·93319✓✓

(b) Area = 0·82754✓✓

19. (a) P(z > – 1·28) = 0·89973✓✓

(b) P(z > u) = 0·33✓

$\Rightarrow$ P(z < u) = 0·67

u = 0·43991✓

20. (a) Area A = 0·67✓

(b) P(z < w) = 0·67

X = 54·399✓✓

21. (a) h = 55.791✓✓

(b) h = 60✓

(c) h = 53.592✓✓

22. (a) P(x > 1460)

= 0·14916✓✓

(b) P(x < 1095)

= 0·33724✓✓

(c) P(730 < z < 1095)

= 0·30718✓✓

(d) P(x < u) = 0·02

u = 686.56✓

$M = \frac{686.56}{30} = 22.89$✓

Smallest integer M = 22✓

23. (a) P(I > 135)

= 0·0038303✓

= 0·38303%✓

(b) P(u < x) = 0·99✓

I = 130·91✓

(c) Dangerous – 1% of 15 000 = 150✓✓

24. (a) (i) P(x < 982) ✓

= 0·066807✓

(ii) P(x > 1030)

= 0·0062096✓✓

(iii) P(982 < x < 1030)

= 0·92698✓✓

(b) P(u < z) = 0·01

u = – 2·3263✓

$-2{\cdot}3263 = \frac{1000 - u}{12}$

u = 1027·9156✓✓

new average 1028 g✓

25. (a) P(x < 4) = 0·091211✓✓

(b) P(x > 10) = 0·0038303✓✓

(c) $P(x > 3 \mid x < 4) = \frac{0.06846}{0.091211}$ ✓✓

= 0·750567✓

26. (a) $P(0{\cdot}485 < x < 0{\cdot}520) = 0{\cdot}91044$✓✓

(b) P (not suitable) $= 1 - 0{\cdot}91044$

$= 0{\cdot}08956$✓

P(too big | not suitable) ✓

$= P(x > 0{\cdot}520 | x > 0{\cdot}520 \cup x < 0{\cdot}485)$

$= \dfrac{0{\cdot}02275}{0{\cdot}08956}$✓

$= 0{\cdot}25402$✓

(c) $P(u > z) = 0{\cdot}01 \Rightarrow u = 2{\cdot}33$✓

$P(u < z) = 0{\cdot}04 \Rightarrow w = -1{\cdot}75$

$2{\cdot}33 = \dfrac{0{\cdot}520 - \mu}{s} - 1{\cdot}75$ ✓

$= \dfrac{0{\cdot}485 - \mu}{\sigma}$✓

$\mu = 0{\cdot}520 - 2{\cdot}33\sigma$

$\mu = 0{\cdot}485 + 1{\cdot}75\sigma$✓

$0{\cdot}520 - 2{\cdot}33\sigma = 0{\cdot}485 + 1{\cdot}75\sigma$

$4{\cdot}08\sigma = 0{\cdot}035$

$\sigma = 0{\cdot}0086$✓

$\therefore \quad \mu = 0{\cdot}5$✓

27. (a) $P(x < 1)$

$= 0{\cdot}02275$✓

$= 2{\cdot}275\%$✓

(b) $P(x < 1 \cup x > 1{\cdot}05)$

$= 0{\cdot}02275 + 0{\cdot}0013498$✓

$= 0{\cdot}0241$✓

$= 2{\cdot}41\%$✓

(c) $P(x > 1{\cdot}05 | x > 1)$

$= \dfrac{0.0013498}{1 - 0.02275}$✓

$= 0{\cdot}001381$✓

$= 0{\cdot}1381\%$✓

(d) P(at least one unsuitable) =

P(one is suitable and one is not suitable)

+ P(both are unsuitable)

$= (1 - 0{\cdot}0241) \times 0{\cdot}0241 \times 2 + (0{\cdot}0241)^2$✓

$= 0{\cdot}0476$✓✓

28. (a) $P(x < 495) = 0{\cdot}02275$✓✓

(b) (i) $0{\cdot}02275 \times 1000 = 22{\cdot}75$✓

23 need topping up. ✓

(ii) Cost $= 23 \times 0{\cdot}40$✓

= \$9.20✓

(c) $P(x > 494 | x < 495)$

$= \dfrac{0.014552}{0.02275}$ ✓✓

$= 0{\cdot}63965$✓

(d) $P(x > 495) = 0.989$

$-2.2903 = \dfrac{495 - 500}{\sigma}$✓✓

$\sigma = \dfrac{-5}{-2.2903}$ ✓

$\sigma = 2.183$✓

$\sigma = 2.2$ (1 d.p.)

