The motion of projectiles.

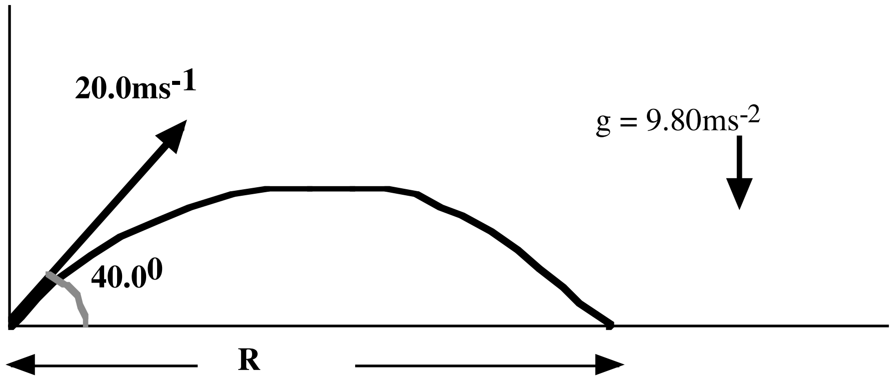
11.1 Components of a vector.

Since vectors have no component perpendicular to their direction, then if we split a vector in to two mutually perpendicular components the two directions can be treated independently. For projectiles this means that the horizontal component of the velocity is constant, since gravity acts vertically and has no effect horizontally. The vertical motion is uniform acceleration downwards.

The link between the horizontal and vertical motion is the time since projection. This is the same in the vertical and horizontal equations for any point along the path of the projectile. Therefore if we write equations for the vertical and horizontal motion and then eliminate time we can usually solve for the unknown or required quantity. See the example that follows.

e.g. A cricket ball is hit at $20.0 ms^{-1}$ at 40.0^0 to the horizontal .
Assume that there is no air resistance and that $g = 9.80 ms^{-2}$
What are the range and time of flight ?

Fig.11[i] The trajectory of a projectile



vertically:

$u = 20.0 \sin 40.0^0$ [vertical component of velocity]
$s = 0.0$ m [on returning to the ground as well as initially]
$a = g = -9.80 ms^{-2}$ [choose up as positive]
find t [when $s = 0$]
$s = ut + {}^1/_2 a t^2$

$0 = 20.0 \sin 40.0 \times t - 0.5 \times 9.80 t^2$

therefore $t = 0.00$s or 2.62 s

horizontally. $a =$ zero [horizontal component of velocity = u cos]

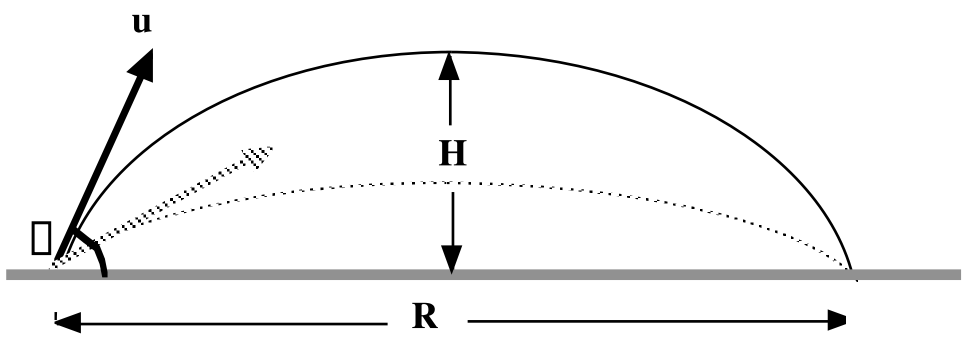
$s = u \cos \, \times t = 20.0 \cos 40.0^0 \times 2.62 = 40.2$ m

Answer: The range is 40.2m and time of flight is 2.62s.

11.2 The general solutions for projectiles.

An object is projected at speed **u** at an angle θ to the horizontal.

Fig.11[ii] A specific range.



