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Specialist Mathematics

Year 11

Cambridge Senior Mathematics
Australian Curriculum

INCLUDES INTERACTIVE TEXTBOOK POWERED BY CAMBRIDGE HOTMATHS

CAMBRIDGE UNIVERSITY PRESS

ISBN 978-1-316-63608-4

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

4843/24, 2nd Floor, Ansari Road, Daryaganj, Delhi – 110002, India

79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

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www.cambridge.org
Information on this title: www.cambridge.org/9781316636084

First printed 2017
20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1

Cover designed by Loupe Studio
Typeset by diacriTech
Printed in China by C&C Offset Printing Co., Ltd.

A Catalogue-in-Publication entry is available from the catalogue of the National Library of Australia at www.nla.gov.au

ISBN 978-1-316-63608-4 Paperback

Additional resources for this publication at www.cambridge.edu.au/GO

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ISBN 978-1-316-63608-4

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Cambridge University Press

Introduction

Cambridge Specialist Mathematics Australian Curriculum Year 11 provides a complete teaching and learning resource for the Australian Curriculum and the State-specific variations. It has been written with understanding as its chief aim and with ample practice offered through the worked examples and exercises. All the work has been trialled in the classroom, and the approaches offered are based on classroom experience and the responses of teachers to earlier versions of this book.

Specialist Mathematics Year 11 offers material on topics from Specialist Mathematics Units 1&2. The topics covered provide excellent background for a student proceeding to Specialist Mathematics Year 12. It also would be very useful for a student proceeding to Mathematical Methods Year 12.

The book has been carefully prepared to reflect the Australian Curriculum. Topics in geometry, proof, statistics, transformations, counting principles and algebra may be new to some, and for these topics particular care has been taken to provide the right depth of coverage and interpretation of the curriculum.

The book contains five revision chapters. These provide technology free, multiple-choice questions and extended-response questions.

The TI-Nspire calculator examples and instructions have been completed by Russell Brown and those for the Casio ClassPad have been completed by Maria Schaffner.

The integration of the features of the textbook and the new digital components of the package, powered by Cambridge HOTmaths, are illustrated in the next two pages.

About Cambridge HOTmaths

Cambridge HOTmaths is a comprehensive, award-winning mathematics learning system - an interactive online maths learning, teaching and assessment resource for students and teachers, for individuals or whole classes, for school and at home. Its digital engine or platform is used to host and power the interactive textbook and the Online Teaching Suite. All this is included in the price of the textbook.

Consultants

The authors and publisher wish to thank Jan Honnens of Christ Church Grammar School, Perth for advice on the preparation of this edition.

ISBN 978-1-316-63608-4

An overview of the Cambridge complete teacher and learning resource

For more detail, see the guide in the online Interactive Textbook

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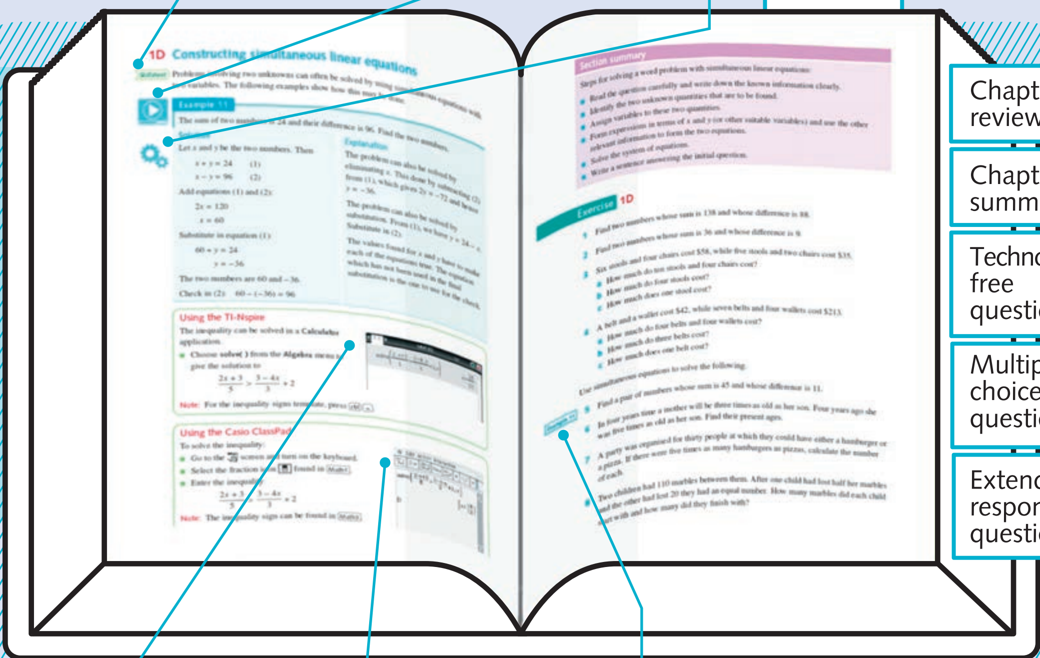

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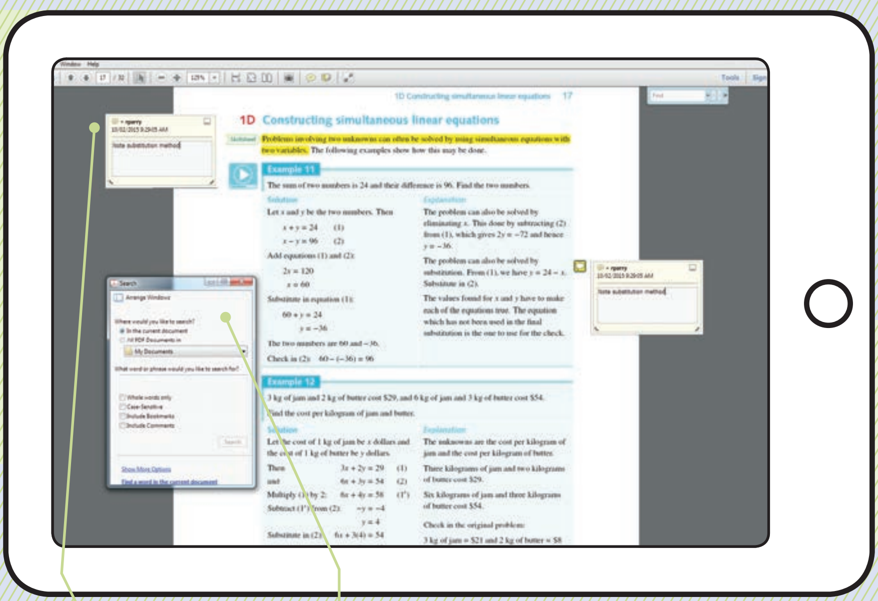

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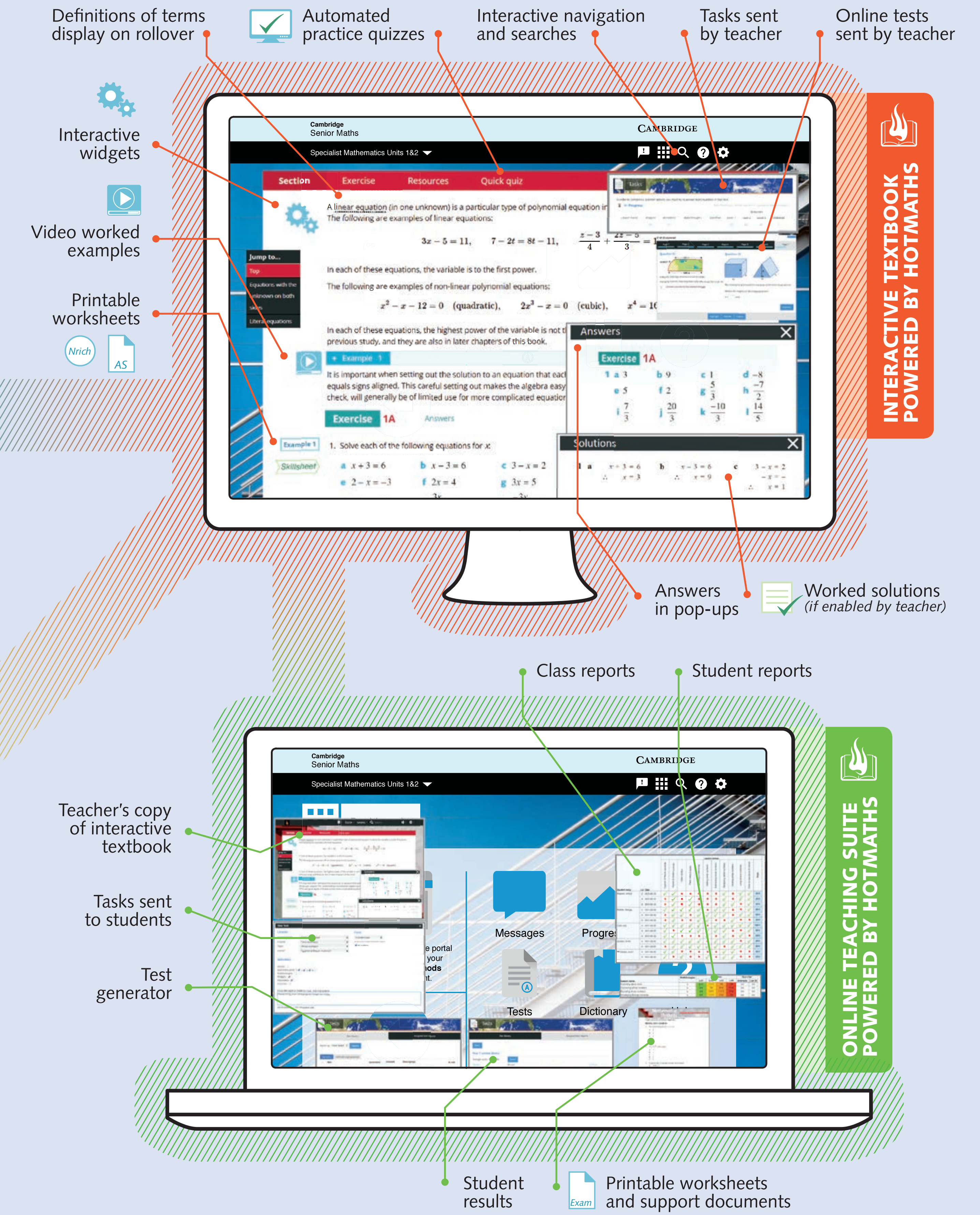

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1 Algebra I

Objectives

- To solve **linear equations**.
- To solve problems with linear equations and **simultaneous linear equations**.
- To use **substitution** and **transposition** with formulas.
- To add and multiply algebraic fractions.
- To solve **literal equations**.
- To solve **simultaneous literal equations**.

Algebra is the language of mathematics. Algebra helps us to state ideas more simply. It also enables us to make general statements about mathematics, and to solve problems that would be difficult to solve otherwise.

We know by basic arithmetic that $9 \times 7 + 2 \times 7 = 11 \times 7$. We could replace the number 7 in this statement by any other number we like, and so we could write down infinitely many such statements. These can all be captured by the algebraic statement $9x + 2x = 11x$, for any number x. Thus algebra enables us to write down general statements.

Formulas enable mathematical ideas to be stated clearly and concisely. An example is the well-known formula for compound interest. Suppose that an initial amount P is invested at an interest rate R, with interest compounded annually. Then the amount, A_n, that the investment is worth after n years is given by $A_n = P(1 + R)^n$.

In this chapter we review some of the techniques which you have met in previous years. Algebra plays a central role in Specialist Mathematics at Years 11 and 12. It is important that you become fluent with the techniques introduced in this chapter and in Chapter 3.

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1A Indices

This section revises algebra involving indices.

Review of index laws

For all non-zero real numbers a and b and all integers m and n:

- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $(ab)^n = a^n b^n$
- $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$
- $a^{-n} = \dfrac{1}{a^n}$
- $\dfrac{1}{a^{-n}} = a^n$
- $a^0 = 1$

Rational indices

If a is a positive real number and n is a natural number, then $a^{\frac{1}{n}}$ is defined to be the nth root of a. That is, $a^{\frac{1}{n}}$ is the positive number whose nth power is a. For example: $9^{\frac{1}{2}} = 3$.

If n is odd, then we can define $a^{\frac{1}{n}}$ when a is negative. If a is negative and n is odd, define $a^{\frac{1}{n}}$ to be the number whose nth power is a. For example: $(-8)^{\frac{1}{3}} = -2$.

In both cases we can write:

$$a^{\frac{1}{n}} = \sqrt[n]{a} \quad \text{with} \quad \left(a^{\frac{1}{n}}\right)^n = a$$

In general, the expression a^x can be defined for rational indices, i.e. when $x = \dfrac{m}{n}$, where m and n are integers, by defining

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$$

To employ this definition, we will always first write the fractional power in simplest form.

Note: The index laws hold for rational indices m and n whenever both sides of the equation are defined (for example, if a and b are positive real numbers).

Example 1

Simplify each of the following:

a $x^2 \times x^3$ **b** $\dfrac{x^4}{x^2}$ **c** $x^{\frac{1}{2}} \div x^{\frac{4}{5}}$ **d** $(x^3)^{\frac{1}{2}}$

Solution	Explanation
a $x^2 \times x^3 = x^{2+3} = x^5$	$a^m \times a^n = a^{m+n}$
b $\dfrac{x^4}{x^2} = x^{4-2} = x^2$	$\dfrac{a^m}{a^n} = a^{m-n}$
c $x^{\frac{1}{2}} \div x^{\frac{4}{5}} = x^{\frac{1}{2}-\frac{4}{5}} = x^{-\frac{3}{10}}$	$\dfrac{a^m}{a^n} = a^{m-n}$
d $(x^3)^{\frac{1}{2}} = x^{\frac{3}{2}}$	$(a^m)^n = a^{mn}$

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Example 2

Evaluate:

a $125^{\frac{2}{3}}$ **b** $\left(\frac{1000}{27}\right)^{\frac{2}{3}}$

Solution

a $125^{\frac{2}{3}} = \left(125^{\frac{1}{3}}\right)^2 = 5^2 = 25$

b $\left(\frac{1000}{27}\right)^{\frac{2}{3}} = \left(\left(\frac{1000}{27}\right)^{\frac{1}{3}}\right)^2 = \left(\frac{10}{3}\right)^2 = \frac{100}{9}$

Explanation

$125^{\frac{1}{3}} = \sqrt[3]{125} = 5$

$\left(\frac{1000}{27}\right)^{\frac{1}{3}} = \sqrt[3]{\frac{1000}{27}} = \frac{10}{3}$

Example 3

Simplify $\dfrac{\sqrt[4]{x^2y^3}}{x^{\frac{1}{2}}y^{\frac{2}{3}}}$.

Solution

$$\frac{\sqrt[4]{x^2y^3}}{x^{\frac{1}{2}}y^{\frac{2}{3}}} = \frac{(x^2y^3)^{\frac{1}{4}}}{x^{\frac{1}{2}}y^{\frac{2}{3}}} = \frac{x^{\frac{2}{4}}y^{\frac{3}{4}}}{x^{\frac{1}{2}}y^{\frac{2}{3}}}$$

$$= x^{\frac{2}{4}-\frac{1}{2}}y^{\frac{3}{4}-\frac{2}{3}}$$

$$= x^0y^{\frac{1}{12}}$$

$$= y^{\frac{1}{12}}$$

Explanation

$(ab)^n = a^nb^n$

$\frac{a^m}{a^n} = a^{m-n}$

$a^0 = 1$

Section summary

- Index laws
  - $a^m \times a^n = a^{m+n}$
  - $a^m \div a^n = a^{m-n}$
  - $(a^m)^n = a^{mn}$
  - $(ab)^n = a^nb^n$
  - $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
  - $a^{-n} = \frac{1}{a^n}$
  - $\frac{1}{a^{-n}} = a^n$
  - $a^0 = 1$
- Rational indices
  - $a^{\frac{1}{n}} = \sqrt[n]{a}$
  - $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$

Exercise 1A

Example 1

1 Simplify each of the following using the appropriate index laws:

a $x^3 \times x^4$ **b** $a^5 \times a^{-3}$ **c** $x^2 \times x^{-1} \times x^2$ **d** $\frac{y^3}{y^7}$

e $\frac{x^8}{x^{-4}}$ **f** $\frac{p^{-5}}{p^2}$ **g** $a^{\frac{1}{2}} \div a^{\frac{2}{3}}$ **h** $(a^{-2})^4$

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i $(y^{-2})^{-7}$ **j** $(x^5)^3$ **k** $(a^{-20})^{\frac{3}{5}}$ **l** $\left(x^{-\frac{1}{2}}\right)^{-4}$

m $(n^{10})^{\frac{1}{5}}$ **n** $2x^{\frac{1}{2}} \times 4x^3$ **o** $(a^2)^{\frac{5}{2}} \times a^{-4}$ **p** $\dfrac{1}{x^{-4}}$

q $\left(2n^{-\frac{2}{5}}\right)^5 \div (4^3 n^4)$ **r** $x^3 \times 2x^{\frac{1}{2}} \times -4x^{-\frac{3}{2}}$

s $(ab^3)^2 \times a^{-2}b^{-4} \times \dfrac{1}{a^2b^{-3}}$ **t** $(2^2p^{-3} \times 4^3p^5 \div (6p^{-3}))^0$

Example 2

2 Evaluate each of the following:

a $25^{\frac{1}{2}}$ **b** $64^{\frac{1}{3}}$ **c** $\left(\dfrac{16}{9}\right)^{\frac{1}{2}}$ **d** $16^{-\frac{1}{2}}$

e $\left(\dfrac{49}{36}\right)^{-\frac{1}{2}}$ **f** $27^{\frac{1}{3}}$ **g** $144^{\frac{1}{2}}$ **h** $64^{\frac{2}{3}}$

i $9^{\frac{3}{2}}$ **j** $\left(\dfrac{81}{16}\right)^{\frac{1}{4}}$ **k** $\left(\dfrac{23}{5}\right)^0$ **l** $128^{\frac{3}{7}}$

3 Use your calculator to evaluate each of the following, correct to two decimal places:

a 4.35^2 **b** 2.4^5 **c** $\sqrt{34.6921}$ **d** $(0.02)^{-3}$ **e** $\sqrt[3]{0.729}$

f $\sqrt[4]{2.3045}$ **g** $(345.64)^{-\frac{1}{3}}$ **h** $(4.568)^{\frac{2}{5}}$ **i** $\dfrac{1}{(0.064)^{-\frac{1}{3}}}$

4 Simplify each of the following, giving your answer with positive index:

a $\dfrac{a^2b^3}{a^{-2}b^{-4}}$ **b** $\dfrac{2a^2(2b)^3}{(2a)^{-2}b^{-4}}$ **c** $\dfrac{a^{-2}b^{-3}}{a^{-2}b^{-4}}$

d $\dfrac{a^2b^3}{a^{-2}b^{-4}} \times \dfrac{ab}{a^{-1}b^{-1}}$ **e** $\dfrac{(2a)^2 \times 8b^3}{16a^{-2}b^{-4}}$ **f** $\dfrac{2a^2b^3}{8a^{-2}b^{-4}} \div \dfrac{16ab}{(2a)^{-1}b^{-1}}$

5 Write $\dfrac{2^n \times 8^n}{2^{2n} \times 16}$ in the form 2^{an+b}.

6 Write $2^{-x} \times 3^{-x} \times 6^{2x} \times 3^{2x} \times 2^{2x}$ as a power of 6.

7 Simplify each of the following:

a $2^{\frac{1}{3}} \times 2^{\frac{1}{6}} \times 2^{-\frac{2}{3}}$ **b** $a^{\frac{1}{4}} \times a^{\frac{2}{5}} \times a^{-\frac{1}{10}}$ **c** $2^{\frac{2}{3}} \times 2^{\frac{5}{6}} \times 2^{-\frac{2}{3}}$

d $\left(2^{\frac{1}{3}}\right)^2 \times \left(2^{\frac{1}{2}}\right)^5$ **e** $\left(2^{\frac{1}{3}}\right)^2 \times 2^{\frac{1}{3}} \times 2^{-\frac{2}{5}}$

Example 3

8 Simplify each of the following:

a $\sqrt[3]{a^3b^2} \div \sqrt[3]{a^2b^{-1}}$ **b** $\sqrt{a^3b^2} \times \sqrt{a^2b^{-1}}$ **c** $\sqrt[5]{a^3b^2} \times \sqrt[5]{a^2b^{-1}}$

d $\sqrt{a^{-4}b^2} \times \sqrt{a^3b^{-1}}$ **e** $\sqrt{a^3b^2c^{-3}} \times \sqrt{a^2b^{-1}c^{-5}}$ **f** $\sqrt[5]{a^3b^2} \div \sqrt[5]{a^2b^{-1}}$

g $\dfrac{\sqrt{a^3b^2}}{a^2b^{-1}c^{-5}} \times \dfrac{\sqrt{a^{-4}b^2}}{a^3b^{-1}} \times \sqrt{a^3b^{-1}}$

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1B Standard form

Often when dealing with real-world problems, the numbers involved may be very small or very large. For example:

- The distance from Earth to the Sun is approximately 150 000 000 kilometres.
- The mass of an oxygen atom is approximately 0.000 000 000 000 000 000 000 026 grams.

To help deal with such numbers, we can use a more convenient way to express them. This involves expressing the number as a product of a number between 1 and 10 and a power of 10 and is called **standard form** or **scientific notation**.

These examples written in standard form are:

- 1.5×10^8 kilometres
- 2.6×10^{-23} grams

Multiplication and division with very small or very large numbers can often be simplified by first converting the numbers into standard form. When simplifying algebraic expressions or manipulating numbers in standard form, a sound knowledge of the index laws is essential.

Example 4

Write each of the following in standard form:

a 3 453 000 **b** 0.00675

Solution

a $3\,453\,000 = 3.453 \times 10^6$ **b** $0.00675 = 6.75 \times 10^{-3}$

Example 5

Find the value of $\dfrac{32\,000\,000 \times 0.000\,004}{16\,000}$.

Solution

$$\begin{aligned}\frac{32\,000\,000 \times 0.000\,004}{16\,000} &= \frac{3.2 \times 10^7 \times 4 \times 10^{-6}}{1.6 \times 10^4} \\ &= \frac{12.8 \times 10^1}{1.6 \times 10^4} \\ &= 8 \times 10^{-3} \\ &= 0.008\end{aligned}$$

▶ Significant figures

When measurements are made, the result is recorded to a certain number of significant figures. For example, if we say that the length of a piece of ribbon is 156 cm to the nearest centimetre, this means that the length is between 155.5 cm and 156.5 cm. The number 156 is said to be correct to three significant figures. Similarly, we may record π as being 3.1416, correct to five significant figures.

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When rounding off to a given number of significant figures, first identify the last significant digit and then:

- if the next digit is 0, 1, 2, 3 or 4, round down
- if the next digit is 5, 6, 7, 8 or 9, round up.

It can help with rounding off if the original number is first written in scientific notation.

So $\pi = 3.141\,592\,653\ldots$ is rounded off to $3,\ 3.1,\ 3.14,\ 3.142,\ 3.1416,\ 3.14159,$ etc. depending on the number of significant figures required.

Writing a number in scientific notation makes it clear how many significant figures have been recorded. For example, it is unclear whether 600 is recorded to one, two or three significant figures. However, when written in scientific notation as 6.00×10^2, 6.0×10^2 or 6×10^2, it is clear how many significant figures are recorded.

Example 6

Evaluate $\dfrac{\sqrt[5]{a}}{b^2}$ if $a = 1.34 \times 10^{-10}$ and $b = 2.7 \times 10^{-8}$.

Solution

$$\begin{aligned}
\frac{\sqrt[5]{a}}{b^2} &= \frac{\sqrt[5]{1.34 \times 10^{-10}}}{(2.7 \times 10^{-8})^2} \\
&= \frac{(1.34 \times 10^{-10})^{\frac{1}{5}}}{2.7^2 \times (10^{-8})^2} \\
&= 1.45443\ldots \times 10^{13} \\
&= 1.45 \times 10^{13} \quad \text{to three significant figures}
\end{aligned}$$

Many calculators can display numbers in scientific notation. The format will vary from calculator to calculator. For example, the number $3\,245\,000 = 3.245 \times 10^6$ may appear as 3.245E6 or 3.245^{06}.

Using the TI-Nspire

Insert a **Calculator** page, then use [on] > **Settings** > **Document Settings** and change the **Exponential Format** field to **Scientific**. If you want this change to apply only to the current page, select OK to accept the change. Select **Current** to return to the current page.

Using the Casio ClassPad

The ClassPad calculator can be set to express decimal answers in various forms. To select a fixed number of decimal places, including specifying scientific notation with a fixed decimal accuracy, go to **Settings** ⚙ and in **Basic format** tap the arrow to select from the various Number formats available.

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Section summary

- A number is said to be in **scientific notation** (or **standard form**) when it is written as a product of a number between 1 and 10 and an integer power of 10.
  For example: $6547 = 6.547 \times 10^3$ and $0.789 = 7.89 \times 10^{-1}$
- Writing a number in scientific notation makes it clear how many **significant figures** have been recorded.
- When rounding off to a given number of significant figures, first identify the last significant digit and then:
  - if the next digit is 0, 1, 2, 3 or 4, round down
  - if the next digit is 5, 6, 7, 8 or 9, round up.

Exercise 1B

Example 4

1 Express each of the following numbers in standard form:

a 47.8 **b** 6728 **c** 79.23 **d** 43 580
e 0.0023 **f** 0.000 000 56 **g** 12.000 34 **h** 50 million
i 23 000 000 000 **j** 0.000 000 0013 **k** 165 thousand **l** 0.000 014 567

2 Express each of the following in scientific notation:

a The mass of a water molecule is 0.000 000 000 000 000 000 000 0299 g.
b X-rays have a wavelength of 0.000 000 01 cm.
c The speed of sound is 343.2 m/s.
d There are 31 536 000 seconds in one year.
e The average distance from Earth to Pluto is 6 090 000 000 km.
f There are 3 057 000 000 000 000 000 000 atoms in one gram of gold.

3 Express each of the following as an ordinary number:

a The diameter of the Sun is 1.39×10^9 m.
b The diameter of a red blood cell is 7.5×10^{-6} m.
c The diameter of an electron is 5.6×10^{-15} m.

4 Write each of the following in scientific notation, correct to the number of significant figures indicated in the brackets:

a 456.89 (4) **b** 34567.23 (2) **c** 5679.087 (5)
d 0.04536 (2) **e** 0.09045 (2) **f** 4568.234 (5)

Example 5

5 Find the value of:

a $\dfrac{324\,000 \times 0.000\,0007}{4000}$ **b** $\dfrac{5\,240\,000 \times 0.8}{42\,000\,000}$

Example 6

6 Evaluate the following correct to three significant figures:

a $\dfrac{\sqrt[3]{a}}{b^4}$ if $a = 2 \times 10^9$ and $b = 3.215$ **b** $\dfrac{\sqrt[4]{a}}{4b^4}$ if $a = 2 \times 10^{12}$ and $b = 0.05$

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Specialist Mathematics Year 11
ISBN 978-1-316-63608-4

1C Solving linear equations and simultaneous linear equations

Many problems may be solved by first translating them into mathematical equations and then solving the equations using algebraic techniques. An equation is solved by finding the value or values of the variables that would make the statement true.

Linear equations are simple equations that can be written in the form $ax + b = 0$. There are a number of standard techniques that can be used for solving linear equations.

Example 7

a Solve $\frac{x}{5} - 2 = \frac{x}{3}$.

b Solve $\frac{x-3}{2} - \frac{2x-4}{3} = 5$.

Solution

a Multiply both sides of the equation by the lowest common multiple of 3 and 5:

$$\frac{x}{5} - 2 = \frac{x}{3}$$
$$\frac{x}{5} \times 15 - 2 \times 15 = \frac{x}{3} \times 15$$
$$3x - 30 = 5x$$
$$3x - 5x = 30$$
$$-2x = 30$$
$$x = \frac{30}{-2}$$
$$\therefore \quad x = -15$$

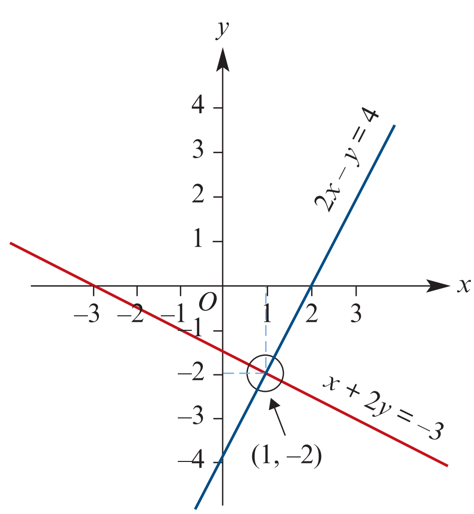
b Multiply both sides of the equation by the lowest common multiple of 2 and 3:

$$\frac{x-3}{2} - \frac{2x-4}{3} = 5$$
$$\frac{x-3}{2} \times 6 - \frac{2x-4}{3} \times 6 = 5 \times 6$$
$$3(x-3) - 2(2x-4) = 30$$
$$3x - 9 - 4x + 8 = 30$$
$$3x - 4x = 30 + 9 - 8$$
$$-x = 31$$
$$x = \frac{31}{-1}$$
$$\therefore \quad x = -31$$

▶ Simultaneous linear equations

The intersection point of two straight lines can be found graphically; however, the accuracy of the solution will depend on the accuracy of the graphs.

Alternatively, the intersection point may be found algebraically by solving the pair of simultaneous equations. We shall consider two techniques for solving simultaneous equations.



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Example 8

Solve the equations $2x - y = 4$ and $x + 2y = -3$.

Solution

Method 1: Substitution

$$2x - y = 4 \qquad (1)$$
$$x + 2y = -3 \qquad (2)$$

Explanation: Using one of the two equations, express one variable in terms of the other variable.

From equation (2), we get $x = -3 - 2y$.

Substitute in equation (1):

$$2(-3 - 2y) - y = 4$$
$$-6 - 4y - y = 4$$
$$-5y = 10$$
$$y = -2$$

Explanation: Then substitute this expression into the other equation (reducing it to an equation in one variable, y). Solve the equation for y.

Substitute the value of y into (2):

$$x + 2(-2) = -3$$
$$x = 1$$

Explanation: Substitute this value for y in one of the equations to find the other variable, x.

Check in (1): LHS $= 2(1) - (-2) = 4$

RHS $= 4$

Explanation: A check can be carried out with the other equation.

Method 2: Elimination

$$2x - y = 4 \qquad (1)$$
$$x + 2y = -3 \qquad (2)$$

To eliminate x, multiply equation (2) by 2 and subtract the result from equation (1).

Explanation: If one of the variables has the same coefficient in the two equations, we can eliminate that variable by subtracting one equation from the other.

It may be necessary to multiply one of the equations by a constant to make the coefficients of x or y the same in the two equations.

When we multiply equation (2) by 2, the pair of equations becomes:

$$2x - y = 4 \qquad (1)$$
$$2x + 4y = -6 \qquad (2')$$

Subtract (2′) from (1):

$$-5y = 10$$
$$y = -2$$

Now substitute for y in equation (2) to find x, and check as in the substitution method.

Note: This example shows that the point $(1, -2)$ is the point of intersection of the graphs of the two linear relations.

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Using the TI-Nspire

Calculator application

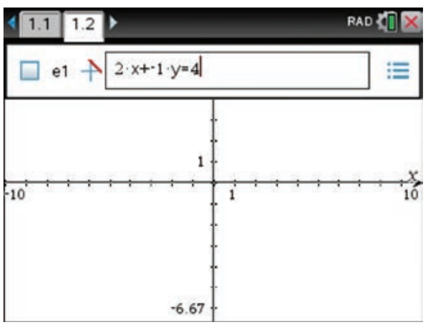
Simultaneous equations can be solved in a **Calculator** application.

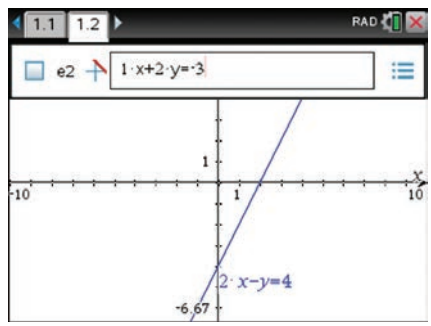
- Use (menu) > **Algebra** > **Solve System of Equations** > **Solve System of Equations**.
- Complete the pop-up screen.
- Enter the equations as shown to give the solution to the simultaneous equations $2x - y = 4$ and $x + 2y = -3$.

Graphs application

Simultaneous equations can also be solved graphically in a **Graphs** application.

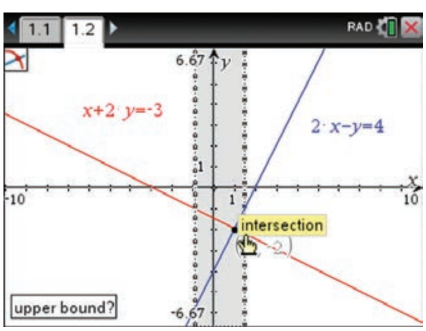
- Equations of the form $ax + by = c$ can be entered directly using (menu) > **Graph Entry/Edit** > **Equation** > **Line** > **Line Standard**.
- Alternatively, rearrange each equation to make y the subject, and enter as a standard function (e.g. $f1(x) = 2x - 4$).

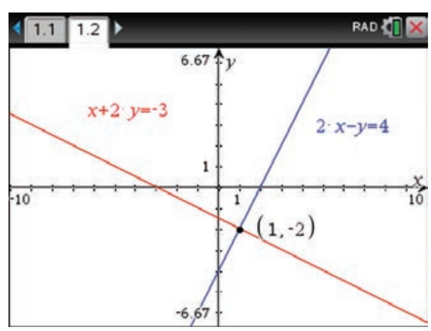




Note: If the Entry Line is not visible, press (tab). Pressing (enter) will hide the Entry Line. If you want to add more equations, use ▼ to add the next equation.

- The intersection point is found using (menu) > **Analyze Graph** > **Intersection**.
- Move the cursor to the left of the intersection point (lower bound), click, and move to the right of the intersection point (upper bound).
- Click to paste the coordinates to the screen.





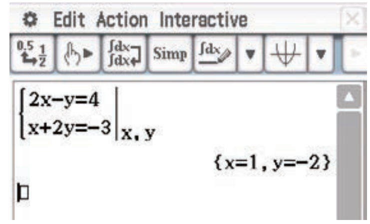
ISBN 978-1-316-63608-4

Cambridge University Press

Using the Casio ClassPad

To solve the simultaneous equations algebraically:

- Open the Main $\sqrt{\alpha}$ application and turn on the keyboard.
- In Math1, tap the simultaneous equations icon.
- Enter the two equations as shown.
- Type x, y in the bottom-right square to indicate the variables.
- Tap EXE.

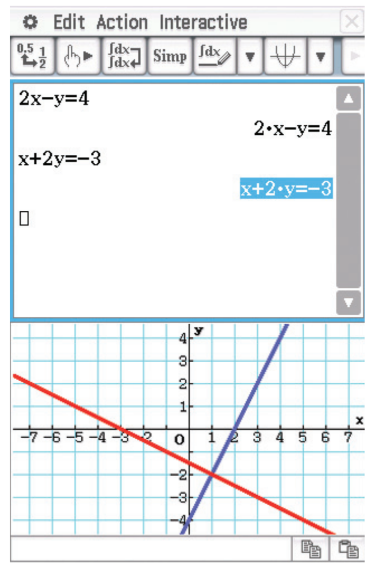


There are two methods for solving simultaneous equations graphically.

Method 1

In the Main $\sqrt{\alpha}$ application:

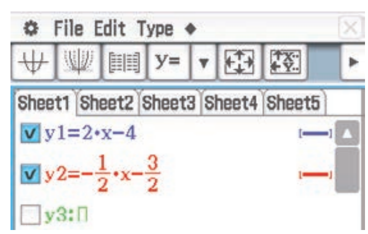
- Enter the equation $2x - y = 4$ and tap EXE.
- Enter the equation $x + 2y = -3$ and tap EXE.
- Select the graph icon from the toolbar to insert a graph window. An appropriate window can be set by selecting **Zoom** > **Quick** > **Quick Standard**.
- Highlight each equation and drag it into the graph window.
- To find the point of intersection, go to **Analysis** > **G-Solve** > **Intersection**.

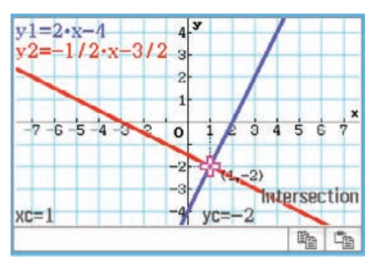


Method 2

For this method, the equations need to be rearranged to make y the subject. In this form, the equations are $y = 2x - 4$ and $y = -\frac{1}{2}x - \frac{3}{2}$.

- Open the menu Menu; select **Graph & Table**.
- Tap in the working line of $y1$ and enter $2x - 4$.
- Tap in the working line of $y2$ and enter $-\frac{1}{2}x - \frac{3}{2}$.
- Tick the boxes for $y1$ and $y2$.
- Select the graph icon from the toolbar.
- Go to **Analysis** > **G-Solve** > **Intersection**.
- If necessary, the view window settings can be adjusted by tapping the view window icon or by selecting **Zoom** > **Quick** > **Quick Standard**.





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Section summary

- An equation is solved by finding the value or values of the variables that would make the statement true.
- A linear equation is one in which the 'unknown' is to the first power.
- There are often several different ways to solve a linear equation. The following steps provide some suggestions:
  1. Expand brackets and, if the equation involves fractions, multiply through by the lowest common denominator of the terms.
  2. Group all of the terms containing a variable on one side of the equation and the terms without the variable on the other side.
- Methods for solving simultaneous linear equations in two variables by hand:

  Substitution
  - Make one of the variables the subject in one of the equations.
  - Substitute for that variable in the other equation.

  Elimination
  - Choose one of the two variables to eliminate.
  - Obtain the same or opposite coefficients for this variable in the two equations. To do this, multiply both sides of one or both equations by a number.
  - Add or subtract the two equations to eliminate the chosen variable.

Exercise 1C

Example 7a

1 Solve the following linear equations:

a $3x + 7 = 15$ **b** $8 - \frac{x}{2} = -16$ **c** $42 + 3x = 22$

d $\frac{2x}{3} - 15 = 27$ **e** $5(2x + 4) = 13$ **f** $-3(4 - 5x) = 24$

g $3x + 5 = 8 - 7x$ **h** $2 + 3(x - 4) = 4(2x + 5)$ **i** $\frac{2x}{5} - \frac{3}{4} = 5x$

j $6x + 4 = \frac{x}{3} - 3$

Example 7b

2 Solve the following linear equations:

a $\frac{x}{2} + \frac{2x}{5} = 16$ **b** $\frac{3x}{4} - \frac{x}{3} = 8$

c $\frac{3x - 2}{2} + \frac{x}{4} = -8$ **d** $\frac{5x}{4} - \frac{4}{3} = \frac{2x}{5}$

e $\frac{x - 4}{2} + \frac{2x + 5}{4} = 6$ **f** $\frac{3 - 3x}{10} - \frac{2(x + 5)}{6} = \frac{1}{20}$

g $\frac{3 - x}{4} - \frac{2(x + 1)}{5} = -24$ **h** $\frac{-2(5 - x)}{8} + \frac{6}{7} = \frac{4(x - 2)}{3}$

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Example 8

3 Solve each of the following pairs of simultaneous equations:

a $3x + 2y = 2$
$2x - 3y = 6$

b $5x + 2y = 4$
$3x - y = 6$

c $2x - y = 7$
$3x - 2y = 2$

d $x + 2y = 12$
$x - 3y = 2$

e $7x - 3y = -6$
$x + 5y = 10$

f $15x + 2y = 27$
$3x + 7y = 45$

1D Solving problems with linear equations

Many problems can be solved by translating them into mathematical language and using an appropriate mathematical technique to find the solution. By representing the unknown quantity in a problem with a symbol (called a pronumeral or a variable) and constructing an equation from the information, the value of the unknown can be found by solving the equation.

