The universal law of gravitation.

The inverse square law and the expression for the gravitational force between two masses can be learnt and the concepts involved in satellite motion, weightlessness and any questions or problems on these answered in simple terms with only a minimum of reading. However, this topic is of huge significance in our understanding of who and where we are in this universe; and also in our understanding of many fundamental concepts.

In the former case we can see through the history how our sense of identity and importance in the scheme of things has changed together with, for some of us, the place of God. In the latter case mass, force, time, inertia, momentum and energy, and the conservation laws spring to mind. Therefore much of this chapter must be read as extension material in that it is really not examinable. There are small sections on satellites and weightlessness that are core material but it is my view that all of the chapter should be required reading. You as the reader would not have this CD if you only wanted to 'cover' the syllabus.

13.1 The force of gravity.

Gravity is the first force we all learn about but it is still the one of the four fundamental forces that has not yet been satisfactorily explained by quantum physics. The electromagnetic, the weak and strong nuclear forces been united in one theory and have had their virtual carrier particles proposed and discovered. However, the force of gravity has not yielded to the power of quantum physics and the proposed mediator of the force of gravity the graviton is still far from being discovered.

Gravity is the reason we fall over as children and it is unavoidable. Until the eighteenth century gravity was thought to be a local Earthly phenomenon. Falling vertically to the ground was natural motion and that was as far as the explanation went. We now know that it acts over vast distances to the extent of the universe. This was not obvious in the seventeenth century and even the great Galileo Galilei did not connect the pull of the Earth's gravity to the motion of the moons of Jupiter or that of the Earth around the sun.

Around 1665 Isaac Newton proposed that the force of gravity extended out into space indefinitely. This was the same force that causes an apple to fall; so the story goes. Newton proposed that the relationship between the force of attraction between two masses and their separation was an inverse square law. This was evident from geometrical considerations. If the influence of the Earth's gravity spreads out equally in all directions then, like sound and light, the intensity should diminish inversely as the area of a sphere.

He checked his theory by comparing the value of the acceleration due to gravity at the surface of the Earth with an estimated value for the acceleration of the Moon towards the Earth. The moon moves in a nearly circular orbit and therefore the acceleration due to gravity is the centripetal acceleration. The time period of the Moon was well known but an accurate value for the radius of it's orbit was required, which was not available in the seventeenth century. However, the value that he calculated for the acceleration was close enough to the predicted value to convince him that he was correct.

i.e. It was proposed that $\mathbf{F} \propto \mathbf{1/d^2}$ where **d** is the separation of the two **spherical** masses.

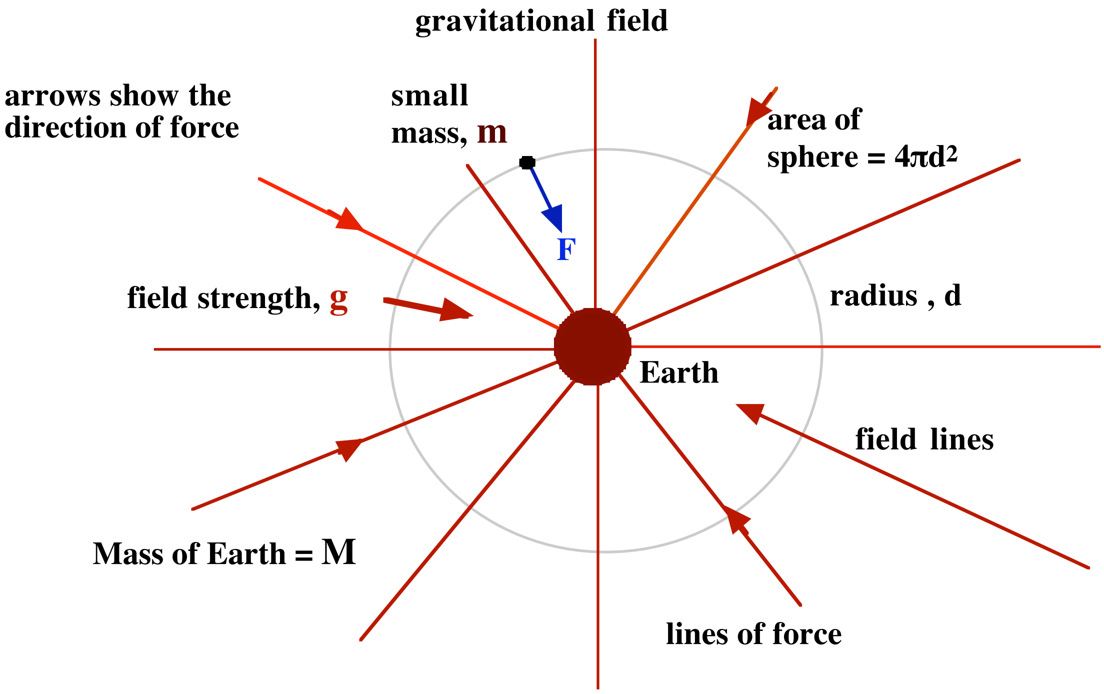
If the masses are not spherical the law is still valid if the value of **d** is much larger than the dimensions of the masses. Newton went to great lengths to verify that a sphere acted as if all the mass was concentrated at the centre. In doing so he invented the calculus or fluxions as it was called then. Flux means flow or change.