29. (a) $P(x > 19) = 0{\cdot}69146$✓✓

(b) $P(16 < x < 21) = 0{\cdot}66871$✓✓

(c) $P(x > x_o) = 0{\cdot}9$

$\Rightarrow x_o = 17.436$✓✓

(d) $0{\cdot}1$✓

(e) $x = 17{\cdot}436$✓

$P(x > 17{\cdot}436) = 0{\cdot}33141$✓

(f) 100 cats with disease

$\Rightarrow$ wrongly diagnosed $= 100 \times 0{\cdot}1 = 10$

100 cats without disease $\Rightarrow$ wrongly diagnosed

$= 100 \times 0{\cdot}33141$✓✓

$= 33.141 \approx 33$✓

Total number of wrong diagnoses = 43✓

(g) Unreliable✓ as too many wrong diagnoses. ✓

30. (a) (i) $P(x < 0{\cdot}45) = 0{\cdot}99533$ ✓✓

(ii) $P(x < 0{\cdot}45) = 0{\cdot}5$ ✓✓

(b) $P(x < 0{\cdot}4 \mid x > 0{\cdot}32)$

$= \dfrac{0.21173}{0.90319} = 0{\cdot}23442$ ✓✓

(c) $P(x < x_o) = 0{\cdot}2$

$x_o = 0{\cdot}27791$ ✓✓

Maximum reaction time

$= 0{\cdot}27791$ seconds. ✓✓

(d) P(reaction time $< 0{\cdot}4$)

$= P_A(x < 0.40) \times 0{\cdot}7 + P_B(x < 0.40) \times 0{\cdot}3$ ✓

$= 0{\cdot}9452 \times 0{\cdot}7 + 0{\cdot}30853 \times 0{\cdot}3$ ✓

$= 0{\cdot}752699$ ✓

(e) $P(\text{one} > 0{\cdot}4 \cap \text{one} < 0{\cdot}4)$

$= 2 \times 0{\cdot}054799 \times 0{\cdot}9452$ ✓✓

$= 0{\cdot}10359$ ✓

31. (a) $P(x > 90) = 0{\cdot}26598$ ✓✓

(b) $P(70 < x < 90) = 0{\cdot}70361$ ✓✓

(c) $\binom{3}{1}(0.26598)(0.73402)^2$ ✓

$= 0{\cdot}429918$ ✓

(d) $P(x > 100) = 0{\cdot}030396$ ✓✓

(e) $P(x > 110 \mid x > 100)$

$= \dfrac{0.091211}{0.30853}$ ✓✓

$= 0{\cdot}29563$ ✓

29 cars are stopped. ✓

32. (a) (i) $P(x > 43{\cdot}5) = 0{\cdot}13378$ ✓✓

(ii) $P(x < 36{\cdot}4) = 0{\cdot}33185$ ✓✓

(b) $P(x < 36{\cdot}4 \mid x < 43{\cdot}5)$

$= \dfrac{0.33185}{0.86622} = 0{\cdot}38310$ ✓✓✓

33. (a) $\mu = 7,\ \sigma^2 = 4$ ✓✓

(b) Wider and thinner, with Area = 1 ✓✓

34. (a) $P(x < b) = 0.65845$ ✓

$\therefore$ $b = 0.4082$ ✓

$\therefore$ $0.5 = 0.4082\sigma$ ✓

$\therefore$ $\sigma = \frac{0.5}{0.4082} = 1.225$ ✓

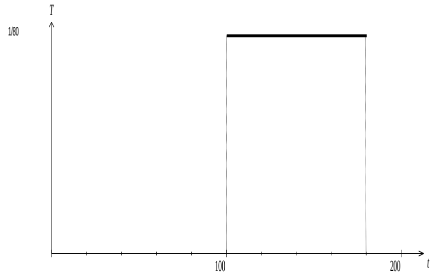
(b) $np = 2$ and $np(1-p) = 1.5$ ✓

$\therefore$ $n = 8$ and $p = 0.25$ ✓✓

(c) $P(x = \mu) = P(x = 2) = 0.3115$ ✓

Chapter 11: Sampling & Inference

1. (a)