Before constructing the equation, each variable and what it stands for (including the units) should be stated. All the elements of the equation must be in units of the same system.

Example 9

For each of the following, form the relevant linear equation and solve it for x:

a The length of the side of a square is $(x - 6)$ cm. Its perimeter is 52 cm.

b The perimeter of a square is $(2x + 8)$ cm. Its area is 100 cm^2.

Solution

a Perimeter = 4 × side length

Therefore

$$4(x - 6) = 52$$
$$x - 6 = 13$$

and so $x = 19$

b The perimeter of the square is $2x + 8$.

The length of one side is $\dfrac{2x + 8}{4} = \dfrac{x + 4}{2}$.

Thus the area is

$$\left(\frac{x + 4}{2}\right)^2 = 100$$

As the side length must be positive, this gives the linear equation

$$\frac{x + 4}{2} = 10$$

Therefore $x = 16$.

ISBN 978-1-316-63608-4

Example 10

An athlete trains for an event by gradually increasing the distance she runs each week over a five-week period. If she runs an extra 5 km each successive week and over the five weeks runs a total of 175 km, how far did she run in the first week?

Solution

Let the distance run in the first week be x km.

Then the distance run in the second week is $x + 5$ km, and the distance run in the third week is $x + 10$ km, and so on.

The total distance run is $x + (x + 5) + (x + 10) + (x + 15) + (x + 20)$ km.

$$\begin{aligned} \therefore \quad 5x + 50 &= 175 \\ 5x &= 125 \\ x &= 25 \end{aligned}$$

The distance she ran in the first week was 25 km.

Example 11

A man bought 14 CDs at a sale. Some cost \$15 each and the remainder cost \$12.50 each. In total he spent \$190. How many \$15 CDs and how many \$12.50 CDs did he buy?

Solution

Let n be the number of CDs costing \$15.
Then $14 - n$ is the number of CDs costing \$12.50.

$$\begin{aligned} \therefore \quad 15n + 12.5(14 - n) &= 190 \\ 15n + 175 - 12.5n &= 190 \\ 2.5n + 175 &= 190 \\ 2.5n &= 15 \\ n &= 6 \end{aligned}$$

He bought 6 CDs costing \$15 and 8 CDs costing \$12.50.

Section summary

Steps for solving a word problem with a linear equation

- Read the question carefully and write down the known information clearly.
- Identify the unknown quantity that is to be found.
- Assign a variable to this quantity.
- Form an expression in terms of x (or the variable being used) and use the other relevant information to form the equation.
- Solve the equation.
- Write a sentence answering the initial question.

ISBN 978-1-316-63608-4

Exercise 1D

Skillsheet

1 For each of the cases below, write down a relevant equation involving the variables defined, and solve the equation for parts **a**, **b** and **c**.

Example 9

a The length of the side of a square is $(x - 2)$ cm. Its perimeter is 60 cm.

b The perimeter of a square is $(2x + 7)$ cm. Its area is 49 cm^2.

c The length of a rectangle is $(x - 5)$ cm. Its width is $(12 - x)$ cm. The rectangle is twice as long as it is wide.

d The length of a rectangle is $(2x + 1)$ cm. Its width is $(x - 3)$ cm. The perimeter of the rectangle is y cm.

e n people each have a meal costing $\$p$. The total cost of the meal is $\$Q$.

f S people each have a meal costing $\$p$. A 10% service charge is added to the cost. The total cost of the meal is $\$R$.

g A machine working at a constant rate produces n bolts in 5 minutes. It produces 2400 bolts in 1 hour.

h The radius of a circle is $(x + 3)$ cm. A sector subtending an angle of 60° at the centre is cut off. The arc length of the minor sector is a cm.

Example 10

2 Bronwyn and Noel have a women's clothing shop in Summerland. Bronwyn manages the shop and her sales are going up steadily over a particular period of time. They are going up by \$500 per week. If over a five-week period her sales total \$17 500, how much did she earn in the first week?

Example 11

3 Bronwyn and Noel have a women's clothing shop in Summerland. Sally, Adam and baby Lana came into the shop and Sally bought dresses and handbags. The dresses cost \$65 each and the handbags cost \$26 each. Sally bought 11 items and in total she spent \$598. How many dresses and how many handbags did she buy?

4 A rectangular courtyard is three times as long as it is wide. If the perimeter of the courtyard is 67 m, find the dimensions of the courtyard.

5 A wine merchant buys 50 cases of wine. He pays full price for half of them, but gets a 40% discount on the remainder. If he paid a total of \$2260, how much was the full price of a single case?

6 A real-estate agent sells 22 houses in six months. He makes a commission of \$11 500 per house on some and \$13 000 per house on the remainder. If his total commission over the six months was \$272 500, on how many houses did he make a commission of \$11 500?

7 Three boys compare their marble collections. The first boy has 14 fewer than the second boy, who has twice as many as the third. If between them they have 71 marbles, how many does each boy have?

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8 Three girls are playing Scrabble. At the end of the game, their three scores add up to 504. Annie scored 10% more than Belinda, while Cassie scored 60% of the combined scores of the other two. What did each player score?

9 A biathlon event involves running and cycling. Kim can cycle 30 km/h faster than she can run. If Kim spends 48 minutes running and a third as much time again cycling in an event that covers a total distance of 60 km, how fast can she run?

10 The mass of a molecule of a certain chemical compound is 2.45×10^{-22} g. If each molecule is made up of two carbon atoms and six oxygen atoms and the mass of an oxygen atom is one-third that of a carbon atom, find the mass of an oxygen atom.

11 Mother's pearl necklace fell to the floor. One-sixth of the pearls rolled under the fridge, one-third rolled under the couch, one-fifth of them behind the book shelf, and nine were found at her feet. How many pearls are there?

12 Parents say they don't have favourites, but everyone knows that's a lie. A father distributes \$96 to his three children according to the following instructions: The middle child receives \$12 less than the oldest, and the youngest receives one-third as much as the middle child. How much does each receive?

13 Kavindi has achieved an average mark of 88% on her first four maths tests. What mark would she need on her fifth test to increase her average to 90%?

14 In a particular class, 72% of the students have black hair. Five black-haired students leave the class, so that now 65% of the students have black hair. How many students were originally in the class?

15 Two tanks are being emptied. Tank A contains 100 litres of water and tank B contains 120 litres of water. Water runs from Tank A at 2 litres per minute, and water runs from tank B at 3 litres per minute. After how many minutes will the amount of water in the two tanks be the same?

16 Suppose that candle A is initially 10 cm tall and burns out after 2 hours. Candle B is initially 8 cm tall and burns out after 4 hours. Both candles are lit at the same time. Assuming 'constant burning rates':

a When is the height of candle A the same as the height of candle B?

b When is the height of candle A half the height of candle B?

c When is candle A 1 cm taller than candle B?

17 A motorist drove 320 km in $\frac{10}{3}$ hours. He drove part of the way at an average speed of 100 km/h and the rest of the way at an average speed of 90 km/h. What is the distance he travelled at 100 km/h?

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18 Jarmila travels regularly between two cities. She takes $\frac{14}{3}$ hours if she travels at her usual speed. If she increase her speed by 3 km/h, she can reduce her time taken by 20 minutes. What is her usual speed?

1E Solving problems with simultaneous linear equations

When the relationship between two quantities is linear, we can find the constants which determine this linear relationship if we are given two sets of information satisfying the relationship. Simultaneous linear equations enable this to be done.

Another situation in which simultaneous linear equations may be used is where it is required to find the point of the Cartesian plane which satisfies two linear relations.

Example 12

There are two possible methods for paying gas bills:

Method A A fixed charge of \$25 per quarter + 50c per unit of gas used

Method B A fixed charge of \$50 per quarter + 25c per unit of gas used

Determine the number of units which must be used before method B becomes cheaper than method A.

Solution

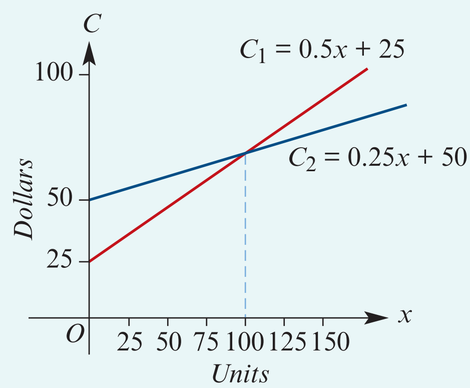
Let C_1 = charge (\$) using method A

C_2 = charge (\$) using method B

x = number of units of gas used

Then $C_1 = 25 + 0.5x$

$C_2 = 50 + 0.25x$



From the graph we see that method B is cheaper if the number of units exceeds 100.

The solution can be obtained by solving simultaneous linear equations:

$$C_1 = C_2$$
$$25 + 0.5x = 50 + 0.25x$$
$$0.25x = 25$$
$$x = 100$$

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Example 13

If 3 kg of jam and 2 kg of butter cost \$29, and 6 kg of jam and 3 kg of butter cost \$54, find the cost per kilogram of jam and butter.

Solution

Let the cost of 1 kg of jam be x dollars and the cost of 1 kg of butter be y dollars.

Then $3x + 2y = 29$ (1)

and $6x + 3y = 54$ (2)

Multiply (1) by 2: $6x + 4y = 58$ (1′)

Subtract (1′) from (2): $-y = -4$

$y = 4$

Substitute in (2): $6x + 3(4) = 54$

$6x = 42$

$x = 7$

Jam costs \$7 per kilogram and butter costs \$4 per kilogram.

Section summary

Steps for solving a word problem with simultaneous linear equations

- Read the question carefully and write down the known information clearly.
- Identify the two unknown quantities that are to be found.
- Assign variables to these two quantities.
- Form expressions in terms of x and y (or other suitable variables) and use the other relevant information to form the two equations.
- Solve the system of equations.
- Write a sentence answering the initial question.

Exercise 1E

Example 12

1 A car hire firm offers the option of paying \$108 per day with unlimited kilometres, or \$63 per day plus 32 cents per kilometre travelled. How many kilometres would you have to travel in a given day to make the unlimited-kilometres option more attractive?

2 Company A will cater for your party at a cost of \$450 plus \$40 per guest. Company B offers the same service for \$300 plus \$43 per guest. How many guests are needed before Company A's charge is less than Company B's?

Example 13

3 A basketball final is held in a stadium which can seat 15 000 people. All the tickets have been sold, some to adults at \$45 and the rest for children at \$15. If the revenue from the tickets was \$525 000, find the number of adults who bought tickets.

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4 A contractor employed eight men and three boys for one day and paid them a total of \$2240. Another day he employed six men and eighteen boys for \$4200. What was the daily rate he paid each man and each boy?

5 The sum of two numbers is 212 and their difference is 42. Find the two numbers.

6 A chemical manufacturer wishes to obtain 700 litres of a 24% acid solution by mixing a 40% solution with a 15% solution. How many litres of each solution should be used?

7 Two children had 220 marbles between them. After one child had lost half her marbles and the other had lost 40 marbles, they had an equal number of marbles. How many did each child start with and how many did each child finish with?

8 An investor received \$31 000 interest per annum from a sum of money, with part of it invested at 10% and the remainder at 7% simple interest. She found that if she interchanged the amounts she had invested she could increase her return by \$1000 per annum. Calculate the total amount she had invested.

9 Each adult paid \$30 to attend a concert and each student paid \$20. A total of 1600 people attended. The total paid was \$37 000. How many adults and how many students attended the concert?

1F Substitution and transposition of formulas

An equation that states a relationship between two or more quantities is called a **formula**; e.g. the area of a circle is given by $A = \pi r^2$. The value of A, the subject of the formula, may be found by substituting a given value of r and the value of π.

Example 14

Using the formula $A = \pi r^2$, find the value of A correct to two decimal places if $r = 2.3$ and $\pi = 3.142$ (correct to three decimal places).

Solution

$$\begin{aligned} A &= \pi r^2 \\ &= 3.142(2.3)^2 \\ &= 16.62118 \end{aligned}$$

$\therefore \quad A = 16.62$ correct to two decimal places

The formula $A = \pi r^2$ can also be transposed to make r the subject.

When transposing a formula, follow a similar procedure to solving a linear equation. Whatever has been done to the variable required is ‘undone’.

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Example 15

a Transpose the formula $A = \pi r^2$ to make r the subject.

b Hence find the value of r correct to three decimal places if $A = 24.58$ and $\pi = 3.142$ (correct to three decimal places).

Solution

a
$$A = \pi r^2$$
$$\frac{A}{\pi} = r^2$$
$$r = \sqrt{\frac{A}{\pi}}$$

b
$$r = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{24.58}{3.142}}$$
$$= 2.79697\ldots$$

$r = 2.797$ correct to three decimal places

Section summary

- A formula relates different quantities: for example, the formula $A = \pi r^2$ relates the radius r with the area A of the circle.
- The variable on the left is called the subject of the formula: for example, in the formula $A = \pi r^2$, the subject is A.
- To calculate the value of a variable which is not the subject of a formula:
  Method 1 Substitute the values for the known variables, then solve the resulting equation for the unknown variable.
  Method 2 Rearrange to make the required variable the subject, then substitute values.

Exercise 1F

Example 14

1 Substitute the specified values to evaluate each of the following, giving the answers correct to two decimal places:

a v if $v = u + at$ and $u = 15, a = 2, t = 5$

b I if $I = \frac{PrT}{100}$ and $P = 600, r = 5.5, T = 10$

c V if $V = \pi r^2 h$ and $r = 4.25, h = 6$

d S if $S = 2\pi r(r + h)$ and $r = 10.2, h = 15.6$

e V if $V = \frac{4}{3}\pi r^2 h$ and $r = 3.58, h = 11.4$

f s if $s = ut + \frac{1}{2}at^2$ and $u = 25.6, t = 3.3, a = -1.2$

g T if $T = 2\pi\sqrt{\frac{\ell}{g}}$ and $\ell = 1.45, g = 9.8$

h f if $\frac{1}{f} = \frac{1}{v} + \frac{1}{u}$ and $v = 3, u = 7$

i c if $c^2 = a^2 + b^2$ and $a = 8.8, b = 3.4$

j v if $v^2 = u^2 + 2as$ and $u = 4.8, a = 2.5, s = 13.6$

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Example 15

2 Transpose each of the following to make the symbol in brackets the subject:

a $v = u + at$ (a)

b $S = \frac{n}{2}(a + \ell)$ (ℓ)

c $A = \frac{1}{2}bh$ (b)

d $P = I^2R$ (I)

e $s = ut + \frac{1}{2}at^2$ (a)

f $E = \frac{1}{2}mv^2$ (v)

g $Q = \sqrt{2gh}$ (h)

h $-xy - z = xy + z$ (x)

i $\frac{ax + by}{c} = x - b$ (x)

j $\frac{mx + b}{x - b} = c$ (x)

3 The formula $F = \frac{9C}{5} + 32$ is used to convert temperatures given in degrees Celsius (C) to degrees Fahrenheit (F).

a Convert 28 degrees Celsius to degrees Fahrenheit.

b Transpose the formula to make C the subject and find C if $F = 135°$.

4 The sum, S, of the interior angles of a polygon with n sides is given by the formula $S = 180(n - 2)$.

a Find the sum of the interior angles of an octagon.

b Transpose the formula to make n the subject and hence determine the number of sides of a polygon whose interior angles add up to 1260°.

5 The volume, V, of a right cone is given by the formula $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height of the cone.

a Find the volume of a cone with radius 3.5 cm and height 9 cm.

b Transpose the formula to make h the subject and hence find the height of a cone with base radius 4 cm and volume 210 cm^3.

c Transpose the formula to make r the subject and hence find the radius of a cone with height 10 cm and volume 262 cm^3.

6 For a particular type of sequence of numbers, the sum (S) of the terms in the sequence is given by the formula

$$S = \frac{n}{2}(a + \ell)$$

where n is the number of terms in the sequence, a is the first term and ℓ is the last term.

a Find the sum of such a sequence of seven numbers whose first term is -3 and whose last term is 22.

b What is the first term of such a sequence of 13 numbers whose last term is 156 and whose sum is 1040?

c How many terms are there in the sequence $25 + 22 + 19 + \cdots + (-5) = 110$?

ISBN 978-1-316-63608-4

1G Algebraic fractions

The principles involved in addition, subtraction, multiplication and division of algebraic fractions are the same as for simple numerical fractions.

▶ Addition and subtraction

To add or subtract, all fractions must be written with a common denominator.

Example 16

Simplify:

a $\dfrac{x}{3} + \dfrac{x}{4}$

b $\dfrac{2}{x} + \dfrac{3a}{4}$

c $\dfrac{5}{x+2} - \dfrac{4}{x-1}$

d $\dfrac{4}{x+2} - \dfrac{7}{(x+2)^2}$

Solution

a $\dfrac{x}{3} + \dfrac{x}{4} = \dfrac{4x+3x}{12}$

$= \dfrac{7x}{12}$

b $\dfrac{2}{x} + \dfrac{3a}{4} = \dfrac{8+3ax}{4x}$

c $\dfrac{5}{x+2} - \dfrac{4}{x-1} = \dfrac{5(x-1)-4(x+2)}{(x+2)(x-1)}$

$= \dfrac{5x-5-4x-8}{(x+2)(x-1)}$

$= \dfrac{x-13}{(x+2)(x-1)}$

d $\dfrac{4}{x+2} - \dfrac{7}{(x+2)^2} = \dfrac{4(x+2)-7}{(x+2)^2}$

$= \dfrac{4x+1}{(x+2)^2}$

▶ Multiplication and division

Before multiplying and dividing algebraic fractions, it is best to factorise numerators and denominators where possible so that common factors can be readily identified.

Example 17

Simplify:

a $\dfrac{3x^2}{10y^2} \times \dfrac{5y}{12x}$

b $\dfrac{2x-4}{x-1} \times \dfrac{x^2-1}{x-2}$

c $\dfrac{x^2-1}{2x-2} \times \dfrac{4x}{x^2+4x+3}$

d $\dfrac{x^2+3x-10}{x^2-x-2} \div \dfrac{x^2+6x+5}{3x+3}$

Solution

a $\dfrac{3x^2}{10y^2} \times \dfrac{5y}{12x} = \dfrac{x}{8y}$

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b $\dfrac{2x-4}{x-1} \times \dfrac{x^2-1}{x-2} = \dfrac{2(x-2)}{x-1} \times \dfrac{(x-1)(x+1)}{x-2}$

$= 2(x+1)$

c $\dfrac{x^2-1}{2x-2} \times \dfrac{4x}{x^2+4x+3} = \dfrac{(x-1)(x+1)}{2(x-1)} \times \dfrac{4x}{(x+1)(x+3)}$

$= \dfrac{2x}{x+3}$

d $\dfrac{x^2+3x-10}{x^2-x-2} \div \dfrac{x^2+6x+5}{3x+3} = \dfrac{(x+5)(x-2)}{(x-2)(x+1)} \times \dfrac{3(x+1)}{(x+1)(x+5)}$

$= \dfrac{3}{x+1}$

► More examples

The following two examples involve algebraic fractions and rational indices.

Example 18

Express $\dfrac{3x^3}{\sqrt{4-x}} + 3x^2\sqrt{4-x}$ as a single fraction.

Solution

$$\begin{aligned}\frac{3x^3}{\sqrt{4-x}} + 3x^2\sqrt{4-x} &= \frac{3x^3 + 3x^2\sqrt{4-x}\,\sqrt{4-x}}{\sqrt{4-x}}\\ &= \frac{3x^3 + 3x^2(4-x)}{\sqrt{4-x}}\\ &= \frac{12x^2}{\sqrt{4-x}}\end{aligned}$$

Example 19

Express $(x-4)^{\frac{1}{5}} - (x-4)^{-\frac{4}{5}}$ as a single fraction.

Solution

$$\begin{aligned}(x-4)^{\frac{1}{5}} - (x-4)^{-\frac{4}{5}} &= (x-4)^{\frac{1}{5}} - \frac{1}{(x-4)^{\frac{4}{5}}}\\ &= \frac{(x-4)^{\frac{1}{5}}(x-4)^{\frac{4}{5}} - 1}{(x-4)^{\frac{4}{5}}}\\ &= \frac{x-5}{(x-4)^{\frac{4}{5}}}\end{aligned}$$

ISBN 978-1-316-63608-4

Section summary

- **Simplifying algebraic fractions**
  - First factorise the numerator and denominator.
  - Then cancel any factors common to the numerator and denominator.
- **Adding and subtracting algebraic fractions**
  - First obtain a common denominator and then add or subtract.
- **Multiplying and dividing algebraic fractions**
  - First factorise each numerator and denominator completely.
  - Then complete the calculation by cancelling common factors.

Exercise 1G

Skillsheet

Example 16

1 Simplify each of the following:

a $\frac{2x}{3}+\frac{3x}{2}$ **b** $\frac{3a}{2}-\frac{a}{4}$ **c** $\frac{3h}{4}+\frac{5h}{8}-\frac{3h}{2}$

d $\frac{3x}{4}-\frac{y}{6}-\frac{x}{3}$ **e** $\frac{3}{x}+\frac{2}{y}$ **f** $\frac{5}{x-1}+\frac{2}{x}$

g $\frac{3}{x-2}+\frac{2}{x+1}$ **h** $\frac{2x}{x+3}-\frac{4x}{x-3}-\frac{3}{2}$ **i** $\frac{4}{x+1}+\frac{3}{(x+1)^2}$

j $\frac{a-2}{a}+\frac{a}{4}+\frac{3a}{8}$ **k** $2x-\frac{6x^2-4}{5x}$ **l** $\frac{2}{x+4}-\frac{3}{x^2+8x+16}$

m $\frac{3}{x-1}+\frac{2}{(x-1)(x+4)}$ **n** $\frac{3}{x-2}-\frac{2}{x+2}+\frac{4}{x^2-4}$

o $\frac{5}{x-2}+\frac{3}{x^2+5x+6}+\frac{2}{x+3}$ **p** $x-y-\frac{1}{x-y}$

q $\frac{3}{x-1}-\frac{4x}{1-x}$ **r** $\frac{3}{x-2}+\frac{2x}{2-x}$

Example 17

2 Simplify each of the following:

a $\frac{x^2}{2y}\times\frac{4y^3}{x}$ **b** $\frac{3x^2}{4y}\times\frac{y^2}{6x}$ **c** $\frac{4x^3}{3}\times\frac{12}{8x^4}$

d $\frac{x^2}{2y}\div\frac{3xy}{6}$ **e** $\frac{4-x}{3a}\times\frac{a^2}{4-x}$ **f** $\frac{2x+5}{4x^2+10x}$

g $\frac{(x-1)^2}{x^2+3x-4}$ **h** $\frac{x^2-x-6}{x-3}$ **i** $\frac{x^2-5x+4}{x^2-4x}$

j $\frac{5a^2}{12b^2}\div\frac{10a}{6b}$ **k** $\frac{x-2}{x}\div\frac{x^2-4}{2x^2}$ **l** $\frac{x+2}{x(x-3)}\div\frac{4x+8}{x^2-4x+3}$

m $\frac{2x}{x-1}\div\frac{4x^2}{x^2-1}$ **n** $\frac{x^2-9}{x+2}\times\frac{3x+6}{x-3}\div\frac{9}{x}$ **o** $\frac{3x}{9x-6}\div\frac{6x^2}{x-2}\times\frac{2}{x+5}$

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3 Express each of the following as a single fraction:

a $\dfrac{1}{x-3}+\dfrac{2}{x-3}$ **b** $\dfrac{2}{x-4}+\dfrac{2}{x-3}$ **c** $\dfrac{3}{x+4}+\dfrac{2}{x-3}$

d $\dfrac{2x}{x-3}+\dfrac{2}{x+4}$ **e** $\dfrac{1}{(x-5)^2}+\dfrac{2}{x-5}$ **f** $\dfrac{3x}{(x-4)^2}+\dfrac{2}{x-4}$

g $\dfrac{1}{x-3}-\dfrac{2}{x-3}$ **h** $\dfrac{2}{x-3}-\dfrac{5}{x+4}$ **i** $\dfrac{2x}{x-3}+\dfrac{3x}{x+3}$

j $\dfrac{1}{(x-5)^2}-\dfrac{2}{x-5}$ **k** $\dfrac{2x}{(x-6)^3}-\dfrac{2}{(x-6)^2}$ **l** $\dfrac{2x+3}{x-4}-\dfrac{2x-4}{x-3}$

Example 18

4 Express each of the following as a single fraction:

a $\sqrt{1-x}+\dfrac{2}{\sqrt{1-x}}$ **b** $\dfrac{2}{\sqrt{x-4}}+\dfrac{2}{3}$ **c** $\dfrac{3}{\sqrt{x+4}}+\dfrac{2}{\sqrt{x+4}}$

d $\dfrac{3}{\sqrt{x+4}}+\sqrt{x+4}$ **e** $\dfrac{3x^3}{\sqrt{x+4}}-3x^2\sqrt{x+4}$ **f** $\dfrac{3x^3}{2\sqrt{x+3}}+3x^2\sqrt{x+3}$

Example 19

5 Simplify each of the following:

a $(6x-3)^{\frac{1}{3}}-(6x-3)^{-\frac{2}{3}}$ **b** $(2x+3)^{\frac{1}{3}}-2x(2x+3)^{-\frac{2}{3}}$ **c** $(3-x)^{\frac{1}{3}}-2x(3-x)^{-\frac{2}{3}}$

1H Literal equations

A literal equation in x is an equation whose solution will be expressed in terms of pronumerals rather than numbers.

For the equation $2x + 5 = 7$, the solution is the number 1.

For the literal equation $ax + b = c$, the solution is $x = \dfrac{c-b}{a}$.

Literal equations are solved in the same way as numerical equations. Essentially, the literal equation is transposed to make x the subject.

Example 20

Solve the following for x:

a $px - q = r$ **b** $ax + b = cx + d$ **c** $\dfrac{a}{x}=\dfrac{b}{2x}+c$

Solution

a
$$\begin{aligned} px - q &= r \\ px &= r + q \\ \therefore\quad x &= \frac{r+q}{p} \end{aligned}$$

b
$$\begin{aligned} ax + b &= cx + d \\ ax - cx &= d - b \\ x(a-c) &= d - b \\ \therefore\quad x &= \frac{d-b}{a-c} \end{aligned}$$

c Multiply both sides by $2x$:
$$\begin{aligned} \frac{a}{x} &= \frac{b}{2x}+c \\ 2a &= b + 2xc \\ 2a - b &= 2xc \\ \therefore\quad x &= \frac{2a-b}{2c} \end{aligned}$$

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Simultaneous literal equations

Simultaneous literal equations are solved by the same methods that are used for solving simultaneous equations, i.e. substitution and elimination.

Example 21

Solve each of the following pairs of simultaneous equations for x and y:

a $y = ax + c$

$y = bx + d$

b $ax - y = c$

$x + by = d$

Solution

a Equate the two expressions for y:

$$ax + c = bx + d$$
$$ax - bx = d - c$$
$$x(a - b) = d - c$$

Thus $$x = \frac{d - c}{a - b}$$

and $$y = a\left(\frac{d - c}{a - b}\right) + c = \frac{ad - ac + ac - bc}{a - b} = \frac{ad - bc}{a - b}$$

b We will use the method of elimination, and eliminate y.

First number the two equations:

$$ax - y = c \qquad (1)$$
$$x + by = d \qquad (2)$$

Multiply (1) by b:

$$abx - by = bc \qquad (1')$$

Add (1′) and (2):

$$abx + x = bc + d$$
$$x(ab + 1) = bc + d$$
$$\therefore \quad x = \frac{bc + d}{ab + 1}$$

Substitute in (1):

$$y = ax - c = a\left(\frac{bc + d}{ab + 1}\right) - c = \frac{ad - c}{ab + 1}$$

Section summary

- An equation for the variable x in which all the coefficients of x, including the constants, are pronumerals is known as a **literal equation**.
- The methods for solving linear literal equations or simultaneous linear literal equations are exactly the same as when the coefficients are given numbers.

ISBN 978-1-316-63608-4

Exercise 1H

Example 20

1 Solve each of the following for x:

a $ax + n = m$

b $ax + b = bx$

c $\frac{ax}{b} + c = 0$

d $px = qx + 5$

e $mx + n = nx - m$

f $\frac{1}{x + a} = \frac{b}{x}$

g $\frac{b}{x - a} = \frac{2b}{x + a}$

h $\frac{x}{m} + n = \frac{x}{n} + m$

i $-b(ax + b) = a(bx - a)$

j $p^2(1 - x) - 2pqx = q^2(1 + x)$

k $\frac{x}{a} - 1 = \frac{x}{b} + 2$

l $\frac{x}{a - b} + \frac{2x}{a + b} = \frac{1}{a^2 - b^2}$

m $\frac{p - qx}{t} + p = \frac{qx - t}{p}$

n $\frac{1}{x + a} + \frac{1}{x + 2a} = \frac{2}{x + 3a}$

2 For the simultaneous equations $ax + by = p$ and $bx - ay = q$, show that $x = \frac{ap + bq}{a^2 + b^2}$ and $y = \frac{bp - aq}{a^2 + b^2}$.

3 For the simultaneous equations $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$, show that $x = y = \frac{ab}{a + b}$.

Example 21

4 Solve each of the following pairs of simultaneous equations for x and y:

a $ax + y = c$
$x + by = d$

b $ax - by = a^2$
$bx - ay = b^2$

c $ax + by = t$
$ax - by = s$

d $ax + by = a^2 + 2ab - b^2$
$bx + ay = a^2 + b^2$

e $(a + b)x + cy = bc$
$(b + c)y + ax = -ab$

f $3(x - a) - 2(y + a) = 5 - 4a$
$2(x + a) + 3(y - a) = 4a - 1$

5 Write s in terms of a only in the following pairs of equations:

a $s = ah$
$h = 2a + 1$

b $s = ah$
$h = a(2 + h)$

c $as = a + h$
$h + ah = 1$

d $as = s + h$
$ah = a + h$

e $s = h^2 + ah$
$h = 3a^2$

f $as = a + 2h$
$h = a - s$

g $s = 2 + ah + h^2$
$h = a - \frac{1}{a}$

h $3s - ah = a^2$
$as + 2h = 3a$

ISBN 978-1-316-63608-4

1I Using a CAS calculator for algebra

Using the TI-Nspire

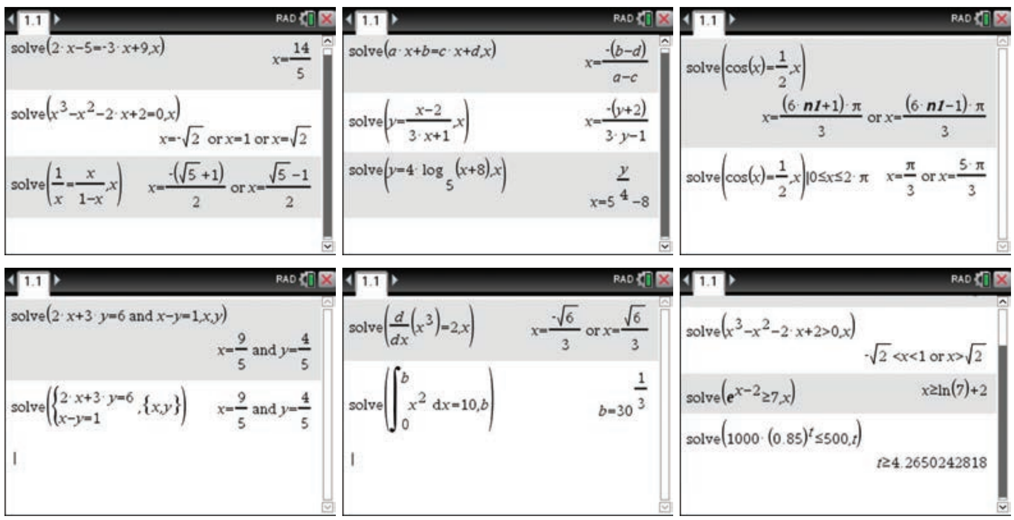
This section demonstrates the basic algebra commands of the TI-Nspire. To access them, open a **Calculator** application (on > **New Document** > **Add Calculator**) and select menu > **Algebra**. The three main commands are solve, factor and expand.

1: Solve

This command is used to solve equations, simultaneous equations and some inequalities.

An approximate (decimal) answer can be obtained by pressing ctrl enter or by including a decimal number in the expression.

The following screens illustrate its use.

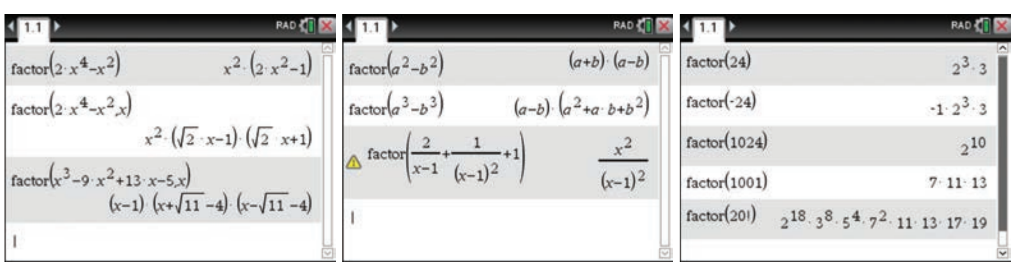


2: Factor

This command is used for factorisation.

Factorisation over the rational numbers is obtained by not specifying the variable, whereas factorisation over the real numbers is obtained by specifying the variable.

The following screens illustrate its use.



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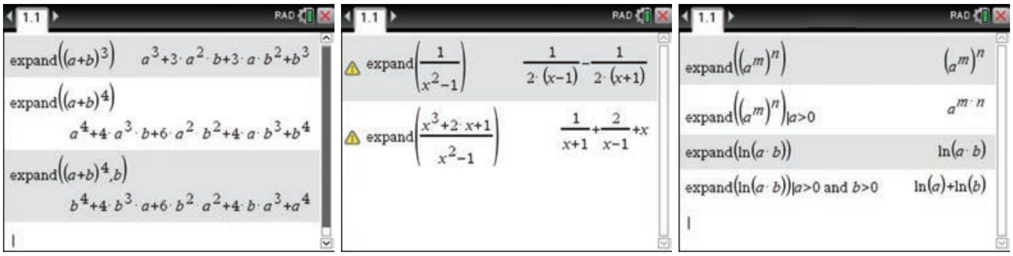
3: Expand

This command is used for expanding out expressions.

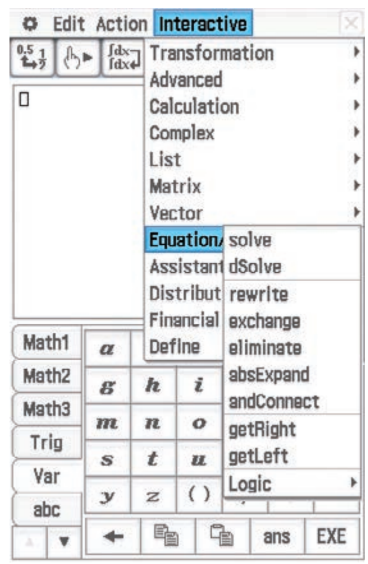
By specifying the variable, the expanded expression will be ordered in decreasing powers of that variable. Symbolic expressions can only be expanded for an appropriate domain.



Using the Casio ClassPad

This section explores the $\sqrt{\alpha}$ (Main) application.

The **Interactive** menu is easiest to use with the stylus and the soft keyboards Math1, Math2 and Math3.



Solve

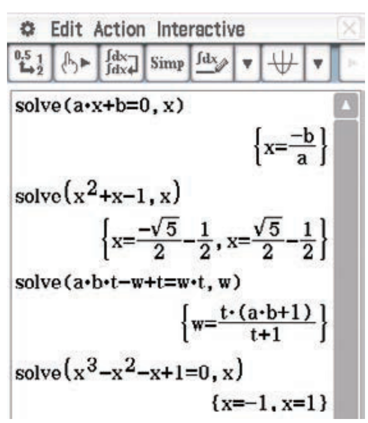
This is used to solve equations and inequalities.

The variables x, y and z are found on the hard keyboard. Other variables may be entered using the Var keyboard. Variables are shown in bold italics.

Note: The abc keyboard allows you to type text; however, the letters are not always recognised as variables. If you use the abc keyboard for variables, then you must type $a \times x$, for example, because ax will be treated as text.

Examples:

- Enter $ax + b = 0$ and highlight it with the stylus. Go to **Interactive** > **Equation/Inequality** > **solve** and ensure the variable selected is x.
- Enter $x^2 + x - 1$ and follow the same instructions as above. Note that '= 0' has been omitted in this example. It is not necessary to enter the right-hand side of an equation if it is zero.
- To solve $abt - w + t = wt$ for w, select w as the variable.
- Solve $x^3 - x^2 - x + 1 = 0$ for x.



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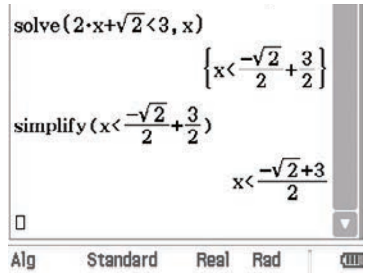
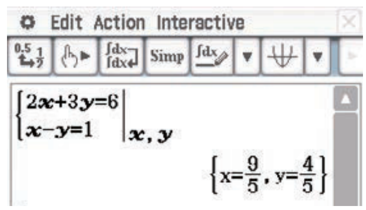
More examples:

- Solve $2x + \sqrt{2} < 3$ for x.



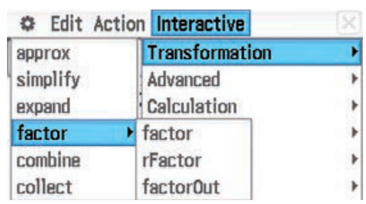
Note: For the square root, use [√■] from [Math1].
The inequality signs (<, >, ≤, ≥) are in [Math3].

- If the answer is not in the form required, it is often possible to cut and paste it into the next entry line and use **Interactive** > **Transformation** > **simplify** as shown on the right.
- To solve a pair of simultaneous equations, tap [{■] from the [Math1] keyboard and enter the equations and variables as shown.
- For more than two equations, tap [{■] until the required number of equations is displayed.



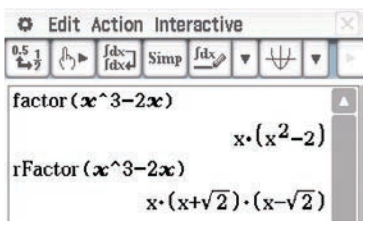
Factor

To factorise is to write an expression as a product of simpler expressions. This command is found in **Interactive** > **Transformation** > **factor**.