13.2 Gravitational field strength.

The region around any mass, such as the Earth, in which forces act on other masses is called a gravitational field. The concept of a field is used to explain or at least account for the action of forces at a distance. The direction of the forces can be shown by lines of force, as in the diagram below, which shows the field of a mass such as the Earth. The strength of the field is called the gravitational field strength. This is defined as the force per unit mass at any point in a gravitational field. It is given the symbol **g** .

gravitational field strength $\mathbf{g = F/m}$

Fig.13[i] **Diagram of the Earth's Field** [or any other spherical mass]



$g = {}^{F}/_{m}$ This is often called **little g** to differentiate it from **big G** the universal gravitational constant.

The units of gravitational field strength are $\mathbf{Nkg^{-1}}$
At the surface of the Earth the mean value is approximately $\mathbf{g = 9.8\ Nkg^{-1}}$

13.3 The inverse square law and geometric considerations.

The strength of the field **g** can be thought of as proportional to the density of the lines. The lines must be radial by symmetry. Suppose there are **N** lines drawn from the Earth and this number **N** is directly proportional to the mass of the Earth **M** thus

$$N \propto M.$$

It is worth noting here that this is the same as saying that the mass is proportional to the force that it causes. Then at the distance **d** from the centre of the Earth the density or number of lines per square metre is related to **d** as follows:

i.e. $g \propto {}^{N}/_{4\pi d^2} \propto {}^{M}/_{4\pi d^2}$ and $g = {}^{GM}/_{d^2}$ where **G** is a constant of proportionality.

$g = {}^{F}/_{m}$ by definition therefore the force on **m** is

$F = mg = {}^{GMm}/_{d^2}$ where **G** is a constant called the Universal Gravitational Constant .

This is known as the **Universal Law of Gravitation.** Although this is accepted as a law of nature, Newton was well aware that it did not explain **how** gravity acted.

"I have not yet been able to discover the cause of these properties of gravity from phenomena, and I frame no hypotheses. It is enough that gravity really exists and acts according to the laws I have explained, and that it abundantly serves to account for the motions of celestial bodies."

I.Newton

In particular Newton was troubled by action at a distance. That is; how does an object influence another through the vacuum of space?

" that a body may act upon another at a distance through a vacuum without the mediation of anything else, by and through which their action and force may be conveyed from one to another, is to me so great an absurdity that, I believe, no man who has in philosophic matters a competent faculty of thinking could ever fall into it."

I. Newton.

13.4 Gravitational Mass.

It seems self evident that the force between two masses must be directly proportional to the product of the two masses and this is taken to be the case in the formula above. However, in doing so we are in fact defining a new property of matter called gravitational mass $\mathbf{m_g}$**.** This is the property of a body that causes the force of gravity on other bodies and is acted on by the gravitational field due to other bodies. It is not necessarily the same as the property that gives matter inertia and is used in Newton's second law of motion.

In assuming that the gravitational force [**F**] is directly proportional to the product of the two masses [$\mathbf{m_1 m_2}$]; we are in fact saying that mass [gravitational, $\mathbf{m_g}$] is directly proportional to this force [**F**]. If we find that the force on a body in a field [**g**] is different then we assume that the 'mass' $\mathbf{m_g}$ of the body changes in direct proportion to the change in the force. Of course mass has been **weighed** this way for thousands of years, and even today most people's concept of inertial mass is inextricably linked to weight.

The value of the Universal Gravitational Constant is very small which means the force between non astronomical objects is very small. In fact the value has been **measured** quite accurately and we now know that the value is

$$\mathbf{G = 6.67 \times 10^{-11}\ Nm^2kg^{-2}}$$

This means that the force between say two one kilo gramme masses one metre apart is incredibly small at about $\mathbf{10^{-10}N}$

13.5 Inertial Mass.

The property of matter called mass which is a measure of inertia is properly called inertial mass. This is used in connection with acceleration and force in Newton's second law of motion.

$$\mathbf{\Sigma F = m_i\, a}$$

Where '$\mathbf{m_i}$' is a measure of the property of an object that resists any change in it's motion, or one could say the property that causes a force to be necessary for any change in the motion. Inertia is not necessarily the same property as that which causes gravity. There seems at first sight no connection between the two quantities as they are defined. Consider the force due to the property we call electric charge [**q**] and the force between say two protons. Clearly the inertial mass [$\mathbf{m_i}$] of the proton has nothing to do with the magnitude of the electric force. Thus there are it seems two types of mass. Newton was aware of the difference at the time but he did not see a way around the problem and consequently he did not stress it.