There are two values for θ that give the same range **R** for the speed **u**. The maximum height **H** is not the same: n.b. the sign of '**g**' is inserted here rather than when the value is used

Vertically. $\mathbf{u = u \sin\theta}$ $\mathbf{a = g}$ $\mathbf{s = 0}$ **up is +ve** $\mathbf{g = -9.80ms^{-2}}$

$\mathbf{s = ut + {}^1/_2\, a\, t^2}$ $\mathbf{0 = u \sin\theta\, t + {}^1/_2\, g\, t^2}$

$\mathbf{t = 0}$ or time of flight is $\mathbf{t = {}^{2u \sin\theta}/_g}$

Horizontally. $\mathbf{v_h = u \cos\theta = constant}$ $\mathbf{a = 0}$

$\mathbf{R = u \cos\theta\, t = u \cos\theta \times {}^{2u \sin\theta}/_g}$ $\mathbf{2 \sin\theta \cos\theta = \sin 2\theta}$

Range $\mathbf{R = u^2/g \sin 2\theta}$ and there are two angles for each range.

It should be noted that the two angles for a given **horizontal** range are complimentary.

$\mathbf{\theta_1 + \theta_2 = 90^o}$ and they are equally each side of 45^o

The **maximum range** is therefore when $\mathbf{\sin 2\theta = 1}$ **;**

$\mathbf{2\theta = 90^o}$ $\mathbf{\theta = 45^o}$ and then

maximum [horizontal] range $\mathbf{R_m = u^2/g}$

Vertically. $\mathbf{v^2 = u^2 + 2\, a\, s}$

$\mathbf{v = 0}$ $\mathbf{u = u \sin\theta}$ $\mathbf{a = g = -9.8ms^{-2}}$ **find H [= s]**

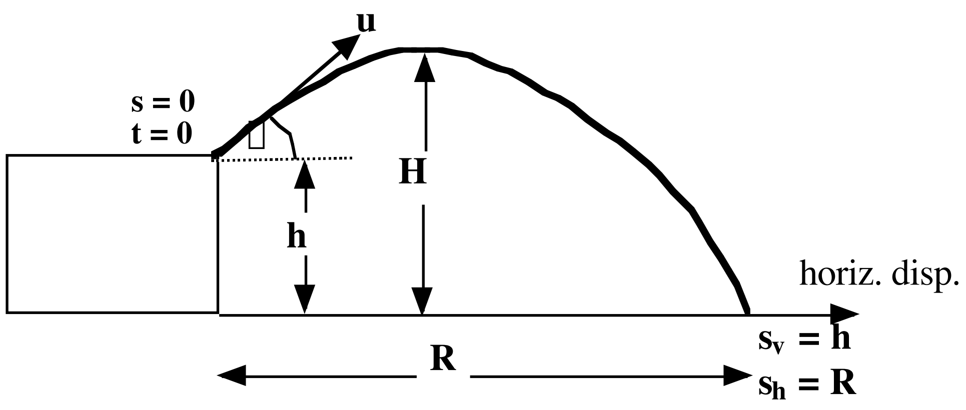
$\mathbf{0 = u^2 \sin^2\theta + 2gH}$

Therefore maximum height $\mathbf{H = -\, u^2/2g\ \sin^2\theta}$ **n.b.** $\mathbf{g = -9.8\ ms\text{-}2}$

This is in itself a maximum for $\theta = 90^o$

It is not worth memorising these formulae since problems are best solved from first principles. The general principles are shown in the example that follows. The range and time of flight depend on the height of the landing surface **h** , as well as the angle of projection and speed.

e.g. A projectile lands a 'height' **h** above or below the launch position **[s=0]** find the range **R.**



Vertically.

$u = u \sin\theta$

$a = g = -9.80 ms^{-2}$ [up is positive]

$s = h$ [at range R]

find t

$s = ut + {}^1/_2 at^2$

therefore $h = u \sin\theta\, t + {}^1/_2 g\, t^2$

solve for t using the quadratic formula [i]

Horizontally.