Examples:

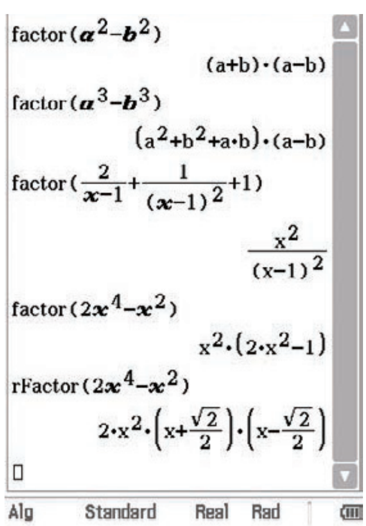
- To factorise $x^3 - 2x$ over the rational numbers, use **factor**.
- To factorise over the real numbers, use **rFactor**.



More examples:

- Factorise $a^2 - b^2$.
- Factorise $a^3 - b^3$.
- Factorise $\dfrac{2}{x-1} + \dfrac{1}{(x-1)^2} + 1$.
- Factorise $2x^4 - x^2$ over the rationals.
- Factorise $2x^4 - x^2$ over the reals.

This command can also be used to give the prime decomposition (factors) of integers.



ISBN 978-1-316-63608-4

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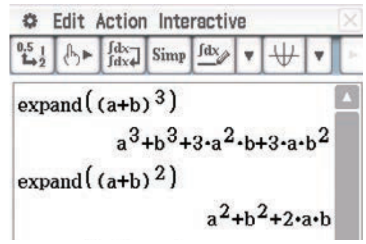
Expand

An expression can be expanded out by using **Interactive** > **Transformation** > **expand**.

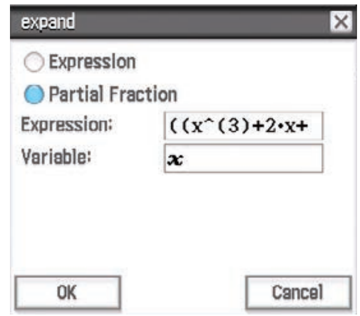
Examples:

- Expand $(a + b)^3$.
- Expand $(a + b)^2$.



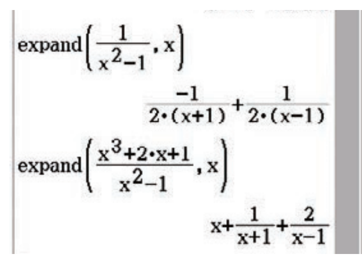
This command can also be used to form partial fractions.

In this case, enter and highlight the expression, go to **Interactive** > **Transformation** > **expand**, select the **Partial Fraction** option as shown on the right, and set the variable as x.



Examples:

- Expand $\dfrac{1}{x^2 - 1}$.
- Expand $\dfrac{x^3 + 2x + 1}{x^2 - 1}$.

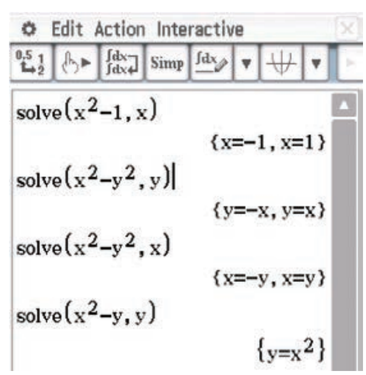


Zeroes

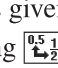
To find the zeroes of an expression in Main $\sqrt{\alpha}$, select **Interactive** > **Equation/Inequality** > **solve** and ensure that you set the variable. The calculator assumes that you are solving an equation for which one side is zero.

Examples:

- Zeroes of $x^2 - 1$ for x.
- Zeroes of $x^2 - y^2$ for y.
- Zeroes of $x^2 - y^2$ for x.
- Zeroes of $x^2 - y$ for y.
- Zeroes of $x^2 - 4x + 8$ for x. No solutions.
- Zeroes of $x^2 - 4x + 1$ for x. Two solutions.
- Zeroes of $x^2 - 4x + 4$ for x. One solution.



Approximate

Switch mode in the status bar to Decimal. If an answer is given in Standard (exact) mode, it can be converted by highlighting the answer and tapping  in the toolbar.

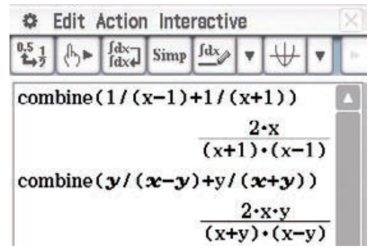
ISBN 978-1-316-63608-4

Combining fractions

This command returns the answer as a single fraction with the denominator in factored form.

Examples:

- Enter and highlight $1/(x-1)+1/(x+1)$. Then select **Interactive** > **Transformation** > **combine**.
- Enter and highlight $y/(x-y)+y/(x+y)$. Then select **Interactive** > **Transformation** > **combine**.



Exercise 1I

This exercise provides practice in some of the skills associated with a CAS calculator. Other exercises in this chapter can be attempted with CAS, but it is recommended that you also use this chapter to develop your 'by hand' skills.

1 Solve each of the following equations for x:

a $\dfrac{a(a-x)}{b}-\dfrac{b(b+x)}{a}=x$

b $2(x-3)+(x-2)(x-4)=x(x+1)-33$

c $\dfrac{x+a}{x+b}=1-\dfrac{x}{x-b}$

d $\dfrac{x+a}{x-c}+\dfrac{x+c}{x-a}=2$

2 Factorise each of the following:

a $x^2y^2-x^2-y^2+1$

b $x^3-2-x+2x^2$

c $a^4-8a^2b-48b^2$

d $a^2+2bc-(c^2+2ab)$

3 Solve each of the following pairs of simultaneous equations for x and y:

a $axy+b=(a+c)y$

$bxy+a=(b+c)y$

b $x(b-c)+by-c=0$

$y(c-a)-ax+c=0$

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Chapter summary

AS

Nrich

- Indices
  - $a^m \times a^n = a^{m+n}$
  - $a^m \div a^n = a^{m-n}$
  - $(a^m)^n = a^{mn}$
  - $(ab)^n = a^n b^n$
  - $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
  - $a^{-n} = \frac{1}{a^n}$
  - $a^0 = 1$
  - $a^{\frac{1}{n}} = \sqrt[n]{a}$
- A number is expressed in **standard form** or **scientific notation** when written as a product of a number between 1 and 10 and an integer power of 10; e.g. 1.5×10^8
- Linear equations

  First identify the steps done to construct an equation; the equation is then solved by 'undoing' these steps. This is achieved by doing 'the opposite' in 'reverse order'.

  e.g.: Solve $3x + 4 = 16$ for x.

  Note that x has been multiplied by 3 and then 4 has been added.

  Subtract 4 from both sides: $3x = 12$

  Divide both sides by 3: $x = 4$
- An equation that states a relationship between two or more quantities is called a **formula**; e.g. the area of a circle is given by $A = \pi r^2$. The value of A, the subject of the formula, may be found by substituting a given value of r and the value of π.

  A formula can be transposed to make a different variable the subject by using a similar procedure to solving linear equations, i.e. whatever has been done to the variable required is 'undone'.
- A **literal equation** is solved using the same techniques as for a numerical equation: transpose the literal equation to make the required variable the subject.

Short-answer questions

1 Simplify the following:

a $(x^3)^4$ **b** $(y^{-12})^{\frac{3}{4}}$ **c** $3x^{\frac{3}{2}} \times -5x^4$ **d** $(x^3)^{\frac{4}{3}} \times x^{-5}$

2 Express the product $23 \times 10^{-6} \times 14 \times 10^{15}$ in standard form.

3 Simplify the following:

a $\frac{3x}{5} + \frac{y}{10} - \frac{2x}{5}$ **b** $\frac{4}{x} - \frac{7}{y}$ **c** $\frac{5}{x+2} + \frac{2}{x-1}$

d $\frac{3}{x+2} + \frac{4}{x+4}$ **e** $\frac{5x}{x+4} + \frac{4x}{x-2} - \frac{5}{2}$ **f** $\frac{3}{x-2} - \frac{6}{(x-2)^2}$

4 Simplify the following:

a $\frac{x+5}{2x-6} \div \frac{x^2+5x}{4x-12}$ **b** $\frac{3x}{x+4} \div \frac{12x^2}{x^2-16}$

c $\frac{x^2-4}{x-3} \times \frac{3x-9}{x+2} \div \frac{9}{x+2}$ **d** $\frac{4x+20}{9x-6} \times \frac{6x^2}{x+5} \div \frac{2}{3x-2}$

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5 Phoebe has bought a 3 terabyte external hard drive (3×10^{12} bytes).

a How many photos could be stored on this hard drive if the average size of a photo is 1.5 megabytes (1.5×10^6 bytes)?

b How long would it take to copy 3 terabytes of data onto the hard drive if the data transfer rate is 120 Mbps (120×10^6 bits per second, where 1 byte equals 8 bits)?

6 Swifts Creek Soccer Team has played 54 matches over the past three seasons. They have drawn one-third of their games and won twice as many games as they have lost. How many games have they lost?

7 A music store specialises in three types of CDs: classical, blues and heavy metal. In one week they sold a total of 420 CDs. They sold 10% more classical than blues, while sales of heavy metal CDs constituted 50% more than the combined sales of classical and blues CDs. How many of each type of CD did they sell?

8 The volume, V, of a cylinder is given by the formula $V = \pi r^2 h$, where r is the radius of the base and h is the height of the cylinder.

a Find the volume of a cylinder with base radius 5 cm and height 12 cm.

b Transpose the formula to make h the subject and hence find the height of a cylinder with a base radius of 5 cm and a volume of 585 cm^3.

c Transpose the formula to make r the subject and hence find the radius of a cylinder with a height of 6 cm and a volume of 768 cm^3.

9 Solve for x:

a $xy + ax = b$

b $\dfrac{a}{x} + \dfrac{b}{x} = c$

c $\dfrac{x}{a} = \dfrac{x}{b} + 2$

d $\dfrac{a - dx}{d} + b = \dfrac{ax + d}{b}$

10 Simplify:

a $\dfrac{p}{p+q} + \dfrac{q}{p-q}$

b $\dfrac{1}{x} - \dfrac{2y}{xy - y^2}$

c $\dfrac{x^2 + x - 6}{x + 1} \times \dfrac{2x^2 + x - 1}{x + 3}$

d $\dfrac{2a}{2a + b} \times \dfrac{2ab + b^2}{ba^2}$

11 A is three times as old as B. In three years' time, B will be three times as old as C. In fifteen years' time, A will be three times as old as C. What are their present ages?

12 **a** Solve the following simultaneous equations for a and b:

$$a - 5 = \frac{1}{7}(b + 3) \qquad b - 12 = \frac{1}{5}(4a - 2)$$

b Solve the following simultaneous equations for x and y:

$$(p - q)x + (p + q)y = (p + q)^2$$
$$qx - py = q^2 - pq$$

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13 A man has to travel 50 km in 4 hours. He does it by walking the first 7 km at x km/h, cycling the next 7 km at $4x$ km/h and motoring the remainder at $(6x + 3)$ km/h. Find x.

14 Simplify each of the following:

a $2n^2 \times 6nk^2 \div (3n)$ **b** $\dfrac{8c^2x^3y}{6a^2b^3c^3} \div \dfrac{\frac{1}{2}xy}{15abc^2}$

15 Solve the equation $\dfrac{x+5}{15} - \dfrac{x-5}{10} = 1 + \dfrac{2x}{15}$.

Multiple-choice questions

1 For non-zero values of x and y, if $5x + 2y = 0$, then the ratio $\dfrac{y}{x}$ is equal to

A $-\dfrac{5}{2}$ **B** $-\dfrac{2}{5}$ **C** $\dfrac{2}{5}$ **D** 1 **E** $\dfrac{5}{4}$

2 The solution of the simultaneous equations $3x + 2y = 36$ and $3x - y = 12$ is

A $x = \dfrac{20}{3},\ y = 8$ **B** $x = 2,\ y = 0$ **C** $x = 1,\ y = -3$

D $x = \dfrac{20}{3},\ y = 6$ **E** $x = \dfrac{3}{2},\ y = -\dfrac{3}{2}$

3 The solution of the equation $t - 9 = 3t - 17$ is

A $t = -4$ **B** $t = \dfrac{11}{2}$ **C** $t = 4$ **D** $t = 2$ **E** $t = -2$

4 If $m = \dfrac{n-p}{n+p}$, then $p =$

A $\dfrac{n(1-m)}{1+m}$ **B** $\dfrac{n(m-1)}{1+m}$ **C** $\dfrac{n(1+m)}{1-m}$ **D** $\dfrac{n(1+m)}{m-1}$ **E** $\dfrac{m(n-1)}{m+1}$

5 $\dfrac{3}{x-3} - \dfrac{2}{x+3} =$

A 1 **B** $\dfrac{x+15}{x^2-9}$ **C** $\dfrac{15}{x-9}$ **D** $\dfrac{x+3}{x^2-9}$ **E** $-\dfrac{1}{9}$

6 $9x^2y^3 \div (15(xy)^3)$ is equal to

A $\dfrac{9x}{15}$ **B** $\dfrac{18xy}{5}$ **C** $\dfrac{3y}{5x}$ **D** $\dfrac{3x}{5}$ **E** $\dfrac{3}{5x}$

7 Transposing the formula $V = \dfrac{1}{3}h(\ell + w)$ gives $\ell =$

A $\dfrac{hw}{3V}$ **B** $\dfrac{3V}{h} - w$ **C** $\dfrac{3V - 2w}{h}$ **D** $\dfrac{3Vh}{2} - w$ **E** $\dfrac{1}{3}h(V + w)$

8 $\dfrac{(3x^2y^3)^2}{2x^2y} =$

A $\dfrac{9}{2}x^2y^7$ **B** $\dfrac{9}{2}x^2y^5$ **C** $\dfrac{9}{2}x^6y^7$ **D** $\dfrac{9}{2}x^6y^6$ **E** $\dfrac{9}{2}x^2y^4$

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Review

9 If X is 50% greater than Y and Y is 20% less than Z, then

A X is 30% greater than Z **B** X is 20% greater than Z **C** X is 20% less than Z
D X is 10% less than Z **E** X is 10% greater than Z

10 The average of two numbers is $5x + 4$. One of the numbers is x. The other number is

A $4x + 4$ **B** $9x + 8$ **C** $9x + 4$ **D** $10x + 8$ **E** $3x + 1$

Extended-response questions

1 Jack cycles home from work, a distance of $10x$ km. Benny leaves at the same time and drives the $40x$ km to his home.

a Write an expression in terms of x for the time taken for Jack to reach home if he cycles at an average speed of 8 km/h.

b Write an expression in terms of x for the time taken for Benny to reach home if he drives at an average speed of 70 km/h.

c In terms of x, find the difference in times of the two journeys.

d If Jack and Benny arrive at their homes 30 minutes apart:

i find x, correct to three decimal places

ii find the distance from work of each home, correct to the nearest kilometre.

2 Sam's plastic dinghy has sprung a leak and water is pouring in the hole at a rate of 27 000 cm^3 per minute. He grabs a cup and frantically starts bailing the water out at a rate of 9000 cm^3 per minute. The dinghy is shaped like a circular prism (cylinder) with a base radius of 40 cm and a height of 30 cm.

a How fast is the dinghy filling with water?

b Write an equation showing the volume of water, V cm^3, in the dinghy after t minutes.

c Find an expression for the depth of water, h cm, in the dinghy after t minutes.

d If Sam is rescued after 9 minutes, is this before or after the dinghy has completely filled with water?

3 Henry and Thomas Wong collect basketball cards. Henry has five-sixths the number of cards that Thomas has. The Wright family also collect cards. George Wright has half as many cards again as Thomas, Sally Wright has 18 fewer than Thomas, and Zeb Wright has one-third the number Thomas has.

a Write an expression for each child's number of cards in terms of the number Thomas has.

b The Wright family owns six more cards than the Wong family. Write an equation representing this information.

c Solve the equation from part **b** and use the result to find the number of cards each child has collected.

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4 The gravitational force between two objects, F N, is given by the formula

$$F = \frac{6.67 \times 10^{-11} m_1 m_2}{r^2}$$

where m_1 and m_2 are the masses (in kilograms) of the two objects and r is the distance (in metres) between them.

a What is the gravitational force between two objects each weighing 200 kg if they are 12 m apart? Express the answer in standard form (to two significant figures).

b Transpose the above formula to make m_1 the subject.

c The gravitational force between a planet and an object 6.4×10^6 m away from the centre of the planet is found to be 2.4×10^4 N. If the object has a mass of 1500 kg, calculate the approximate mass of the planet, giving the answer in standard form (to two significant figures).

5 A water storage reservoir is 3 km wide, 6 km long and 30 m deep. (The water storage reservoir is assumed to be a cuboid.)

a Write an equation to show the volume of water, V m^3, in the reservoir when it is d metres full.

b Calculate the volume of water, V_F m^3, in the reservoir when it is completely filled.

The water flows from the reservoir down a long pipe to a hydro-electric power station in a valley below. The amount of energy, E J, that can be obtained from a full reservoir is given by the formula

$$E = kV_F h$$

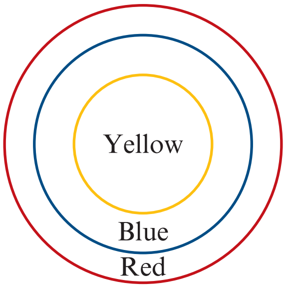
where k is a constant and h m is the length of the pipe.

c Find k, given that $E = 1.06 \times 10^{15}$ when $h = 200$, expressing the answer in standard form correct to three significant figures.

d How much energy could be obtained from a full reservoir if the pipe was 250 m long?

e If the rate of water falling through the pipe is 5.2 m^3/s, how many days without rain could the station operate before emptying an initially full reservoir?

6 A new advertising symbol is to consist of three concentric circles as shown, with the outer circle having a radius of 10 cm. It is desired that the three coloured regions cover the same area. Find the radius of the innermost circle in the figure shown.



7 Temperatures in Fahrenheit (F) can be converted to Celsius (C) by the formula

$$C = \frac{5}{9}(F - 32)$$

Find the temperature which has the same numerical value in both scales.

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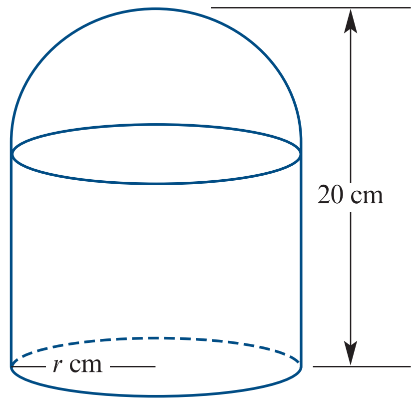
Review

8 A cyclist goes up a long slope at a constant speed of 15 km/h. He turns around and comes down the slope at a constant speed of 40 km/h. Find his average speed over a full circuit.

9 A container has a cylindrical base and a hemispherical top, as shown in the figure. The height of the container is 20 cm and its capacity is to be exactly 2 litres. Let r cm be the radius of the base.



a Express the height of the cylinder, h cm, in terms of r.

b **i** Express the volume of the container in terms of r.

ii Find r and h if the volume is 2 litres.

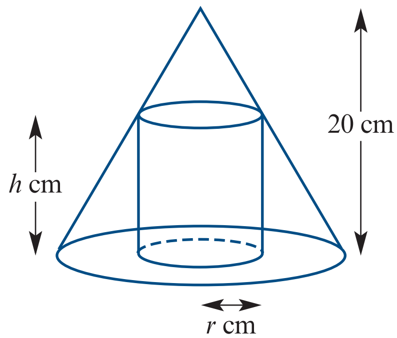
10 **a** Two bottles contain mixtures of wine and water. In bottle A there is two times as much wine as water. In bottle B there is five times as much water as wine. Bottle A and bottle B are used to fill a third bottle, which has a capacity of 1 litre. How much liquid must be taken from each of bottle A and bottle B if the third bottle is to contain equal amounts of wine and water?

b Repeat for the situation where the ratio of wine to water in bottle A is $1 : 2$ and the ratio of wine to water in bottle B is $3 : 1$.

c Generalise the result for the ratio $m : n$ in bottle A and $p : q$ in bottle B.

11 A cylinder is placed so as to fit into a cone as shown in the diagram.

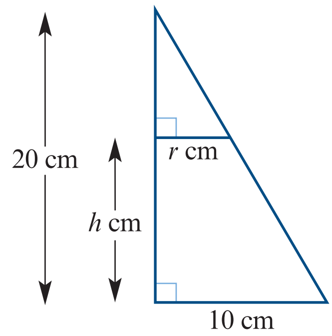
The cone has a height of 20 cm and a base radius of 10 cm. The cylinder has a height of h cm and a base radius of r cm.



a Use similar triangles to find h in terms of r.

b The volume of the cylinder is given by the formula $V = \pi r^2 h$. Find the volume of the cylinder in terms of r.

c Use a CAS calculator to find the values of r and h for which the volume of the cylinder is 500 cm^3.



ISBN 978-1-316-63608-4

2 Number systems and sets

Objectives

- To understand and use **set notation**, including the symbols ∈, ⊆, ∪, ∩, ∅ and ξ.
- To be able to identify sets of numbers, including the natural numbers, integers, rational numbers, irrational numbers and real numbers.
- To interpret subsets of the real numbers defined using the **modulus function**.
- To know and apply the rules for working with **surds**, including:
  - simplification of surds
  - rationalisation of surds.
- To know and apply the definitions of **factor**, **prime**, **highest common factor** and **lowest common multiple**.
- To be able to solve **linear Diophantine equations**.
- To be able to solve problems with sets.

This chapter introduces set notation and discusses sets of numbers and their properties. Set notation is used widely in mathematics and in this book it is employed where appropriate. In this chapter we discuss natural numbers, integers and rational numbers, and then continue on to consider irrational numbers.

Irrational numbers such as $\sqrt{2}$ naturally arise when applying Pythagoras' theorem. When solving a quadratic equation, using the method of completing the square or the quadratic formula, we obtain answers such as $x = \frac{1}{2}(1 \pm \sqrt{5})$. These numbers involve surds.

Since these numbers are irrational, we cannot express them in exact form using decimals or fractions. Sometimes we may wish to approximate them using decimals, but mostly we prefer to leave them in exact form. Thus we need to be able to manipulate these types of numbers and to simplify combinations of them which arise when solving a problem.

ISBN 978-1-316-63608-4

2A Set notation

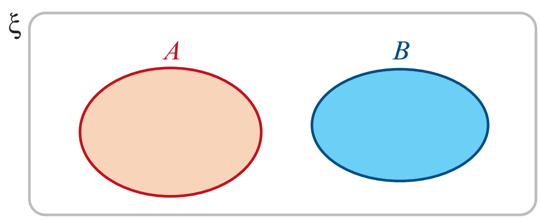
A **set** is a general name for any collection of things or numbers. There must be a way of deciding whether any particular object is a member of the set or not. This may be done by referring to a list of the members of the set or a statement describing them.

For example: $A = \{-3, 3\} = \{\, x : x^2 = 9 \,\}$

Note: $\{\, x : \ldots \}$ is read as 'the set of all x such that ...'.

- The symbol $\in$ means 'is a member of' or 'is an element of'.
  For example: $3 \in$ {prime numbers} is read '3 is a member of the set of prime numbers'.
- The symbol $\notin$ means 'is not a member of' or 'is not an element of'.
  For example: $4 \notin$ {prime numbers} is read '4 is not a member of the set of prime numbers'.
- Two sets are **equal** if they contain exactly the same elements, not necessarily in the same order. For example: if A = {prime numbers less than 10} and $B = \{2, 3, 5, 7\}$, then $A = B$.
- The set with no elements is called the **empty set** and is denoted by $\varnothing$.
- The **universal set** will be denoted by ξ. The universal set is the set of all elements which are being considered.
- If all the elements of a set B are also elements of a set A, then the set B is called a **subset** of A. This is written $B \subseteq A$. For example: $\{a, b, c\} \subseteq \{a, b, c, d, e, f, g\}$ and $\{3, 9, 27\} \subseteq$ {multiples of 3}. We note also that $A \subseteq A$ and $\varnothing \subseteq A$.

Venn diagrams are used to illustrate sets. For example, the diagram on the right shows two subsets A and B of a universal set ξ such that A and B have no elements in common. Two such sets are said to be **disjoint**.



► The union of two sets

The set of all the elements that are members of set A or set B (or both) is called the **union** of A and B. The union of A and B is written $A \cup B$.

Example 1

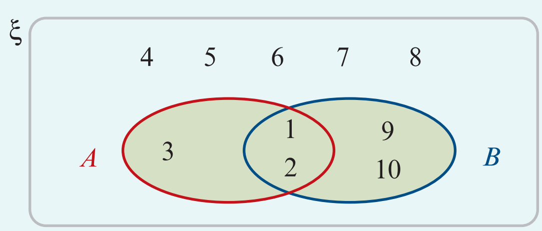
Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3\}$ and $B = \{1, 2, 9, 10\}$.

Find $A \cup B$ and illustrate on a Venn diagram.

Solution

$A \cup B = \{1, 2, 3, 9, 10\}$

The shaded area illustrates $A \cup B$.



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► The intersection of two sets

The set of all the elements that are members of both set A and set B is called the **intersection** of A and B. The intersection of A and B is written $A \cap B$.

Example 2

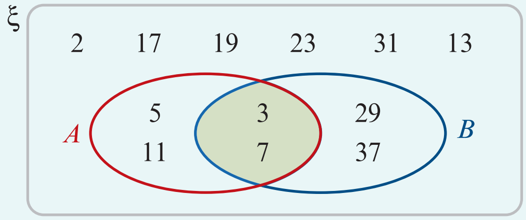
Let ξ = {prime numbers less than 40}, $A = \{3, 5, 7, 11\}$ and $B = \{3, 7, 29, 37\}$.

Find $A \cap B$ and illustrate on a Venn diagram.

Solution

$A \cap B = \{3, 7\}$

The shaded area illustrates $A \cap B$.



► The complement of a set

The **complement** of a set A is the set of all elements of ξ that are not members of A. The complement of A is denoted by A'.

If ξ = {students at Highland Secondary College} and A = {students with blue eyes}, then A' is the set of all students at Highland Secondary College who do not have blue eyes.

Similarly, if $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{1, 3, 5, 7, 9\}$, then $A' = \{2, 4, 6, 8, 10\}$.

Example 3

Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

A = {odd numbers} = $\{1, 3, 5, 7, 9\}$

B = {multiples of 3} = $\{3, 6, 9\}$

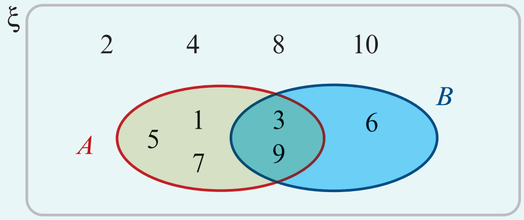
a Show these sets on a Venn diagram.

b Use the diagram to list the following sets:

i A' **ii** B' **iii** $A \cup B$ **iv** the complement of $A \cup B$, i.e. $(A \cup B)'$ **v** $A' \cap B'$

Solution

a



b From the diagram:

i $A' = \{2, 4, 6, 8, 10\}$

ii $B' = \{1, 2, 4, 5, 7, 8, 10\}$

iii $A \cup B = \{1, 3, 5, 6, 7, 9\}$

iv $(A \cup B)' = \{2, 4, 8, 10\}$

v $A' \cap B' = \{2, 4, 8, 10\}$

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▶ Finite and infinite sets

When all the elements of a set may be counted, the set is called a **finite** set. For example, the set A = {months of the year} is finite. The number of elements of a set A will be denoted $|A|$. In this example, $|A| = 12$. If C = {letters of the alphabet}, then $|C| = 26$.

Sets which are not finite are called **infinite** sets. For example, the set of real numbers, $\mathbb{R}$, and the set of integers, $\mathbb{Z}$, are infinite sets.

Section summary

- If x is an element of a set A, we write $x \in A$.
- If x is not an element of a set A, we write $x \notin A$.
- The **empty set** is denoted by $\varnothing$ and the **universal set** by ξ.
- If every element of B is an element of A, we say B is a **subset** of A and write $B \subseteq A$.
- The set $A \cup B$ is the **union** of A and B, where $x \in A \cup B$ if and only if $x \in A$ or $x \in B$.
- The set $A \cap B$ is the **intersection** of A and B, where $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.
- The **complement** of A, denoted by A', is the set of all elements of ξ that are not in A.
- If two sets A and B have no elements in common, we say that they are **disjoint** and write $A \cap B = \varnothing$.

Exercise 2A

Example 1 **1** Let $\xi = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3, 5\}$ and $B = \{2, 4\}$.
Show these sets on a Venn diagram and use the diagram to find:

a A' **b** B' **c** $A \cup B$ **d** $(A \cup B)'$ **e** $A' \cap B'$

Example 2 **2** Let ξ = {natural numbers less than 17}, P = {multiples of 3} and Q = {even numbers}.
Show these sets on a Venn diagram and use it to find:

a P' **b** Q' **c** $P \cup Q$ **d** $(P \cup Q)'$ **e** $P' \cap Q'$

Example 3 **3** Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, A = {multiples of 4} and B = {even numbers}.
Show these sets on a Venn diagram and use this diagram to list the sets:

a A' **b** B' **c** $A \cup B$ **d** $(A \cup B)'$ **e** $A' \cap B'$

4 Let ξ = {natural numbers from 10 to 25}, P = {multiples of 4} and Q = {multiples of 5}.
Show these sets on a Venn diagram and use this diagram to list the sets:

a P' **b** Q' **c** $P \cup Q$ **d** $(P \cup Q)'$ **e** $P' \cap Q'$

5 Let $\xi = \{p, q, r, s, t, u, v, w\}$, $X = \{r, s, t, w\}$ and $Y = \{q, s, t, u, v\}$.
Show ξ, X and Y on a Venn diagram, entering all members. Hence list the sets:

a X' **b** Y' **c** $X' \cap Y'$ **d** $X' \cup Y'$ **e** $X \cup Y$ **f** $(X \cup Y)'$

Which two sets are equal?

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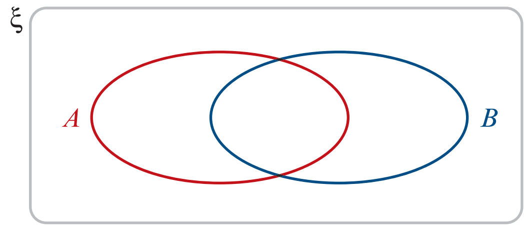
6 Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, X = {factors of 12} and Y = {even numbers}. Show ξ, X and Y on a Venn diagram, entering all members. Hence list the sets:

a X' **b** Y' **c** $X' \cup Y'$ **d** $(X \cap Y)'$ **e** $X \cup Y$ **f** $(X \cup Y)'$

Which two sets are equal?

7 Draw this diagram six times. Use shading to illustrate each of the following sets:

a A' **b** B' **c** $A' \cap B'$
d $A' \cup B'$ **e** $A \cup B$ **f** $(A \cup B)'$



8 Let ξ = {different letters in the word *GENERAL*},
A = {different letters in the word *ANGEL*},
B = {different letters in the word *LEAN*}

Show these sets on a Venn diagram and use this diagram to list the sets:

a A' **b** B' **c** $A \cap B$ **d** $A \cup B$ **e** $(A \cup B)'$ **f** $A' \cup B'$

9 Let ξ = {different letters in the word *MATHEMATICS*}
A = {different letters in the word *ATTIC*}
B = {different letters in the word *TASTE*}

Show ξ, A and B on a Venn diagram, entering all the elements. Hence list the sets:

a A' **b** B' **c** $A \cap B$ **d** $(A \cup B)'$ **e** $A' \cup B'$ **f** $A' \cap B'$

2B Sets of numbers

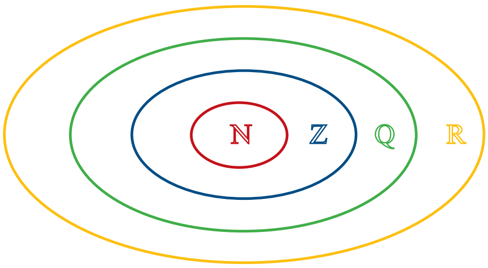
Recall that the elements of $\{1, 2, 3, 4, \ldots\}$ are called **natural numbers**, and the elements of $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ are called **integers**.

The numbers of the form $\frac{p}{q}$, with p and q integers, $q \neq 0$, are called **rational numbers**.

The real numbers which are not rational are called **irrational**. Some examples of irrational numbers are $\sqrt{2}$, $\sqrt{3}$, π, $\pi + 2$ and $\sqrt{6} + \sqrt{7}$. These numbers cannot be written in the form $\frac{p}{q}$, for integers p, q; the decimal representations of these numbers do not terminate or repeat.

- The set of real numbers is denoted by $\mathbb{R}$.
- The set of rational numbers is denoted by $\mathbb{Q}$.
- The set of integers is denoted by $\mathbb{Z}$.
- The set of natural numbers is denoted by $\mathbb{N}$.

It is clear that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, and this may be represented by the diagram on the right.



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We can use set notation to describe subsets of the real numbers.

For example:

- $\{ x : 0 < x < 1 \}$ is the set of all real numbers strictly between 0 and 1
- $\{ x : x > 0,\ x \in \mathbb{Q} \}$ is the set of all positive rational numbers
- $\{ 2n : n = 0, 1, 2, \dots \}$ is the set of all non-negative even numbers.

The set of all ordered pairs of real numbers is denoted by $\mathbb{R}^2$. That is,

$$\mathbb{R}^2 = \{ (x, y) : x, y \in \mathbb{R} \}$$

This set is known as the **Cartesian product** of $\mathbb{R}$ with itself.

Rational numbers

Every rational number can be expressed as a terminating or recurring decimal.

To find the decimal representation of a rational number $\frac{m}{n}$, perform the division $m \div n$.

For example, to find the decimal representation of $\frac{3}{7}$, divide $3.0000000\dots$ by 7.

$$\begin{array}{r|l} & 0.\ 4\ \ 2\ \ 8\ \ 5\ \ 7\ \ 1\ \ 4\ \dots \\ \hline 7 & 3.\ {}^{3}0\ {}^{2}0\ {}^{6}0\ {}^{4}0\ {}^{5}0\ {}^{1}0\ {}^{3}0\ \dots \end{array}$$

Therefore $\frac{3}{7} = 0.\dot{4}2857\dot{1}$.

Theorem

Every rational number can be written as a terminating or recurring decimal.

Proof Consider any two natural numbers m and n. At each step in the division of m by n, there is a remainder. If the remainder is 0, then the division algorithm stops and the decimal is a terminating decimal.

If the remainder is never 0, then it must be one of the numbers $1, 2, 3, \dots, n - 1$. (In the above example, $n = 7$ and the remainders can only be 1, 2, 3, 4, 5 and 6.) Hence the remainder must repeat after at most $n - 1$ steps.

Further examples:

$$\frac{1}{2} = 0.5, \quad \frac{1}{5} = 0.2, \quad \frac{1}{10} = 0.1, \quad \frac{1}{3} = 0.\dot{3}, \quad \frac{1}{7} = 0.\dot{1}4285\dot{7}$$

Theorem

A real number has a terminating decimal representation if and only if it can be written as

$$\frac{m}{2^{\alpha} \times 5^{\beta}}$$

for some $m \in \mathbb{Z}$ and some $\alpha, \beta \in \mathbb{N} \cup \{0\}$.

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Proof Assume that $x = \dfrac{m}{2^\alpha \times 5^\beta}$ with $\alpha \geq \beta$. Multiply the numerator and denominator by $5^{\alpha-\beta}$. Then

$$x = \frac{m \times 5^{\alpha-\beta}}{2^\alpha \times 5^\alpha} = \frac{m \times 5^{\alpha-\beta}}{10^\alpha}$$

and so x can be written as a terminating decimal. The case $\alpha < \beta$ is similar.

Conversely, if x can be written as a terminating decimal, then there is $m \in \mathbb{Z}$ and $\alpha \in \mathbb{N} \cup \{0\}$ such that $x = \dfrac{m}{10^\alpha} = \dfrac{m}{2^\alpha \times 5^\alpha}$.

The method for finding a rational number $\dfrac{m}{n}$ from its decimal representation is demonstrated in the following example.

Example 4

Write each of the following in the form $\dfrac{m}{n}$, where m and n are integers:

a 0.05 **b** $0.\dot{4}2857\dot{1}$

Solution

a $0.05 = \dfrac{5}{100} = \dfrac{1}{20}$

b We can write

$$0.\dot{4}2857\dot{1} = 0.428571428571\ldots \qquad (1)$$

Multiply both sides by 10^6:

$$0.\dot{4}2857\dot{1} \times 10^6 = 428571.428571428571\ldots \qquad (2)$$

Subtract (1) from (2):

$$0.\dot{4}2857\dot{1} \times (10^6 - 1) = 428571$$

$$\therefore \quad 0.\dot{4}2857\dot{1} = \frac{428571}{10^6 - 1} = \frac{3}{7}$$

► Real numbers

The set of real numbers is made up of two important subsets: the **algebraic numbers** and the **transcendental numbers**.

An algebraic number is a solution to a polynomial equation of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0, \quad \text{where } a_0, a_1, \ldots, a_n \text{ are integers}$$

Every rational number is algebraic. The irrational number $\sqrt{2}$ is algebraic, as it is a solution of the equation

$$x^2 - 2 = 0$$

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It can be shown that π is not an algebraic number; it is a transcendental number. The proof is too difficult to be given here.

The proof that $\sqrt{2}$ is irrational is presented in Chapter 6.

Interval notation

Among the most important subsets of $\mathbb{R}$ are the **intervals**. The following is an exhaustive list of the various types of intervals and the standard notation for them. We suppose that a and b are real numbers with $a < b$.

$$(a, b) = \{ x : a < x < b \} \qquad [a, b] = \{ x : a \le x \le b \}$$
$$(a, b] = \{ x : a < x \le b \} \qquad [a, b) = \{ x : a \le x < b \}$$
$$(a, \infty) = \{ x : a < x \} \qquad [a, \infty) = \{ x : a \le x \}$$
$$(-\infty, b) = \{ x : x < b \} \qquad (-\infty, b] = \{ x : x \le b \}$$

Intervals may be represented by diagrams as shown in Example 5.

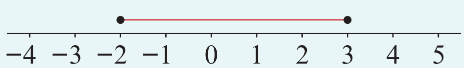
Example 5

Illustrate each of the following intervals of real numbers:

a $[-2, 3]$ **b** $(-3, 4]$ **c** $(-\infty, 2]$ **d** $(-2, 4)$ **e** $(-3, \infty)$

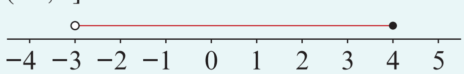
Solution

a $[-2, 3]$



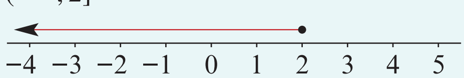
Explanation: The square brackets indicate that the endpoints are included; this is shown with closed circles.

b $(-3, 4]$



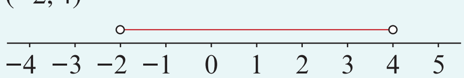
The round bracket indicates that the left endpoint is not included; this is shown with an open circle. The right endpoint is included.

c $(-\infty, 2]$



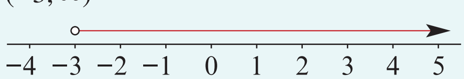
The symbol $-\infty$ indicates that the interval continues indefinitely (i.e. forever) to the left; it is read as 'negative infinity'. The right endpoint is included.

d $(-2, 4)$



Both brackets are round; the endpoints are not included.

e $(-3, \infty)$



The symbol ∞ indicates that the interval continues indefinitely (i.e. forever) to the right; it is read as 'infinity'. The left endpoint is not included.