13.6 Principle of Equivalence

Weight is the force of gravity on an object due to the pull of the Earth. There is some confusion if we also assume that weight is the force that we exert on a weighing device such as bathroom scales. This is not the weight or force due to gravity. It is in fact just what was stated, which was the force that you exert on the scales. These two forces are rarely the same for several different reasons which are dealt with elsewhere in the book.

$\mathbf{W}$ or $\mathbf{F_g} = \mathbf{m_g\, g}$ [where $\mathbf{m_g}$ is gravitational mass]

If an object is allowed to fall freely under the action of it's own weight then the acceleration will be given by Newton's second law:-

$\mathbf{\Sigma F = m_i\, a}$ [where $\mathbf{m_i}$ is inertial mass] and $\therefore F = W = m_g\, g = m_i\, a$

Are inertial and gravitational mass are identical? No experimental has ever been able to detect any difference. A.Einstein postulated that these are in fact the same in his, which states that an observer can not tell the difference [by any means] between being in a gravitational field and being in an accelerated frame of reference. This means that $\mathbf{m_i}$ is the same as, and indistinguishable from, $\mathbf{m_g}$. The General Theory of Relativity is based on this.

"We must remember that the equality of the two masses, gravitational and inertial, was quite accidental from the point of view of classical mechanics and played no role in its structure. Here, however, this equality reflected in the equal acceleration of all falling bodies is essential and forms the basis of our whole argument."

A.Einstein.

Today the distinction is rarely mentioned and matter and energy are said to have mass. This is why **g** is also often called the acceleration due to gravity.

The units of **g** which are Nkg^{-1} can also be $\mathbf{ms^{-2}}$.

13.7 The acceleration due to gravity at the surface of the Earth.

The mass of the Earth is usually denoted by $\mathbf{M_e}$ and the radius by $\mathbf{R_e}$.

On Earth at the surface the **weight** of an object of mass **m** is $\mathbf{F_g}$

$\mathbf{F_g = W = m\, g}$ The force of gravity is also given by the law of gravitation:-

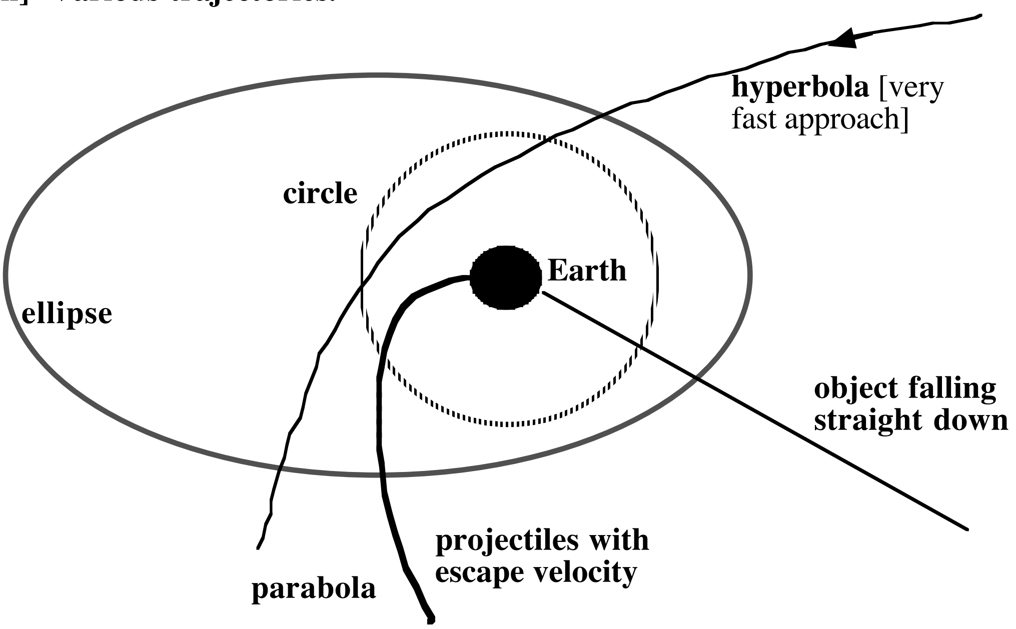
$$F_g = W = m\,g = \frac{G\,M_e\,m}{R_e^2}$$

i.e.

$$\boxed{g = \frac{G\,M_e}{R_e^2}}$$

13.8 Motion under the action of gravity.

Fig.13[ii] Various trajectories.



Depending on the position and the velocity circular motion the trajectory can be circular, elliptical, parabolic, a hyperbola or of course a straight line. These are **all conic sections** and the latter is only a special case of a conic section. We have seen in the section on projectiles that near the surface and for reasonably small velocities the path is a parabola. Indeed a satellite with a speed much lower than necessary falls to Earth like any other projectile. Comets have very elliptical orbits around the sun. Mars has an orbit that is quite elliptical, or eccentric, but the Earth and the moon have orbits that are nearly circular.

[i] Circular orbits.