$R = u \cos\theta \times t$

use time t from [i]

solve for R

[$a = g = -9.8\ ms^{-2}$; h is negative as shown but h can be +ve]

11.3 The parabolic equation for the trajectory.

The equation of the path [trajectory] for zero friction is as follows:-

vertically $y = [u \sin\theta]\, t + [{}^g/_2]\, t^2$

horizontally $x = [u \cos\theta]\, t$ therefore $t = {}^x/[u\cos\theta]$

eliminating time we have

$$y = [\tan\theta\ x] + \frac{g\,x^2}{2u^2\cos^2\theta}$$

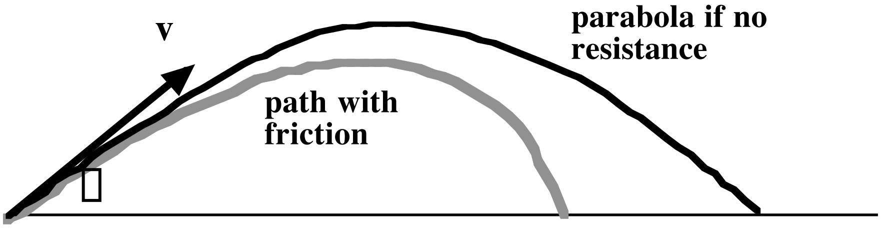
equation **[1]**

This is an equation of the form $ax^2 + bx + c = 0$ and is a **parabola.** However, it is **NOT** a good idea to memorise all these formulae and to solve problems always work from **first principles.**

11.4 The effects of friction.

Friction acts both horizontally and vertically directly opposing the motion and it is roughly proportional to the square of the speed. However, we cannot treat problems quantitatively using this unless we use calculus. The effect of air resistance on a projectile is to decrease the range and the maximum height. It also makes the trajectory non symmetrical, being steeper on the way down.

Fig.11[iii] Air resistance.



The effect of air resistance on the path of the projectile is much greater in the horizontal direction for angles of projection up to about 60^o. This is because the speed is reduced in the vertical direction by the force of gravity and since resistance is proportional to the square of the speed the effect diminishes rapidly as the object rises. Thus the reduction in maximum height is less the the reduction in the range.

Terminal velocity.

As speed increases so does air resistance and for a falling object eventually the net force on a projectile will be zero. In this case the acceleration is zero and the velocity is constant. Of course the velocity will be vertically down and the friction will be equal in magnitude to the weight.

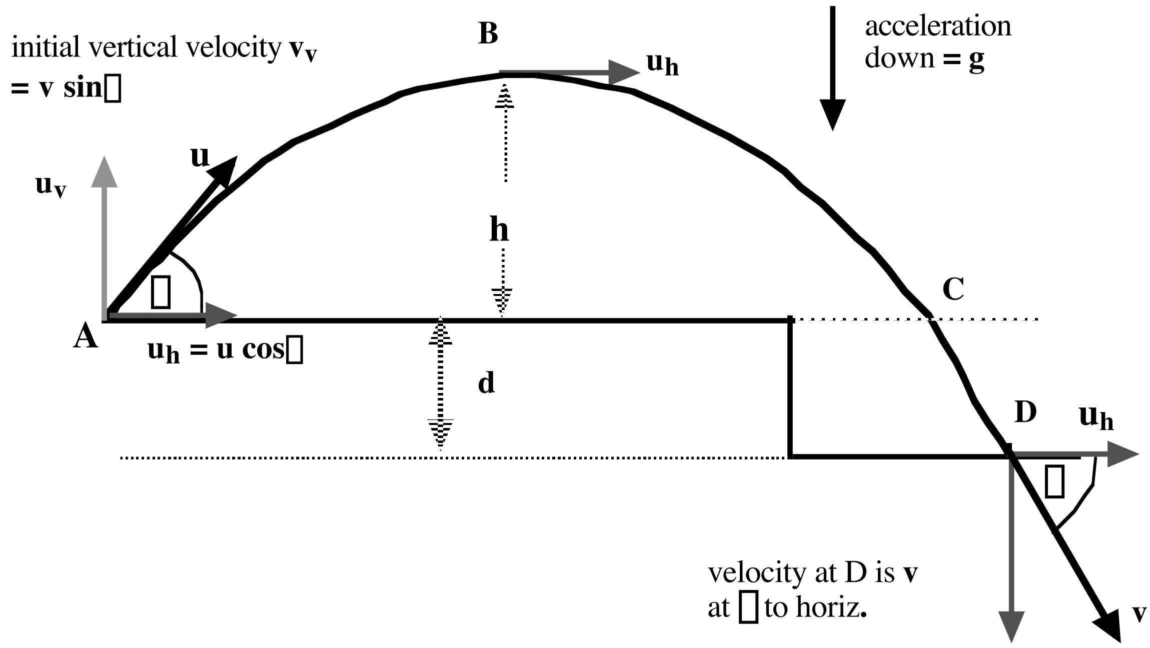
11.5 Conservation of energy.