✓

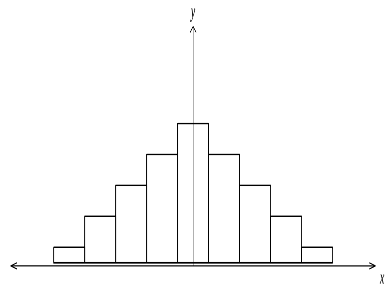
$$T(t) = \begin{cases} \dfrac{1}{80} & 100 \le t \le 180 \\ 0 & \text{Otherwise} \end{cases}$$ ✓

(b) 0.375 ✓

(c) $X \sim B(5, 0.375)$

$P(X = 3) = 0.206$ ✓✓

(d) (i)



✓

(ii) 140 mins ✓

2. (a) $\mu = 4$ ✓

$\sigma = \sqrt{100(0.04)(0.96)} = 1.96$ ✓

(b) $n = 40$ $\therefore$ $X \approx N\left(4, \dfrac{3.84}{40}\right)$ ✓

$\therefore$ $P(X > 4.5) = 0.147$ ✓

3. (a) $\binom{25}{2}\binom{20}{2}\binom{16}{2} = 6\,840\,000$ ✓

(b) No, since different class sizes. ✓

4. 90% confidence interval means that 90% of the samples would capture the true mean. ✓ ie. 10% will not include μ.

$\therefore$ 20 intervals will not contain μ ✓

5. (a) X is uniformly distributed ✓

(b) $\mu = 3.5,\ \sigma^2 = \dfrac{2.916}{100} = 0.02916$ ✓✓

6. Because the graphs are becoming more “normal” when viewed from left to right. This is the basis for the Central Limit Theorem. ✓

7. (a) A 90% CI goes from $\mu - 1.645\sigma$ to $\mu + 1.645\sigma$ or width of 3.29σ ✓

A 50% CI goes from $\mu - \frac{2}{3}\sigma$ to $\mu + \frac{2}{3}\sigma$

$\therefore$ 90% CI is $\frac{3.29}{\frac{4}{3}}$ times as wide as 50% or 2.47 times✓✓

(b) A 99% CI has width of 2 x $2.57\sigma = 5.04\sigma$✓

90% CI has a width of 2 x $1.645\sigma = 3.29\sigma$

$\therefore$ ratio of $\frac{5.04}{3.29} = 1.53$ ✓✓

8. (a) The number of successes must be >15, and so must the number of failures. Both conditions are satisfied. ✓✓

(b) $\hat{p} = \frac{17}{57} = 0.3$✓

(c) p is between $0.3 \pm 1.645s$ where

$s = \sqrt{\frac{\frac{17}{57} \times \frac{40}{57}}{57}}$ ie 0.3 ± 0.1

ie p is between 0.2 and 0.4 ✓✓

(d) Between 20% and 40% of 60 000 fans is between 12 000 and 24 000 fans would bring their own food. ✓

(e) The information would help the caterers to plan for those who would buy food at the game. This is very simplistic, since not all the 60 000 fans will be Dockers (home town) fans.

Visitors from interstate would behave differently.

The weather and the time of the game would be big factors in food purchasing.✓

9. $0.5 - 1.96\sqrt{(0.5)\frac{0.5}{50}} < p < 0.5 + 1.96\sqrt{(0.5)\frac{0.5}{50}}$

ie $0.36 < p < 0.64$✓✓

His method has an effectiveness between 36 % and 64 % of times. His method could well be better than the existing method. ✓

10. Interval is

$0.05 - 1.96\sqrt{(0.05)\frac{0.95}{400}} < p < 0.05 + 1.96\sqrt{(0.05)\frac{0.95}{400}}$✓✓

ie $0.029 < p < 0.071$✓

11. $\hat{p} = \frac{21}{59} = 0.36$

$0.36 - 1.96(0.062) < p < 0.36 + 1.96 (0.62)$

$0.23 < p < 0.36$✓✓

We can be 95% confident that the majority of footballers are not above their playing weight when pre–season training began. ✓

12. While $\hat{p}$ is a normally distributed random variable, and hence we can make such statements about $\hat{p}$, p is not a random variable and is a constant, so we cannot make probability statements about p. ✓✓

13. (a) $\hat{p} = \frac{15}{100} = 0.15$✓

(b) mean = $\frac{1}{6}$ ie the population mean

SD = $\sqrt{\frac{\frac{1}{6} \times \frac{5}{6}}{100}} = \frac{\sqrt{5}}{60}$✓✓

(c)