Since the force of gravity is a central force it can give rise to uniform circular motion. If the initial tangential speed is the required value a body will follow a circular path or orbit around the Earth. This means that the speed would have to satisfy $\mathbf{g = v^2/r}$ where the value of the acceleration due to gravity is the value for that height and also be perpendicular to the radius vector **r**.

[ii] The mathematics of circular orbits.

The motion of satellites in circular orbits is reasonably simple since the speed is constant and the only force on the satellite is the force of gravity.

The speed $\mathbf{v = 2\pi r/T}$ where **T** is the time period, for one orbit.

$$\Sigma F = \frac{G M_e m}{r^2} \qquad \Sigma F = m a_c \qquad a_c = [v^2/r]$$

These equations can be used to determine various quantities. Some special cases are shown.

[iii] Geosynchronous orbits.

These keep the satellite above the same place on the surface of the Earth. Thus the period is the same as one revolution of the Earth. This is known as a sidereal day and is about 4.0 min less than a solar day which is from noon to noon and is 24.0 hour by definition. This is overlooked in problem solving since the difference is not significant. Do not use sidereal day unless it is mentioned. If you are not sure why there is a difference ask your teacher to explain. The height of such a satellite is calculated as follows.

The orbit, like all orbits, must be centred on the centre of the Earth since this is the direction of the centripetal force. Thus a geostationary satellite must orbit over the equator. A satellite can have a time period of one day and orbit in any plane but only an equatorial orbit will stay over one place on the surface. All other orbits must move north and then south of the equator and they do not even stay over a line of longitude if they are not circular.

$$\mathbf{T} = 24 \times 60 \times 60 = \mathbf{8.64 \times 10^4\ s} \text{ and the only force } \mathbf{F_g = \Sigma F}$$

$$\mathbf{v = 2\pi r/T} \qquad \mathbf{\Sigma F = m a_c = m [v^2/r]}$$

$$\frac{G M_e m}{R^2} = \frac{m 4\pi^2 R^2}{R T^2}$$

$$R^3 = \frac{6.67 \times 10^{-11} \times 5.96 \times 10^{24} \times [8.64 \times 10^4]^2}{4\pi^2}$$

$$\mathbf{R = 4.22 \times 10^4 km} \qquad \mathbf{Height\ is\ 3.58 \times 10^4 km}$$

Consider a polar orbit with a twenty four hour time period and you will see that inclined orbits vary in longitude as well as latitude. Luckily geostationary satellites orbit at a great distance from the Earth and therefore countries near the poles are still able to obtain a direct line of sight and use them. Which way round the Earth do such satellites rotate ?

[iv] Weather and survey satellites.

These orbit **lower** and thus faster. They are often in **polar** orbits. This enables them to cover large areas of the surface, if not all of it, in a few orbits. Not all satellites are in circular orbits and many are very elliptical. The advantage is that the angular velocity varies with altitude and this is sometimes desirable. This would place the satellite over a region for a large part of the orbital time period but also bring it back over other places.

13.9 Kepler's laws for planetary motion.

The following section on Kepler's Laws is not on the syllabus, but it is well worth reading. Some years before Newton a German astronomer J. Kepler produced three **empirical** laws of planetary motion that had been deduced from thousands of **observations of Mars** and the four other bright planets. Newton was able to derive these laws from his theory of gravitation. Newton also showed that because of the fact that many planets are orbiting together Kepler's laws had to be modified.

[i] The First Law.

This simply states that the orbits of the planets are ellipses about the sun as a focus. In fact the general solution for all velocities and distances is a conic section, as seen in figure [ii].

[ii] The Second Law.

This states that equal areas are traced out by a displacement vector **r** in equal times. See fig. [iii] in the next section.

[iii] The Third Law.

The third law states that for all the planets the ratio of the square of the time period to the cube of the major axis [of the ellipse] is a constant. For simplicity this is called the mean radius of the orbit and in a circular orbit the major axis is the radius.

[a] Circular Orbits. [This is a relationship that could be examined].

Newton's theory is applied for a planet speed **v** and mass **m** orbiting the sun mass M_S in a circular orbit of radius **r** with a time period **T:-**

$$\frac{G M_S m}{r^2} = m\,[v^2/r] \quad \text{therefore} \quad \frac{G M_S}{r^2} = \frac{4\pi^2 r}{T^2}$$

$$\frac{G M_S}{4\pi^2} = \frac{r^3}{T^2} = \text{a constant}$$

Thus r^3/T^2 is a constant for any number of satellites orbiting the same central body in circular orbits.

[b] Elliptical Orbits.

This is more difficult to show for elliptical orbits but it can be done using the laws of conservation of angular momentum and of energy. The proof is covered in 13.15.

13.10 Weightlessness.

We have seen in the previous chapter that all measurements of any quantities are made from a particular point of view or frame of reference. The values obtained are different for observers who are moving relative to each other. Observers in accelerating frames of reference which of course includes rotating frames do not observe the same laws of physics. Accelerating frames of reference are called non inertial frames of reference. In a rotating frame such as a fairground ride or a spinning space station the laws of physics are very strange indeed. If the observers refer all observations and measurements to their rotating frame objects seem to have forces acting on them, which have no apparent cause. These are called fictitious forces. Projectiles follow curved paths for no obvious reason. The so called weight or artificial gravity in a rotating space station varies with distance from the axis of rotation.