The law of conservation of energy is stated as follows

$\Delta E_k + \Delta E_p = 0$ **or** $\Sigma\, [E_k + E_p] = \text{constant}$

Thus for projectiles with **no friction** we can say the total mechanical energy is constant throughout the flight. Also the velocity is horizontal at the highest point. The potential energy is maximum at the top and the kinetic energy is minimum at the top. It is common to refer to the launch height as zero for potential energy.

Fig.11[iv] Values at various heights



Since there are no horizontal forces then the horizontal velocity is constant. The kinetic energy is all due to the horizontal component at the top. Therefore, if we can find the kinetic energy at the top we can find the speed and hence the initial horizontal component of the launch velocity. Indeed this is the horizontal component at **any** time.

More difficult problems are often made easier if energy is considered.

$\Sigma\, [E_k + E_p] = \text{constant}$

$$\tfrac{1}{2} mu^2 = \tfrac{1}{2} mu_h^2 + mgh = \tfrac{1}{2} mv^2 + mgd$$

[n.b. d is negative]

11.6 Problem solving in general.

In addition to the energy relationships above the following are all true. These are true for all projectiles and any of them can be used to solve problems.

vertically $\mathbf{d = u \sin\theta\, t + {}^{1}/_{2}\, g\, t^{2}}$ **[a = g]**

vertically $\mathbf{v_v = u \sin\theta + g\, t}$ [n.b. **g** is down and negative if up is positive]

horizontally $\mathbf{R = v_h\, t = u \cos\theta\, t}$

horizontally the component $\mathbf{v_h}$ of the velocity is constant therefore

horizontally $\mathbf{v_h = u \cos\theta = v \cos\theta}$

In solving problems follow the procedure outlined below.

[i] always sketch a **large** simple line **diagram**.

[ii] put all the **information** on the diagram.

[iii] choose a direction as **positive.** $\mathbf{a = g = -9.8 ms^{-2}}$ **up**

[iv] state what is required, give it a **symbol** and put it in the diagram**.**

[v] write the relevant **equations**.

[vi] **resolve** the initial velocity of the projectile into vertical [a = g] and horizontal components [a = 0] . mark them on the diagram.

[vii] **substitute values** in the equations, **before** rearranging them-.

[viii] **solve** for required quantity - eliminate any unknowns that are not required - **trust** your maths.

[ix] remember the **physical principles** and use them.