Notes:

- The 'closed' circle (●) indicates that the number is included.
- The 'open' circle (○) indicates that the number is not included.

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The following are subsets of the real numbers for which we have special notation:

- Positive real numbers: $\mathbb{R}^+ = \{ x : x > 0 \}$
- Negative real numbers: $\mathbb{R}^- = \{ x : x < 0 \}$
- Real numbers excluding zero: $\mathbb{R} \setminus \{0\}$

Section summary

- Sets of numbers
  - Real numbers: $\mathbb{R}$
  - Rational numbers: $\mathbb{Q}$
  - Integers: $\mathbb{Z}$
  - Natural numbers: $\mathbb{N}$
- For real numbers a and b with $a < b$, we can consider the following **intervals**:

$$(a, b) = \{ x : a < x < b \} \qquad [a, b] = \{ x : a \le x \le b \}$$
$$(a, b] = \{ x : a < x \le b \} \qquad [a, b) = \{ x : a \le x < b \}$$
$$(a, \infty) = \{ x : a < x \} \qquad [a, \infty) = \{ x : a \le x \}$$
$$(-\infty, b) = \{ x : x < b \} \qquad (-\infty, b] = \{ x : x \le b \}$$

Exercise 2B

1 **a** Is the sum of two rational numbers also rational?
b Is the product of two rational numbers also rational?
c Is the quotient of two rational numbers also rational (if defined)?

2 **a** Is the sum of two irrational numbers always irrational?
b Is the product of two irrational numbers always irrational?
c Is the quotient of two irrational numbers always irrational?

Example 4

3 Write each of the following in the form $\frac{m}{n}$, where m and n are integers:

a 0.45 **b** $0.\dot{2}\dot{7}$ **c** 0.12
d $0.\dot{2}8571\dot{4}$ **e** $0.\dot{3}\dot{6}$ **f** $0.\dot{2}$

4 Give the decimal representation of each of the following rational numbers:

a $\frac{2}{7}$ **b** $\frac{5}{11}$ **c** $\frac{7}{20}$ **d** $\frac{4}{13}$ **e** $\frac{1}{17}$

Example 5

5 Illustrate each of the following intervals of real numbers:

a $[-1, 4]$ **b** $(-2, 2]$ **c** $(-\infty, 3]$ **d** $(-1, 5)$ **e** $(-2, \infty)$

6 Write each of the following sets using interval notation:

a $\{ x : x < 3 \}$ **b** $\{ x : x \ge -3 \}$ **c** $\{ x : x \le -3 \}$
d $\{ x : x > 5 \}$ **e** $\{ x : -2 \le x < 3 \}$ **f** $\{ x : -2 \le x \le 3 \}$
g $\{ x : -2 < x \le 3 \}$ **h** $\{ x : -5 < x < 3 \}$

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2C The modulus function

The **modulus** or **absolute value** of a real number x is denoted by $|x|$ and is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

It may also be defined as $|x| = \sqrt{x^2}$. For example: $|5| = 5$ and $|-5| = 5$.

Example 6

Evaluate each of the following:

a **i** $|-3 \times 2|$ **ii** $|-3| \times |2|$

b **i** $\left|\dfrac{-4}{2}\right|$ **ii** $\dfrac{|-4|}{|2|}$

c **i** $|-6 + 2|$ **ii** $|-6| + |2|$

Solution

a **i** $|-3 \times 2| = |-6| = 6$ **ii** $|-3| \times |2| = 3 \times 2 = 6$ Note: $|-3 \times 2| = |-3| \times |2|$

b **i** $\left|\dfrac{-4}{2}\right| = |-2| = 2$ **ii** $\dfrac{|-4|}{|2|} = \dfrac{4}{2} = 2$ Note: $\left|\dfrac{-4}{2}\right| = \dfrac{|-4|}{|2|}$

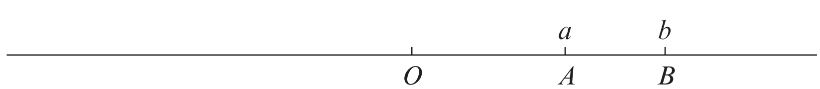
c **i** $|-6 + 2| = |-4| = 4$ **ii** $|-6| + |2| = 6 + 2 = 8$ Note: $|-6 + 2| \neq |-6| + |2|$

Properties of the modulus function

- $|ab| = |a|\,|b|$ and $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$
- $|x| = a$ implies $x = a$ or $x = -a$
- $|a + b| \leq |a| + |b|$
- If a and b are both positive or both negative, then $|a + b| = |a| + |b|$.
- If $a \geq 0$, then $|x| \leq a$ is equivalent to $-a \leq x \leq a$.
- If $a \geq 0$, then $|x - k| \leq a$ is equivalent to $k - a \leq x \leq k + a$.

▶ The modulus function as a measure of distance

Consider two points A and B on a number line:



On a number line, the distance between points A and B is $|a - b| = |b - a|$.

Thus $|x - 2| \leq 3$ can be read as 'on the number line, the distance of x from 2 is less than or equal to 3', and $|x| \leq 3$ can be read as 'on the number line, the distance of x from the origin is less than or equal to 3'. Note that $|x| \leq 3$ is equivalent to $-3 \leq x \leq 3$ or $x \in [-3, 3]$.

ISBN 978-1-316-63608-4

Cambridge University Press

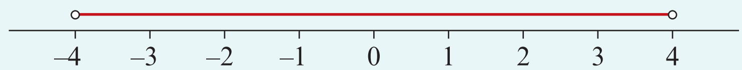
Example 7

Illustrate each of the following sets on a number line and represent the sets using interval notation:

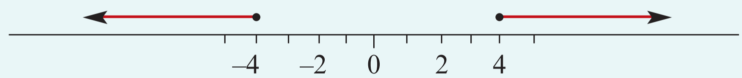
a $\{x : |x| < 4\}$ **b** $\{x : |x| \geq 4\}$ **c** $\{x : |x - 1| \leq 4\}$

Solution

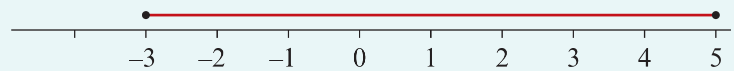
a $(-4, 4)$



b $(-\infty, -4] \cup [4, \infty)$



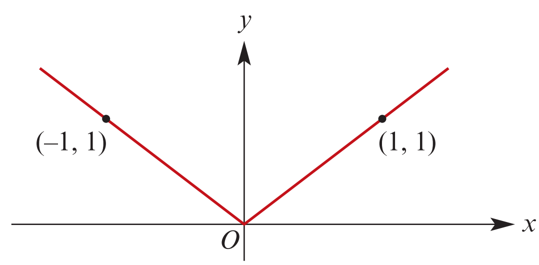
c $[-3, 5]$



The graph of $y = |x|$

The graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ is shown here.

This graph is symmetric about the y-axis, since $|x| = |-x|$.



Example 8

For each of the following functions, sketch the graph and state the range:

a $f(x) = |x - 3| + 1$ **b** $f(x) = -|x - 3| + 1$

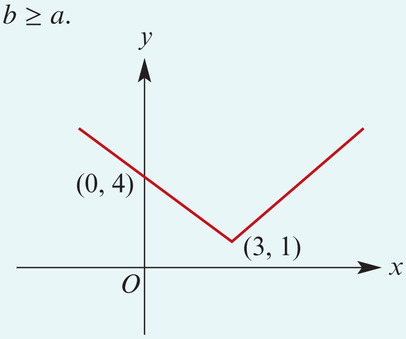
Solution

Note that $|a - b| = a - b$ if $a \geq b$, and $|a - b| = b - a$ if $b \geq a$.

a $$f(x) = |x - 3| + 1 = \begin{cases} x - 3 + 1 & \text{if } x \geq 3 \\ 3 - x + 1 & \text{if } x < 3 \end{cases}$$

$$= \begin{cases} x - 2 & \text{if } x \geq 3 \\ 4 - x & \text{if } x < 3 \end{cases}$$

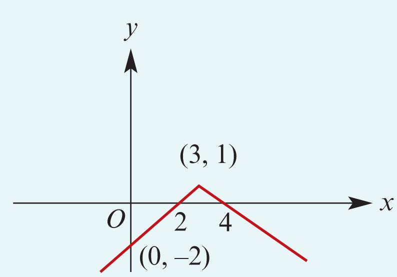
Range = $[1, \infty)$



b $$f(x) = -|x - 3| + 1 = \begin{cases} -(x - 3) + 1 & \text{if } x \geq 3 \\ -(3 - x) + 1 & \text{if } x < 3 \end{cases}$$

$$= \begin{cases} -x + 4 & \text{if } x \geq 3 \\ -2 + x & \text{if } x < 3 \end{cases}$$

Range = $(-\infty, 1]$



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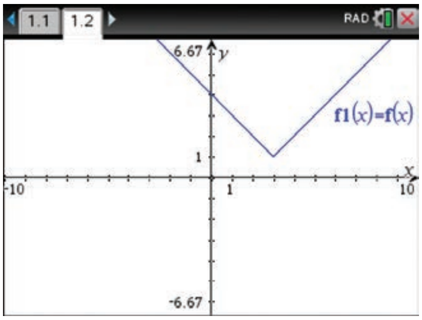
Using the TI-Nspire

- Use (menu) > **Actions** > **Define** to define the function $f(x) = \text{abs}(x - 3) + 1$.

Note: The absolute value function can be obtained by typing **abs( )** or using the 2D-template palette.

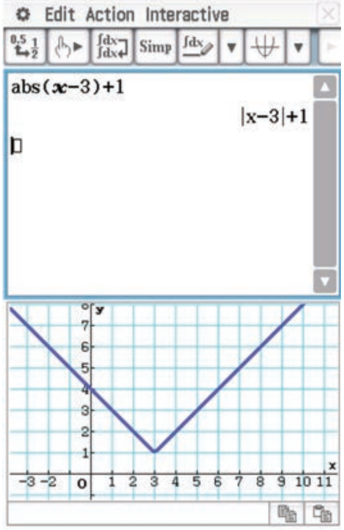
- Open a **Graphs** application ((ctrl) (I) > **Graphs**) and let $f1(x) = f(x)$.
- Press (enter) to obtain the graph.



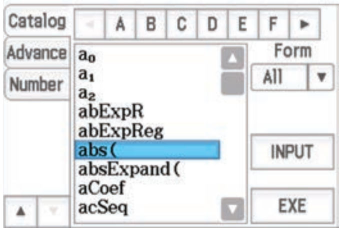
Note: The expression $\text{abs}(x - 3) + 1$ could have been entered directly for $f1(x)$.

Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter the expression $|x - 3| + 1$.



Note: To obtain the absolute value function, either choose **abs(** from the catalog (as shown below) or select [▣] from the (Math1) keyboard.



- Tap [graph icon] to open the graph window.
- Highlight $|x - 3| + 1$ and drag into the graph window.
- Select **Zoom** > **Initialize** or use [window icon] to adjust the window manually.

Note: Alternatively, the function can be graphed using the **Graph & Table** application. Enter the expression in $y1$, tick the box, and tap [graph icon].

Exercise 2C

Skillsheet

Example 6

1 Evaluate each of the following:

a $|-5| + 3$ **b** $|-5| + |-3|$ **c** $|-5| - |-3|$

d $|-5| - |-3| - 4$ **e** $|-5| - |-3| - |-4|$ **f** $|-5| + |-3| - |-4|$

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2 Solve each of the following equations for x:

a $|x - 1| = 2$ **b** $|2x - 3| = 4$ **c** $|5x - 3| = 9$ **d** $|x - 3| - 9 = 0$

e $|3 - x| = 4$ **f** $|3x + 4| = 8$ **g** $|5x + 11| = 9$

Example 7 **3** Illustrate each of the following sets on a number line and represent the sets using interval notation:

a $\{ x : |x| < 3 \}$ **b** $\{ x : |x| \geq 5 \}$ **c** $\{ x : |x - 2| \leq 1 \}$

d $\{ x : |x - 2| < 3 \}$ **e** $\{ x : |x + 3| \geq 5 \}$ **f** $\{ x : |x + 2| \leq 1 \}$

Example 8 **4** For each of the following functions, sketch the graph and state the range:

a $f(x) = |x - 4| + 1$ **b** $f(x) = -|x + 3| + 2$

c $f(x) = |x + 4| - 1$ **d** $f(x) = 2 - |x - 1|$

5 Solve each of the following inequalities, giving your answer using set notation:

a $\{ x : |x| \leq 5 \}$ **b** $\{ x : |x| \geq 2 \}$ **c** $\{ x : |2x - 3| \leq 1 \}$

d $\{ x : |5x - 2| < 3 \}$ **e** $\{ x : |-x + 3| \geq 7 \}$ **f** $\{ x : |-x + 2| \leq 1 \}$

6 Solve each of the following for x:

a $|x - 4| - |x + 2| = 6$ **b** $|2x - 5| - |4 - x| = 10$ **c** $|2x - 1| + |4 - 2x| = 10$

7 If $f(x) = |x - a| + b$ with $f(3) = 3$ and $f(-1) = 3$, find the values of a and b.

8 Prove that $|x - y| \leq |x| + |y|$.

9 Prove that $|x| - |y| \leq |x - y|$.

10 Prove that $|x + y + z| \leq |x| + |y| + |z|$.

2D Surds

A **quadratic surd** is a number of the form $\sqrt{a}$, where a is a rational number which is not the square of another rational number.

Note: $\sqrt{a}$ is taken to mean the positive square root.

In general, a **surd of order *n*** is a number of the form $\sqrt[n]{a}$, where a is a rational number which is not a perfect nth power.

For example:

- $\sqrt{7}, \sqrt{24}, \sqrt{\frac{9}{7}}, \sqrt{\frac{1}{2}}$ are quadratic surds
- $\sqrt{9}, \sqrt{16}, \sqrt{\frac{9}{4}}$ are *not* surds
- $\sqrt[3]{7}, \sqrt[3]{15}$ are surds of order 3
- $\sqrt[4]{100}, \sqrt[4]{26}$ are surds of order 4

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Quadratic surds hold a prominent position in school mathematics. For example, the solutions of quadratic equations often involve surds:

$$x = \frac{1+\sqrt{5}}{2} \text{ is a solution of the quadratic equation } x^2 - x - 1 = 0.$$

Some well-known values of trigonometric functions involve surds. For example:

$$\sin 60^\circ = \frac{\sqrt{3}}{2}, \qquad \sin 15^\circ = \frac{\sqrt{6}-\sqrt{2}}{4}$$

Exact solutions are often required in Mathematical Methods Year 12 and Specialist Mathematics Year 12.

Properties of square roots

The following properties of square roots are often used.

For positive numbers a and b:

- $\sqrt{ab} = \sqrt{a}\sqrt{b}$ e.g. $\sqrt{50} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$
- $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$ e.g. $\sqrt{\dfrac{7}{9}} = \dfrac{\sqrt{7}}{\sqrt{9}} = \dfrac{\sqrt{7}}{3}$

Properties of surds

Simplest form

If possible, a factor which is the square of a rational number is 'taken out' of a square root. When the number under the square root has no factors which are squares of a rational number, the surd is said to be in **simplest form**.

Example 9

Write each of the following in simplest form:

a $\sqrt{72}$ **b** $\sqrt{28}$ **c** $\sqrt{\dfrac{700}{117}}$ **d** $\sqrt{\dfrac{99}{64}}$

Solution

a $\sqrt{72} = \sqrt{36 \times 2} = 6\sqrt{2}$

b $\sqrt{28} = \sqrt{4 \times 7} = 2\sqrt{7}$

c $$\sqrt{\frac{700}{117}} = \frac{\sqrt{700}}{\sqrt{117}} = \frac{\sqrt{7 \times 100}}{\sqrt{9 \times 13}} = \frac{10}{3}\sqrt{\frac{7}{13}}$$

d $$\sqrt{\frac{99}{64}} = \frac{\sqrt{99}}{\sqrt{64}} = \frac{\sqrt{9 \times 11}}{8} = \frac{3\sqrt{11}}{8}$$

Like surds

Surds which have the same 'irrational factor' are called **like surds**.

For example: $3\sqrt{7}$, $2\sqrt{7}$ and $\sqrt{7}$ are like surds.

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The sum or difference of two like surds can be simplified:

- $m\sqrt{p} + n\sqrt{p} = (m+n)\sqrt{p}$
- $m\sqrt{p} - n\sqrt{p} = (m-n)\sqrt{p}$

Example 10

Express each of the following as a single surd in simplest form:

a $\sqrt{147} + \sqrt{108} - \sqrt{363}$

b $\sqrt{3} + \sqrt{5} + \sqrt{20} + \sqrt{27} - \sqrt{45} - \sqrt{48}$

c $\sqrt{\frac{1}{8}} - \sqrt{\frac{1}{18}} - 5\sqrt{\frac{1}{72}}$

d $\sqrt{50} + \sqrt{2} - 2\sqrt{18} + \sqrt{8}$

Solution

a $\sqrt{147} + \sqrt{108} - \sqrt{363}$

$$= \sqrt{7^2 \times 3} + \sqrt{6^2 \times 3} - \sqrt{11^2 \times 3}$$
$$= 7\sqrt{3} + 6\sqrt{3} - 11\sqrt{3}$$
$$= 2\sqrt{3}$$

b $\sqrt{3} + \sqrt{5} + \sqrt{20} + \sqrt{27} - \sqrt{45} - \sqrt{48}$

$$= \sqrt{3} + \sqrt{5} + 2\sqrt{5} + 3\sqrt{3} - 3\sqrt{5} - 4\sqrt{3}$$
$$= 0\sqrt{3} + 0\sqrt{5}$$
$$= 0$$

c $\sqrt{\frac{1}{8}} - \sqrt{\frac{1}{18}} - 5\sqrt{\frac{1}{72}}$

$$= \sqrt{\frac{1}{4 \times 2}} - \sqrt{\frac{1}{9 \times 2}} - 5\sqrt{\frac{1}{36 \times 2}}$$
$$= \frac{1}{2}\sqrt{\frac{1}{2}} - \frac{1}{3}\sqrt{\frac{1}{2}} - \frac{5}{6}\sqrt{\frac{1}{2}}$$
$$= \frac{3}{6}\sqrt{\frac{1}{2}} - \frac{2}{6}\sqrt{\frac{1}{2}} - \frac{5}{6}\sqrt{\frac{1}{2}}$$
$$= \frac{-4}{6}\sqrt{\frac{1}{2}}$$
$$= \frac{-2}{3}\sqrt{\frac{1}{2}}$$

d $\sqrt{50} + \sqrt{2} - 2\sqrt{18} + \sqrt{8}$

$$= 5\sqrt{2} + \sqrt{2} - 2 \times 3\sqrt{2} + 2\sqrt{2}$$
$$= 8\sqrt{2} - 6\sqrt{2}$$
$$= 2\sqrt{2}$$

► Rationalising the denominator

In the past, a labour-saving procedure with surds was to **rationalise** any surds in the denominator of an expression. This is still considered to be a neat way of expressing final answers.

For $\sqrt{5}$, a rationalising factor is $\sqrt{5}$, as $\sqrt{5} \times \sqrt{5} = 5$.

For $1 + \sqrt{2}$, a rationalising factor is $1 - \sqrt{2}$, as $(1 + \sqrt{2})(1 - \sqrt{2}) = 1 - 2 = -1$.

For $\sqrt{3} + \sqrt{6}$, a rationalising factor is $\sqrt{3} - \sqrt{6}$, as $(\sqrt{3} + \sqrt{6})(\sqrt{3} - \sqrt{6}) = 3 - 6 = -3$.

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Example 11

Rationalise the denominator of each of the following:

a $\dfrac{1}{2\sqrt{7}}$ **b** $\dfrac{1}{2-\sqrt{3}}$ **c** $\dfrac{1}{\sqrt{3}-\sqrt{6}}$ **d** $\dfrac{3+\sqrt{8}}{3-\sqrt{8}}$

Solution

a $\dfrac{1}{2\sqrt{7}} \times \dfrac{\sqrt{7}}{\sqrt{7}} = \dfrac{\sqrt{7}}{14}$

b $\dfrac{1}{2-\sqrt{3}} \times \dfrac{2+\sqrt{3}}{2+\sqrt{3}} = \dfrac{2+\sqrt{3}}{4-3}$

$= 2+\sqrt{3}$

c $\dfrac{1}{\sqrt{3}-\sqrt{6}} \times \dfrac{\sqrt{3}+\sqrt{6}}{\sqrt{3}+\sqrt{6}} = \dfrac{\sqrt{3}+\sqrt{6}}{3-6}$

$= \dfrac{-1}{3}(\sqrt{3}+\sqrt{6})$

d $\dfrac{3+\sqrt{8}}{3-\sqrt{8}} = \dfrac{3+2\sqrt{2}}{3-2\sqrt{2}} \times \dfrac{3+2\sqrt{2}}{3+2\sqrt{2}}$

$= \dfrac{9+12\sqrt{2}+8}{9-8}$

$= 17+12\sqrt{2}$

Example 12

Expand the brackets in each of the following and collect like terms, expressing surds in simplest form:

a $(3-\sqrt{2})^2$ **b** $(3-\sqrt{2})(1+\sqrt{2})$

Solution

a $(3-\sqrt{2})^2$

$= (3-\sqrt{2})(3-\sqrt{2})$

$= 3(3-\sqrt{2}) - \sqrt{2}(3-\sqrt{2})$

$= 9-3\sqrt{2}-3\sqrt{2}+2$

$= 11-6\sqrt{2}$

b $(3-\sqrt{2})(1+\sqrt{2})$

$= 3(1+\sqrt{2}) - \sqrt{2}(1+\sqrt{2})$

$= 3+3\sqrt{2}-\sqrt{2}-2$

$= 1+2\sqrt{2}$

Using the TI-Nspire

A CAS calculator can be used to work with irrational numbers.

Expressions on the screen can be selected using the up arrow ▲. This returns the expression to the entry line and modifications can be made.

For example:

- Evaluate $\dfrac{2^3 \cdot 2^2}{5} \cdot 2^{\frac{8}{5}}$ as shown.

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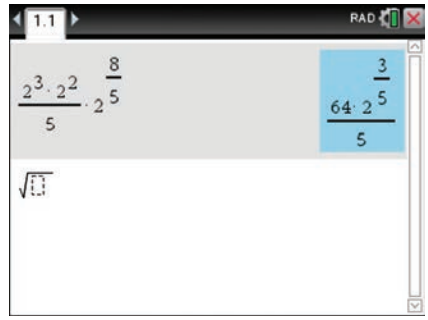
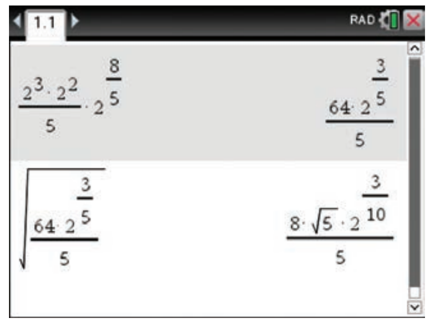
- To find the square root of this expression, first type (ctrl)(x²). Then move upwards by pressing the up arrow ▲, so that the expression is highlighted.



- Press (enter) to paste this expression into the square root sign.
- Press (enter) once more to evaluate the square root of this expression.

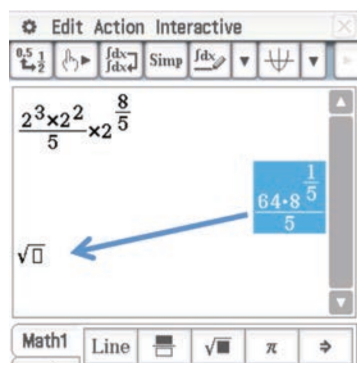


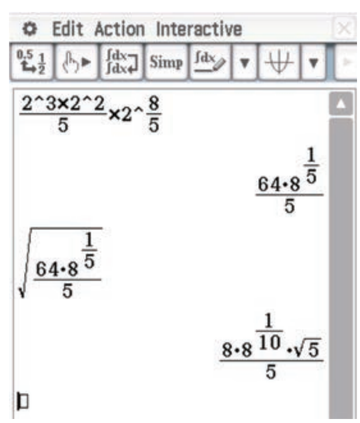
Using the Casio ClassPad

Expressions on the screen can be selected using the stylus. Highlight and drag the expression to the next entry line, where modifications can be made.

For example:

- Evaluate $\dfrac{2^3 \cdot 2^2}{5} \cdot 2^{\frac{8}{5}}$ as shown.
- In the next entry line, tap [√■] from the (Math1) keyboard.
- Highlight the expression and drag to the square root sign.
- Tap (EXE) to evaluate.
- Alternatively, highlight the expression and select **Edit** > **Copy**. Then tap the cursor in the desired position and select **Edit** > **Paste**.





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Exercise 2D

Skillsheet

Example 9

1 Express each of the following in terms of the simplest possible surds:

a $\sqrt{8}$ **b** $\sqrt{12}$ **c** $\sqrt{27}$ **d** $\sqrt{50}$
e $\sqrt{45}$ **f** $\sqrt{1210}$ **g** $\sqrt{98}$ **h** $\sqrt{108}$
i $\sqrt{25}$ **j** $\sqrt{75}$ **k** $\sqrt{512}$

Example 10

2 Simplify each of the following:

a $\sqrt{8} + \sqrt{18} - 2\sqrt{2}$ **b** $\sqrt{75} + 2\sqrt{12} - \sqrt{27}$
c $\sqrt{28} + \sqrt{175} - \sqrt{63}$ **d** $\sqrt{1000} - \sqrt{40} - \sqrt{90}$
e $\sqrt{512} + \sqrt{128} + \sqrt{32}$ **f** $\sqrt{24} - 3\sqrt{6} - \sqrt{216} + \sqrt{294}$

3 Simplify each of the following:

a $\sqrt{75} + \sqrt{108} + \sqrt{14}$ **b** $\sqrt{847} - \sqrt{567} + \sqrt{63}$
c $\sqrt{720} - \sqrt{245} - \sqrt{125}$ **d** $\sqrt{338} - \sqrt{288} + \sqrt{363} - \sqrt{300}$
e $\sqrt{12} + \sqrt{8} + \sqrt{18} + \sqrt{27} + \sqrt{300}$ **f** $2\sqrt{18} + 3\sqrt{5} - \sqrt{50} + \sqrt{20} - \sqrt{80}$

Example 11

4 Express each of the following with rational denominators:

a $\dfrac{1}{\sqrt{5}}$ **b** $\dfrac{1}{\sqrt{7}}$ **c** $-\dfrac{1}{\sqrt{2}}$ **d** $\dfrac{2}{\sqrt{3}}$ **e** $\dfrac{3}{\sqrt{6}}$

f $\dfrac{1}{2\sqrt{2}}$ **g** $\dfrac{1}{\sqrt{2}+1}$ **h** $\dfrac{1}{2-\sqrt{3}}$ **i** $\dfrac{1}{4-\sqrt{10}}$ **j** $\dfrac{2}{\sqrt{6}+2}$

k $\dfrac{1}{\sqrt{5}-\sqrt{3}}$ **l** $\dfrac{3}{\sqrt{6}-\sqrt{5}}$ **m** $\dfrac{1}{3-2\sqrt{2}}$

Example 12

5 Express each of the following in the form $a + b\sqrt{c}$:

a $\dfrac{2}{3-2\sqrt{2}}$ **b** $(\sqrt{5}+2)^2$ **c** $(1+\sqrt{2})(3-2\sqrt{2})$ **d** $(\sqrt{3}-1)^2$

e $\sqrt{\dfrac{1}{3}} - \dfrac{1}{\sqrt{27}}$ **f** $\dfrac{\sqrt{3}+2}{2\sqrt{3}-1}$ **g** $\dfrac{\sqrt{5}+1}{\sqrt{5}-1}$ **h** $\dfrac{\sqrt{8}+3}{\sqrt{18}+2}$

6 Expand and simplify each of the following:

a $(2\sqrt{a}-1)^2$ **b** $(\sqrt{x+1}+\sqrt{x+2})^2$

7 For real numbers a and b, we have $a > b$ if and only if $a - b > 0$. Use this to state the larger of:

a $5 - 3\sqrt{2}$ and $6\sqrt{2} - 8$ **b** $2\sqrt{6} - 3$ and $7 - 2\sqrt{6}$

8 For positive real numbers a and b, we have $a > b$ if and only if $a^2 - b^2 > 0$. Use this to state the larger of:

a $\dfrac{2}{\sqrt{3}}$ and $\dfrac{3}{\sqrt{2}}$ **b** $\dfrac{\sqrt{7}}{3}$ and $\dfrac{\sqrt{5}}{2}$ **c** $\dfrac{\sqrt{3}}{7}$ and $\dfrac{\sqrt{5}}{5}$ **d** $\dfrac{\sqrt{10}}{2}$ and $\dfrac{8}{\sqrt{3}}$

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9 Find the values of b and c for a quadratic function $f(x) = x^2 + bx + c$ such that the solutions of the equation $f(x) = 0$ are:

a $\sqrt{3},\ -\sqrt{3}$ **b** $2\sqrt{3},\ -2\sqrt{3}$ **c** $1 - \sqrt{2},\ 1 + \sqrt{2}$

d $2 - \sqrt{3},\ 2 + \sqrt{3}$ **e** $3 - 2\sqrt{2},\ 3 + 2\sqrt{2}$ **f** $4 - 7\sqrt{5},\ 3 + 2\sqrt{5}$

10 Express $\dfrac{1}{\sqrt{2} + \sqrt{3} + \sqrt{5}}$ with a rational denominator.

11 **a** Show that $a - b = \left(a^{\frac{1}{3}} - b^{\frac{1}{3}}\right)\left(a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}\right)$.

b Express $\dfrac{1}{1 - 2^{\frac{1}{3}}}$ with a rational denominator.

2E Natural numbers

▶ Factors and composites

The factors of 8 are 1, 2, 4 and 8.

The factors of 24 are 1, 2, 3, 4, 6, 8, 12 and 24.

The factors of 5 are 1 and 5.

> A natural number a is a **factor** of a natural number b if there exists a natural number k such that $b = ak$.

> A natural number greater than 1 is called a **prime number** if its only factors are itself and 1.

The prime numbers less than 100 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

> A natural number m is called a **composite number** if it can be written as a product $m = a \times b$, where a and b are natural numbers greater than 1 and less than m.

▶ Prime decomposition

Expressing a composite number as a product of powers of prime numbers is called **prime decomposition**. For example:

$$3000 = 3 \times 2^3 \times 5^3$$

$$2294 = 2 \times 31 \times 37$$

This is useful for finding the factors of a number. For example, the prime decomposition of 12 is given by $12 = 2^2 \times 3$. The factors of 12 are

$$1, \quad 2, \quad 2^2 = 4, \quad 3, \quad 2 \times 3 = 6 \quad \text{and} \quad 2^2 \times 3 = 12$$

ISBN 978-1-316-63608-4

Cambridge University Press

This property of natural numbers is described formally by the following theorem.

Fundamental theorem of arithmetic

Every natural number greater than 1 either is a prime number or can be represented as a product of prime numbers. Furthermore, this representation is unique apart from rearrangement of the order of the prime factors.

Example 13

Give the prime decomposition of 17 248 and hence list the factors of this number.

Solution

The prime decomposition can be found by repeated division, as shown on the right.

2	17248
2	8624
2	4312
2	2156
2	1078
7	539
7	77
11	11
	1

The prime decomposition of 17 248 is

$$17\,248 = 2^5 \times 7^2 \times 11$$

Therefore each factor must be of the form

$$2^\alpha \times 7^\beta \times 11^\gamma$$

where $\alpha = 0, 1, 2, 3, 4, 5$, $\beta = 0, 1, 2$ and $\gamma = 0, 1$.

The factors of 17 248 can be systematically listed as follows:

1	2	2^2	2^3	2^4	2^5
7	2×7	$2^2 \times 7$	$2^3 \times 7$	$2^4 \times 7$	$2^5 \times 7$
7^2	2×7^2	$2^2 \times 7^2$	$2^3 \times 7^2$	$2^4 \times 7^2$	$2^5 \times 7^2$
11	2×11	$2^2 \times 11$	$2^3 \times 11$	$2^4 \times 11$	$2^5 \times 11$
7×11	$2 \times 7 \times 11$	$2^2 \times 7 \times 11$	$2^3 \times 7 \times 11$	$2^4 \times 7 \times 11$	$2^5 \times 7 \times 11$
$7^2 \times 11$	$2 \times 7^2 \times 11$	$2^2 \times 7^2 \times 11$	$2^3 \times 7^2 \times 11$	$2^4 \times 7^2 \times 11$	$2^5 \times 7^2 \times 11$

▶ Highest common factor

The **highest common factor** of two natural numbers a and b is the largest natural number that is a factor of both a and b. It is denoted by HCF(a, b).

For example, the highest common factor of 15 and 24 is 3. We write HCF$(15, 24) = 3$.

Note: The highest common factor is also called the **greatest common divisor**.

ISBN 978-1-316-63608-4

Using prime decomposition to find HCF

Prime decomposition can be used to find the highest common factor of two numbers.

For example, consider the numbers 140 and 110. Their prime factorisations are

$$140 = 2^2 \times 5 \times 7 \quad \text{and} \quad 110 = 2 \times 5 \times 11$$

A number which is a factor of both 140 and 110 must have prime factors which occur in both these factorisations. The highest common factor of 140 and 110 is $2 \times 5 = 10$.

Next consider the numbers

$$396\,000 = 2^5 \cdot 3^2 \cdot 5^3 \cdot 11 \quad \text{and} \quad 1\,960\,200 = 2^3 \cdot 3^4 \cdot 5^2 \cdot 11^2$$

To obtain the highest common factor, we take the lower power of each prime factor:

$$\text{HCF}(396\,000, 1\,960\,200) = 2^3 \cdot 3^2 \cdot 5^2 \cdot 11$$

Example 14

a Find the highest common factor of 528 and 3168.

b Find the highest common factor of 3696 and 3744.

Solution

a $528 = 2^4 \times 3 \times 11$

$3168 = 2^5 \times 3^2 \times 11$

$\therefore \text{HCF}(528, 3168) = 2^4 \times 3 \times 11$

$= 528$

b $3696 = 2^4 \times 3 \times 7 \times 11$

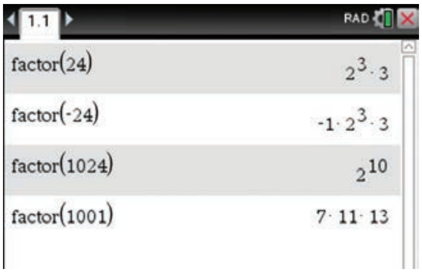
$3744 = 2^5 \times 3^2 \times 13$

$\therefore \text{HCF}(3696, 3744) = 2^4 \times 3$

$= 48$

Using the TI-Nspire

- The prime decomposition of a natural number can be obtained using (menu) > **Algebra** > **Factor** as shown.



- The highest common factor of two numbers (also called their *greatest common divisor*) can be found by using the command **gcd( )** from (menu) > **Number** > **Greatest Common Divisor**, or by just typing it in, as shown.

Note: Nested **gcd( )** commands may be used to find the greatest common divisor of several numbers.

ISBN 978-1-316-63608-4

Using the Casio ClassPad

- To find the highest common factor of two numbers, go to **Interactive** > **Calculation** > **gcd/lcm** > **gcd**.
- Enter the required numbers in the two lines provided, and tap OK.

Lowest common multiple

A natural number a is a **multiple** of a natural number b if there exists a natural number k such that $a = kb$.

The **lowest common multiple** of two natural numbers a and b is the smallest natural number that is a multiple of both a and b. It is denoted by LCM(a, b).

For example: LCM(24, 36) = 72 and LCM(256, 100) = 6400.

Using prime decomposition to find LCM

Consider again the numbers

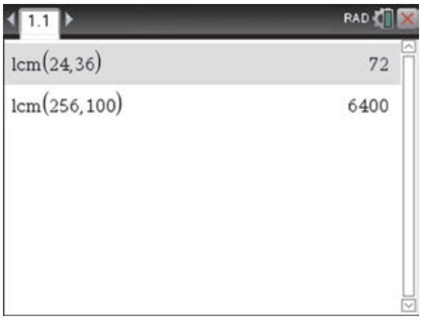
$$396\,000 = 2^5 \cdot 3^2 \cdot 5^3 \cdot 11 \quad \text{and} \quad 1\,960\,200 = 2^3 \cdot 3^4 \cdot 5^2 \cdot 11^2$$

To obtain the lowest common multiple, we take the higher power of each prime factor:

$$\text{LCM}(396\,000, 1\,960\,200) = 2^5 \cdot 3^4 \cdot 5^3 \cdot 11^2 = 39\,204\,000$$

Using the TI-Nspire

The lowest common multiple of two numbers (also called their *least common multiple*) can be found by using the command **lcm( )** from menu > **Number** > **Least Common Multiple**, or by just typing it in, as shown.



ISBN 978-1-316-63608-4

Using the Casio ClassPad

- To find the lowest common multiple of two numbers, go to **Interactive** > **Calculation** > **gcd/lcm** > **lcm**.
- Enter the required numbers in the two lines provided, and tap OK.

Section summary

- A natural number a is a **factor** of a natural number b if there exists a natural number k such that $b = ak$.
- A natural number greater than 1 is a **prime number** if its only factors are itself and 1.
- A natural number m is a **composite number** if it can be written as a product $m = a \times b$, where a and b are natural numbers greater than 1 and less than m.
- Every composite number can be expressed as a product of powers of prime numbers; this is called **prime decomposition**. For example: $1300 = 2^2 \times 5^2 \times 13$
- The **highest common factor** of two natural numbers a and b, denoted by HCF(a, b), is the largest natural number that is a factor of both a and b.
- The **lowest common multiple** of two natural numbers a and b, denoted by LCM(a, b), is the smallest natural number that is a multiple of both a and b.

Exercise 2E

Example 13

1 Give the prime decomposition of each of the following numbers:

a 60 **b** 676 **c** 228 **d** 900 **e** 252
f 6300 **g** 68 640 **h** 96 096 **i** 32 032 **j** 544 544

Example 14

2 Find the highest common factor of each of the following pairs of numbers:

a 4361, 9281 **b** 999, 2160 **c** 5255, 716 845 **d** 1271, 3875 **e** 804, 2358

3 **a** List all the factors of 18 and all the factors of 36.
b Why does 18 have an even number of factors and 36 an odd number of factors?
c Find the smallest number greater than 100 with exactly three factors.

4 A woman has three children and two of them are teenagers, aged between 13 and 19. The product of their three ages is 1050. How old is each child?