Since any object in orbit is in freefall it will not appear to have any weight in the frame of reference of the satellite. That is it will not exert any force on a weighing device in the satellite with it. Any observer in the satellite will be falling freely with the object and if an object is dropped it will not fall any faster than the observer. Thus it floats in the space inside the satellite.

In the frame of reference of the Earth, which can be considered stationary with respect to the much faster satellite, the objects in the satellite and the astronauts all have weight and it is this force that is causing them to orbit around the Earth. When referring to a frame of reference one must put oneself, figuratively speaking, completely in that frame. All observations and measurements must be with respect to that set of co-ordinates. Imagine that you are inside the satellite with no windows and you do not even know that you are in a satellite.

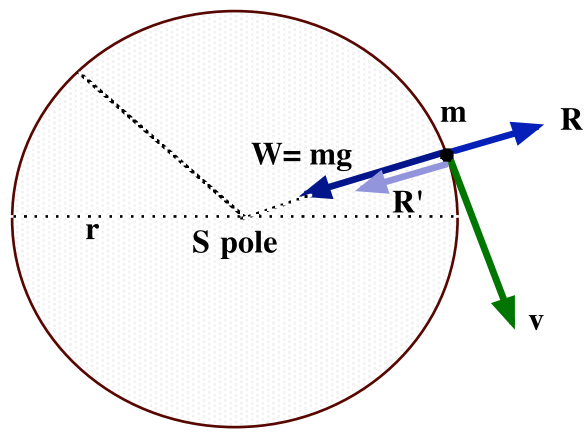
The astronauts feel weightless, which feels like falling naturally, and of course they do not exert any force on a seat or the floor. To all intents and purposes the astronauts and anything in the spacecraft are indeed weightless. Within the frame of reference that they are in all objects seem to have zero weight. This is sometimes called apparent weightlessness incorrectly since there is nothing apparent about the observations of the observers in the spacecraft. All objects in the spacecraft do have mass, inertia, momentum, etc. but they cannot be weighed. To the observer in the spacecraft everything is weightless.

13.11 The effect of the Earth's rotation on g.

At the equator and any latitude other than at the poles the rotation of the Earth means that objects at the surface require a centripetal force to keep them moving in the circle. The simplest place to calculate the effect of this is at the equator as follows.

Fig.13[iii] Apparent Weight.

equator



An objects **measured** weight is determined by the force that it exerts on the ground **R'**

R' < W [true weight]

R [ground on object] = -**R'** [object on ground]

.

Newton's third law states that action and reaction are equal and opposite.

$\Sigma \mathbf{F} = \mathbf{mg} + \mathbf{R} = \mathbf{m\,a_c} = \mathbf{m\,[v^2/r]}$ [down is +ve.]

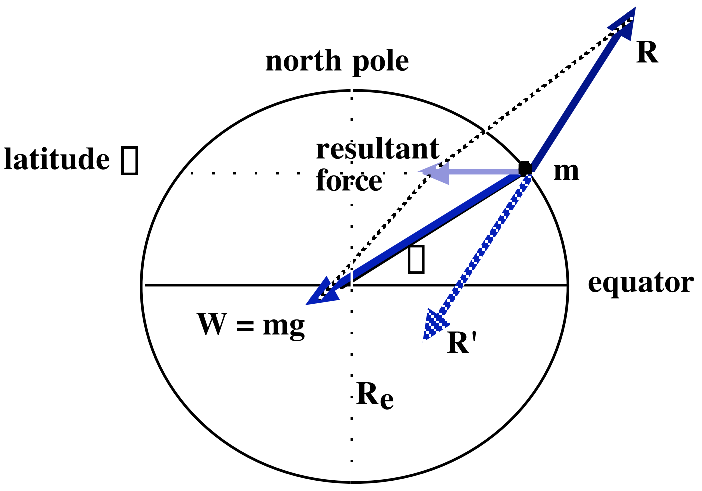
$\mathbf{R} = \mathbf{m\,[v^2/r] - mg}$ **R** is negative as it is upwards.

$\mathbf{R'} = \mathbf{-\,R}$ The force that the object exerts on ground which is **less** than at the poles.

[i] If weight is the force of gravity then it is not your weight that has changed. It is the force that you exert on the ground **R'**, your apparent weight, that is less. We must stay in an inertial frame for simplicity.

[ii] At all places other than the poles and the equator the vectors **R** and consequently **R'** are **not vertical**. This is because the resultant of the forces acting on a body must provide the centripetal force which is **not** towards the centre of the Earth.

Fig.13[iv] Apparent weight in general.