11.7 A harder example

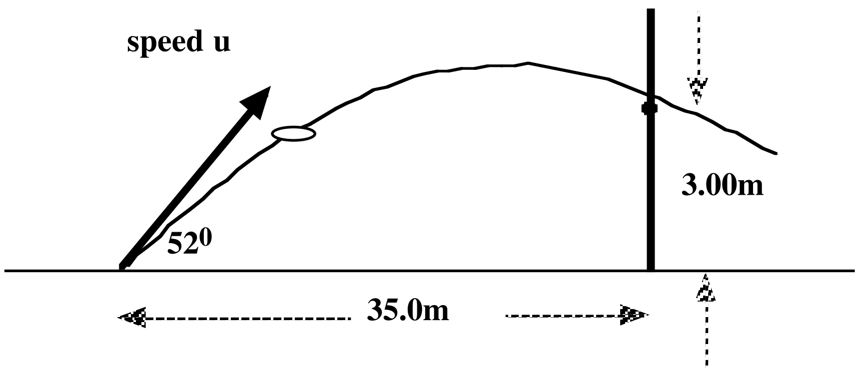
Cyril, who usually takes the penalties for his club, takes a place kick from 35.0m directly in front of the posts. The cross bar is 3.00m high and his kick just clears the bar. Place kicks are taken with the ball on the ground.

[a] If we ignore air resistance and he kicks the ball at 52.0^0 to the horizontal what is the speed of the ball off his boot?

[b] **Explain** why Cyril has to kick the ball at an angle greater than 45^0 if he is at his maximum range for clearing the bar.

Solution.

[a] Diagram.



acceleration **a = -9.80ms^{-2}** [up is chosen as positive]

Let the time of flight be **t.** **Find v.**

Horizontally. [right is positive]

$v_h = u \cos 52^0$ $R = 35 = u \cos 52^0 \times t$

Therefore $t = {}^{35}/_{u \cos 52}$

Vertically. [up is positive]

$u = v_v = u \sin 52^0$ $s = ut + {}^1/_2\, a\, t^2$

$3 = u \sin 52\ t\ -\ 0.5 \times 9.8 \times t^2$

Solving for **v** by eliminating **t.**

$3 = u \sin 52^o \times [{}^{35}/_{u \cos 52^o}] - 0.5 \times 9.8 \times [{}^{35}/_{u \cos 52^o}]^2$

This is a quadratic in **v** but it can be solved easily since there is only a u^2 term and it is not necessary to use the quadratic formula.

$u^2\,[35\tan 52^o - 3] = 4.9 \times 35^2/\cos^2 52^o$

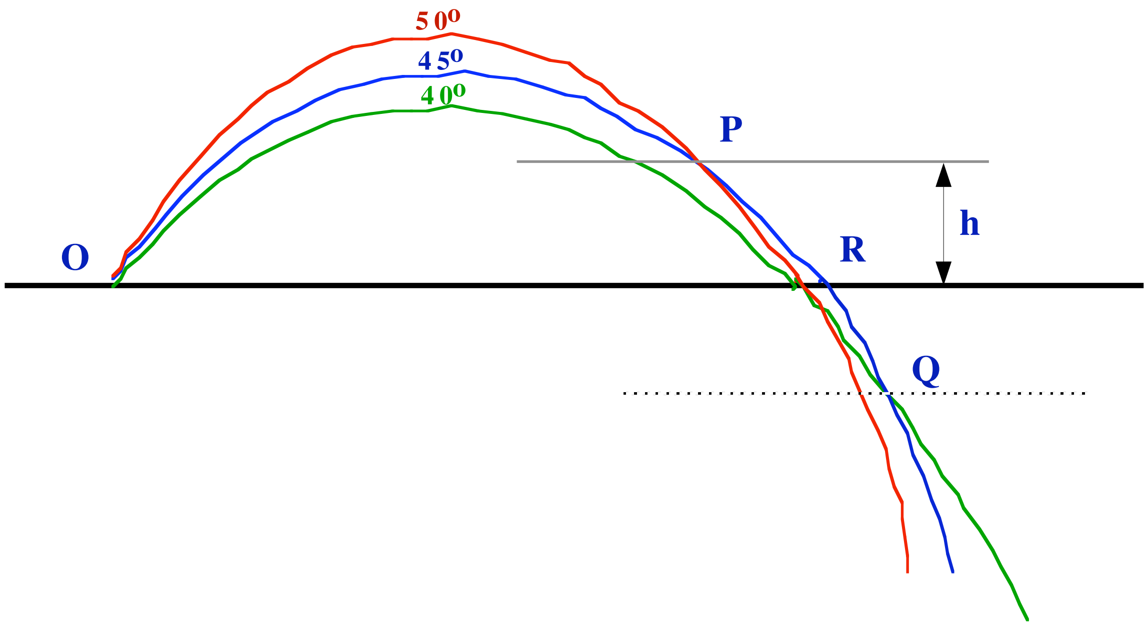
$u = 19.5\ ms^{-1}$

[b] *What follows is a detailed explanation that is much more than required for the exam It is often very useful to take cases to the extreme as in this case. The physics of the problem is obvious.*