ISBN 978-1-316-63608-4

5 By using prime decomposition, find a natural number n such that $22^2 \times 55^2 = 10^2 \times n^2$.

6 The natural number n has exactly eight different factors. Two of these factors are 15 and 21. What is the value of n?

7 Let n be the smallest of three natural numbers whose product is 720. What is the largest possible value of n?

8 When all eight factors of 30 are multiplied together, the product is 30^k. What is the value of k?

9 A bell rings every 36 minutes and a buzzer rings every 42 minutes. If they sound together at 9 a.m., when will they next sound together?

10 The LCM of two numbers is $2^5 \times 3^3 \times 5^3$ and the HCF is $2^3 \times 3 \times 5^2$. Find all the possible numbers.

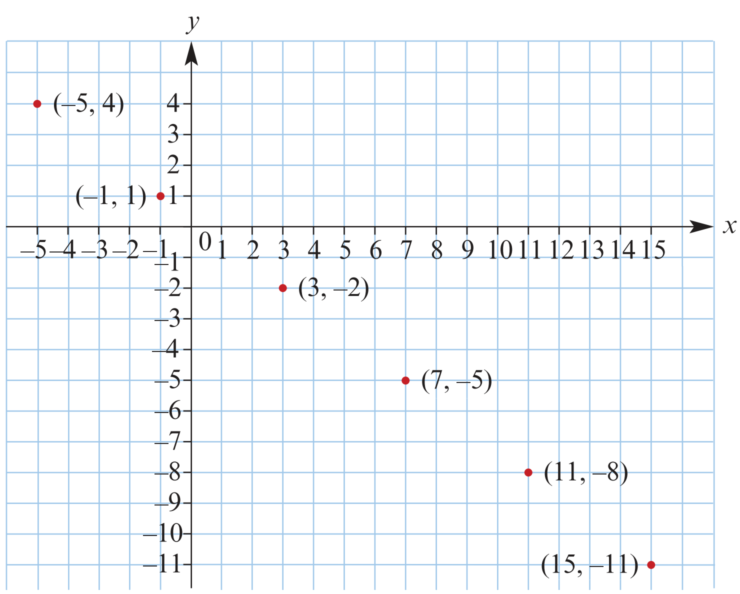
2F Linear Diophantine equations

A **Diophantine equation** is an equation in which only integer solutions are allowed.

The equation $x^2 + y^2 = z^2$ can be considered as a Diophantine equation. There are infinitely many integer solutions. For example: $3^2 + 4^2 = 5^2$ and $5^2 + 12^2 = 13^2$. Of course, these solutions correspond to right-angled triangles with these side lengths.

An equation of the form $ax + by = c$, where the coefficients a, b, c are integers, is called a **linear Diophantine equation** when the intention is to find integer solutions for x, y.

For example, consider the equation $3x + 4y = 1$. This equation defines a straight line. A family of integer solutions to this equation is illustrated on the following graph.



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From the graph, we can see that as the integer solutions for x increase by 4, the corresponding integer solutions for y decrease by 3.

The solutions may be built up in the following way using $(-1, 1)$ as the starting point.

x	y
$-1 + 1 \times 4$	$1 - 1 \times 3$
$-1 + 2 \times 4$	$1 - 2 \times 3$
$-1 + 3 \times 4$	$1 - 3 \times 3$

i.e.

x	y
3	−2
7	−5
11	−8

The family of solutions may be described as

$$x = -1 + 4t, \quad y = 1 - 3t \qquad \text{for } t \in \mathbb{Z}$$

The solution set is

$$\{ (x, y) : x = -1 + 4t,\ y = 1 - 3t,\ t \in \mathbb{Z} \}$$

If a linear Diophantine equation has one solution, then it has infinitely many:

Theorem

If $ax + by = c$ is a linear Diophantine equation and (x_0, y_0) is found to be one solution, then the general solution is given by

$$x = x_0 + \frac{b}{d}t, \quad y = y_0 - \frac{a}{d}t \qquad \text{for } t \in \mathbb{Z}$$

where d is the highest common factor of a and b.

Proof Assume that (x_1, y_1) is another solution to the equation. Then

$$ax_1 + by_1 = c \qquad (1)$$

$$ax_0 + by_0 = c \qquad (2)$$

Subtract (2) from (1):

$$a(x_1 - x_0) = b(y_0 - y_1) \qquad (3)$$

Divide both sides by d:

$$\frac{a}{d}(x_1 - x_0) = \frac{b}{d}(y_0 - y_1)$$

The integers $\frac{a}{d}$ and $\frac{b}{d}$ have no common factor. Hence $x_1 - x_0$ must be divisible by $\frac{b}{d}$ and so

$$x_1 - x_0 = \frac{b}{d}t, \quad \text{for some } t \in \mathbb{Z}$$

Therefore $x_1 = x_0 + \frac{b}{d}t$ and from (3) it follows that $y_1 = y_0 - \frac{a}{d}t$.

It can be checked by substitution that $x = x_0 + \frac{b}{d}t$ and $y = y_0 - \frac{a}{d}t$ is a solution of the equation for any $t \in \mathbb{Z}$.

ISBN 978-1-316-63608-4

Cambridge University Press

Example 15

A man has \$200 in his wallet, made up of \$50 and \$20 notes. What are the possible numbers of each of these types of notes?

Solution

Let x and y be the numbers of \$50 and \$20 notes respectively.

The linear Diophantine equation is

$$50x + 20y = 200$$

$$\therefore \quad 5x + 2y = 20$$

By inspection, a solution is

$$x = 4, \ y = 0$$

Therefore the general solution is

$$x = 4 + 2t, \ y = 0 - 5t, \quad \text{for } t \in \mathbb{Z}$$

We are only interested in the case where $x \geq 0$ and $y \geq 0$, that is, $4 + 2t \geq 0$ and $-5t \geq 0$.

Hence $-2 \leq t \leq 0$.

For $t = -2$: $\quad x = 0, \quad y = 10$

For $t = -1$: $\quad x = 2, \quad y = 5$

For $t = 0$: $\quad x = 4, \quad y = 0$

The man could have ten \$20 notes, or two \$50 notes and five \$20 notes, or four \$50 notes.

Theorem

A linear Diophantine equation $ax + by = c$ has integer solutions if and only if HCF(a, b) divides c.

This means, for example, that there are no integer solutions to $2x + 6y = 3$. In the next section we will see a method for finding a solution to $ax + by = c$ when HCF(a, b) divides c.

Section summary

- A **Diophantine equation** is an equation in which only integer solutions are allowed.
- An equation of the form $ax + by = c$, where the coefficients a, b, c are integers, is called a **linear Diophantine equation** when the intention is to find integer solutions for x, y.
- If a linear Diophantine equation has one solution, then it has infinitely many:
  If $ax + by = c$ is a linear Diophantine equation and (x_0, y_0) is found to be one solution, then the general solution is given by
  $$x = x_0 + \frac{b}{d}t, \quad y = y_0 - \frac{a}{d}t \qquad \text{for } t \in \mathbb{Z}$$
  where d is the highest common factor of a and b.

ISBN 978-1-316-63608-4

Exercise 2F

1 Find all solutions to each of the following Diophantine equations:

a $11x + 3y = 1$ **b** $2x + 7y = 2$ **c** $24x + 63y = 99$

d $22x + 6y = 2$ **e** $2x + 7y = 22$ **f** $10x + 35y = 110$

2 For **e** in Question 1, find the solution(s) with both x and y positive.

3 Prove that, if $ax + by = c$ and the highest common factor of a and b does not divide c, then there is no solution to the Diophantine equation.

Example 15

4 A student puts a number of spiders (with eight legs each) and a number of beetles (with six legs each) in a box. She counted 54 legs in all.

a Form a Diophantine equation.

b Find the number of spiders and the number of beetles in the box.

5 Helena has a number of coins in her purse. They are all either 20c or 50c coins. The total value of the coins is \$5. What are the possible numbers of each type of coin?

6 One of the solutions of the equation $19x + 83y = 1983$ in positive integers x and y is obviously $x = 100$, $y = 1$. Show that there is only one other pair of positive integers which satisfies this equation and find it. Consider the equation $19x + 98y = 1998$.

7 A man has \$500 in his wallet, made up of \$50 and \$10 notes. Find the possible combinations of notes that he could have.

8 There are seven coconuts and 63 heaps of pineapples, where each heap has exactly the same number of pineapples. The fruit is to be divided equally between 23 people. Let x be the number of pineapples in each heap and let y be the number of pieces of fruit that each person receives. Form a Diophantine equation and find the possible values for x and y.

9 A dealer spent \$10 000 buying cattle, some at \$410 each and the rest at \$530 each. How many of each sort did she buy?

10 Find the smallest positive number which when divided by 7 leaves a remainder of 6, and when divided by 11 leaves a remainder of 9. Also find the general form of such numbers.

11 Given a 3 litre jug and a 5 litre jug, can I measure exactly 7 litres of water? If it is possible, explain how this may be done as efficiently as possible.

12 The Guadeloupe Post Office has only 3c and 5c stamps. What amounts of postage can the post office sell?

13 A man spent \$29.60 buying party hats. There were two types of party hat: type A cost \$1.70, while type B cost \$1.00. How many of each type did he buy?

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ISBN 978-1-316-63608-4

14 Why has the equation $6x - 9y = 10$ no integer solutions?

15 Find the smallest multiple of 13 which when divided by 18 leaves 5 as a remainder.

16 A two-digit number exceeds five times the sum of its digits by 17. Find two such numbers.

2G The Euclidean algorithm

The Euclidean algorithm provides a method for finding the highest common factor of two numbers and also a method for solving linear Diophantine equations.

From your earlier work on arithmetic, you will know that dividing one natural number by another gives a result such as:

- $65 \div 7$ gives $65 = 7 \times 9 + 2$
- $92 \div 3$ gives $92 = 3 \times 30 + 2$

We can formalise this observation as follows.

Division algorithm

If a and b are integers with $a > 0$, then there are unique integers q and r such that

$$b = aq + r \quad \text{and} \quad 0 \le r < a$$

Proof Suppose there is another pair of integers q_1 and r_1 with $b = aq_1 + r_1$ and $0 \le r_1 < a$.

Then $aq_1 + r_1 = aq + r$

$\therefore \quad a(q_1 - q) = r - r_1 \qquad (1)$

First suppose that $r > r_1$. The left-hand side of equation (1) is an integer which is a multiple of a. Therefore a divides the right-hand side. But the right-hand side is an integer which is greater than 0 and less than a. This is a contradiction, and so the assumption that $r > r_1$ must be false.

Next suppose that $r < r_1$. Multiplying both sides of equation (1) by -1, we obtain

$$a(q - q_1) = r_1 - r$$

and a similar contradiction will arise.

Thus we have shown that $r = r_1$, and so equation (1) gives $a(q_1 - q) = 0$. Hence $q_1 = q$ and $r_1 = r$, and the uniqueness of the integers q and r has been proved.

Example 16

Express -45 in the form $6q + r$, where $0 \le r < 6$.

Solution

$-45 = 6(-8) + 3$

Note: The answer $-45 = 6(-7) - 3$ is not correct, since the remainder -3 is less than zero.

ISBN 978-1-316-63608-4

The following theorem is useful for determining the highest common factor of any two given integers. We use HCF(a, b) to denote the highest common factor of two integers a and b.

Theorem

Let a and b be two integers with $a \neq 0$. If $b = aq + r$, where q and r are integers, then HCF(a, b) = HCF(a, r).

Proof If d is a common factor of a and r, then d divides the right-hand side of the equation $b = aq + r$, and so d divides b.

This proves that all common factors of a and r are also common factors of a and b.

But HCF(a, r) is a common factor of a and r, and therefore HCF(a, r) must divide a and b. It follows that HCF(a, r) must divide HCF(a, b). That is,

$$\text{HCF}(a, b) = m \cdot \text{HCF}(a, r) \qquad \text{for some integer } m \qquad (1)$$

Now rewrite the equation $b = aq + r$ as $r = b - aq$.

If d is a common factor of a and b, then d divides the right-hand side of the equation $r = b - aq$, and so d divides r.

This proves that all common factors of a and b are also common factors of a and r. It follows that HCF(a, b) must divide HCF(a, r). That is,

$$\text{HCF}(a, r) = n \cdot \text{HCF}(a, b) \qquad \text{for some integer } n \qquad (2)$$

From equations (1) and (2), we obtain

$$\text{HCF}(a, r) = mn \cdot \text{HCF}(a, r)$$

$$\therefore \quad 1 = mn$$

This equation in integers m, n is possible only if $m = n = 1$ or $m = n = -1$.

Hence HCF(a, b) = HCF(a, r), since both must be positive.

Example 17

Find HCF(1271, 3875).

Solution

At each step, we use the division algorithm and the previous theorem:

$$\underline{3875} = 3 \times \underline{1271} + \underline{62} \qquad \text{and so} \qquad \text{HCF}(1271, 3875) = \text{HCF}(1271, 62)$$

$$\underline{1271} = 20 \times \underline{62} + \underline{31} \qquad \text{and so} \qquad \text{HCF}(62, 1271) = \text{HCF}(62, 31)$$

$$62 = 2 \times 31 + 0 \qquad \text{and so} \qquad \text{HCF}(31, 62) = \text{HCF}(31, 0) = 31$$

Hence it follows that HCF(1271, 3875) = 31.

Note: Keep using the division algorithm until you get remainder zero. Then the HCF is the last non-zero remainder.

This procedure is called the **Euclidean algorithm**.

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► Method for finding a solution of a linear Diophantine equation

The method presented here uses the Euclidean algorithm.

Example 18

Find $a, b \in \mathbb{Z}$ such that $22a + 6b = 2$.

Solution

Apply the Euclidean algorithm to 22 and 6:

$$\begin{aligned} \underline{22} &= 3 \times \underline{6} + \underline{4} && (1) \\ \underline{6} &= 1 \times \underline{4} + \underline{2} && (2) \\ 4 &= 2 \times 2 + 0 && (3) \end{aligned}$$

Hence HCF(22, 6) = 2.

Now use these equations backwards:

$$\begin{aligned} \underline{2} &= \underline{6} - 1 \times \underline{4} && \text{from (2)} \\ &= \underline{6} - 1 \times (\underline{22} - 3 \times \underline{6}) && \text{from (1)} \\ &= \underline{6} - 1 \times \underline{22} + 3 \times \underline{6} \\ &= 4 \times \underline{6} - 1 \times \underline{22} \end{aligned}$$

Therefore

$$-1 \times 22 + 4 \times 6 = 2$$

and so one solution is $a = -1$ and $b = 4$.

The general solution is $a = -1 + 3t$ and $b = 4 - 11t$, where $t \in \mathbb{Z}$.

Example 19

Find $a, b \in \mathbb{Z}$ such that $125a + 90b = 5$.

Solution

First divide by 5:

$$25a + 18b = 1$$

Apply the Euclidean algorithm to 25 and 18:

$$\begin{aligned} \underline{25} &= 1 \times \underline{18} + \underline{7} && (1) \\ \underline{18} &= 2 \times \underline{7} + \underline{4} && (2) \\ \underline{7} &= 1 \times \underline{4} + \underline{3} && (3) \\ \underline{4} &= 1 \times \underline{3} + \underline{1} && (4) \\ 3 &= 3 \times 1 + 0 && (5) \end{aligned}$$

Hence HCF(25, 18) = 1.

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Now use the equations backwards:

$$\begin{aligned}
\underline{1} &= \underline{4} - 1 \times \underline{3} && \text{from (4)} \\
&= \underline{4} - 1 \times (\underline{7} - 1 \times \underline{4}) && \text{from (3)} \\
&= 2 \times \underline{4} - 1 \times \underline{7} \\
&= 2 \times (\underline{18} - 2 \times \underline{7}) - 1 \times \underline{7} && \text{from (2)} \\
&= 2 \times \underline{18} - 5 \times \underline{7} \\
&= 2 \times \underline{18} - 5 \times (\underline{25} - 1 \times \underline{18}) && \text{from (1)} \\
&= 7 \times \underline{18} - 5 \times \underline{25}
\end{aligned}$$

Therefore

$$-5 \times 25 + 7 \times 18 = 1$$

and so one solution is $a = -5$ and $b = 7$.

The general solution is $a = -5 + 18t$ and $b = 7 - 25t$, where $t \in \mathbb{Z}$.

Section summary

- **Division algorithm**
  If a and b are integers with $a > 0$, then there are unique integers q and r such that $b = aq + r$ and $0 \leq r < a$.
- **Euclidean algorithm**
  Let a and b be integers with $a \neq 0$. If $b = aq + r$, where q and r are integers, then $\text{HCF}(a, b) = \text{HCF}(a, r)$.
  The repeated application of this result can be used to find the highest common factor of two natural numbers and to solve linear Diophantine equations.

Exercise 2G

Example 16

1 For each of the following, express b in the form $b = aq + r$ with $0 \leq r < a$, and show that $\text{HCF}(a, b) = \text{HCF}(a, r)$:

a $a = 5,\ b = 43$ **b** $a = 13,\ b = 39$

c $a = 17,\ b = 37$ **d** $a = 16,\ b = 128$

2 If d is a common factor of a and b, prove that d is also a common factor of $a + b$ and $a - b$.

Example 17

3 Use the Euclidean algorithm to find:

a $\text{HCF}(4361, 9284)$ **b** $\text{HCF}(999, 2160)$

c $\text{HCF}(-372, 762)$ **d** $\text{HCF}(5255, 716\,485)$

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Example 18, 19

4 Solve each equation in the integers:

a $804x + 2358y = 6$ **b** $18x + 24y = 6$ **c** $3x + 4y = 478$

d $3x - 5y = 38$ **e** $804x + 2688y = 12$ **f** $1816x + 2688y = 8$

2H Problems involving sets

Sets can be used to help sort information, as each of the following examples demonstrates. Recall that, if A is a finite set, then the number of elements in A is denoted by $|A|$.

Example 20

Two hundred and eighty students each travel to school by either train or bus or both. Of these students, 150 travel by train, and 60 travel by both train and bus.

a Show this information on a Venn diagram.

b Hence find the number of students who travel by:

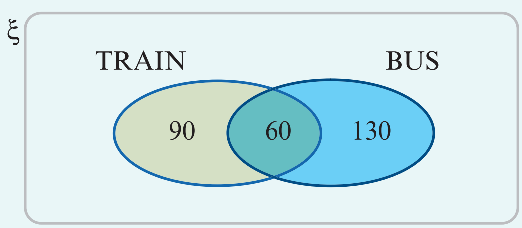
i bus

ii train but not bus

iii just one of these modes of transport.

Solution

a



$|\xi| = 280$

b **i** $|\text{BUS}| = 130 + 60 = 190$

ii $|\text{TRAIN} \cap (\text{BUS})'| = 90$

iii $|\text{TRAIN} \cap (\text{BUS})'| + |(\text{TRAIN})' \cap \text{BUS}| = 90 + 130 = 220$

Example 21

An athletics team has 18 members. Each member competes in at least one of three events: sprints (S), jumps (J) and hurdles (H). Every hurdler also jumps or sprints. The following additional information is available:

$$|S| = 11, \quad |J| = 10, \quad |J \cap H' \cap S'| = 5, \quad |J' \cap H' \cap S| = 5 \quad \text{and} \quad |J \cap H'| = 7$$

a Draw a Venn diagram.

b Find:

i $|H|$ **ii** $|S \cap H \cap J|$ **iii** $|S \cup J|$ **iv** $|S \cap J \cap H'|$

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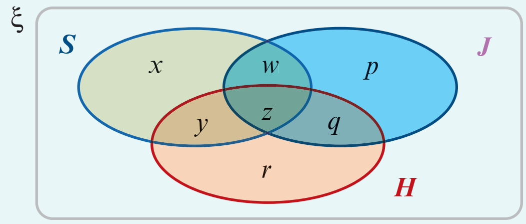
Solution

a Assign a variable to the number of members in each region of the Venn diagram.



The information in the question can be summarised in terms of these variables:

$x + y + z + w = 11$	as $\lvert S \rvert = 11$	(1)
$p + q + z + w = 10$	as $\lvert J \rvert = 10$	(2)
$x + y + z + w + p + q + r = 18$	as all members compete	(3)
$p = 5$	as $\lvert J \cap H' \cap S' \rvert = 5$	(4)
$x = 5$	as $\lvert J' \cap H' \cap S \rvert = 5$	(5)
$r = 0$	as every hurdler also jumps or sprints	(6)
$w + p = 7$	as $\lvert J \cap H' \rvert = 7$	(7)

From (4) and (7): $w = 2$.
Equation (3) now becomes

$$5 + y + z + 2 + 5 + q = 18$$

$$\therefore \quad y + z + q = 6 \qquad (8)$$

Equation (1) becomes

$$y + z = 4$$

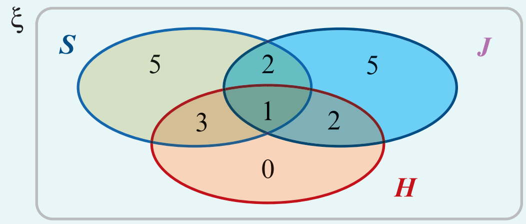
Therefore from (8): $q = 2$.
Equation (2) becomes

$$5 + 2 + z + 2 = 10$$

$$\therefore \quad z = 1$$

$$\therefore \quad y = 3$$

The Venn diagram can now be completed as shown.



b **i** $\lvert H \rvert = 6$ **ii** $\lvert S \cap H \cap J \rvert = 1$ **iii** $\lvert S \cup J \rvert = 18$ **iv** $\lvert S \cap J \cap H' \rvert = 2$

Exercise 2H

Skillsheet
Example 20

1 There are 28 students in a class, all of whom take either History or Economics or both. Of the 14 students who take History, five also take Economics.

a Show this information on a Venn diagram.

b Hence find the number of students who take:

i Economics **ii** History but not Economics **iii** just one of these subjects.

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2 **a** Draw a Venn diagram to show three sets A, B and C in a universal set ξ. Enter numbers in the correct parts of the diagram using the following information:

$$|A \cap B \cap C| = 2, \quad |A \cap B| = 7, \quad |B \cap C| = 6,$$
$$|A \cap C| = 8, \quad |A| = 16, \quad |B| = 20, \quad |C| = 19, \quad |\xi| = 50$$

b Use the diagram to find:

i $|A' \cap C'|$ **ii** $|A \cup B'|$ **iii** $|A' \cap B \cap C'|$

3 In a border town in the Balkans, 60% of people speak Bulgarian, 40% speak Greek and 20% speak neither. What percentage of the town speak both Bulgarian and Greek?

4 A survey of a class of 40 students showed that 16 own at least one dog and 25 at least one cat. Six students have neither. How many students own both?

5 At an international conference there were 105 delegates. Seventy spoke English, 50 spoke French and 50 spoke Japanese. Twenty-five spoke English and French, 15 spoke French and Japanese and 30 spoke Japanese and English.

a How many delegates spoke all three languages?

b How many spoke Japanese only?

6 A restaurant serves lunch to 350 people. It offers three desserts: profiteroles, gelati and fruit. Forty people have all three desserts, 70 have gelati only, 50 have profiteroles only and 60 have fruit only. Forty-five people have fruit and gelati only, 30 people have gelati and profiteroles only and 10 people have fruit and profiteroles only. How many people do not have a dessert?

Example 21 **7** Forty travellers were questioned about the various methods of transport they had used the previous day. Every traveller used at least one of the following methods: car (C), bus (B), train (T). Of these travellers:

- eight had used all three methods of transport
- four had travelled by bus and car only
- two had travelled by car and train only
- the number (x) who had travelled by train only was equal to the number who had travelled by bus and train only.

If 20 travellers had used a train and 33 had used a bus, find:

a the value of x

b the number who travelled by bus only

c the number who travelled by car only.

8 Let ξ be the set of all integers and let

$$X = \{ x : 21 < x < 37 \}, \quad Y = \{ 3y : 0 < y \le 13 \}, \quad Z = \{ z^2 : 0 < z < 8 \}$$

a Draw a Venn diagram representing these sets.

b **i** Find $X \cap Y \cap Z$. **ii** Find $|X \cap Y|$.

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9 A number of students bought red, green and black pens. Three bought one of each colour. Of the students who bought two colours, three did not buy red, five not green and two not black. The same number of students bought red only as bought red with other colours. The same number bought black only as bought green only. More students bought red and black but not green than bought black only. More bought only green than bought green and black but not red. How many students were there and how many pens of each colour were sold?

10 For three subsets B, M and F of a universal set ξ,

$$|B \cap M| = 12, \quad |M \cap F \cap B| = |F'|, \quad |F \cap B| > |M \cap F|,$$

$$|B \cap F' \cap M'| = 5, \quad |M \cap B' \cap F'| = 5, \quad |F \cap M' \cap B'| = 5, \quad |\xi| = 28$$

Find $|M \cap F|$.

11 A group of 80 students were interviewed about which sports they play. It was found that 23 do athletics, 22 swim and 18 play football. If 10 students do athletics and swim only, 11 students do athletics and play football only, six students swim and play football only and 46 students do none of these activities on a regular basis, how many students do all three?

12 At a certain secondary college, students have to be proficient in at least one of the languages Italian, French and German. In a particular group of 33 students, two are proficient in all three languages, three in Italian and French only, four in French and German only and five in German and Italian only. The number of students proficient in Italian only is x, in French only is x and in German only is $x + 1$. Find x and then find the total number of students proficient in Italian.

13 At a certain school, 201 students study one or more of Mathematics, Physics and Chemistry. Of these students: 35 take Chemistry only, 50% more students study Mathematics only than study Physics only, four study all three subjects, 25 study both Mathematics and Physics but not Chemistry, seven study both Mathematics and Chemistry but not Physics, and 20 study both Physics and Chemistry but not Mathematics. Find the number of students studying Mathematics.

ISBN 978-1-316-63608-4

Review

AS

Nrich

Chapter summary

Sets

- Set notation

$x \in A$	x is an element of A
$x \notin A$	x is not an element of A
ξ	the universal set
$\varnothing$	the empty set
$A \subseteq B$	A is a subset of B
$A \cup B$	the union of A and B consists of all elements that are in either A or B or both
$A \cap B$	the intersection of A and B consists of all elements that are in both A and B
A'	the complement of A consists of all elements of ξ that are not in A

- Sets of numbers

$\mathbb{N}$	Natural numbers	$\mathbb{Z}$	Integers
$\mathbb{Q}$	Rational numbers	$\mathbb{R}$	Real numbers

The modulus function

- The **modulus** or **absolute value** of a real number x is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

For example: $|5| = 5$ and $|-5| = 5$.

- On the number line, the distance between two numbers a and b is given by $|a - b| = |b - a|$. For example: $|x - 2| < 5$ can be read as 'the distance of x from 2 is less than 5'.

Surds

- A **quadratic surd** is a number of the form $\sqrt{a}$, where a is a rational number which is not the square of another rational number.
- A **surd of order n** is a number of the form $\sqrt[n]{a}$, where a is a rational number which is not a perfect nth power.
- When the number under the square root has no factors which are squares of a rational number, the surd is said to be in **simplest form**.
- Surds which have the same 'irrational factor' are called **like surds**. The sum or difference of two like surds can be simplified:

$$m\sqrt{p} + n\sqrt{p} = (m + n)\sqrt{p} \quad \text{and} \quad m\sqrt{p} - n\sqrt{p} = (m - n)\sqrt{p}$$

Natural numbers

- A natural number a is a **factor** of a natural number b if there exists a natural number k such that $b = ak$.
- A natural number greater than 1 is a **prime number** if its only factors are itself and 1.
- A natural number m is a **composite number** if it can be written as a product $m = a \times b$, where a and b are natural numbers greater than 1 and less than m.

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- Every composite number can be expressed as a product of powers of prime numbers; this is called **prime decomposition**. For example: $1300 = 2^2 \times 5^2 \times 13$
- The **highest common factor** of two natural numbers a and b, denoted by $\text{HCF}(a, b)$, is the largest natural number that is a factor of both a and b.
- The **lowest common multiple** of two natural numbers a and b, denoted by $\text{LCM}(a, b)$, is the smallest natural number that is a multiple of both a and b.

Diophantine equations and the Euclidean algorithm

- A **Diophantine equation** is an equation in which only integer solutions are allowed.
- An equation of the form $ax + by = c$, where the coefficients a, b, c are integers, is called a **linear Diophantine equation** when the intention is to find integer solutions for x, y.
- If $ax + by = c$ is a linear Diophantine equation and (x_0, y_0) is found to be one solution, then the general solution is given by

  $$x = x_0 + \frac{b}{d}t, \quad y = y_0 - \frac{a}{d}t \qquad \text{for } t \in \mathbb{Z}$$

  where d is the highest common factor of a and b.
- A linear Diophantine equation $ax + by = c$ has integer solutions if and only if $\text{HCF}(a, b)$ divides c.
- Division algorithm If a and b are integers with $a > 0$, then there are unique integers q and r such that $b = aq + r$ and $0 \le r < a$.
- Euclidean algorithm Let a and b be integers with $a \neq 0$. If $b = aq + r$, where q and r are integers, then $\text{HCF}(a, b) = \text{HCF}(a, r)$.

  The repeated application of this result can be used to find the highest common factor of two natural numbers and to solve linear Diophantine equations.

Short-answer questions

1 Express the following as fractions in their simplest form:

a $0.0\dot{7}$ **b** $0.\dot{4}\dot{5}$ **c** 0.005 **d** 0.405 **e** $0.2\dot{6}$ **f** $0.1\dot{7}1428\dot{5}$

2 Express 504 as a product of powers of prime numbers.

3 **a** Find the four integer values of n such that $|n^2 - 9|$ is a prime number.

b Solve each equation for x:

i $x^2 + 5|x| - 6 = 0$ **ii** $x + |x| = 0$

c Solve the inequality $5 - |x| < 4$ for x.

4 Express each of the following with a rational denominator:

a $\dfrac{2\sqrt{3} - 1}{\sqrt{2}}$ **b** $\dfrac{\sqrt{5} + 2}{\sqrt{5} - 2}$ **c** $\dfrac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}}$

5 Express $\dfrac{3 + 2\sqrt{75}}{3 - \sqrt{12}}$ in the form $a + b\sqrt{3}$, where $a, b \in \mathbb{Q} \setminus \{0\}$.

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Review

6 Express each of the following with a rational denominator:

a $\dfrac{6\sqrt{2}}{3\sqrt{2} - 2\sqrt{3}}$ **b** $\dfrac{\sqrt{a+b} - \sqrt{a-b}}{\sqrt{a+b} + \sqrt{a-b}}$

7 In a class of 100 students, 55 are girls, 45 have blue eyes, 40 are blond, 25 are blond girls, 15 are blue-eyed blonds, 20 are blue-eyed girls, and five are blue-eyed blond girls. Find:

a the number of blond boys

b the number of boys who are neither blond nor blue-eyed.

8 A group of 30 students received prizes in at least one of the subjects of English, Mathematics and French. Two students received prizes in all three subjects. Fourteen received prizes in English and Mathematics but not French. Two received prizes in English alone, two in French alone and five in Mathematics alone. Four received prizes in English and French but not Mathematics.

a How many received prizes in Mathematics and French but not English?

b How many received prizes in Mathematics?

c How many received prizes in English?

9 Fifty people are interviewed. Twenty-three people like Brand X, 25 like Brand Y and 19 like Brand Z. Eleven like X and Z. Eight like Y and Z. Five like X and Y. Two like all three. How many like none of them?

10 Three rectangles A, B and C overlap (intersect). Their areas are 20 cm^2, 10 cm^2 and 16 cm^2 respectively. The area common to A and B is 3 cm^2, that common to A and C is 6 cm^2 and that common to B and C is 4 cm^2. How much of the area is common to all three if the total area covered is 35 cm^2?

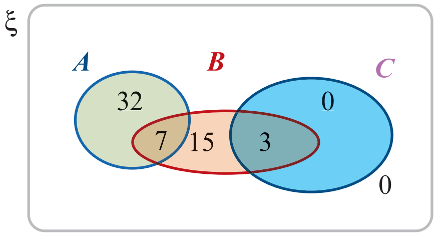
11 Express $\sqrt{112} - \sqrt{63} - \dfrac{224}{\sqrt{28}}$ in simplest form.

12 If $\dfrac{\sqrt{7} - \sqrt{3}}{x} = \dfrac{x}{\sqrt{7} + \sqrt{3}}$, find the values of x.

13 Express $\dfrac{1 + \sqrt{2}}{\sqrt{5} + \sqrt{3}} + \dfrac{1 - \sqrt{2}}{\sqrt{5} - \sqrt{3}}$ in the form $a\sqrt{5} + b\sqrt{6}$.

14 Simplify $\sqrt{27} - \sqrt{12} + 2\sqrt{75} - \sqrt{\dfrac{48}{25}}$.

15 A, B and C are three sets and $\xi = A \cup B \cup C$. The number of elements in the regions of the Venn diagram are as shown. Find:



a the number of elements in $A \cup B$

b the number of elements in C

c the number of elements in $B' \cap A$.

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Review

16 Using the result that $(\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab}$, determine the square root of $17 + 6\sqrt{8}$.

17 Find the highest common factor of 1885 and 365 using the Euclidean algorithm.

18 Consider $9x + 43y = 7$. Solve for x and y where:

a $x \in \mathbb{Z}, y \in \mathbb{Z}$ **b** $x \in \mathbb{N}, y \in \mathbb{N}$

19 Prove that the product of two odd integers is odd. (You may assume that the sum and product of any two integers is an integer.)

20 Using the Euclidean algorithm, find the highest common factor of 10 659 and 12 121.

21 **a** Solve the Diophantine equation $5x + 7y = 1$.

b Hence solve the Diophantine equation $5x + 7y = 100$.

c Find $\{ (x, y) : 5x + 7y = 1,\ y \geq x \text{ and } x, y \in \mathbb{Z} \}$.

22 The sum of the ages of Tom and Fred is 63. Tom is twice as old as Fred was when Tom was as old as Fred is now. What are the ages of Tom and Fred?

Multiple-choice questions

1 $\dfrac{4}{3 + 2\sqrt{2}}$ expressed in the form $a + b\sqrt{2}$ is

A $12 - 8\sqrt{2}$ **B** $3 + 2\sqrt{2}$ **C** $\dfrac{3}{17} - \dfrac{8}{17}\sqrt{2}$

D $\dfrac{3}{17} + \dfrac{8}{17}\sqrt{2}$ **E** $12 + 8\sqrt{2}$

2 The prime decomposition of 86 400 is

A $2^5 \times 3^2 \times 5$ **B** $2^6 \times 3^3 \times 5^2$ **C** $2^7 \times 3^3 \times 5$

D $2^7 \times 3^3 \times 5^2$ **E** $2^6 \times 3^3 \times 5^3$

3 $(\sqrt{6} + 3)(\sqrt{6} - 3)$ is equal to

A $3 - 12\sqrt{6}$ **B** $-3 - 6\sqrt{6}$ **C** $-3 + 6\sqrt{6}$ **D** -3 **E** 3

4 For the Venn diagram shown, ξ is the set of natural numbers less than 20, A is the set of natural numbers less than 10, and B is the set of natural numbers less than 20 that are divisible by 3. The set $B' \cap A$ is

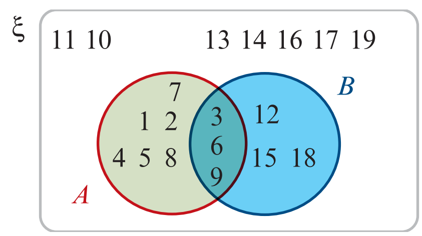
A $\{3, 6, 9\}$

B $\{12, 15, 18\}$

C $\{10, 11, 13, 14, 16, 17, 19\}$

D $\{1, 2, 4, 5, 7, 8\}$

E $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 15, 18\}$



5 $(3, \infty) \cap (-\infty, 5] =$

A $(-\infty, 3)$ **B** $(-\infty, 5]$ **C** $(3, 5]$ **D** $\mathbb{R}$ **E** $[3, 5]$

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6 A bell is rung every 6 minutes and a gong is sounded every 14 minutes. If these occur together at a particular time, then the smallest number of minutes until the bell and the gong are again heard simultaneously is

A 10 **B** 20 **C** 72 **D** 42 **E** 84

7 If X is the set of multiples of 2, Y the set of multiples of 7, and Z the set of multiples of 5, then $X \cap Y \cap Z$ can be described as

A the set of multiples of 2
B the set of multiples of 70
C the set of multiples of 35
D the set of multiples of 14
E the set of multiples of 10

8 In a class of students, 50% play football, 40% play tennis and 30% play neither. The percentage that plays both is

A 10 **B** 20 **C** 30 **D** 50 **E** 40

9 $\dfrac{\sqrt{7}-\sqrt{6}}{\sqrt{7}+\sqrt{6}} =$

A $5+2\sqrt{7}$ **B** $13+2\sqrt{6}$ **C** $13-2\sqrt{42}$ **D** $1+2\sqrt{42}$ **E** $13-2\sqrt{13}$

10 There are 40 students in a class, all of whom take either Literature or Economics or both. Twenty take Literature and five of these also take Economics. The number of students who take only Economics is

A 20 **B** 5 **C** 10 **D** 15 **E** 25

11 The number of solutions of the Diophantine equation $3x + 5y = 1008$, where x and y are positive integers, is

A 1 **B** 134 **C** 68 **D** 67 **E** infinite

12 The number of factors that the integer $2^p 3^q 5^r$ has is

A $\dfrac{(p+q+r)!}{p!\,q!\,r!}$ **B** pqr **C** $p+q+r$

D $(p+1)(q+1)(r+1)$ **E** $p+q+r+1$

13 The number of pairs of integers (m, n) which satisfy the equation $m + n = mn$ is

A 1 **B** 2 **C** 3 **D** 4 **E** more than 4

Extended-response questions

1 **a** Show that $(\sqrt{x}+\sqrt{y})^2 = x + y + 2\sqrt{xy}$.

b Substitute $x = 3$ and $y = 5$ in the identity from part **a** to show that

$$\sqrt{3}+\sqrt{5} = \sqrt{8+2\sqrt{15}}$$

c Use this technique to find the square root of:

i $14+2\sqrt{33}$ (Hint: Use $x = 11$ and $y = 3$.) **ii** $15-2\sqrt{56}$ **iii** $51-36\sqrt{2}$

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2 In this question, we consider the set $\{ a + b\sqrt{3} : a, b \in \mathbb{Q} \}$. In Chapter 13, the set $\mathbb{C}$ of complex numbers is introduced, where $\mathbb{C} = \{ a + b\sqrt{-1} : a, b \in \mathbb{R} \}$.

a If $(2 + 3\sqrt{3}) + (4 + 2\sqrt{3}) = a + b\sqrt{3}$, find a and b.

b If $(2 + 3\sqrt{3})(4 + 2\sqrt{3}) = p + q\sqrt{3}$, find p and q.

c If $\dfrac{1}{3 + 2\sqrt{3}} = a + b\sqrt{3}$, find a and b.

d Solve each of the following equations for x:

i $(2 + 5\sqrt{3})x = 2 - \sqrt{3}$ **ii** $(x - 3)^2 - 3 = 0$ **iii** $(2x - 1)^2 - 3 = 0$

e Explain why every rational number is a member of $\{ a + b\sqrt{3} : a, b \in \mathbb{Q} \}$.