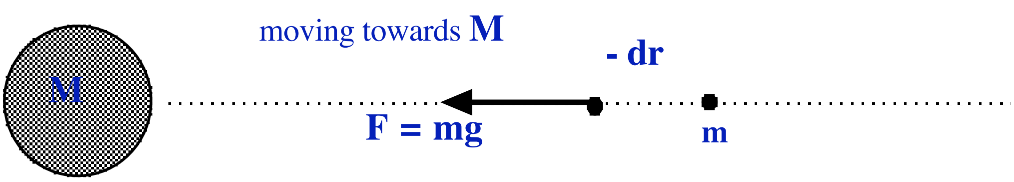
The resultant of **W** and **R** provides the centripetal force necessary for the object to remain on the surface. The forces **R** and **R'** are not vertical but the angle to the vertical is too small to be seen in everyday circumstances. You might calculate a value for the angle at say 45^{o} of latitude.

13.12. Gravitational Potential. [V]

This is defined as the work done per unit mass in bringing a very small mass from infinity to that point in the field. Since the force varies then we must use calculus to find the expression. We take the potential to be zero at points outside the field or at infinity, since a mass in that position is not in a field and clearly has no potential to do work. The force due to gravity on a mass [**m**] towards the mass **M**. The displacement **dr** is also in the negative direction and causes a decrease in the value of **r**. The field of a spherical mass **M** is given by

$$g = -\frac{GM}{r^2}$$

Fig.13[v] Gravitational potential.



$$dV = [\text{work done per unit mass } \textbf{by} \text{ the system}] = \frac{F}{m}\,ds = g \times -dr$$

$$V = \int_{\infty}^{r} -g\,dr = \int_{\infty}^{r} \frac{GM}{r^2}\,dr = \left[-\frac{GM}{r}\right]_{\infty}^{r} = -\frac{GM}{r}$$

Therefore the potential energy of a mass at a great distance from any other masses is **zero** and for a mass [**m**] distance [**r**] from a planet mass [**M**] is

$$E_p = -\frac{GMm}{r}$$

13.13 Changes to altitude of satellites in orbit.

The total energy of a satellite mass **m** in orbit is found as follows using

$\Sigma F = ma$ $\qquad \frac{GMm}{r^2} = \frac{mv^2}{r}$

The kinetic energy in an orbit radius **r** is $E_k = \frac{1}{2}mv^2 = +\frac{GMm}{2r}$

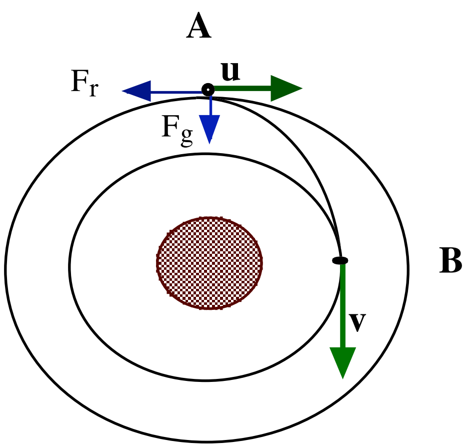
The total energy is $E_k + E_p = \frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{2r}$

[i] Unless rockets fire or friction acts, the total energy is constant and in an elliptical orbit or any trajectory the kinetic and potential energy change but the total is constant.

[ii] The energy needed to project a satellite up to a higher orbit is only about 50% of the increase in potential energy as it will need to have less kinetic energy.

[iii] Most importantly as friction tends to slow the satellite down and energy is transformed to internal energy [heat] the satellite does not actually slow down. Rather it loses potential energy and falls in to a lower orbit where it finishes up with a greater speed. This is because the loss of potential energy in falling to a lower orbit is more than the required gain in kinetic energy to orbit at the lower altitude. This is more difficult to explain in terms of forces since they are vectors and we initially have perpendicular forces acting on the satellite. The subsequent motion is quite complex and at first sight it seems illogical that friction can result in the satellite speeding up.

Fig.13[vi] Changes in altitude.



In going from A to B the loss of potential energy is equal to the gain of kinetic energy plus the work done against friction.

A satellite returning to Earth must initially slow down and then slow again or it would finish up going faster in a lower orbit.

The tangential velocity is in fact reduced but at the same time the inward radial velocity increases. In the case of a small retarding force the increase in the latter is greater than the decrease in the former and the magnitude of the **resultant** velocity increases. Thus the satellite speeds up as it 'falls' inwards.

If, however, a very large retarding force is applied, by say the retro rockets of a satellite returning to Earth, then there can and usually will be a reduction in speed. The satellite then falls towards the Earth initially gaining speed and kinetic energy, but as the friction increases with the thickening atmosphere a further retardation by the rockets is usually not necessary.