It is well known that the maximum horizontal range, to a target above the point of projection is achieved using an angle greater than 45^{o}. Imagine a really high target that one would only just be able to clear.
The initial velocity would be at nearly 90^{o} to the horizontal and the target would have to be very close. It is also obvious that the maximum range to reach a target below the point of projection is obtained with an angle less than 45^{o}. Imagine throwing a stone from a cliff top. If the cliff is particularly high one would throw the stone almost horizontally.

The above is illustrated if we draw the trajectories for a particular projection speed **u** at say, 40^{o} ; 50^{o} and the 45^{o} projection for maximum horizontal range **R.** Consider the diagram below in which it can be seen that all trajectories launched at angles greater than or less than 45^{o} do not reach **R**.

Fig.11[v] Targets at different heights.



Consider the point **P**, height **h** above **R.** The diagram shows that for all heights greater than **h** the 50^{o} trajectory goes further than 45^{o}..

Now consider a point **Q** below the launch height. This is on the trajectory for 40^{o} and 45^{o}. Clearly the 40^{o} trajectory goes further than the 45^{o} trajectory for all points below Q. It should be obvious from the diagram, that all trajectories launched at angles less than 45^{o} cross the 45^{o} trajectory below the horizontal and beyond **R**.

11.8 Full mathematical treatment [extension]

Horizontally. $\mathbf{X = u \cos\theta\, t}$

Vertically. $\mathbf{Y = u\sin\theta\, t + {}^{1}/{}_{2}gt^{2}}$ **[sign for g substituted later]**

eliminating time we have

$$Y = \tan\theta X + \frac{g X^2}{2 u^2 \cos^2\theta} = \tan\theta X + \frac{g X^2 \sec^2\theta}{2 u^2}$$

using $\sec^2\theta = 1 + \tan^2\theta$

$$Y = \tan\theta X + \frac{g X^2 [1 + \tan^2\theta]}{2 u^2}$$

For any given values for **v, X, and Y** ; the equation is rearranged and solved for θ.

$$g X^2 \tan^2\theta + 2u^2 X \tan\theta - 2u^2 Y + gX^2 = 0$$

This equation can be used to find any one of the variables :- [n.b. g = -9.80ms^{-2}]

X [range] ; Y [height] ; θ [angle of projection] ; and u [initial speed of projectile]

Be careful in finding θ as there are two solutions to any quadratic.

Check 1:

For maximum horizontal range

$\theta = 45^o$, $\tan\theta = 1$, **Y** = 0 and **u** = 30ms^{-1} [say] **g** = -10ms^{-2} .

$$Y = X + \frac{g X^2}{u^2} = 0$$ hence $X = -u^2/g = 90m$

Which is the formula for maximum horizontal range and hence the correct outcome.

Check 2:

For **X** = 70m ; **Y** = -20m ; **g** = -10ms^{-2} [say] and **u** = 30ms^{-1} we have :-

$$[-10] \times 70^2 \times \tan^2\theta + 2 \times 30^2 \times 70 \times \tan\theta - 2 \times 30^2 \times [-20] + 10 \times 70^2$$

$$-10^3 [49\tan^2\theta - 126 \tan\theta - 85] = 0$$ and hence

$$\tan\theta = \frac{126 \pm [126^2 + 4 \times 49 \times 85]^{1/2}}{2 \times 49}$$ and $\theta = 72.3^o$ or -29.0^o