3 **a** Show that $\dfrac{1}{2 + \sqrt{3}} = 2 - \sqrt{3}$.

b Use the substitution $t = \left(\sqrt{2 + \sqrt{3}}\right)^x$ and part **a** to show that the equation

$$\left(\sqrt{2 + \sqrt{3}}\right)^x + \left(\sqrt{2 - \sqrt{3}}\right)^x = 4$$

can be written as $t + \dfrac{1}{t} = 4$.

c Show that the solutions of the equation are $t = 2 - \sqrt{3}$ and $t = 2 + \sqrt{3}$.

d Use this result to solve the equation $\left(\sqrt{2 + \sqrt{3}}\right)^x + \left(\sqrt{2 - \sqrt{3}}\right)^x = 4$.

4 Use Venn diagrams to illustrate:

a $|A \cup B| = |A| + |B| - |A \cap B|$

b $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

5 A quadratic equation with integer coefficients $x^2 + bx + c = 0$ has a solution $x = 2 - \sqrt{3}$.

a Find the values of b and c.

Hint: Use the result that, for m, n rational, if $m + n\sqrt{3} = 0$, then $m = 0$ and $n = 0$.

b Find the other solution to this quadratic equation.

c Now consider a quadratic equation with integer coefficients $x^2 + bx + c = 0$ that has a solution $x = m - n\sqrt{q}$, where q is not a perfect square. Show that:

i $b = -2m$ **ii** $c = m^2 - n^2 q$

Hence show that:

iii $x^2 + bx + c = (x - (m - n\sqrt{q}))(x - (m + n\sqrt{q}))$

6 A **Pythagorean triple** (x, y, z) consists of three natural numbers x, y, z such that $x^2 + y^2 = z^2$. For example: $(3, 4, 5)$ and $(5, 12, 13)$ are Pythagorean triples.
A Pythagorean triple is in simplest form if x, y, z have no common factor. Up to swapping x and y, all Pythagorean triples in simplest form may be generated by:

$$x = 2mn, \quad y = m^2 - n^2, \quad z = m^2 + n^2 \qquad \text{where } m, n \in \mathbb{N}$$

For example, if $m = 2$ and $n = 1$, then $x = 4$, $y = 3$ and $z = 5$.

a Find the Pythagorean triple for $m = 5$ and $n = 2$.

b Verify that, if $x = 2mn$, $y = m^2 - n^2$ and $z = m^2 + n^2$, where $m, n \in \mathbb{N}$, then $x^2 + y^2 = z^2$.

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7 The factors of 12 are 1, 2, 3, 4, 6, 12.

a How many factors does each of the following numbers have?

i 2^3 **ii** 3^7

b How many factors does 2^n have?

c How many factors does each of the following numbers have?

i $2^3 \cdot 3^7$ **ii** $2^n \cdot 3^m$

d Every natural number greater than 1 may be expressed as a product of powers of primes; this is called prime decomposition. For example: $1080 = 2^3 \times 3^3 \times 5$.
Let x be a natural number greater than 1 and let

$$x = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_n^{\alpha_n}$$

be its prime decomposition, where each $\alpha_i \in \mathbb{N}$ and each p_i is a prime number.
How many factors does x have? (Answer to be given in terms of α_i.)

e Find the smallest number which has eight factors.

8 **a** Give the prime decompositions of 1080 and 25 200.

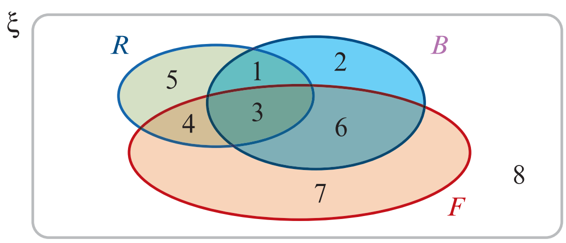
b Use your answer to part **a** to find the lowest common multiple of 1080 and 25 200.

c Carefully explain why, if m and n are integers, then $mn = \text{LCM}(m, n) \times \text{HCF}(m, n)$.

d **i** Find four consecutive even numbers such that the smallest is a multiple of 5, the second a multiple of 7, the third a multiple of 9 and the largest a multiple of 11.

ii Find four consecutive natural numbers such that the smallest is a multiple of 5, the second a multiple of 7, the third a multiple of 9 and the largest a multiple of 11.

9 **a** The Venn diagram shows the set ξ of all students enrolled at Argos Secondary College. Set R is the set of all students with red hair. Set B is the set of all students with blue eyes. Set F is the set of all female students.



The numbers on the diagram are to label the eight different regions.

i Identify the region in the Venn diagram which represents male students who have neither red hair nor blue eyes.

ii Describe the gender, hair colour and eye colour of students represented in region 1 of the diagram.

iii Describe the gender, hair colour and eye colour of students represented in region 2 of the diagram.

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b It is known that, at Argos Secondary College, 250 students study French (F), Greek (G) or Japanese (J). Forty-one students do not study French. Twelve students study French and Japanese but not Greek. Thirteen students study Japanese and Greek but not French. Thirteen students study only Greek. Twice as many students study French and Greek but not Japanese as study all three. The number studying only Japanese is the same as the number studying both French and Greek.

  i How many students study all three languages?

  ii How many students study only French?

10 Consider the universal set ξ as the set of all students enrolled at Sounion Secondary College. Let B denote the set of students taller than 180 cm and let A denote the set of female students.

a Give a brief description of each of the following sets:

  i B' **ii** $A \cup B$ **iii** $A' \cap B'$

b Use a Venn diagram to show $(A \cup B)' = A' \cap B'$.

c Hence show that $A \cup B \cup C = (A' \cap B' \cap C')'$, where C is the set of students who play sport.

11 In a certain city, three Sunday newspapers (A, B and C) are available. In a sample of 500 people from this city, it was found that:

- nobody regularly reads both A and C
- a total of 100 people regularly read A
- 205 people regularly read only B
- of those who regularly read C, exactly half of them also regularly read B
- 35 people regularly read A and B but not C
- 35 people don't read any of the papers at all.

a Draw a Venn diagram showing the number of regular readers for each possible combination of A, B and C.

b How many people in the sample were regular readers of C?

c How many people in the sample regularly read A only?

d How many people are regular readers of A, B and C?

12 You have an inexhaustible supply of 5c and 8c stamps.

a List all possible ways of obtaining a total value of 38c with these stamps.

b List all possible ways of obtaining a total of \$1.20 with these stamps.

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Chapter 3

3 Algebra II

Objectives

- To understand equality of **polynomials**.
- To use **equating coefficients** to solve problems.
- To solve **quadratic equations** by various methods.
- To use quadratic equations to solve problems involving **rates**.
- To resolve rational algebraic expressions into **partial fractions**.
- To find the coordinates of the points of intersection of straight lines with parabolas, circles and rectangular hyperbolas.

In this chapter we first consider equating coefficients of polynomial functions, and then apply this technique to establish partial fractions.

In Chapter 1 we added and subtracted algebraic fractions such as

$$\frac{2}{x+3} + \frac{4}{x-3} = \frac{6(x+1)}{x^2-9}$$

In this chapter we learn how to go from right to left in similar equations. This process is sometimes called **partial fraction decomposition**. Another example is

$$\frac{4x^2+2x+6}{(x^2+3)(x-3)} = \frac{2}{x^2+3} + \frac{4}{x-3}$$

This is a useful tool in integral calculus, and partial fractions are applied this way in Specialist Mathematics Year 12.

This chapter also includes further study of quadratic functions: solving quadratic equations, using the discriminant, applying quadratic functions to problems involving rates and using quadratic equations to find the intersection of straight lines with parabolas, circles and rectangular hyperbolas.

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3A Polynomial identities

Polynomials are introduced in Mathematical Methods Year 11.

- A **polynomial function** is a function that can be written in the form

  $$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

  where n is a natural number or zero, and the coefficients $a_0, \ldots, a_n$ are real numbers with $a_n \neq 0$.
- The number 0 is called the **zero polynomial**.
- The **leading term**, $a_n x^n$, of a polynomial is the term of highest index among those terms with a non-zero coefficient.
- The **degree of a polynomial** is the index n of the leading term.
- A **monic polynomial** is a polynomial whose leading term has coefficient 1.
- The **constant term** is the term of index 0. (This is the term not involving x.)

Two polynomials are **equal** if they give the same value for all x. It can be proved that, if two polynomials are equal, then they have the same degree and corresponding coefficients are equal. For example:

- If $ax + b = cx^2 + dx + e$, then $c = 0$, $d = a$ and $e = b$.
- If $x^2 - x - 12 = x^2 + (a + b)x + ab$, then $a + b = -1$ and $ab = -12$.

This process is called **equating coefficients**.

Example 1

If the expressions $(a + 2b)x^2 - (a - b)x + 8$ and $3x^2 - 6x + 8$ are equal for all x, find the values of a and b.

Solution

Assume that

$$(a + 2b)x^2 - (a - b)x + 8 = 3x^2 - 6x + 8 \quad \text{for all } x$$

Then by equating coefficients:

$$a + 2b = 3 \qquad (1)$$

$$-(a - b) = -6 \qquad (2)$$

Solve as simultaneous equations.

Add (1) and (2):

$$3b = -3$$

$$\therefore \quad b = -1$$

Substitute into (1):

$$a - 2 = 3$$

$$\therefore \quad a = 5$$

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Example 2

Express x^2 in the form $c(x-3)^2 + a(x-3) + d$.

Solution

Let
$$\begin{aligned} x^2 &= c(x-3)^2 + a(x-3) + d \\ &= c(x^2 - 6x + 9) + a(x-3) + d \\ &= cx^2 + (a-6c)x + 9c - 3a + d \end{aligned}$$

This implies that

$$c = 1 \qquad (1)$$
$$a - 6c = 0 \qquad (2)$$
$$9c - 3a + d = 0 \qquad (3)$$

From (2): $a = 6$

From (3): $9 - 18 + d = 0$

i.e. $d = 9$

Hence $x^2 = (x-3)^2 + 6(x-3) + 9$.

Example 3

Find the values of a, b, c and d such that

$$x^3 = a(x+2)^3 + b(x+1)^2 + cx + d \quad \text{for all } x$$

Solution

Expand the right-hand side and collect like terms:

$$\begin{aligned} x^3 &= a(x^3 + 6x^2 + 12x + 8) + b(x^2 + 2x + 1) + cx + d \\ &= ax^3 + (6a+b)x^2 + (12a + 2b + c)x + (8a + b + d) \end{aligned}$$

Equate coefficients:

$$a = 1 \qquad (1)$$
$$6a + b = 0 \qquad (2)$$
$$12a + 2b + c = 0 \qquad (3)$$
$$8a + b + d = 0 \qquad (4)$$

Substituting $a = 1$ into (2) gives

$$6 + b = 0$$
$$\therefore \quad b = -6$$

Substituting $a = 1$ and $b = -6$ into (3) gives

$$12 - 12 + c = 0$$
$$\therefore \quad c = 0$$

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Substituting $a = 1$ and $b = -6$ into (4) gives

$$8 - 6 + d = 0$$

$$\therefore \quad d = -2$$

Hence $x^3 = (x+2)^3 - 6(x+1)^2 - 2$.

Example 4

Show that $2x^3 - 5x^2 + 4x + 1$ cannot be expressed in the form $a(x+b)^3 + c$.

Solution

Suppose that

$$2x^3 - 5x^2 + 4x + 1 = a(x+b)^3 + c$$

for some constants a, b and c.

Then expanding the right-hand side gives

$$\begin{aligned} 2x^3 - 5x^2 + 4x + 1 &= a(x^3 + 3bx^2 + 3b^2x + b^3) + c \\ &= ax^3 + 3abx^2 + 3ab^2x + ab^3 + c \end{aligned}$$

Equating coefficients:

$$\begin{aligned} a &= 2 & (1) \\ 3ab &= -5 & (2) \\ 3ab^2 &= 4 & (3) \\ ab^3 + c &= 1 & (4) \end{aligned}$$

From (2), we have $b = -\frac{5}{6}$. But from (3), we have $b = \pm\sqrt{\frac{2}{3}} = \pm\frac{\sqrt{6}}{3}$.

This is a contradiction, and therefore we have shown that $2x^3 - 5x^2 + 4x + 1$ cannot be expressed in the form $a(x+b)^3 + c$.

Section summary

- A **polynomial function** can be written in the form

  $$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

  where n is a natural number or zero, and the coefficients $a_0, \ldots, a_n$ are real numbers with $a_n \neq 0$. The **leading term** is $a_n x^n$ (the term of highest index) and the **constant term** is a_0 (the term not involving x).
- The **degree** of a polynomial is the index n of the leading term.
- Equating coefficients

  Two polynomials are equal if they give the same value for all x. If two polynomials are equal, then they have the same degree and corresponding coefficients are equal.

  For example: if $x^2 - x - 12 = x^2 + (a+b)x + ab$, then $a + b = -1$ and $ab = -12$.

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Exercise 3A

1 If $ax^2 + bx + c = 10x^2 - 7$, find the values of a, b and c.

Example 1 **2** If $(2a - b)x^2 + (a + 2b)x + 8 = 4x^2 - 3x + 8$, find the values of a and b.

3 If $(2a - 3b)x^2 + (3a + b)x + c = 7x^2 + 5x + 7$, find the values of a, b and c.

4 If $2x^2 + 4x + 5 = a(x + b)^2 + c$, find the values of a, b and c.

Example 2 **5** Express x^2 in the form $c(x + 2)^2 + a(x + 2) + d$.

6 Express x^3 in the form $(x + 1)^3 + a(x + 1)^2 + b(x + 1) + c$.

Example 3 **7** Find the values of a, b and c such that $x^2 = a(x + 1)^2 + bx + c$.

Example 4 **8** **a** Show that $3x^3 - 9x^2 + 8x + 2$ cannot be expressed in the form $a(x + b)^3 + c$.

b If $3x^3 - 9x^2 + 9x + 2$ can be expressed in the form $a(x + b)^3 + c$, then find the values of a, b and c.

9 Show that constants a, b, c and d can be found such that

$$n^3 = a(n + 1)(n + 2)(n + 3) + b(n + 1)(n + 2) + c(n + 1) + d$$

10 **a** Show that no constants a and b can be found such that

$$n^2 = a(n + 1)(n + 2) + b(n + 2)(n + 3)$$

b Express n^2 in the form $a(n + 1)(n + 2) + b(n + 1) + c$.

11 **a** Express $a(x + b)^2 + c$ in expanded form.

b Express $ax^2 + bx + c$ in completed-square form.

12 Prove that, if $ax^3 + bx^2 + cx + d = (x - 1)^2(px + q)$, then $b = d - 2a$ and $c = a - 2d$.

13 If $3x^2 + 10x + 3 = c(x - a)(x - b)$ for all values of x, find the values of a, b and c.

14 For any number n, show that n^2 can be expressed as $a(n - 1)^2 + b(n - 2)^2 + c(n - 3)^2$, and find the values of a, b and c.

15 If $x^3 + 3x^2 - 9x + c$ can be expressed in the form $(x - a)^2(x - b)$, show that either $c = 5$ or $c = -27$, and find a and b for each of these cases.

16 A polynomial P is said to be **even** if $P(-x) = P(x)$ for all x. A polynomial P is said to be **odd** if $P(-x) = -P(x)$ for all x.

a Show that, if $P(x) = ax^4 + bx^3 + cx^2 + dx + e$ is even, then $b = d = 0$.

b Show that, if $P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ is odd, then $b = d = f = 0$.

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3B Quadratic equations

A polynomial function of degree 2 is called a **quadratic function**. The general quadratic function can be written as $P(x) = ax^2 + bx + c$, where $a \neq 0$.

Quadratic functions are studied extensively in Mathematical Methods Year 11. In this section we provide further practice exercises.

A quadratic equation $ax^2 + bx + c = 0$ may be solved by factorising, by completing the square or by using the general quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The following example demonstrates each method.

Example 5

Solve the following quadratic equations for x:

a $2x^2 + 5x = 12$ **b** $3x^2 + 4x = 2$ **c** $9x^2 + 6x + 1 = 0$

Solution	*Explanation*
a $2x^2 + 5x - 12 = 0$	Rearrange the quadratic equation.
$(2x - 3)(x + 4) = 0$	Factorise.
$2x - 3 = 0$ or $x + 4 = 0$	Use the null factor theorem.
Therefore $x = \frac{3}{2}$ or $x = -4$.	
b $3x^2 + 4x - 2 = 0$	Rearrange the quadratic equation.
$x^2 + \frac{4}{3}x - \frac{2}{3} = 0$	Divide both sides by 3.
$x^2 + \frac{4}{3}x + \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^2 - \frac{2}{3} = 0$	Add and subtract $\left(\frac{b}{2}\right)^2$ to 'complete the square'.
$\left(x + \frac{2}{3}\right)^2 - \frac{4}{9} - \frac{2}{3} = 0$	
$\left(x + \frac{2}{3}\right)^2 = \frac{10}{9}$	
$x + \frac{2}{3} = \pm\frac{\sqrt{10}}{3}$	
$x = -\frac{2}{3} \pm \frac{\sqrt{10}}{3}$	
Therefore $x = \frac{-2 + \sqrt{10}}{3}$ or $x = \frac{-2 - \sqrt{10}}{3}$.	

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c If $9x^2 + 6x + 1 = 0$, then

$$x = \frac{-6 \pm \sqrt{6^2 - 4 \times 9 \times 1}}{2 \times 9}$$

$$= \frac{-6 \pm \sqrt{0}}{18}$$

$$= -\frac{1}{3}$$

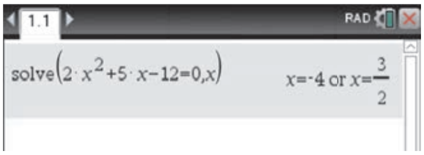
Use the general quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Alternatively, the equation can be solved by noting that $9x^2 + 6x + 1 = (3x + 1)^2$.

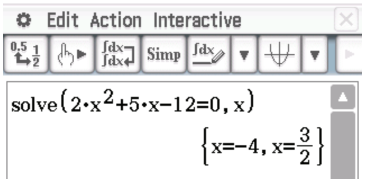
Using the TI-Nspire

Use (menu) > **Algebra** > **Solve** as shown.



Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight the equation $2x^2 + 5x - 12 = 0$.
- Select **Interactive** > **Equation/Inequality** > **solve** and ensure the variable is set to x. Tap OK.



The discriminant: real solutions

The number of solutions to a quadratic equation $ax^2 + bx + c = 0$ can be determined by the **discriminant** Δ, where $\Delta = b^2 - 4ac$.

- If $\Delta > 0$, the equation has two real solutions.
- If $\Delta = 0$, the equation has one real solution.
- If $\Delta < 0$, the equation has no real solutions.

Note: In parts **a** and **b** of Example 5, we have $\Delta > 0$ and so there are two real solutions. In part **c**, we have $\Delta = 6^2 - 4 \times 9 \times 1 = 0$ and so there is only one real solution.

The discriminant: rational solutions

For a quadratic equation $ax^2 + bx + c = 0$ such that a, b and c are rational numbers:

- If Δ is a perfect square and $\Delta \neq 0$, then the equation has two rational solutions.
- If $\Delta = 0$, then the equation has one rational solution.
- If Δ is not a perfect square and $\Delta > 0$, then the equation has two irrational solutions.

Note: In part **a** of Example 5, we have $\Delta = 121$, which is a perfect square.

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Example 6

Consider the quadratic equation $x^2 - 4x = t$. Make x the subject and give the values of t for which real solution(s) to the equation can be found.

Solution

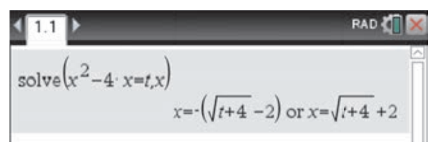
$$\begin{aligned} x^2 - 4x &= t \\ x^2 - 4x + 4 &= t + 4 \qquad \text{(completing the square)} \\ (x-2)^2 &= t + 4 \\ x - 2 &= \pm\sqrt{t+4} \\ x &= 2 \pm \sqrt{t+4} \end{aligned}$$

For real solutions to exist, we must have $t + 4 \geq 0$, i.e. $t \geq -4$.

Note: In this case the discriminant is $\Delta = 16 + 4t$. There are real solutions when $\Delta \geq 0$.

Using the TI-Nspire

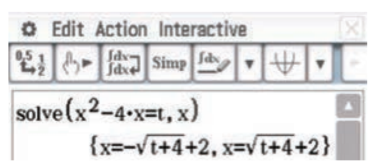
Use (menu) > **Algebra** > **Solve** as shown.



Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight the equation $x^2 - 4x = t$.
- Select **Interactive** > **Equation/Inequality** > **solve** and ensure the variable is set to x.

Note: The variable t is found in the (Var) keyboard.



Example 7

a Find the discriminant of the quadratic $x^2 + px - \frac{25}{4}$ in terms of p.

b Solve the quadratic equation $x^2 + px - \frac{25}{4} = 0$ in terms of p.

c Prove that there are two solutions for all values of p.

d Find the values of p, where p is a non-negative integer, for which the quadratic equation has rational solutions.

Solution

Here we have $a = 1$, $b = p$ and $c = -\frac{25}{4}$.

a $\Delta = b^2 - 4ac = p^2 + 25$

b The quadratic formula gives $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-p \pm \sqrt{p^2 + 25}}{2}$.

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c We have $\Delta = p^2 + 25 > 0$, for all values of p. Thus there are always two solutions.

d If there are rational solutions, then $\Delta = p^2 + 25$ is a perfect square. Since p is an integer, we can write $p^2 + 25 = k^2$, where k is an integer with $k \geq 0$.

Rearranging, we have

$$k^2 - p^2 = 25$$

$$\therefore \quad (k - p)(k + p) = 25$$

We can factorise 25 as 5×5 or 1×25.

Note: We do not need to consider negative factors of 25, as p and k are non-negative, and so $k + p \geq 0$. Since p is non-negative, we also know that $k - p \leq k + p$.

The table on the right shows the values of k and p in each of the two cases.

Hence $p = 0$ and $p = 12$ are the only values for which the solutions are rational.

$k - p$	$k + p$	k	p
5	5	5	0
1	25	13	12

Example 8

A rectangle has an area of 288 cm^2. If the width is decreased by 1 cm and the length increased by 1 cm, the area would be decreased by 3 cm^2. Find the original dimensions of the rectangle.

Solution

Let w and ℓ be the width and length, in centimetres, of the original rectangle.

Then $\quad w\ell = 288 \quad (1)$

The dimensions of the new rectangle are $w - 1$ and $\ell + 1$, and the area is 285 cm^2.

Thus $\quad (w - 1)(\ell + 1) = 285 \quad (2)$

Rearranging (1) to make ℓ the subject and substituting in (2) gives

$$(w - 1)\left(\frac{288}{w} + 1\right) = 285$$

$$288 - \frac{288}{w} + w - 1 = 285$$

$$w - \frac{288}{w} + 2 = 0$$

$$w^2 + 2w - 288 = 0$$

Using the general quadratic formula gives

$$w = \frac{-2 \pm \sqrt{2^2 - 4 \times (-288)}}{2}$$

$$= -18 \text{ or } 16$$

But $w > 0$, and so $w = 16$. The original dimensions of the rectangle are 16 cm by 18 cm.

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Example 9

Solve the equation $x - 4\sqrt{x} - 12 = 0$ for x.

Solution

For $\sqrt{x}$ to be defined, we must have $x \geq 0$.

Let $x = a^2$, where $a \geq 0$.

The equation becomes

$$a^2 - 4\sqrt{a^2} - 12 = 0$$
$$a^2 - 4a - 12 = 0$$
$$(a - 6)(a + 2) = 0$$
$$\therefore \quad a = 6 \text{ or } a = -2$$

But $a \geq 0$. Hence $a = 6$ and so $x = 36$.

Section summary

- Quadratic equations can be solved by completing the square. This method allows us to deal with all quadratic equations, even though there may be no solution for some quadratic equations.
- To complete the square of $x^2 + bx + c$:
  - Take half the coefficient of x (that is, $\frac{b}{2}$) and add and subtract its square $\frac{b^2}{4}$.
- To complete the square of $ax^2 + bx + c$:
  - First take out a as a factor and then complete the square inside the bracket.
- The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- The **discriminant** Δ of a quadratic polynomial $ax^2 + bx + c$ is

$$\Delta = b^2 - 4ac$$

  For the equation $ax^2 + bx + c = 0$:
  - If $\Delta > 0$, there are two solutions.
  - If $\Delta = 0$, there is one solution.
  - If $\Delta < 0$, there are no real solutions.
- For the equation $ax^2 + bx + c = 0$, where a, b and c are rational numbers:
  - If Δ is a perfect square and $\Delta \neq 0$, there are two rational solutions.
  - If $\Delta = 0$, there is one rational solution.
  - If Δ is not a perfect square and $\Delta > 0$, there are two irrational solutions.

ISBN 978-1-316-63608-4

Exercise 3B

Skillsheet
Example 5

1 Solve the following quadratic equations for x:

a $x^2 + 2x = -1$ **b** $x^2 - 6x + 9 = 0$ **c** $5x^2 - 10x = 1$

d $-2x^2 + 4x = 1$ **e** $2x^2 + 4x = 7$ **f** $6x^2 + 13x + 1 = 0$

2 The following equations have the number of solutions shown in brackets. Find the possible values of m.

a $x^2 + 3x + m = 0$ (0) **b** $x^2 - 5x + m = 0$ (2) **c** $mx^2 + 5x - 8 = 0$ (1)

d $x^2 + mx + 9 = 0$ (2) **e** $x^2 - mx + 4 = 0$ (0) **f** $4x^2 - mx - m = 0$ (1)

Example 6

3 Make x the subject in each of the following and give the values of t for which real solution(s) to the equation can be found:

a $2x^2 - 4t = x$ **b** $4x^2 + 4x - 4 = t - 2$

c $5x^2 + 4x + 10 = t$ **d** $tx^2 + 4tx + 10 = t$

Example 7

4 **a** Solve the quadratic equation $x^2 + px - 16 = 0$ in terms of p.

b Find the values of p, where p is an integer with $0 \leq p \leq 10$, for which the quadratic equation in **a** has rational solutions.

5 **a** Show that the solutions of the equation $2x^2 - 3px + (3p - 2) = 0$ are rational for all integer values of p.

b Find the value of p for which there is only one solution.

c Solve the equation when:

i $p = 1$ **ii** $p = 2$ **iii** $p = -1$

6 **a** Show that the solutions of the equation $4(4p - 3)x^2 - 8px + 3 = 0$ are rational for all integer values of p.

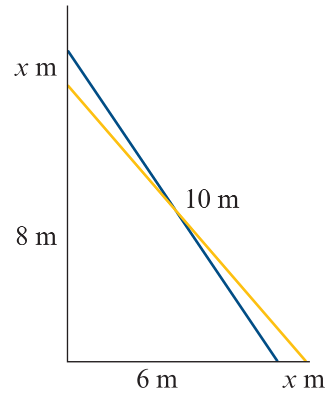
b Find the value of p for which there is only one solution.

c Solve the equation when:

i $p = 1$ **ii** $p = 2$ **iii** $p = -1$

Example 8

7 A pole 10 m long leans against a wall. The bottom of the pole is 6 m from the wall. If the bottom of the pole is pulled away x m so that the top slides down by the same amount, find x.



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8 A wire of length 200 cm is cut into two parts and each part is bent to form a square. If the area of the larger square is 9 times the area of the smaller square, find the length of the sides of the larger square.

Example 9 **9** Solve each of the following equations for x:

a $x - 8\sqrt{x} + 12 = 0$

b $x - 8 = 2\sqrt{x}$

c $x - 5\sqrt{x} - 14 = 0$

d $\sqrt[3]{x^2} - 9\sqrt[3]{x} + 8 = 0$

e $\sqrt[3]{x^2} - \sqrt[3]{x} - 6 = 0$

f $x - 29\sqrt{x} + 100 = 0$

10 Find constants a, b and c such that

$$3x^2 - 5x + 1 = a(x + b)^2 + c$$

for all values of x. Hence find the minimum value of $3x^2 - 5x + 1$.

11 Show that the graphs of $y = 2 - 4x - x^2$ and $y = 24 + 8x + x^2$ do not intersect.

12 Solve the quadratic equation $(b - c)x^2 + (c - a)x + (a - b) = 0$ for x.

13 Given that the two solutions of the equation $2x^2 - 6x - m = 0$ differ by 5, find the value of m.

14 For the equation $(b^2 - 2ac)x^2 + 4(a + c)x = 8$:

a Prove that there are always (real) solutions of the equation.

b Find the conditions that there is only one solution.

15 The equation $\frac{1}{2} + \frac{1}{x + k} = \frac{1}{x}$ has no solutions. Find the possible values of k.

16 Find the smallest positive integer p for which the equation $3x^2 + px + 7 = 0$ has solutions.

3C Applying quadratic equations to rate problems

A **rate** describes how a certain quantity changes with respect to the change in another quantity (often time). An example of a rate is ‘speed’. A speed of 60 km/h gives us a measure of how fast an object is travelling. A further example is ‘flow’, where a rate of 20 L/min is going to fill an empty swimming pool faster than a rate of 6 L/min.

Many problems are solved using rates, which can be expressed as fractions. For example, a speed of 60 km/h can be expressed in fraction form as

$$\frac{\text{distance (km)}}{\text{time taken (h)}} = \frac{60}{1}$$

When solving rate problems, it is often necessary to add two or more fractions with different denominators, as shown in the following examples.

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Example 10

a Express $\dfrac{6}{x} + \dfrac{6}{x+8}$ as a single fraction.

b Solve the equation $\dfrac{6}{x} + \dfrac{6}{x+8} = 2$ for x.

Solution

a
$$\begin{aligned}\frac{6}{x} + \frac{6}{x+8} &= \frac{6(x+8)}{x(x+8)} + \frac{6x}{x(x+8)}\\ &= \frac{6x+48+6x}{x(x+8)}\\ &= \frac{12(x+4)}{x(x+8)}\end{aligned}$$

b Since $\dfrac{6}{x} + \dfrac{6}{x+8} = \dfrac{12(x+4)}{x(x+8)}$, we have

$$\begin{aligned}\frac{12(x+4)}{x(x+8)} &= 2\\ 12(x+4) &= 2x(x+8)\\ 6(x+4) &= x(x+8)\\ 6x+24 &= x^2+8x\\ x^2+2x-24 &= 0\\ (x+6)(x-4) &= 0\end{aligned}$$

Therefore $x = -6$ or $x = 4$.

Example 11

A car travels 500 km at a constant speed. If it had travelled at a speed of 10 km/h less, it would have taken 1 hour more to travel the distance. Find the speed of the car.

Solution

Let x km/h be the speed of the car.

It takes $\dfrac{500}{x}$ hours for the journey.

If the speed is 10 km/h less, then the new speed is $(x - 10)$ km/h.

The time taken is $\dfrac{500}{x} + 1$ hours.

We can now write:

$$\begin{aligned}500 &= (x-10) \times \left(\frac{500}{x} + 1\right)\\ 500x &= (x-10)(500+x)\\ 500x &= 490x - 5000 + x^2\end{aligned}$$

Thus

$$x^2 - 10x - 5000 = 0$$

and so

$$\begin{aligned}x &= \frac{10 \pm \sqrt{100 + 4 \times 5000}}{2}\\ &= 5(1 \pm \sqrt{201})\end{aligned}$$

The speed is $5(1 + \sqrt{201}) \approx 75.887$ km/h.

Explanation

For an object travelling at a constant speed in one direction:

$$\text{speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

and so

$$\text{time taken} = \frac{\text{distance travelled}}{\text{speed}}$$

and

$$\text{distance travelled} = \text{speed} \times \text{time taken}$$

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ISBN 978-1-316-63608-4

Example 12

A tank is filled by two pipes. The smaller pipe alone will take 24 minutes longer than the larger pipe alone, and 32 minutes longer than when both pipes are used. How long will each pipe take to fill the tank alone? How long will it take for both pipes used together to fill the tank?

Solution

Let C cubic units be the capacity of the tank, and let x minutes be the time it takes for the larger pipe alone to fill the tank.

Then the average rate of flow for the larger pipe is $\frac{C}{x}$ cubic units per minute.

Since the smaller pipe alone takes $(x + 24)$ minutes to fill the tank, the average rate of flow for the smaller pipe is $\frac{C}{x+24}$ cubic units per minute.

The average rate of flow when both pipes are used together is the sum of these two rates:

$$\frac{C}{x} + \frac{C}{x+24} \text{ cubic units per minute}$$

Expressed as a single fraction:

$$\begin{aligned}\frac{C}{x} + \frac{C}{x+24} &= \frac{C(x+24) + Cx}{x(x+24)} \\ &= \frac{2C(x+12)}{x(x+24)}\end{aligned}$$

The time taken to fill the tank using both pipes is

$$\begin{aligned}C \div \frac{2C(x+12)}{x(x+24)} &= C \times \frac{x(x+24)}{2C(x+12)} \\ &= \frac{x(x+24)}{2(x+12)}\end{aligned}$$

Therefore the time taken for the smaller pipe alone to fill the tank can be also be expressed as $\frac{x(x+24)}{2(x+12)} + 32$ minutes.

Thus

$$\begin{aligned}\frac{x(x+24)}{2(x+12)} + 32 &= x + 24 \\ \frac{x(x+24)}{2(x+12)} &= x - 8 \\ x(x+24) &= 2(x+12)(x-8) \\ x^2 + 24x &= 2x^2 + 8x - 192 \\ x^2 - 16x - 192 &= 0 \\ (x-24)(x+8) &= 0\end{aligned}$$

But $x > 0$, and hence $x = 24$.

It takes 24 minutes for the larger pipe alone to fill the tank, 48 minutes for the smaller pipe alone to fill the tank, and 16 minutes for both pipes together to fill the tank.

ISBN 978-1-316-63608-4

Exercise 3C

Skillshet

Example 10

1 **a** Express $\dfrac{6}{x} - \dfrac{6}{x+3}$ as a single fraction.

b Solve the equation $\dfrac{6}{x} - \dfrac{6}{x+3} = 1$ for x.

2 Solve the equation $\dfrac{300}{x+5} = \dfrac{300}{x} - 2$ for x.

3 The sum of the reciprocals of two consecutive odd numbers is $\dfrac{36}{323}$. Form a quadratic equation and hence determine the two numbers.

Example 11

4 A cyclist travels 40 km at a speed of x km/h.

a Find the time taken in terms of x.

b Find the time taken when his speed is reduced by 2 km/h.

c If the difference between the times is 1 hour, find his original speed.

5 A car travels from town A to town B, a distance of 600 km, in x hours. A plane, travelling 220 km/h faster than the car, takes $5\frac{1}{2}$ hours less to cover the same distance.

a Express, in terms of x, the average speed of the car and the average speed of the plane.

b Find the actual average speed of each of them.

6 A car covers a distance of 200 km at a speed of x km/h. A train covers the same distance at a speed of $(x + 5)$ km/h. If the time taken by the car is 2 hours more than that taken by the train, find x.

7 A man travels 108 km, and finds that he could have made the journey in $4\frac{1}{2}$ hours less had he travelled at an average speed 2 km/h faster. What was the man's average speed when he made the trip?

8 A bus is due to reach its destination 75 km away at a certain time. The bus usually travels with an average speed of x km/h. Its start is delayed by 18 minutes but, by increasing its average speed by 12.5 km/h, the driver arrives on time.

a Find x. **b** How long did the journey actually take?

9 Ten minutes after the departure of an express train, a slow train starts, travelling at an average speed of 20 km/h less. The slow train reaches a station 250 km away 3.5 hours after the arrival of the express. Find the average speed of each of the trains.

10 When the average speed of a car is increased by 10 km/h, the time taken for the car to make a journey of 105 km is reduced by 15 minutes. Find the original average speed.

11 A tank can be filled with water by two pipes running together in $11\frac{1}{9}$ minutes. If the larger pipe alone takes 5 minutes less to fill the tank than the smaller pipe, find the time that each pipe will take to fill the tank.

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Example 12 **12** At first two different pipes running together will fill a tank in $\frac{20}{3}$ minutes. The rate that water runs through each of the pipes is then adjusted. If one pipe, running alone, takes 1 minute less to fill the tank at its new rate, and the other pipe, running alone, takes 2 minutes more to fill the tank at its new rate, then the two running together will fill the tank in 7 minutes. Find in what time the tank will be filled by each pipe running alone at the new rates.

13 The journey between two towns by one route consists of 233 km by rail followed by 126 km by sea. By a second route the journey consists of 405 km by rail followed by 39 km by sea. If the time taken for the first route is 50 minutes longer than for the second route, and travelling by rail is 25 km/h faster than travelling by sea, find the average speed by rail and the average speed by sea.

14 A sea freighter travelling due north at 12 km/h sights a cruiser straight ahead at an unknown distance and travelling due east at unknown speed. After 15 minutes the vessels are 10 km apart and then, 15 minutes later, they are 13 km apart. (Assume that both travel at constant speeds.) How far apart are the vessels when the cruiser is due east of the freighter?

15 Cask A, which has a capacity of 20 litres, is filled with wine. A certain quantity of wine from cask A is poured into cask B, which also has a capacity of 20 litres. Cask B is then filled with water. After this, cask A is filled with some of the mixture from cask B. A further $\frac{20}{3}$ litres of the mixture now in A is poured back into B, and the two casks now have the same amount of wine. How much wine was first taken out of cask A?

16 Two trains travel between two stations 80 km apart. If train A travels at an average speed of 5 km/h faster than train B and completes the journey 20 minutes faster, find the average speeds of the two trains, giving your answers correct to two decimal places.

17 A tank is filled by two pipes. The smaller pipe running alone will take 24 minutes longer than the larger pipe alone, and a minutes longer than when both pipes are running together.

a Find, in terms of a, how long each pipe takes to fill the tank.

b Find how long each pipe takes to fill the tank when:

i $a = 49$ **ii** $a = 32$ **iii** $a = 27$ **iv** $a = 25$

18 Train A leaves Armadale and travels at constant speed to Bundong, which is a town 300 km from Armadale. At the same time, train B leaves Bundong and travels at constant speed to Armadale. They meet at a town Yunga, which is between the two towns. Nine hours after leaving Yunga, train A reaches Bundong, and four hours after leaving Yunga, train B reaches Armadale.

a Find the distance of Yunga from Armadale.

b Find the speed of each of the trains.

ISBN 978-1-316-63608-4

3D Partial fractions

A **rational function** is the quotient of two polynomials. If $g(x)$ and $h(x)$ are polynomials, then $f(x) = \dfrac{g(x)}{h(x)}$ is a rational function; e.g. $f(x) = \dfrac{4x+2}{x^2-1}$.

- If the degree of $g(x)$ is less than the degree of $h(x)$, then $f(x)$ is a **proper fraction**.
- If the degree of $g(x)$ is greater than or equal to the degree of $h(x)$, then $f(x)$ is an **improper fraction**.

By convention, we consider a rational function for its maximal domain. For example, the function $f(x) = \dfrac{4x+2}{x^2-1}$ is only considered for $x \in \mathbb{R} \setminus \{-1, 1\}$.

A rational function may be expressed as a sum of simpler functions by resolving it into what are called **partial fractions**. For example:

$$\frac{4x+2}{x^2-1} = \frac{3}{x-1} + \frac{1}{x+1}$$

This technique can help when sketching the graphs of rational functions or when performing other mathematical procedures such as integration.