13.14 A Few Thoughts on Cosmology.

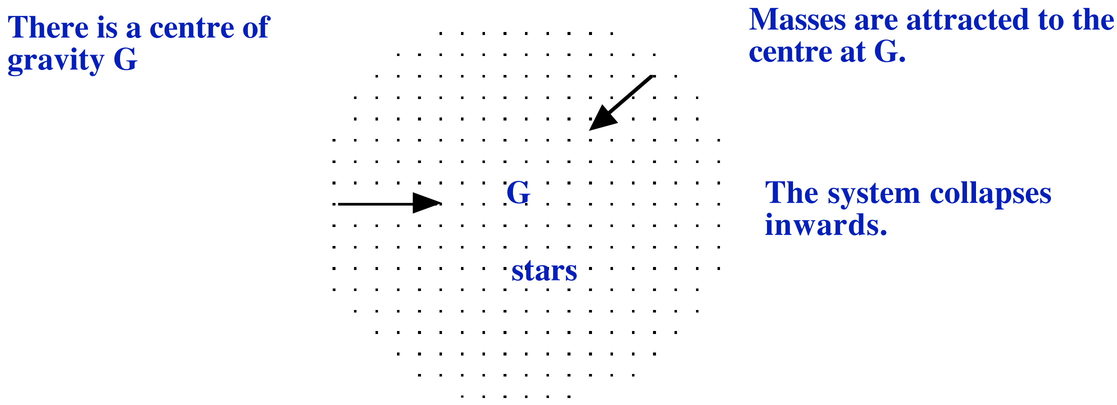
Newton's theory of gravitation was soon widely accepted and it changed the whole approach to astronomy and cosmology. The size of the universe had been thought about by **natural philosophers** [physicists] for thousands of years. The general conclusion was that it was not very big at all. The sky was a few kilometres up, using todays units. The stars were also very close and all the same distance away. The sun went round the Earth as did the moon and inner visible planets.

The Earth was at the centre and in a sense the Earth **was** the universe. Man held a very important place in the order of things. Humankind was '**meant**' to be there and God [whichever one was worshipped at the time] had put him [her] there. Then it was found that the Earth went round the sun with the other planets and the Earth was **not at the centre**. Also it was discovered that things were much bigger than previously thought. Gravity suddenly created several problems and the first problem was the **size** of the universe.

[i] The infinite universe.

This is because if gravity extends outwards and can reach everything then if the universe is finite it should collapse. Since it clearly did not collapse then it must be because it is infinite and has no centre. If the universe has no centre because it is infinite then where does that put man or more to the point where does that put God ?

Fig.13[vii] A Finite Universe.



Another aspect of infinity was Olber's paradox. Why not look this up ?

[ii] Religion

The second problem was, and still is, that as scientific knowledge increases, mankind is relegated to specks of biological material swarming over the surface of a planet, that is itself smaller than one grain of sand in all the beaches of all the oceans on Earth compared to our galaxy, the milky way. The milky way is in turn is unimaginably small compared to the known universe. So where does this leave mankind. Well there is no scientific explanation of our awareness of our own existence or **consciousness**. We can explain away everyone else's existence and consciousness as reactions to the environment, but not our own. This is still the foundation stone of all religions. However, we seem a great deal less important in the great scheme of things today than a few hundred years ago, if there is a 'plan'.

[iii] Spacetime.

If the universe is not infinite then is it finite? The solution is not absolutely certain today but we are fairly sure that it is **unbounded and finite**. This is not too difficult to envisage if one thinks of a two dimensional **analogy**. A flat surface can only be infinite or bounded but the surface of a sphere is finite and unbounded. This is only an analogy and in this 2D world the space inside and outside the sphere does not exist. On the curved surface of the sphere the shortest distance between two points would be a great circle not a so called straight line. Light would follow this path. In our 3D [or 4D] world we see along the path light takes and what we assume are straight lines are in fact curves along the 4D 'surface' of spacetime.

The universe is expanding and was once much smaller. Indeed we believe today that it started from nothing at all in an incredible **'big bang'**. This is really not any more difficult to accept than the idea that the universe has existed for ever. In the beginning there was **no space and no time** and everything came into existence about 15 000 million years ago. Recent measurements indicate a lower value but as some stars are older than this reduced theoretical age the theory is in doubt. This does not mean that the big bang theory is wrong. It does mean that it is in need of some refining.

Space is curved and inextricably linked with time and we are limited to our own individual universes as observers. Space is intimately linked with time in that distances depend on speed and so does time. Gravity slows time and bends space. Time itself is not absolute and it is ruled by the speed of light or information transfer. It is possible to imagine travelling at the speed of light and visiting the whole universe. Such a journey would take no time at all for the traveller and [he] would be truly God like since [he] could be **simultaneously everywhere at all times.** Time does not exist at the speed of light. The universe is indeed a strange place.

Spacetime is **warped** by gravity in regions near large masses and in some extreme cases in **black holes** it is warped and twisted to such an extent that it may lead to new laws of physics. Time seems to stop near black holes and who knows what happens inside them or through them. Wormholes, which involve extreme bending or warping of space by say a gravitational [or magnetic] field, are even stranger perhaps allowing travel from one place in the universe to another by a short cut that goes outside spacetime. Could these be the warp drive of the starship 'Enterprise'?