Maximum range

The maximum r[illegible]tance below the horizontal [-20m] is found when there is only **one solution** to the equation in **tan**θ. This mathematics is much more involved than anything you can usually expect. but only uses simple trigonometry and algebra. None of the mathematics should be memorised, only the method is worth study.

$$g X^2 \tan^2\theta + 2v^2 X \tan\theta - 2v^2 Y + gX^2 = 0$$

For maximum range [X] in quadratic equation **$b^2 - 4ac = 0$**

i.e. **$[2u^2X]^2 = 4\ gX^2\ [gX^2 - 2u^2Y]$**

hence **$[4u^4 + 8gv^2Y] = 4g^2\ X^2$**

If we check the angle for this trajectory and maximum range it is less than 45^o. After some simplification we get :-

and **$X = \frac{-u^2}{g}\sqrt{1 + {}^{2gY}/_{u2}}$ = 108m** **n.b. Y = -20m g = $-10ms^{-2}$**

$\tan\theta = [{}^{-b}/2a] = 2u^2X/2gX^2 = -u^2/gX = 0.832$ thus **$\theta = 39.8^o$**

The angle is less than 45^o . The range is greater than for a level surface
If you check the values for a height Y = +20m above the horizontal and find the maximum range and launch angle, they should be :-

Range **X = 67.1m** [which is less than 90m]
Angle **$\theta = 53.3^o$** [which is greater than 45^o]

Alternatively we can use calculus to find the maximum range at which **${}^{dx}/_{d\theta} = 0$**

differentiating the equation for the parabola we and assuming that **${}^{dx}/_{d\theta} = 0$** we get

$g\ X^2\ 2\tan\theta\sec^2\theta + 2u^2X\sec^2\theta = 0$

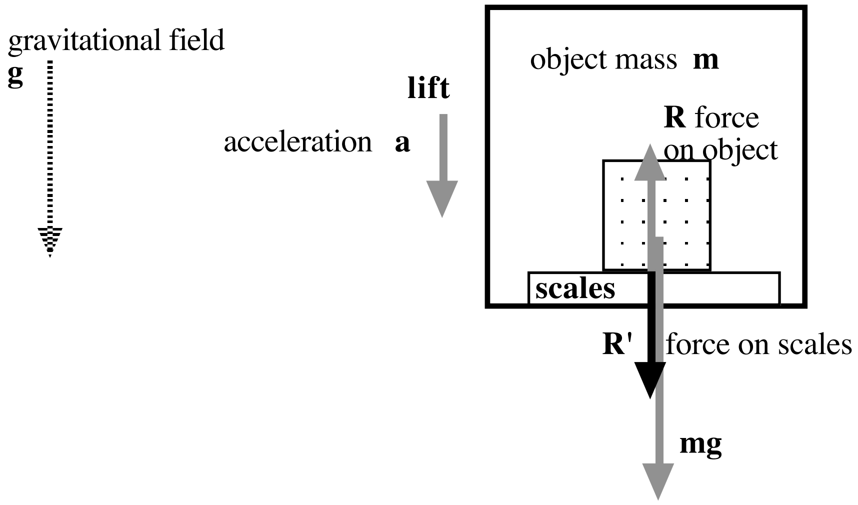
thus $X = -u^2/g\tan\theta$ which is the same result as for the quadratic condition for equal roots above.

also $\tan\theta = \frac{1}{\sqrt{1 + {}^{2gY}/_{u2}}}$

11.9 Weightlessness.

Consider an object or person in an elevator moving with an acceleration downwards. In the frame of reference of the lift, weight is the force that the object exerts on a weighing device. The experience of an observer in the lift is indistinguishable from a gravitational field. In the inertial frame of reference outside the lift and for an observer who is not accelerating, weight is the force of gravity due to the Earth.

Fig.11[vi] Weightlessness



R and **R'** are action and reaction and hence equal and opposite.

Working in the inertial frame and considering downwards as positive.

Using Newton's Second Law.