Proper fractions

For proper fractions, the technique used for obtaining partial fractions depends on the type of factors in the denominator of the original algebraic fraction. We only consider examples where the denominators have factors that are either degree 1 (linear) or degree 2 (quadratic).

- For every linear factor $ax + b$ in the denominator, there will be a partial fraction of the form $\dfrac{A}{ax+b}$.
- For every repeated linear factor $(cx + d)^2$ in the denominator, there will be partial fractions of the form $\dfrac{B}{cx+d}$ and $\dfrac{C}{(cx+d)^2}$.
- For every irreducible quadratic factor $ax^2 + bx + c$ in the denominator, there will be a partial fraction of the form $\dfrac{Dx+E}{ax^2+bx+c}$.

Note: A quadratic expression is said to be **irreducible** if it cannot be factorised over $\mathbb{R}$. For example, both $x^2 + 1$ and $x^2 + 4x + 10$ are irreducible.

To resolve an algebraic fraction into its partial fractions:

Step 1 Write a statement of identity between the original fraction and a sum of the appropriate number of partial fractions.

Step 2 Express the sum of the partial fractions as a single fraction, and note that the numerators of both sides are equivalent.

Step 3 Find the values of the introduced constants $A, B, C, \ldots$ by substituting appropriate values for x or by equating coefficients.

ISBN 978-1-316-63608-4

Example 13

Resolve $\dfrac{3x+5}{(x-1)(x+3)}$ into partial fractions.

Solution

Method 1

Let

$$\frac{3x+5}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3} \qquad (1)$$

for all $x \in \mathbb{R} \setminus \{1, -3\}$.

Express the right-hand side as a single fraction:

$$\frac{3x+5}{(x-1)(x+3)} = \frac{A(x+3)+B(x-1)}{(x-1)(x+3)}$$

$$\therefore \quad \frac{3x+5}{(x-1)(x+3)} = \frac{(A+B)x+3A-B}{(x-1)(x+3)}$$

$$\therefore \quad 3x+5 = (A+B)x + 3A - B$$

Equate coefficients:

$$A + B = 3$$

$$3A - B = 5$$

Solving these equations simultaneously gives

$$4A = 8$$

and so $A = 2$ and $B = 1$.

Therefore

$$\frac{3x+5}{(x-1)(x+3)} = \frac{2}{x-1} + \frac{1}{x+3}$$

Method 2

From equation (1) we can write:

$$3x+5 = A(x+3) + B(x-1) \qquad (2)$$

Substitute $x = 1$ in equation (2):

$$8 = 4A$$

$$\therefore \quad A = 2$$

Substitute $x = -3$ in equation (2):

$$-4 = -4B$$

$$\therefore \quad B = 1$$

Explanation

Since the denominator has two linear factors, there will be two partial fractions of the form $\dfrac{A}{x-1}$ and $\dfrac{B}{x+3}$.

We know that equation (2) is true for all $x \in \mathbb{R} \setminus \{1, -3\}$.

But if this is the case, then it also has to be true for $x = 1$ and $x = -3$.

Note: You could substitute any values of x to find A and B in this way, but these values simplify the calculations.

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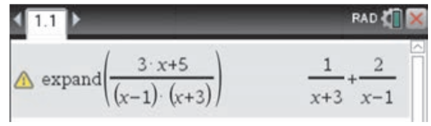
Using the TI-Nspire

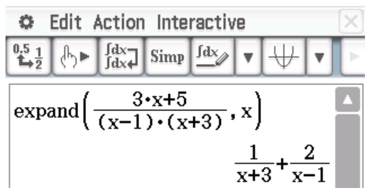
Use menu > **Algebra** > **Expand** as shown.

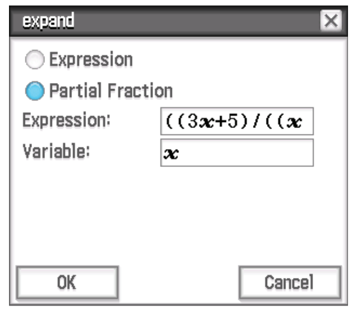
Note: You can access the fraction template using ctrl ÷.



Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight $\dfrac{3x+5}{(x-1)(x+3)}$.
- Go to **Interactive** > **Transformation** > **expand** and select the **Partial Fraction** option.
- Enter the variable and tap OK.





Example 14

Resolve $\dfrac{2x+10}{(x+1)(x-1)^2}$ into partial fractions.

Solution

Since the denominator has a repeated linear factor and a single linear factor, there are three partial fractions:

$$\frac{2x+10}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$\therefore \quad \frac{2x+10}{(x+1)(x-1)^2} = \frac{A(x-1)^2 + B(x+1)(x-1) + C(x+1)}{(x+1)(x-1)^2}$$

This gives the equation

$$2x+10 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

We will use a combination of methods to find A, B and C.

Let $x = 1$: $\quad 2(1) + 10 = C(1+1)$

$$12 = 2C$$

$$\therefore \quad C = 6$$

Let $x = -1$: $\quad 2(-1) + 10 = A(-1-1)^2$

$$8 = 4A$$

$$\therefore \quad A = 2$$

ISBN 978-1-316-63608-4

Cambridge University Press

Now substitute these values for A and C:

$$\begin{aligned}2x + 10 &= 2(x-1)^2 + B(x+1)(x-1) + 6(x+1) \qquad (1)\\ &= 2(x^2 - 2x + 1) + B(x^2 - 1) + 6(x+1)\\ &= (2+B)x^2 + 2x + 8 - B\end{aligned}$$

Equate coefficients:

$$2 + B = 0$$

$$8 - B = 10$$

Therefore $B = -2$ and hence

$$\frac{2x+10}{(x+1)(x-1)^2} = \frac{2}{x+1} - \frac{2}{x-1} + \frac{6}{(x-1)^2}$$

Alternatively, the value of B could be found by substituting $x = 0$ into equation (1).

Note: In Exercise 3D, you will show that it is impossible to find A and C such that

$$\frac{2x+10}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{C}{(x-1)^2}$$

Example 15

Resolve $\dfrac{x^2 + 6x + 5}{(x-2)(x^2 + x + 1)}$ into partial fractions.

Solution

Since the denominator has a single linear factor and an irreducible quadratic factor (i.e. cannot be reduced to linear factors), there are two partial fractions:

$$\frac{x^2 + 6x + 5}{(x-2)(x^2 + x + 1)} = \frac{A}{x-2} + \frac{Bx + C}{x^2 + x + 1}$$

$$\therefore \quad \frac{x^2 + 6x + 5}{(x-2)(x^2 + x + 1)} = \frac{A(x^2 + x + 1) + (Bx + C)(x-2)}{(x-2)(x^2 + x + 1)}$$

This gives the equation

$$x^2 + 6x + 5 = A(x^2 + x + 1) + (Bx + C)(x-2) \qquad (1)$$

Subsituting $x = 2$:

$$\begin{aligned}2^2 + 6(2) + 5 &= A(2^2 + 2 + 1)\\ 21 &= 7A\\ \therefore \quad A &= 3\end{aligned}$$

We can rewrite equation (1) as

$$\begin{aligned}x^2 + 6x + 5 &= A(x^2 + x + 1) + (Bx + C)(x-2)\\ &= A(x^2 + x + 1) + Bx^2 - 2Bx + Cx - 2C\\ &= (A+B)x^2 + (A - 2B + C)x + A - 2C\end{aligned}$$

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Since $A = 3$, this gives

$$x^2 + 6x + 5 = (3 + B)x^2 + (3 - 2B + C)x + 3 - 2C$$

Equate coefficients:

$$3 + B = 1 \qquad \text{and} \qquad 3 - 2C = 5$$
$$\therefore \quad B = -2 \qquad\qquad \therefore \quad C = -1$$

Check: $3 - 2B + C = 3 - 2(-2) + (-1) = 6$

Therefore

$$\frac{x^2 + 6x + 5}{(x-2)(x^2 + x + 1)} = \frac{3}{x-2} + \frac{-2x - 1}{x^2 + x + 1}$$
$$= \frac{3}{x-2} - \frac{2x + 1}{x^2 + x + 1}$$

Note: The values of B and C could also be found by substituting $x = 0$ and $x = 1$ in equation (1).

► Improper fractions

Improper algebraic fractions can be expressed as a sum of partial fractions by first dividing the denominator into the numerator to produce a quotient and a proper fraction. This proper fraction can then be resolved into its partial fractions using the techniques just introduced.

Example 16

Express $\dfrac{x^5 + 2}{x^2 - 1}$ as partial fractions.

Solution

Dividing through:

$$\begin{array}{r|l} & x^3 + x \\ \hline x^2 - 1 & x^5 + 2 \\ & x^5 - x^3 \\ \hline & \quad x^3 + 2 \\ & \quad x^3 - x \\ \hline & \qquad x + 2 \end{array}$$

Therefore

$$\frac{x^5 + 2}{x^2 - 1} = x^3 + x + \frac{x + 2}{x^2 - 1}$$

By expressing $\dfrac{x + 2}{x^2 - 1} = \dfrac{x + 2}{(x-1)(x+1)}$ as partial fractions, we obtain

$$\frac{x^5 + 2}{x^2 - 1} = x^3 + x - \frac{1}{2(x+1)} + \frac{3}{2(x-1)}$$

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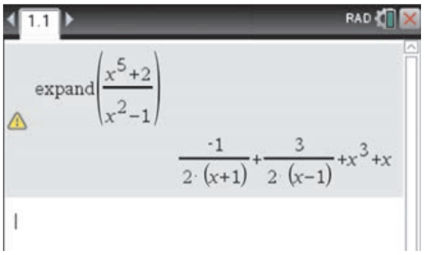
Using the TI-Nspire

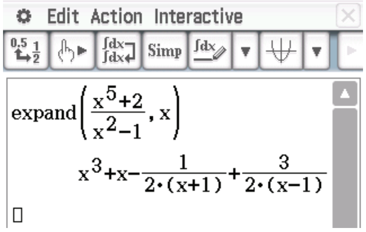
Use (menu) > **Algebra** > **Expand** as shown.

Note: You can access the fraction template using (ctrl) (÷).



Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight $\dfrac{x^5+2}{x^2-1}$.
- Go to **Interactive** > **Transformation** > **expand** and choose the **Partial Fraction** option.
- Enter the variable and tap OK.



Section summary

- A rational function may be expressed as a sum of simpler functions by resolving it into **partial fractions**. For example:

$$\frac{4x+2}{x^2-1} = \frac{3}{x-1} + \frac{1}{x+1}$$

- Examples of resolving a proper fraction into partial fractions:
  - Single linear factors
$$\frac{3x-4}{(2x-3)(x+5)} = \frac{A}{2x-3} + \frac{B}{x+5}$$
  - Repeated linear factor
$$\frac{3x-4}{(2x-3)(x+5)^2} = \frac{A}{2x-3} + \frac{B}{x+5} + \frac{C}{(x+5)^2}$$
  - Irreducible quadratic factor
$$\frac{3x-4}{(2x-3)(x^2+5)} = \frac{A}{2x-3} + \frac{Bx+C}{x^2+5}$$

- A quadratic polynomial is **irreducible** if it cannot be factorised over $\mathbb{R}$. For example, the quadratics x^2+5 and $x^2+4x+10$ are irreducible.
- If $f(x) = \dfrac{g(x)}{h(x)}$ is an improper fraction, i.e. if the degree of $g(x)$ is greater than or equal to the degree of $h(x)$, then the division must be performed first.

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Exercise 3D

Example 13

1 Resolve the following rational expressions into partial fractions:

a $\dfrac{5x+1}{(x-1)(x+2)}$ **b** $\dfrac{-1}{(x+1)(2x+1)}$ **c** $\dfrac{3x-2}{x^2-4}$

d $\dfrac{4x+7}{x^2+x-6}$ **e** $\dfrac{7-x}{(x-4)(x+1)}$

Example 14

2 Resolve the following rational expressions into partial fractions:

a $\dfrac{2x+3}{(x-3)^2}$ **b** $\dfrac{9}{(1+2x)(1-x)^2}$ **c** $\dfrac{2x-2}{(x+1)(x-2)^2}$

Example 15

3 Resolve the following rational expressions into partial fractions:

a $\dfrac{3x+1}{(x+1)(x^2+x+1)}$ **b** $\dfrac{3x^2+2x+5}{(x^2+2)(x+1)}$ **c** $\dfrac{x^2+2x-13}{2x^3+6x^2+2x+6}$

Example 16

4 Resolve $\dfrac{3x^2-4x-2}{(x-1)(x-2)}$ into partial fractions.

5 Show that it is not possible to find values of A and C such that

$$\frac{2x+10}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{C}{(x-1)^2}$$

6 Express each of the following as partial fractions:

a $\dfrac{1}{(x-1)(x+1)}$ **b** $\dfrac{x}{(x-2)(x+3)}$ **c** $\dfrac{3x+1}{(x-2)(x+5)}$

d $\dfrac{1}{(2x-1)(x+2)}$ **e** $\dfrac{3x+5}{(3x-2)(2x+1)}$ **f** $\dfrac{2}{x^2-x}$

g $\dfrac{3x+1}{x^3+x}$ **h** $\dfrac{3x^2+8}{x(x^2+4)}$ **i** $\dfrac{1}{x^2-4x}$

j $\dfrac{x+3}{x^2-4x}$ **k** $\dfrac{x^3-x^2-1}{x^2-x}$ **l** $\dfrac{x^3-x^2-6}{2x-x^2}$

m $\dfrac{x^2-x}{(x+1)(x^2+2)}$ **n** $\dfrac{x^2+2}{x^3-3x-2}$ **o** $\dfrac{2x^2+x+8}{x(x^2+4)}$

p $\dfrac{1-2x}{2x^2+7x+6}$ **q** $\dfrac{3x^2-6x+2}{(x-1)^2(x+2)}$ **r** $\dfrac{4}{(x-1)^2(2x+1)}$

s $\dfrac{x^3-2x^2-3x+9}{x^2-4}$ **t** $\dfrac{x^3+3}{(x+1)(x-1)}$ **u** $\dfrac{2x-1}{(x+1)(3x+2)}$

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3E Simultaneous equations

In this section, we look at methods for finding the coordinates of the points of intersection of a linear graph with different non-linear graphs: parabolas, circles and rectangular hyperbolas. We also consider the intersections of two parabolas. These types of graphs are studied further in Mathematical Methods Year 11.

Example 17

Find the coordinates of the points of intersection of the parabola with equation $y = x^2 - 2x - 2$ and the straight line with equation $y = x + 4$.

Solution

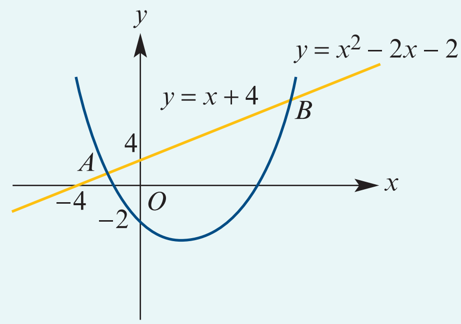
Equate the two expressions for y:

$$x^2 - 2x - 2 = x + 4$$

$$x^2 - 3x - 6 = 0$$

$$\therefore \quad x = \frac{3 \pm \sqrt{9 - 4 \times (-6)}}{2}$$

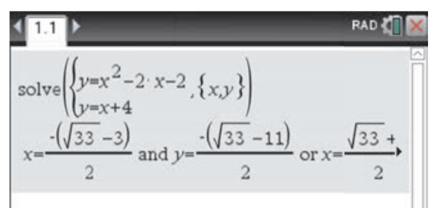
$$= \frac{3 \pm \sqrt{33}}{2}$$



The points of intersection are $A\left(\frac{3 - \sqrt{33}}{2}, \frac{11 - \sqrt{33}}{2}\right)$ and $B\left(\frac{3 + \sqrt{33}}{2}, \frac{11 + \sqrt{33}}{2}\right)$.

Using the TI-Nspire

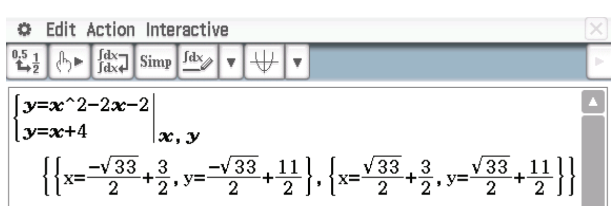
- Use menu > **Algebra** > **Solve System of Equations** > **Solve System of Equations** as shown.
- Use the touchpad to move the cursor up to the solution and see all the solutions.



Using the Casio ClassPad

The exact coordinates of the points of intersection can be obtained in the Main $\sqrt{\alpha}$ application.

- To select the simultaneous equations template, tap from the Math1 keyboard.
- Enter the two equations and the variables x, y in the spaces provided. Then tap EXE.
- Tap Rotate from the icon panel and ► on the touch screen to view the entire solution.



ISBN 978-1-316-63608-4

Cambridge University Press

Example 18

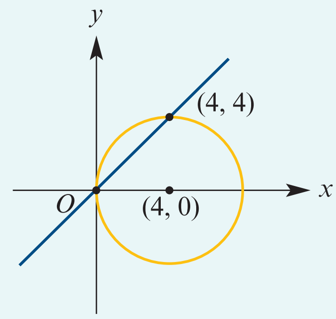
Find the points of intersection of the circle with equation $(x-4)^2 + y^2 = 16$ and the line with equation $x - y = 0$.

Solution

Rearrange $x - y = 0$ to make y the subject.

Substitute $y = x$ into the equation of the circle:

$$\begin{aligned}(x-4)^2 + x^2 &= 16\\ x^2 - 8x + 16 + x^2 &= 16\\ 2x^2 - 8x &= 0\\ 2x(x-4) &= 0\\ \therefore \quad x = 0 \text{ or } x &= 4\end{aligned}$$



The points of intersection are $(0,0)$ and $(4,4)$.

Example 19

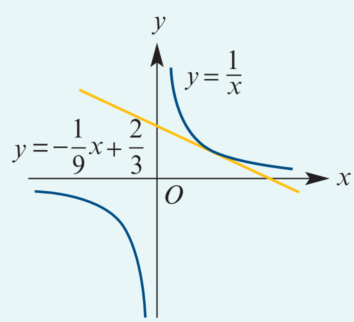
Find the point of contact of the straight line with equation $\frac{1}{9}x + y = \frac{2}{3}$ and the curve with equation $xy = 1$.

Solution

Rewrite the equations as $y = -\frac{1}{9}x + \frac{2}{3}$ and $y = \frac{1}{x}$.

Equate the expressions for y:

$$\begin{aligned}-\frac{1}{9}x + \frac{2}{3} &= \frac{1}{x}\\ -x^2 + 6x &= 9\\ x^2 - 6x + 9 &= 0\\ (x-3)^2 &= 0\\ \therefore \quad x &= 3\end{aligned}$$

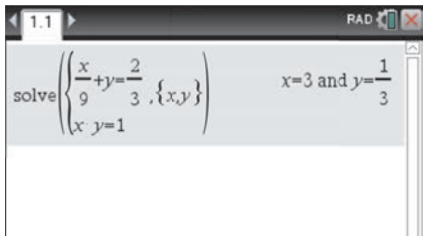


The point of intersection is $(3, \frac{1}{3})$.

Using the TI-Nspire

Use (menu) > **Algebra** > **Solve System of Equations** > **Solve System of Equations** as shown.

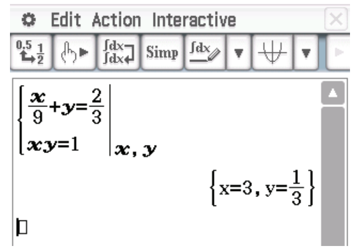
Note: The multiplication sign between x and y is required, as the calculator will consider xy to be a single variable.



ISBN 978-1-316-63608-4

Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, select the simultaneous equations template by tapping {☐ from the Math1 keyboard.
- Enter the two equations and the variables x, y in the spaces provided; tap EXE.

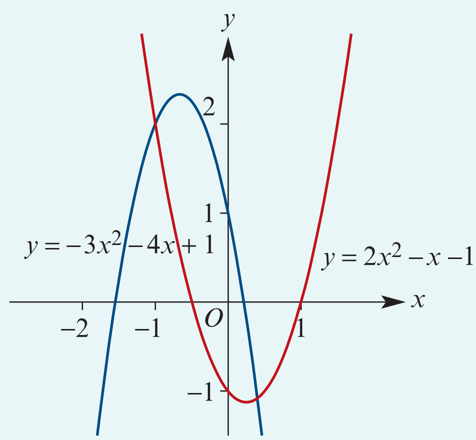


Example 20

Find the coordinates of the points of intersection of the graphs of $y = -3x^2 - 4x + 1$ and $y = 2x^2 - x - 1$.

Solution

$$-3x^2 - 4x + 1 = 2x^2 - x - 1$$
$$-5x^2 - 3x + 2 = 0$$
$$5x^2 + 3x - 2 = 0$$
$$(5x - 2)(x + 1) = 0$$
$$\therefore \quad x = \frac{2}{5} \text{ or } x = -1$$



Substitute in $y = 2x^2 - x - 1$:

When $x = -1$, $y = 2$.

When $x = \frac{2}{5}$, $y = 2 \times \frac{4}{25} - \frac{2}{5} - 1 = -\frac{27}{25}$.

The points of intersection are $(-1, 2)$ and $\left(\frac{2}{5}, -\frac{27}{25}\right)$.

Exercise 3E

Skillsheet

Example 17

1 Find the coordinates of the points of intersection for each of the following:

a $y = x^2$, $y = x$

b $y - 2x^2 = 0$, $y - x = 0$

c $y = x^2 - x$, $y = 2x + 1$

Example 18

2 Find the coordinates of the points of intersection for each of the following:

a $x^2 + y^2 = 178$, $x + y = 16$

b $x^2 + y^2 = 125$, $x + y = 15$

c $x^2 + y^2 = 185$, $x - y = 3$

d $x^2 + y^2 = 97$, $x + y = 13$

e $x^2 + y^2 = 106$, $x - y = 4$

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Example 19

3 Find the coordinates of the points of intersection for each of the following:

a $x + y = 28$
$xy = 187$

b $x + y = 51$
$xy = 518$

c $x - y = 5$
$xy = 126$

4 Find the coordinates of the points of intersection of the straight line with equation $y = 2x$ and the circle with equation $(x - 5)^2 + y^2 = 25$.

5 Find the coordinates of the points of intersection of the curves with equations $y = \dfrac{1}{x - 2} + 3$ and $y = x$.

6 Find the coordinates of the points A and B where the line with equation $x - 3y = 0$ meets the circle with equation $x^2 + y^2 - 10x - 5y + 25 = 0$.

7 Find the coordinates of the points of intersection of the line with equation $\dfrac{y}{4} - \dfrac{x}{5} = 1$ and the circle with equation $x^2 + 4x + y^2 = 12$.

8 Find the coordinates of the points of intersection of the curve with equation $y = \dfrac{1}{x + 2} - 3$ and the line with equation $y = -x$.

9 Find the point where the line $4y = 9x + 4$ touches the parabola $y^2 = 9x$.

10 Find the coordinates of the point where the line with equation $y = 2x + 3\sqrt{5}$ touches the circle with equation $x^2 + y^2 = 9$.

11 Find the coordinates of the point where the straight line with equation $y = \dfrac{1}{4}x + 1$ touches the curve with equation $y = -\dfrac{1}{x}$.

12 Find points of intersection of the curve $y = \dfrac{2}{x - 2}$ and the line $y = x - 1$.

Example 20

13 Find the coordinates of the points of intersection of the graphs of the following pairs of quadratic functions:

a $y = 2x^2 - 4x + 1,\ y = 2x^2 - x - 1$

b $y = -2x^2 + x + 1,\ y = 2x^2 - x - 1$

c $y = x^2 + x + 1,\ y = x^2 - x - 2$

d $y = 3x^2 + x + 2,\ y = x^2 - x + 2$

14 In each of the following, use the discriminant of the resulting quadratic equation:

a Find the possible values of k for which the straight line $y = k(1 - 2x)$ touches but does not cross the parabola $y = x^2 + 2$.

b Find the possible values of c for which the line $y = 2x + c$ intersects the circle $x^2 + y^2 = 20$ in two distinct points.

c Find the value of p for which the line $y = 6$ meets the parabola $y = x^2 + (1 - p)x + 2p$ at only one point.

ISBN 978-1-316-63608-4

Chapter summary

AS

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Polynomials

- A **polynomial function** can be written in the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $n \in \mathbb{N} \cup \{0\}$ and the coefficients $a_0, \dots, a_n$ are real numbers with $a_n \neq 0$.

- The **degree** of a polynomial is the index n of the leading term (the term of highest index among those terms with a non-zero coefficient).
- Equating coefficients

  Two polynomials are equal if they give the same value for all x. If two polynomials are equal, then they have the same degree and corresponding coefficients are equal.

  For example: if $x^2 - x - 12 = x^2 + (a + b)x + ab$, then $a + b = -1$ and $ab = -12$.

Quadratics

- A quadratic function can be written in the form $y = ax^2 + bx + c$, where $a \neq 0$.
- A quadratic equation $ax^2 + bx + c = 0$ may be solved by:
  - Factorising
  - Completing the square
  - Using the **general quadratic formula** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- The number of solutions of a quadratic equation $ax^2 + bx + c = 0$ can be found from the **discriminant** $\Delta = b^2 - 4ac$:
  - If $\Delta > 0$, the quadratic equation has two real solutions.
  - If $\Delta = 0$, the quadratic equation has one real solution.
  - If $\Delta < 0$, the quadratic equation has no real solutions.

Partial fractions

- A **rational function** has the form $f(x) = \dfrac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials.

  For example: $f(x) = \dfrac{2x + 10}{x^3 - x^2 - x + 1}$

- Some rational functions may be expressed as a sum of **partial fractions**:
  - For every linear factor $ax + b$ in the denominator, there will be a partial fraction of the form $\dfrac{A}{ax + b}$.
  - For every repeated linear factor $(cx + d)^2$ in the denominator, there will be partial fractions of the form $\dfrac{B}{cx + d}$ and $\dfrac{C}{(cx + d)^2}$.
  - For every irreducible quadratic factor $ax^2 + bx + c$ in the denominator, there will be a partial fraction of the form $\dfrac{Dx + E}{ax^2 + bx + c}$.

  For example: $\dfrac{2x + 10}{(x + 1)(x - 1)^2} = \dfrac{A}{x + 1} + \dfrac{B}{x - 1} + \dfrac{C}{(x - 1)^2}$, where $A = 2$, $B = -2$ and $C = 6$

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Short-answer questions

1 If $(3a + b)x^2 + (a - 2b)x + b + 2c = 11x^2 - x + 4$, find the values of a, b and c.

2 Express x^3 in the form $(x - 1)^3 + a(x - 1)^2 + b(x - 1) + c$.

3 Prove that, if $ax^3 + bx^2 + cx + d = (x + 1)^2(px + q)$, then $b = 2a + d$ and $c = a + 2d$.

4 Prove that, if $ax^3 + bx^2 + cx + d = (x - 2)^2(px + q)$, then $b = -4a + \frac{1}{4}d$ and $c = 4a - d$.

5 Solve the following quadratic equations for x:

a $x^2 + x = 12$ **b** $x^2 - 2 = x$ **c** $-x^2 + 3x + 11 = 1$

d $2x^2 - 4x + 1 = 0$ **e** $3x^2 - 2x + 5 = t$ **f** $tx^2 + 4 = tx$

6 Solve the equation $\dfrac{2}{x - 1} - \dfrac{3}{x + 2} = \dfrac{1}{2}$ for x.

7 Express each of the following as partial fractions:

a $\dfrac{-3x + 4}{(x - 3)(x + 2)}$ **b** $\dfrac{7x + 2}{x^2 - 4}$ **c** $\dfrac{7 - x}{x^2 + 2x - 15}$

d $\dfrac{3x - 9}{x^2 - 4x - 5}$ **e** $\dfrac{3x - 4}{(x + 3)(x + 2)^2}$ **f** $\dfrac{6x^2 - 5x - 16}{(x - 1)^2(x + 4)}$

g $\dfrac{x^2 - 6x - 4}{(x^2 + 2)(x + 1)}$ **h** $\dfrac{-x + 4}{(x - 1)(x^2 + x + 1)}$ **i** $\dfrac{-4x + 5}{(x + 4)(x - 3)}$

j $\dfrac{-2x + 8}{(x + 4)(x - 3)}$

8 Express each of the following as partial fractions:

a $\dfrac{14(x - 2)}{(x - 3)(x^2 + x + 2)}$ **b** $\dfrac{1}{(x + 1)(x^2 - x + 2)}$ **c** $\dfrac{3x^3}{x^2 - 5x + 4}$

9 Find the coordinates of the points of intersection for each of the following:

a $y = x^2$, $y = -x$ **b** $x^2 + y^2 = 16$, $x + y = 4$ **c** $x + y = 5$, $xy = 4$

10 Find the coordinates of the points of intersection of the line with equation $3y - x = 1$ and the circle with equation $x^2 + 2x + y^2 = 9$.

11 A motorist makes a journey of 135 km at an average speed of x km/h.

a Write an expression for the number of hours taken for the journey.

b Owing to road works, on a certain day his average speed for the journey is reduced by 15 km/h. Write an expression for the number of hours taken on that day.

c If the second journey takes 45 minutes longer than the first, form an equation in x and solve it.

d Find his average speed for each journey.

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Multiple-choice questions

1 If x^2 is written in the form $(x + 1)^2 + b(x + 1) + c$, then the values of b and c are

A $b = 0,\ c = 0$ **B** $b = -2,\ c = 0$ **C** $b = -2,\ c = 1$
D $b = 1,\ c = 2$ **E** $b = 1,\ c = -2$

2 If $x^3 = a(x + 2)^3 + b(x + 2)^2 + c(x + 2) + d$, then the values of a, b, c and d are

A $a = 0,\ b = -8,\ c = 10,\ d = -6$ **B** $a = 0,\ b = -6,\ c = 10,\ d = -8$
C $a = 1,\ b = -8,\ c = 10,\ d = -6$ **D** $a = 1,\ b = -6,\ c = 12,\ d = -8$
E $a = 1,\ b = -8,\ c = 12,\ d = -6$

3 The quadratic equation $3x^2 - 6x + 3 = 0$ has

A two real solutions, $x = \pm 1$ **B** one real solution, $x = -1$
C no real solutions **D** one real solution, $x = 1$
E two real solutions, $x = 1$ and $x = 2$

4 The quadratic equation whose solutions are 4 and -6 is

A $(x + 4)(x - 6) = 0$ **B** $x^2 - 2x - 24 = 0$ **C** $2x^2 + 4x = 48$
D $-x^2 + 2x - 24 = 0$ **E** $x^2 + 2x + 24 = 0$

5 $\dfrac{3}{x + 4} - \dfrac{5}{x - 2}$ is equal to

A $\dfrac{-2}{(x + 4)(x - 2)}$ **B** $\dfrac{2(x + 1)}{(x + 4)(x - 2)}$ **C** $\dfrac{-2(x - 7)}{(x + 4)(x - 2)}$

D $\dfrac{2(4x + 13)}{(x + 4)(x - 2)}$ **E** $\dfrac{-2(x + 13)}{(x + 4)(x - 2)}$

6 $\dfrac{4}{(x + 3)^2} + \dfrac{2x}{x + 1}$ is equal to

A $\dfrac{8x}{(x + 3)^2(x + 1)}$ **B** $\dfrac{2(3x^2 + x + 18)}{(x + 3)^2(x + 1)}$ **C** $\dfrac{3x^2 + 13x + 18}{(x + 3)^2(x + 1)}$

D $\dfrac{2(3x^2 + 13x + 18)}{(x + 3)^2(x + 1)}$ **E** $\dfrac{2(x^3 + 6x^2 + 11x + 2)}{(x + 3)^2(x + 1)}$

7 If $\dfrac{7x^2 + 13}{(x - 1)(x^2 + x + 2)}$ is expressed in the form $\dfrac{a}{x - 1} + \dfrac{bx + c}{x^2 + x + 2}$, then

A $a = 5,\ b = 0,\ c = -13$ **B** $a = 5,\ b = 0,\ c = -10$ **C** $a = 5,\ b = 2,\ c = -3$
D $a = 7,\ b = 2,\ c = 3$ **E** $a = 7,\ b = 3,\ c = 13$

8 $\dfrac{4x - 3}{(x - 3)^2}$ is equal to

A $\dfrac{3}{x - 3} + \dfrac{1}{x - 3}$ **B** $\dfrac{4x}{x - 3} - \dfrac{3}{x - 3}$ **C** $\dfrac{9}{x - 3} + \dfrac{4}{(x - 3)^2}$

D $\dfrac{4}{x - 3} + \dfrac{9}{(x - 3)^2}$ **E** $\dfrac{4}{x - 3} - \dfrac{15}{(x - 3)^2}$

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9 $\dfrac{8x+7}{2x^2+5x+2}$ is equal to

A $\dfrac{2}{2x+1} - \dfrac{3}{x+2}$ **B** $\dfrac{2}{2x+1} + \dfrac{3}{x+2}$ **C** $\dfrac{-4}{2x+2} - \dfrac{1}{x+1}$

D $\dfrac{-4}{2x+2} + \dfrac{1}{x+1}$ **E** $\dfrac{4}{2x+2} - \dfrac{1}{x+1}$

10 $\dfrac{-3x^2+2x-1}{(x^2+1)(x+1)}$ is equal to

A $\dfrac{2}{x^2+1} + \dfrac{3}{x+1}$ **B** $\dfrac{2}{x^2+1} - \dfrac{3}{x+1}$ **C** $\dfrac{5}{x^2+1} + \dfrac{2}{x+1}$

D $\dfrac{3}{x^2+1} - \dfrac{2}{x+1}$ **E** $\dfrac{3}{x^2+1} + \dfrac{2}{x+1}$

Extended-response questions

1 A train completes a journey of 240 km at a constant speed.

a If the train had travelled 4 km/h slower, it would have taken two hours more for the journey. Find the actual speed of the train.

b If the train had travelled a km/h slower and still taken two hours more for the journey of 240 km, what would have been the actual speed? (Answer in terms of a.) Discuss the practical possible values of a and also the possible values for the speed of the train.

c If the train had travelled a km/h slower and taken a hours more for the journey of 240 km, and if a is an integer and the speed is an integer, find the possible values for a and the speed of the train.

2 Two trains are travelling at constant speeds. The slower train takes a hours longer to cover b km. It travels 1 km less than the faster train in c hours.

a What is the speed of the faster train, in terms of a, b and c?

b If a, b, c and the speeds of the trains are all rational numbers, find five sets of values for a, b and c. Choose and discuss two sensible sets of values.

3 A tank can be filled using two pipes. The smaller pipe alone will take a minutes longer than the larger pipe alone to fill the tank. Also, the smaller pipe will take b minutes longer to fill the tank than when both pipes are used.

a In terms of a and b, how long will each of the pipes take to fill the tank?

b If $a = 24$ and $b = 32$, how long will each of the pipes take to fill the tank?

c If a and b are consecutive positive integers, find five pairs of values of a and b such that $b^2 - ab$ is a perfect square. Interpret these results in the context of this problem.

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Revision

4 Revision of Chapters 1–3

4A Short-answer questions

1 Rewrite each fraction with an integer denominator:

a $\dfrac{1}{\sqrt{2}-3}$ **b** $\dfrac{3}{\sqrt{5}-1}$ **c** $\dfrac{2}{2\sqrt{2}-1}$

d $\dfrac{3}{\sqrt{5}-\sqrt{3}}$ **e** $\dfrac{1}{\sqrt{7}-\sqrt{2}}$ **f** $\dfrac{1}{2\sqrt{5}-\sqrt{3}}$

2 **a** If the equations $x^2 + x - 1 = 0$ and $x^2 + bx + 1 = 0$ have a common solution, show that $b = \pm\sqrt{5}$.

b Find the common solution when:

i $b = \sqrt{5}$ **ii** $b = -\sqrt{5}$

3 Find constants a, b and c such that $(n + 1)(n - 7) = a + bn + cn(n - 1)$ for all n.

4 Prove that, if $n = \text{HCF}(a, b)$, then n divides $a - b$.

5 Write down the prime factorisation of each of the following numbers, and hence determine the square root of each number:

a 576 **b** 1225 **c** 1936 **d** 1296

6 Solve the equation $\dfrac{x+b}{x-c} = 1 - \dfrac{x}{x-c}$ for x.

7 Solve the equation $\dfrac{1}{x-a} + \dfrac{1}{x-b} = \dfrac{2}{x}$ for x.

8 Find the positive integer solutions of $5x + 12y = 193$.

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Revision

9 A man bought a number of books at \$25 each and a number at \$35 each. He spent \$190 in total. How many books did he buy at \$25 and how many at \$35?

10 Find two sets of values of λ, a, b such that, for all values of x,

$$x^2 - 4x - 8 + \lambda(x^2 - 2x - 5) = a(x - b)^2$$

11 Solve each of the following equations for x:

a $|x - 3| = 2$
b $|3x - 4| = 4$
c $|5x - 6| = 9$
d $|x - 4| - 10 = 0$
e $|5 - x| = 4$
f $|3x - 4| = 8$
g $|2x + 5| = 10$

12 Solve each of the following inequalities, giving your answer using set notation:

a $\{ x : |x| \le 2 \}$
b $\{ x : |x| \ge 1 \}$
c $\{ x : |2x - 5| \le 4 \}$
d $\{ x : |2x - 1| < 3 \}$
e $\{ x : |-2x + 3| \ge 4 \}$
f $\{ x : |-3x + 2| \le 3 \}$

13 Transpose each of the following to make x the subject:

a $y = 3 + \sqrt{2x - 1}$
b $y = \dfrac{2}{\sqrt{3x + 1}} - 2$

14 Two bushwalkers are 20 km apart on a track and travelling towards each other. One walks at 3 km/h and the other at 5 km/h. After how many minutes will they meet?

15 Solve each pair of simultaneous equations:

a $\dfrac{x}{3} + \dfrac{y}{4} = 1$
$3x - 4y = 1$

b $\dfrac{x}{a} + \dfrac{y}{b} = 1$
$ax - by = 1$

16 A group of 50 students were interviewed about the types of movies that they watch: 25 of the students like action movies, 26 like comedy, 17 like drama, 11 like action and comedy, 5 like action and drama, 8 like comedy and drama, and 3 like all three. How many of these students:

a like none of these types of movies
b like action movies only
c like action and comedy but not drama?

17 Consider the quadratic polynomial $p(x) = ax^2 - 2ax + 1$, where $a \ne 0$.

a Find the discriminant of $p(x)$.
b Find the values of a for which the graph of $y = p(x)$:
 i touches the x-axis once only
 ii crosses the x-axis twice
 iii does not intersect the x-axis.