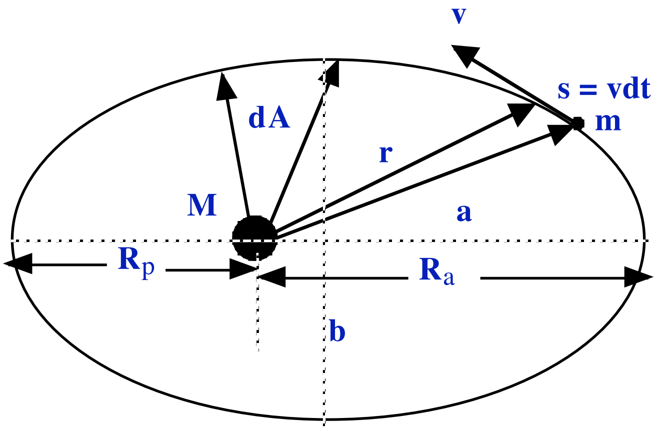
Gravity may one day halt the expansion and bring about **gravitational collapse** just as predicted in Newton's day. If that happens living in a collapsing universe would be very different and of course impossible in the later stages. Measurements of the density of mass in the universe are inaccurate but the present belief is that the figure is that which would **just not** cause collapse, but it would stop the expansion in an infinite time. This would mean that the total energy in the universe is zero. Remember, gravitational potential energy is negative. At the end of such a universe or indeed a continually expanding one it would be very quiet. There would be no stars, no life , no heat, no movement, just black emptiness. We call it the **heat death of the universe**. In this universe time would pass very slowly **if at all**. It seems either way that man, life, the world of matter and movement, time and the universe itself will come to an end. It will either disappear into nothingness at a singularity or becomes a vast expanse of cold, still and dark empty nothingness of another sort.

Luckily for us all this is a long way off in the future but it should be thought provoking. The preceding few paragraphs on space and time are only meant to wet your appetite and encourage you to read more. The library will help you find out about the latest books. They are all very readable and written for the general public. You will find that there is plenty of really easy to read literature on cosmology and nearly every issue of Scientific American and the New Scientist have articles on this or particle physics which is closely related. Also there are many popular books by world famous cosmologists such as Paul Davies, Steven Hawking, Roger Penrose, and many others that are very readable with a level of understanding of physics comparable with year twelve.

13.15 The derivation of Kepler's laws.

The following mathematics is complete but some simple steps in the algebra have been omitted. You will find it quite easy to fill in the gaps as all the physics is shown. Equation [i] is the **definition** of angular momentum, which is a vector product. Equation [vi] is the expression for gravitational potential energy taking zero to be at infinity, see appendix 13B. The second law regarding equal areas in equal times is also derived in this proof.

Fig.13[viii] An elliptical orbit.



Ellipse eccentricity **e**

$b^2 = a^2[1 - e^2]$

$R_p = a[1 - e]$
$R_a = a[1 + e]$

Area $A = \pi ab$

For small **dt** then $dA = {}^1/_2 v dt \times r$

Angular momentum of planet $L = r \times mv$ [i]

Angular momentum in a closed system is constant [conserved]

Therefore $dA/dt = L/2m = \text{constant}$ [and hence the 2nd law]

And integrating we have $A = LT/2m = \pi ab$

Thus $L = 2\pi mab/T$ [ii]

At perihelion and at aphelion v_p and v_a are perpendicular to **R** thus sin□ = 1 for the vector products.

and $L_p = mR_pv_p = L_a = mR_av_a$ [iii]

Thus ${}^1/_2 L^2 = [{}^1/_2 mv_a^2]\, mR_a^2$

and minimum $E_k = {}^1/_2 L^2/mR_a^2$ [iv]

Using conservation of energy □ $[E_k + E_p] = \text{constant}$ [v]

Gravitational Potential Energy is $E_p = -GMm/r$ [vi]
Thus

$$\frac{L^2}{2mR_a^2} - \frac{GMm}{R_a} = \frac{L^2}{2mR_p^2} - \frac{GMm}{R_p}$$

$L^2/2m\,[R_a + R_p] = GMm\,R_aR_p$ and using the properties of an ellipse

$[R_a + R_p] = 2a$ and $R_aR_p = a\,[1 + e] \times a[1 - e] = b^2$

Therefore $L^2 = GMm^2b^2/a$ and using eqn. [iii] gives

$$\frac{4\pi^2m^2a^2b^2}{T^2} = \frac{GMm^2b^2}{a}$$

and hence $$T^2 = \frac{4\pi^2}{GM}\,a^3$$

This is Kepler's Third Law.

What is amazing is that Kepler determined the three laws empirically from astronomical observations made with quite crude [but very large] instruments. Kepler had observations taken over many decades to work with, but he was observing the motion of Mars from the frame of reference of the spinning [daily] and orbiting [annually] Earth. Mars appears as a dim speck of light seen against the back ground of the stars. These laws apply to any satellites, such as all planets around the sun; all moons of a planet; all satellites of the Earth. Try going outside one menials night, find Mars, and see for yourself the enormity of the challenge.