$\Sigma \mathbf{F = m\,a}$ thus $\Sigma \mathbf{F = mg - R}$ [n.b. R is up]

$\mathbf{mg - R = m\,a}$ $\mathbf{R = [mg - ma]}$ upwards

$\mathbf{R' = - R = [mg - ma]}$ downwards

Thus **R'** is less than mg and downwards. The apparent weight to the observer in the lift is less than the true weight or force of gravity.

If the lift was **falling freely** then $\mathbf{a = g}$ and $\mathbf{R' = 0}$

This means that to a person in the falling lift the object is apparently **weightless**. The sensation experienced by the person is that of falling, which of course they are doing. The same laws apply to projectiles which are moving freely under the action of gravity and of course to **satellites** which are also in a sense falling. Anything in the elevator [or satellite] will be **impossible to weigh** since it does not exert a force on anything that is falling freely with it. In the frame of reference of the elevator objects **are in fact weightless.** That is, to a person in the elevator referring all observations to the surrounding walls and floor of the elevator everything is weightless. Of course to a stationary observer outside the lift watching from the ground, the lift and the objects in it all have weight and this is **why** they are accelerating downwards.

There is **no detectable difference** between the falling elevator and being out in space far from any stars or planets that could cause a force on the object. It is not possible to do any experiment within the elevator that would tell whether one was falling in a uniform field **or** far from any astronomical bodies. An object can only be weightless for an inertial observer if it is far away from any other masses.

Similarly it is not possible to tell the difference between being inside an accelerating spacecraft and standing still in a gravitational field. It is this proposal put forward by **A.Einstein** that is called the **Principle of Equivalence** that leads to the General Theory of Relativity and then the curvature of space-time. *That is another story. Read chapter 23*

Rotating Frames of Reference.

The situation in a satellite that is falling around the Earth or any other planet is different. An observer in the satellite can do experiments that will show that the frame of reference that he is in is rotating. The question that the observer must then ask is "*What is the frame of reference that I am rotating with respect to ?* "

This is discussed again in the section on gravity and satellites.

11.10 The effect of air resistance on height.

The force due to the air acts in the opposite direction to the velocity at all times.

$$F_r = k\,v^2$$

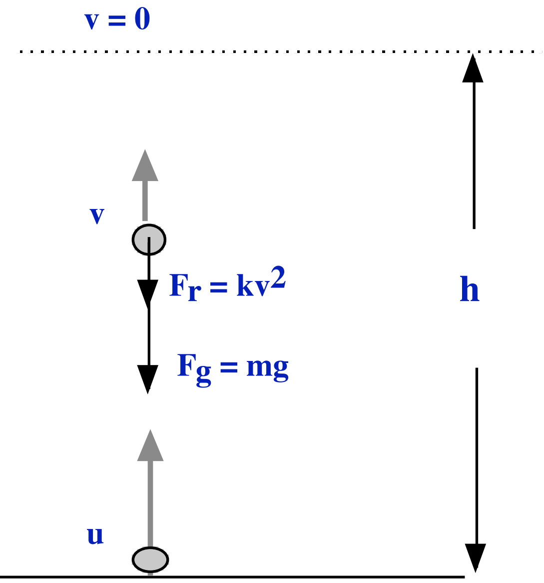
Fig.11[vii] The effect of air resistance on maximum height.

$$\Sigma F = m\,a = m\,{}^{dv}/_{dt}$$

$$= m\,{}^{dv}/_{ds}\,{}^{ds}/_{dt}$$

Therefore

$$-[mg + kv^2] = mv\,{}^{dv}/_{ds}$$



$$\int_0^h ds = \int_u^0 \frac{-\,mv\,dv}{[mg + kv^2]}$$

$$h = {}^{m}/_{2k}\,\Big[\ln[mg + kv^2]\Big]_0^u$$

$$h = {}^{m}/_{2k}\,\ln[1 + {}^{k}/_{mg}\,u^2]$$

Questions. [for interest only]

1. If $k = 0$ then we should get $mgh = {}^{1}/_{2}\,mu^2$
Can you use your maths [calculus] to show this ?

2. How can you determine the velocity of the ball when it hits the ground ?