$0.15 - 1.645\sqrt{0.15\frac{0.85}{1000}} < p < 0.15 + 1.645\sqrt{0.15\frac{0.85}{1000}}$✓✓

$0.1314 < p < 0.1686$✓

14. $E = 0.1 = 1.96\sqrt{(0.8)\frac{0.2}{n}}$

N = 61.46 $\therefore$ 62 are required in the sample. ✓✓

15. $E = 1.96\sqrt{(0.5)\frac{0.5}{6000}}$✓✓

= 0.01265 ie 1.3%✓

16. (a) a = $\frac{5}{36}$, b = $\frac{6}{36}$, c = $\frac{5}{36}$, d = $\frac{4}{36}$ ✓✓✓✓

(b) $\hat{p} = \frac{80}{500} = 0.16$ ✓

(c) (i)

$0.16 - 1.96\sqrt{(0.16)\frac{0.84}{500}} < p < 0.16 + 1.96\sqrt{(0.16)\frac{0.84}{500}}$

ie $0.128 < p < 0.192$✓✓✓

(ii) E (X) = $\frac{6}{36}$ = 0.166.

This is in the interval.✓

(d) $n = \hat{p}\left(1 - \hat{p}\right)\left(\frac{Z}{E}\right)^2 = (0.16)(0.84).\left(\frac{6}{0.99}\right)^2$

= 4.93✓✓

So, samples of size 5 would satisfy this situation.✓

17. (a) $\hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < p < \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

$0.42 - 1.96\sqrt{(0.42)\frac{0.58}{980}} < p < 0.42 + 1.96\sqrt{(0.42)\frac{0.58}{980}}$ ✓✓

$0.39 < p < 0.45$ ✓

(b) We can be 95% confident that the population proportion is between 39% and 45% of the population.
So, we can be very confident that the majority of the population are not in favour. ✓✓

18. If the sample proportion is not known, we take that proportion to be 0.5, ✓since $\hat{p}(1-\hat{p})$ has a maximum value for $\hat{p} = 0.5$ ✓✓

19. $\hat{p} = 0.1, \; 1 - \hat{p} = 0.9$

Upper limit = 0.13 = $0.1 + x\sqrt{(0.1)\frac{0.9}{400}}$

$0.03 = x\left(\frac{0.3}{20}\right)$ ✓✓

x = 2, but 2 standard deviations above the mean (and below) corresponds to 95% confidence level. ✓

20. (a) Not likely, since 2 standard deviations below would be negative.✓

(b) Central Limit Theorem, and $n > 30$ ✓

(c) $\bar{T} \sim N(0{\cdot}65, 0{\cdot}0754)$ ✓✓

$\therefore$ $P(\bar{T} > 0{\cdot}7) = 0.2536$✓

21. (a) $X \sim N(1, 0.2^2)$

$P(0.9 < X < 1.2) = 0.5328 = 53.28\%$✓✓

(b) $X \sim B(6, 0.5328)$

$\therefore$ $P(X = 4) = 0.2638$✓✓✓

(c) $P(z > 1) = 0.1586$

$\therefore$ 1 586 or 1 587 trout✓✓

(d) $P(X < k) = 0.1$ $\therefore$ $k = 0.7437$✓

$\therefore$ Largest weight is 0.7437 kg ✓

(e)

$\bar{x} \pm z(\sigma_{\bar{x}}) = 0{\cdot}64 \pm 2{\cdot}57(0{\cdot}096) = 0{\cdot}64 \pm 0{\cdot}2467$ ✓✓

$\therefore$Confidence Interval is 0.393 to 0.887✓

(f) $0.1 = 1.96\sqrt{(0.64)\frac{0.36}{n}}$ ✓

N = 184.4✓✓

185 are required in the sample. ✓✓

22. (a) $\mu = 0{\cdot}43$ and $\sigma^2 = \frac{(0{\cdot}43)(0{\cdot}57)}{100}$ ✓✓

so $\sigma = 0{\cdot}05$

95% confidence interval is $\mu \pm 2\sigma$

$\therefore x = 0{\cdot}43 - 0{\cdot}1 = 0{\cdot}33$ ✓

and $y = 0{\cdot}43 + 0{\cdot}1 = 0{\cdot}53$ ✓

(b) Margin of error = 0.1 = 10%✓