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4B Multiple-choice questions

1 In algebraic form, five is seven less than three times one more than x can be written as

A $5 = 7 - 3(x + 1)$ **B** $3x + 1 = 5 - 7$ **C** $(x + 1) - 7 = 5$
D $5 = 7 - 3x + 1$ **E** $5 = 3x - 4$

2 $\dfrac{3}{x-3} - \dfrac{2}{x+3}$ is equal to

A 1 **B** $\dfrac{x+15}{x^2-9}$ **C** $\dfrac{15}{x-9}$ **D** $\dfrac{x-3}{x^2-9}$ **E** $-\dfrac{1}{6}$

3 If $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 4, 5, 6\}$ and $C = \{3, 4, 5, 6, 7\}$, then $A \cap (B \cup C)$ is equal to

A $\{1, 2, 3, 4, 5, 6, 7\}$ **B** $\{1, 2, 3, 4, 5, 6\}$ **C** $\{2, 3, 4\}$
D $\{3, 4\}$ **E** $\{2, 3, 4, 5, 6, 7\}$

4 The recurring decimal $0.\dot{7}\dot{2}$ is equal to

A $\dfrac{72}{101}$ **B** $\dfrac{72}{100}$ **C** $\dfrac{72}{99}$ **D** $\dfrac{72}{90}$ **E** $\dfrac{73}{90}$

5 $\dfrac{-4}{x-1} - \dfrac{3}{1-x} + \dfrac{x}{x-1}$ is equal to

A 1 **B** -1 **C** $\dfrac{7x}{x-1}$ **D** $\dfrac{1}{1-x}$ **E** none of these

6 $\dfrac{x+2}{3} - \dfrac{5}{6}$ is equal to

A $\dfrac{x-3}{6}$ **B** $\dfrac{2x+4}{6}$ **C** $\dfrac{2x-1}{6}$ **D** $\dfrac{2x-5}{6}$ **E** $\dfrac{x-3}{3}$

7 If $a = 1 + \dfrac{1}{1+b}$, then b equals

A $1 - \dfrac{1}{a-1}$ **B** $1 + \dfrac{1}{a-1}$ **C** $\dfrac{1}{a-1} - 1$ **D** $\dfrac{1}{a+1} + 1$ **E** $\dfrac{1}{a+1} - 1$

8 When the repeating decimal $0.\dot{3}\dot{6}$ is written in simplest fractional form, the sum of the numerator and denominator is

A 15 **B** 45 **C** 114 **D** 135 **E** 150

9 If $\dfrac{2x-y}{2x+y} = \dfrac{3}{4}$, then $\dfrac{x}{y}$ equals

A $\dfrac{2}{7}$ **B** $\dfrac{7}{2}$ **C** $\dfrac{3}{4}$ **D** $\dfrac{4}{3}$ **E** cannot be determined

10 If $\dfrac{3}{3+y} = 4$, then y equals

A $\dfrac{1}{4}$ **B** $-\dfrac{9}{4}$ **C** $\dfrac{9}{4}$ **D** 0 **E** $-\dfrac{4}{9}$

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Revision

11 The coordinates of the point where the lines with equations $3x + y = -7$ and $2x + 5y = 4$ intersect are

A $(3, -16)$ **B** $(-3, 2)$ **C** $(3, -2)$ **D** $(-2, 3)$ **E** no solution

12 If $\frac{m+2}{4} - \frac{2-m}{4} = \frac{1}{2}$, then m is equal to

A 1 **B** -1 **C** $\frac{1}{2}$ **D** 0 **E** $-\frac{1}{2}$

13 The number 46 200 can be written as

A $2 \times 3 \times 5 \times 7 \times 11$ **B** $2^2 \times 3^2 \times 5^2 \times 7 \times 11$ **C** $2 \times 3^2 \times 5 \times 7^2 \times 11$
D $2^3 \times 3 \times 5^2 \times 7 \times 11$ **E** $2^2 \times 3 \times 5^3 \times 7 \times 11$

14 If the positive integers $n + 1, n - 1, n - 6, n - 5, n + 4$ are arranged in increasing order of magnitude, then the middle number is

A $n + 1$ **B** $n - 1$ **C** $n - 6$ **D** $n - 5$ **E** $n + 4$

15 The expression $\frac{4}{n+1} + \frac{3}{n-1}$ is equal to

A $\frac{7n-1}{1-n^2}$ **B** $\frac{1-7n}{1-n^2}$ **C** $\frac{7n-1}{n^2+1}$ **D** $\frac{7}{n^2-1}$ **E** $\frac{7}{n}$

16 The second number is twice the first number; the third number is half the first number; the three numbers sum to 28. These numbers are

A $(8, 16, 4)$ **B** $(2, 3, 12)$ **C** $(7, 9, 11)$ **D** $(6, 8, 16)$ **E** $(12, 14, 2)$

17 $(\sqrt{7} + 3)(\sqrt{7} - 3)$ is equal to

A -2 **B** 10 **C** $\sqrt{14} - 19$ **D** $2\sqrt{7} - 9$ **E** 45

18 If $\frac{13x - 10}{2x^2 - 9x + 4} = \frac{P}{x-4} + \frac{Q}{2x-1}$, then the values of P and Q are

A $P = 1$ and $Q = 1$ **B** $P = -1$ and $Q = 1$ **C** $P = 6$ and $Q = 1$
D $P = -6$ and $Q = 1$ **E** $P = 1$ and $Q = -6$

19 If $\frac{5x}{(x+2)(x-3)} = \frac{P}{x+2} + \frac{Q}{x-3}$, then

A $P = 2$ and $Q = 3$ **B** $P = 2$ and $Q = -3$ **C** $P = -2$ and $Q = 3$
D $P = -2$ and $Q = -3$ **E** $P = 1$ and $Q = 1$

20 If the natural number n is a perfect square, then the next perfect square is

A $n + 1$ **B** $n^2 + 1$ **C** $n^2 + 2n + 1$ **D** $n^2 + n$ **E** $n + 2\sqrt{n} + 1$

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21 Which of the following is *not* a rational number?

A 0.4 **B** $\frac{3}{8}$ **C** $\sqrt{5}$ **D** $\sqrt{16}$ **E** 4.125

22 If $\frac{1}{x} = \frac{a}{b}$ and $\frac{1}{y} = a - b$, then $x + y$ equals

A $\frac{2}{a}$ **B** $\frac{a^2 - b^2}{a}$ **C** $\frac{ba - b^2 + a}{a(a - b)}$ **D** $\frac{2a}{a^2 - b^2}$ **E** $\frac{-2b}{a^2 - b^2}$

23 $9x^2 - 4mx + 4$ is a perfect square when m equals

A 5 **B** ± 12 **C** 2 **D** ± 1 **E** ± 3

24 If $x = (n + 1)(n + 2)(n + 3)$, for some positive integer n, then x is not always divisible by

A 1 **B** 2 **C** 3 **D** 5 **E** 6

25 If both n and p are odd numbers, which one of the following numbers must be even?

A $n + p$ **B** np **C** $np + 2$ **D** $n + p + 1$ **E** $2n + p$

26 $4a^2b^4 \times 3(ab^3)^{-2}$ is equal to

A $12b^{-2}$ **B** $12ab^{-2}$ **C** $12a^{-4}b^{-2}$ **D** $12a^3b^5$ **E** $12ab^5$

27 $\frac{3 \times 10^8}{\sqrt{0.144 \times 10^5}}$ is equal to

A 3.6×10^6 **B** 8×10^3 **C** 8×10^{-3} **D** 2.5×10^6 **E** 2.5×10^5

28 If $\mathrm{LCM}(12, n) = 60$ and $\mathrm{HCF}(12, n) = 6$, then $n =$

A 10 **B** 15 **C** 20 **D** 30 **E** 60

29 If x^2 is written in the form $(x - 2)^2 + b(x - 2) + c$, then the values of b and c are

A $b = 2,\ c = 0$ **B** $b = -4,\ c = -4$ **C** $b = 4,\ c = 4$
D $b = 2,\ c = 2$ **E** $b = 0,\ c = 2$

30 A car covers a distance of 50 km at an average speed of x km/h. Over the same period of time, a train covers a distance of 70 km at an average speed of $(x + 25)$ km/h. An equation that can be used to find x is

A $50x = 70(x + 25)$ **B** $70x = 50(x + 25)$ **C** $50x = 70x + 25$
D $70x = 50x + 25$ **E** $70 = 50 + 25x$

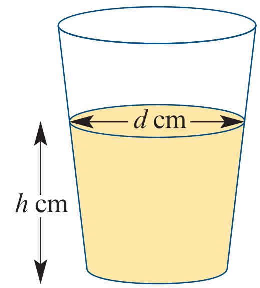
 ISBN 978-1-316-63608-4 Cambridge University Press

Revision

4C Extended-response questions

1 The diagram represents a glass containing milk. When the height of the milk in the glass is h cm, the diameter, d cm, of the surface of the milk is given by the formula

$$d = \frac{h}{5} + 6$$



a Find d when $h = 10$.

b Find d when $h = 8.5$.

c What is the diameter of the bottom of the glass?

d The diameter of the top of the glass is 9 cm. What is the height of the glass?

2 The cost, $\$C$, of manufacturing each jacket of a particular type is given by the formula

$$C = an + b \quad \text{for } 0 < n \leq 300$$

where a and b are constants and n is the size of the production run of this type of jacket. For making 100 jackets, the cost is \$108 each. For 120 jackets, the cost is \$100 each.

a Find the values of a and b.

b Sketch the graph of C against n for $0 < n \leq 300$.

c Find the cost of manufacturing each jacket if 200 jackets are made.

d If the cost of making each jacket is \$48.80, find the size of the production run.

3 The formula $A = 180 - \dfrac{360}{n}$ gives the size of each interior angle, $A°$, of a regular polygon with n sides.

a Find the value of A when n equals:

i 180 **ii** 360 **iii** 720 **iv** 7200

b As n becomes very large:

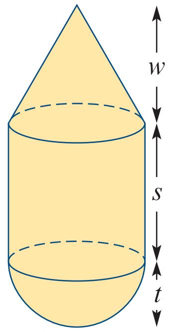
i What value does A approach? **ii** What shape does the polygon approach?

c Find the value of n when $A = 162$.

d Make n the subject of the formula.

e Three regular polygons, two of which are octagons, meet at a point so that they fit together without any gaps. Describe the third polygon.

4 The figure shows a solid consisting of three parts – a cone, a cylinder and a hemisphere – all of the same base radius.



a Find, in terms of w, s, t and π, the volume of each part.

b **i** If the volume of each of the three parts is the same, find the ratio $w : s : t$.

ii If also $w + s + t = 11$, find the total volume in terms of π.

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5 At the beginning of 2007, Andrew and John bought a small catering business. The profit, $\$P$, in a particular year is given by

$$P = an + b$$

where n is the number of years of operation and a and b are constants.

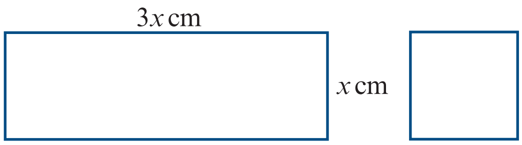
a Given the table, find the values of a and b.

Year	2007	2011
Number of years of operation (n)	1	5
Profit (P)	−9000	15 000

b Find the profit when $n = 12$.

c In which year was the profit \$45 000?

6 A piece of wire 28 cm long is cut into two parts: one to make a rectangle three times as long as it is wide, and the other to make a square.



a What is the perimeter of the rectangle in terms of x?

b What is the perimeter of the square in terms of x?

c What is the length of each side of the square in terms of x?

Let A be the sum of the areas of the two figures.

d Show that $A = 7(x^2 - 4x + 7)$.

e Use a CAS calculator to help sketch the graph of $A = 7(x^2 - 4x + 7)$ for $0 < x < 5$.

f Find the minimum value that A can take and the corresponding value of x.

7 A particular plastic plate manufactured at a factory sells at \$1.50. The cost of production consists of an initial cost of \$3500 and then \$0.50 per plate. Let x be the number of plates produced.

a Let $\$C$ be the cost of production of x plates. Write an expression for C in terms of x.

b Let $\$I$ be the income from selling x plates. Write an expression for I in terms of x.

c On the one set of axes, sketch the graphs of I against x and C against x.

d How many plates must be sold for the income to equal the cost of production?

e How many plates must be sold for a profit of \$2000 to be made?

f Let $P = I - C$. Sketch the graph of P against x. What does P represent?

ISBN 978-1-316-63608-4

Revision

8 **a** **i** For the equation $\sqrt{7x-5}-\sqrt{2x}=\sqrt{15-7x}$, square both sides to show that this equation implies

$$8x-10=\sqrt{14x^2-10x}$$

ii Square both sides of this new equation and simplify to form the equation

$$x^2-3x+2=0 \qquad (1)$$

iii The solutions to equation (1) are $x=1$ and $x=2$. Test these solutions for the equation

$$\sqrt{7x-5}-\sqrt{2x}=\sqrt{15-7x}$$

and hence show that $x=2$ is the only solution to the original equation.

b Use the techniques of part **a** to solve the equations:

i $\sqrt{x+2}-2\sqrt{x}=\sqrt{x+1}$

ii $2\sqrt{x+1}+\sqrt{x-1}=3\sqrt{x}$

9 Let n be a natural number less than 50 such that $n+25$ is a perfect square.

a Show that there exists an integer a such that $n=a(a+10)$.

b Any natural number less than 100 can be written in the form $10p+q$, where p and q are digits. For this representation of n, show that $q=p^2$.

c Give all possible values of n.

10 If an object of mass m kg is at a height of h m above the ground, then its potential energy (P joules) is given by the formula $P=mgh$, where g is a constant.

a **i** Find the value of the constant g given that $P=980$ when $h=20$ and $m=5$.

ii Sketch the graph of P against h for an object of mass 5 kg.

iii Find m if $P=2058$ and $h=30$.

b **i** What is the effect on the potential energy if the height (h m) is doubled and the mass remains constant?

ii What is the effect on the potential energy if the object has one-quarter of the original height (h m) and double the original mass (m kg)?

c If an object is dropped from a height (h m) above ground level, its speed (V m/s) when it reaches the ground is given by $V=\sqrt{19.6h}$.

i Find V when $h=10$.

ii Find V when $h=90$.

d In order to double the speed that a given object has when it hits the ground, by what factor must the height from which it is dropped be increased?

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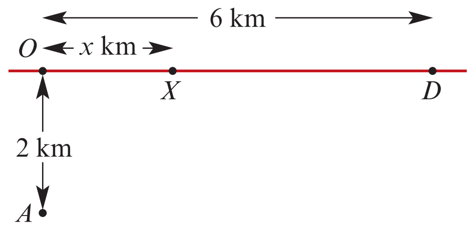
Revision

11 The diagram shows a straight road OD, where $OD = 6$ km.



A hiker is at A, which is 2 km from O. The hiker walks directly to X and then walks along the road to D. The hiker can walk at 3 km/h off-road, but at 8 km/h along the road.

a If $OX = 3$ km, calculate the total time taken for the hiker to walk from A to D via X in hours and minutes, correct to the nearest minute.

b If the total time taken was $1\frac{1}{2}$ hours, calculate the distance OX in kilometres, correct to one decimal place.

12 Seventy-six photographers submitted work for a photographic exhibition in which they were permitted to enter not more than one photograph in each of three categories: black and white (B), colour prints (C), transparencies (T). Eighteen entrants had all their work rejected, while 30 B, 30 T and 20 C were accepted.

- From the exhibitors, as many showed T only as showed T and C.
- There were three times as many exhibitors showing B only as showing C only.
- Four exhibitors showed B and T but not C.

a Write the last three sentences in symbolic form.

b Draw a Venn diagram representing the information.

c **i** Find $|B \cap C \cap T|$.

ii Find $|B \cap C \cap T'|$.

ISBN 978-1-316-63608-4

Cambridge University Press

Chapter 5

5 Principles of counting

Objectives

- To solve problems using the **addition** and **multiplication principles**.
- To solve problems involving **permutations**.
- To solve problems involving **combinations**.
- To establish and use identities associated with **Pascal's triangle**.
- To solve problems using the **pigeonhole principle**.
- To understand and apply the **inclusion–exclusion principle**.

Take a deck of 52 playing cards. This simple, familiar deck can be arranged in so many ways that if you and every other living human were to shuffle a deck once per second from the beginning of time, then by now only a tiny fraction of all possible arrangements would have been obtained. So, remarkably, every time you shuffle a deck you are likely to be the first person to have created that particular arrangement of cards!

To see this, note that we have 52 choices for the first card, and then 51 choices for the second card, and so on. This gives a total of

$$52 \times 51 \times \cdots \times 2 \times 1 \approx 8.1 \times 10^{67}$$

arrangements. This is quite an impressive number, especially in light of the fact that the universe is estimated to be merely 1.4×10^{10} years old.

Combinatorics is concerned with counting the number of ways of doing something. Our goal is to find clever ways of doing this without explicitly listing all the possibilities. This is particularly important in the study of probability. For instance, we can use combinatorics to explain why certain poker hands are more likely to occur than others without considering all 2 598 960 possible hands.

ISBN 978-1-316-63608-4

Cambridge University Press

5A Basic counting methods

▶ Tree diagrams

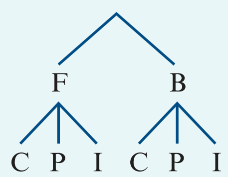
In most combinatorial problems, we are interested in the *number of solutions* to a given problem, rather than the solutions themselves. Nonetheless, for simple counting problems it is sometimes practical to list and then count all the solutions. Tree diagrams provide a systematic way of doing this, especially when the problem involves a small number of steps.

Example 1

A restaurant has a fixed menu, offering a choice of fish or beef for the main meal, and cake, pudding or ice-cream for dessert. How many different meals can be chosen?

Solution

We illustrate the possibilities on a tree diagram:



This gives six different meals, which we can write as

FC, FP, FI, BC, BP, BI

▶ The multiplication principle

In the above example, for each of the two ways of selecting the main meal, there were three ways of selecting the dessert. This gives a total of $2 \times 3 = 6$ ways of choosing a meal. This is an example of the **multiplication principle**, which will be used extensively throughout this chapter.

Multiplication principle

If there are m ways of performing one task and then there are n ways of performing another task, then there are $m \times n$ ways of performing *both* tasks.

Example 2

Sandra has three different skirts, four different tops and five different pairs of shoes. How many choices does she have for a complete outfit?

Solution	*Explanation*
$3 \times 4 \times 5 = 60$	Using the multiplication principle, we multiply the number of ways of making each choice.

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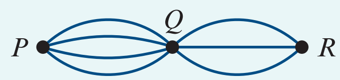
Example 3

How many paths are there from point P to point R travelling from left to right?



Solution

$4 \times 3 = 12$

Explanation

For each of the four paths from P to Q, there are three paths from Q to R.

The addition principle

In some instances, we have to count the number of ways of choosing between two alternative tasks. In this case, we use the **addition principle**.

Addition principle

Suppose there are m ways of performing one task and n ways of performing another task. If we cannot perform both tasks, then there are $m + n$ ways to perform one of the tasks.

Example 4

To travel from Canberra to Sydney tomorrow, Kara has a choice between three different flights and two different trains. How many choices does she have?

Solution

$3 + 2 = 5$

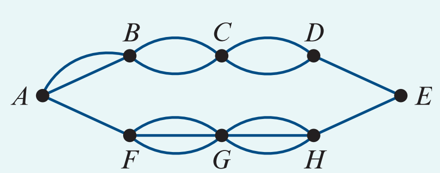
Explanation

The addition principle applies because Kara cannot travel by both plane and train. Therefore, we add the number of ways of making each choice.

Some problems will require use of both the multiplication and the addition principles.

Example 5

How many paths are there from point A to point E travelling from left to right?



Solution

We can take *either* an upper path *or* a lower path:

- Going from A to B to C to D to E there are $2 \times 2 \times 2 \times 1 = 8$ paths.
- Going from A to F to G to H to E there are $1 \times 3 \times 3 \times 1 = 9$ paths.

Using the addition principle, there is a total of $8 + 9 = 17$ paths from A to E.

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▶ Harder problems involving tree diagrams

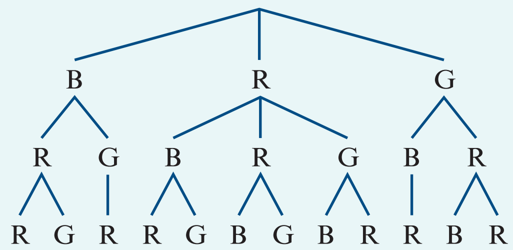
For some problems, a straightforward application of the multiplication and addition principles is not possible.

Example 6

A bag contains one blue token, two red tokens and one green token. Three tokens are removed from the bag and placed in a row. How many arrangements are possible?

Solution

The three tokens are selected without replacement. So once a blue or green token is taken, these cannot appear again. We use a tree diagram to systematically find every arrangement.



The complete set of possible arrangements can be read by tracing out each path from top to bottom of the diagram. This gives 12 different arrangements:

BRR, BRG, BGR, RBR, RBG, RRB, RRG, RGB, RGR, GBR, GRB, GRR

Section summary

Three useful approaches to solving simple counting problems:

- **Tree diagrams**
  These can be used to systematically list all solutions to a problem.
- **Multiplication principle**
  If there are m ways of performing one task and then there are n ways of performing another task, then there are $m \times n$ ways of performing *both* tasks.
- **Addition principle**
  Suppose there are m ways of performing one task and n ways of performing another task. If we cannot perform both tasks, there are $m + n$ ways to perform one of the tasks.

Some problems require use of both the addition and the multiplication principles.

Exercise 5A

Skillsheet
Example 2

1 Sam has five T-shirts, three pairs of pants and three pairs of shoes. How many different outfits can he assemble using these clothes?

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Example 4

2 A restaurant offers five beef dishes and three chicken dishes. How many selections of one main meal does a customer have?

3 Each of the 10 boys at a party shakes hands with each of the 12 girls. How many handshakes take place?

4 Draw a tree diagram showing all the two-digit numbers that can be formed using the digits 7, 8 and 9 if each digit:

a cannot be repeated **b** can be repeated.

5 How many different three-digit numbers can be formed using the digits 2, 4 and 6 if each digit can be used:

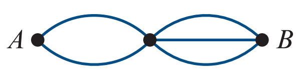
a as many times as you would like **b** at most once?

6 Jack wants to travel from Sydney to Perth via Adelaide. There are four flights and two trains from Sydney to Adelaide. There are two flights and three trains from Adelaide to Perth. How many ways can Jack travel from Sydney to Perth?

7 Travelling from left to right, how many paths are there from point A to point B in each of the following diagrams?

Example 3

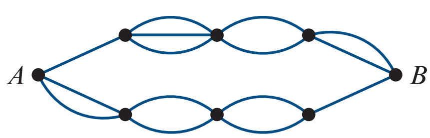
a



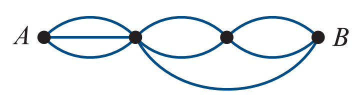
b

Example 5

c



d



Example 6

8 A bag contains two blue, one red and two green tokens. Two tokens are removed from the bag and placed in a row. With the help of a tree diagram, list all the different arrangements.

9 How many ways can you make change for 50 cents using 5, 10 and 20 cent pieces?

10 Four teachers decide to swap desks at work. How many ways can this be done if no teacher is to sit at their previous desk?

11 Three runners compete in a race. In how many ways can the runners complete the race assuming:

a there are no tied places **b** the runners can tie places?

12 A six-sided die has faces labelled with the numbers 0, 2, 3, 5, 7 and 11. If the die is rolled twice and the two results are multiplied, how many different answers can be obtained?

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5B Factorial notation and permutations

► Factorial notation

Factorial notation provides a convenient way of expressing products of consecutive natural numbers. For each natural number n, we define

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$$

where the notation $n!$ is read as 'n factorial'.

We also define $0! = 1$. Although it might seem strange at first, this definition will turn out to be very convenient, as it is compatible with formulas that we will establish shortly.

Another very useful identity is

$$n! = n \cdot (n-1)!$$

Example 7

Evaluate:

a $3!$ **b** $\dfrac{50!}{49!}$ **c** $\dfrac{10!}{2!\,8!}$

Solution

a $3! = 3 \cdot 2 \cdot 1$
$= 6$

b $\dfrac{50!}{49!} = \dfrac{50 \cdot 49!}{49!}$
$= 50$

c $\dfrac{10!}{2!\,8!} = \dfrac{10 \cdot 9 \cdot 8!}{2! \cdot 8!}$
$= \dfrac{10 \cdot 9}{2 \cdot 1}$
$= 45$

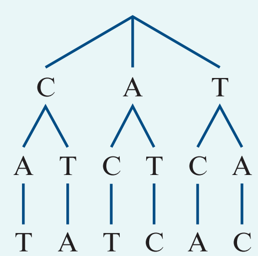
► Permutations of n objects

A **permutation** is an ordered arrangement of a collection of objects.

Example 8

Using a tree diagram, list all the permutations of the letters in the word CAT.

Solution



There are six permutations:

CAT, CTA, ACT, ATC, TCA, TAC

Explanation

There are three choices for the first letter. This leaves only two choices for the second letter, and then one for the third.

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Another way to find the number of permutations for the previous example is to draw three boxes, corresponding to the three positions. In each box, we write the number of choices we have for that position.

- We have 3 choices for the first letter (C, A or T).
- We have 2 choices for the second letter (because we have already used one letter).
- We have 1 choice for the third letter (because we have already used two letters).

3	2	1

By the multiplication principle, the total number of arrangements is

$$3 \times 2 \times 1 = 3!$$

So three objects can be arranged in 3! ways. More generally:

> The number of permutations of n objects is $n!$.

Proof The reason for this is simple:

- The first item can be chosen in n ways.
- The second item can be chosen in $n - 1$ ways, since only $n - 1$ objects remain.
- The third item can be chosen in $n - 2$ ways, since only $n - 2$ objects remain.

⋮

- The last item can be chosen in 1 way, since only 1 object remains.

Therefore, by the multiplication principle, there are

$$n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1 = n!$$

permutations of n objects.

Example 9

How many ways can six different books be arranged on a shelf?

Solution

$$\begin{aligned} 6! &= 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 720 \end{aligned}$$

Explanation

Six books can be arranged in 6! ways.

Example 10

Using your calculator, find how many ways 12 students can be lined up in a row.

Using the TI-Nspire

Evaluate 12! as shown.

Note: The factorial symbol (!) can be accessed using (?!▸), the Symbols palette ((ctrl)(📖)) or (menu) > **Probability** > **Factorial**.

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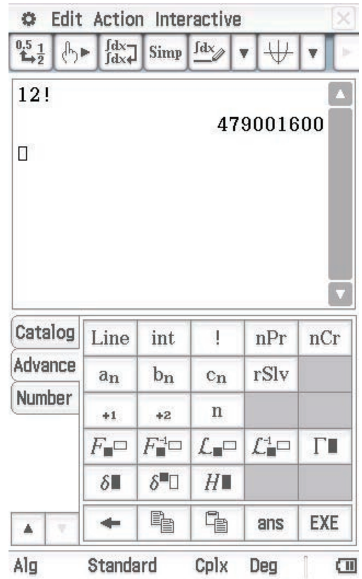
Using the Casio ClassPad

- In $\sqrt[\text{Main}]{\alpha}$, open the keyboard.
- Enter the number 12, followed by the factorial symbol. Tap EXE.

Note: The factorial symbol (!) is found in the Advance keyboard; you need to scroll down to see this.



Example 11

How many four-digit numbers can be formed using the digits 1, 2, 3 and 4 if:

a they cannot be repeated

b they can be repeated?

Solution

a $4! = 4 \times 3 \times 2 \times 1 = 24$

Explanation: Four numbers can be arranged in 4! ways.

b $4^4 = 4 \times 4 \times 4 \times 4 = 256$

Explanation: Using the multiplication principle, there are 4 choices for each of the 4 digits.

▶ Permutations of *n* objects taken *r* at a time

Imagine a very small country with very few cars. Licence plates consist of a sequence of four digits, and repetitions of the digits are not allowed. How many such licence plates are there?

Here, we are asking for the number of permutations of 10 digits taken four at a time. We will denote this number by ${}^{10}P_4$.

To solve this problem, we draw four boxes. In each box, we write the number of choices we have for that position. For the first digit, we have a choice of 10 digits. Once chosen, we have only 9 choices for the second digit, then 8 choices for the third and 7 choices for the fourth.

10	9	8	7

By the multiplication principle, the total number of licence plates is

$$10 \times 9 \times 8 \times 7$$

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There is a clever way of writing this product as a fraction involving factorials:

$$\begin{aligned}{}^{10}P_4 &= 10 \cdot 9 \cdot 8 \cdot 7 \\ &= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\ &= \frac{10!}{6!} \\ &= \frac{10!}{(10-4)!}\end{aligned}$$

More generally:

Number of permutations

The number of permutations of n objects taken r at a time is denoted by ${}^nP_r$ and is given by the formula

$$ {}^nP_r = \frac{n!}{(n-r)!} $$

Proof To establish this formula we note that:

- The 1st item can be chosen in n ways.
- The 2nd item can be chosen in $n - 1$ ways.

⋮

- The rth item can be chosen in $n - r + 1$ ways.

Therefore, by the multiplication principle, the number of permutations of n objects taken r at a time is

$$\begin{aligned}{}^nP_r &= n \cdot (n-1) \cdot \dots \cdot (n-r+1) \\ &= \frac{n \cdot (n-1) \cdot \dots \cdot (n-r+1) \cdot (n-r) \cdot \dots \cdot 2 \cdot 1}{(n-r) \cdot \dots \cdot 2 \cdot 1} \\ &= \frac{n!}{(n-r)!}\end{aligned}$$

Notes:

- If $r = n$, then we have ${}^nP_n$, which is simply the number of permutations of n objects and so must equal $n!$. The formula still works in this instance, since

$$\begin{aligned}{}^nP_n &= \frac{n!}{(n-n)!} \\ &= \frac{n!}{0!} \\ &= n!\end{aligned}$$

  Note that this calculation depends crucially on our decision to define $0! = 1$.
- If $r = 1$, then we obtain ${}^nP_1 = n$. Given n objects, there are n choices of one object, and each of these can be arranged in just one way.

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Example 12

a Using the letters A, B, C, D and E without repetition, how many different two-letter arrangements are there?

b Six runners compete in a race. In how many ways can the gold, silver and bronze medals be awarded?

Solution

a There are five letters to arrange in two positions:

$$\begin{aligned} {}^5P_2 &= \frac{5!}{(5-2)!} \\ &= \frac{5!}{3!} \\ &= \frac{5 \cdot 4 \cdot 3!}{3!} \\ &= 20 \end{aligned}$$

b There are six runners to arrange in three positions:

$$\begin{aligned} {}^6P_3 &= \frac{6!}{(6-3)!} \\ &= \frac{6!}{3!} \\ &= \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!} \\ &= 120 \end{aligned}$$

Although the formula developed for ${}^nP_r$ will have an important application later in this chapter, you do not actually have to use it when solving problems. It is often more convenient to simply draw boxes corresponding to the positions, and to write in each box the number of choices for that position.

Example 13

How many ways can seven friends sit along a park bench with space for only four people?

Solution

7	6	5	4

By the multiplication principle, the total number of arrangements is

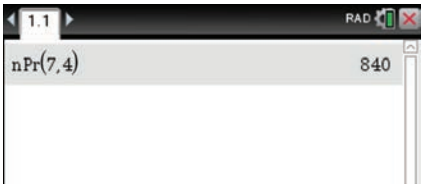
$$7 \times 6 \times 5 \times 4 = 840$$

Explanation

We draw four boxes, representing the positions to be filled. In each box we write the number of ways we can fill that position.

Using the TI-Nspire

- To evaluate ${}^7P_4$, use menu > **Probability** > **Permutations** as shown.



Note: Alternatively, you can simply type npr(7, 4). The command is not case sensitive.

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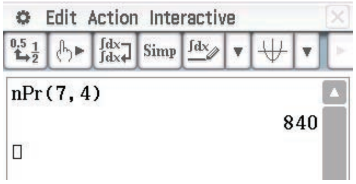
Using the Casio ClassPad

To evaluate ${}^7P_4$:

- In Main $\sqrt{\alpha}$, select nPr from the Advance keyboard. (You need to scroll down to find this keyboard.)
- In the brackets, enter the numbers 7 and 4, separated by a comma. Then tap EXE.



Section summary

- $n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1$ and $0! = 1$
- $n! = n \cdot (n-1)!$
- A **permutation** is an ordered arrangement of objects.
- The number of permutations of n objects is $n!$.
- The number of permutations of n objects taken r at a time is given by

$$ {}^nP_r = \frac{n!}{(n-r)!} $$

Exercise 5B

1 Evaluate $n!$ for $n = 0, 1, 2, \dots, 10$.

Example 7 **2** Evaluate each of the following:

a $\dfrac{5!}{4!}$ **b** $\dfrac{10!}{8!}$ **c** $\dfrac{12!}{10!\,2!}$ **d** $\dfrac{100!}{97!\,3!}$

3 Simplify the following expressions:

a $\dfrac{(n+1)!}{n!}$ **b** $\dfrac{(n+2)!}{(n+1)!}$ **c** $\dfrac{n!}{(n-2)!}$ **d** $\dfrac{1}{n!} + \dfrac{1}{(n+1)!}$

4 Evaluate ${}^4P_r$ for $r = 0, 1, 2, 3, 4$.

Example 8 **5** Use a tree diagram to find all the permutations of the letters in the word DOG.

Example 9 **6** How many ways can five books on a bookshelf be arranged?

7 How many ways can the letters in the word HYPERBOLA be arranged?

Example 12 **8** Write down all the two-letter permutations of the letters in the word FROG.

Example 13 **9** How many ways can six students be arranged along a park bench if the bench has:

a six seats **b** five seats **c** four seats?

10 Using the digits 1, 2, 5, 7 and 9 without repetition, how many numbers can you form that have:

a five digits **b** four digits **c** three digits?

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11 How many ways can six students be allocated to eight vacant desks?

12 How many ways can three letters be posted in five mailboxes if each mailbox can receive:

a more than one letter **b** at most one letter?

13 Using six differently coloured flags without repetition, how many signals can you make using:

a three flags in a row **b** four flags in a row **c** five flags in a row?

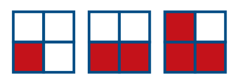
14 You are in possession of four flags, each coloured differently. How many signals can you make using at least two flags arranged in a row?

15 Many Australian car licence plates consist of a sequence of three letters followed by a sequence of three digits.

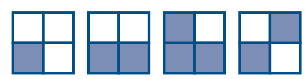
a How many different car licence plates have letters and numbers arranged this way?

b How many of these have no repeated letters or numbers?

16 **a** The three tiles shown are to be arranged in a row, and can be rotated. How many different ways can this be done?



b The four tiles shown are to be arranged in a row, and can be rotated. How many different ways can this be done?



17 Find all possible values of m and n if $m! \cdot n! = 720$ and $m > n$.

18 Show that $n! = (n^2 - n) \cdot (n - 2)!$ for $n \geq 2$.

19 Given six different colours, how many ways can you paint a cube so that all the faces have different colours? Two colourings are considered to be the same when one can be obtained from the other by rotating the cube.

5C Permutations with restrictions

Suppose we want to know how many three-digit numbers have no repeated digits. The answer is *not* simply ${}^{10}P_3$, the number of permutations of 10 digits taken three at a time. This is because the digit 0 cannot be used in the hundreds place.

- There are 9 choices for the first digit (1, 2, 3, . . . , 9).
- There are 9 choices for the second digit (0 and the eight remaining non-zero digits).
- This leaves 8 choices for the third digit.

100s	10s	units
9	9	8

By the multiplication principle, there are $9 \times 9 \times 8 = 648$ different three-digit numbers.

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When considering permutations with restrictions, we deal with the restrictions first.

Example 14

a How many arrangements of the word DARWIN begin and end with a vowel?

b Using the digits 0, 1, 2, 3, 4 and 5 without repetition, how many odd four-digit numbers can you form?

Solution

a We draw six boxes. In each box, we write the number of choices we have for that position. We first consider restrictions. There are two choices of vowel (A or I) for the first letter, leaving only one choice for the last letter.

2					1

This leaves four choices for the second letter, three for the next, and so on.

2	4	3	2	1	1

By the multiplication principle, the number of arrangements is

$$2 \times 4 \times 3 \times 2 \times 1 \times 1 = 48$$

b We draw four boxes. Again, we first consider restrictions. The last digit must be odd (1, 3 or 5), giving three choices. We cannot use 0 in the first position, so this leaves four choices for that position.

4			3

Once these two digits have been chosen, this leaves four choices and then three choices for the remaining two positions.

4	4	3	3

Thus the number of arrangements is

$$4 \times 4 \times 3 \times 3 = 144$$

▶ Permutations with items grouped together

For some arrangements, we may want certain items to be grouped together. In this case, the trick is to initially treat each group of items as a single object. We then multiply by the numbers of arrangements within each group.

Example 15

a How many arrangements of the word EQUALS are there if the vowels are kept together?

b How many ways can two chemistry, four physics and five biology books be arranged on a shelf if the books of each subject are kept together?

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Solution	Explanation
a $4! \times 3! = 144$	We group the three vowels together so that we have four items to arrange: (E, U, A), Q, L, S. They can be arranged in 4! ways. Then the three vowels can be arranged among themselves in 3! ways. We use the multiplication principle.
b $3! \times 2! \times 4! \times 5! = 34\,560$	There are three groups and so they can be arranged in 3! ways. The two chemistry books can be arranged among themselves in 2! ways, the four physics books in 4! ways and the five biology books in 5! ways. We use the multiplication principle.

Section summary

- To count permutations that are subject to restrictions, we draw a series of boxes. In each box, we write the number of choices we have for that position. We always consider the restrictions first.
- When items are to be grouped together, we initially treat each group as a single object. We find the number of arrangements of the groups, and then multiply by the numbers of arrangements within each group.

Exercise 5C

Skillsheet

Example 14

1 Using the digits 1, 2, 3, 4 and 5 without repetition, how many five-digit numbers can you form:

a without restriction
b that are odd
c that begin with 5
d that do not begin with 5?

Example 15

2 In how many ways can three girls and two boys be arranged in a row:

a without restriction
b if the two boys sit together
c if the two boys do not sit together
d if girls and boys alternate?

3 How many permutations of the word QUEASY:

a begin with a vowel
b begin and end with a vowel
c keep the vowels together
d keep the vowels and consonants together?

4 How many ways can four boys and four girls be arranged in a row if:

a boys and girls sit in alternate positions
b boys sit together and girls sit together?

5 The digits 0, 1, 2, 3, 4 and 5 can be combined without repetition to form new numbers. In how many ways can you form:

a a six-digit number
b a four-digit number divisible by 5
c a number less than 6000
d an even three-digit number?

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6 Two parents and four children are seated in a cinema along six consecutive seats. How many ways can this be done:

a without restriction
b if the two parents sit at either end
c if the children sit together
d if the parents sit together and the children sit together
e if the youngest child must sit between and next to both parents?

7 12321 is a **palindromic number** because it reads the same backwards as forwards. How many palindromic numbers have:

a five digits
b six digits?

8 How many arrangements of the letters in VALUE do not begin and end with a vowel?

9 Using each of the digits 1, 2, 3 and 4 at most once, how many even numbers can you form?

10 How many ways can six girls be arranged in a row so that two of the girls, A and B:

a do not sit together
b have one person between them?

11 How many ways can three girls and three boys be arranged in a row if no two girls sit next to each other?

5D Permutations of like objects

The name for the Sydney suburb of WOOLLOOMOOLOO has the unusual distinction of having 13 letters in total, of which only four are different. Finding the number of permutations of the letters in this word is not as simple as evaluating 13!. This is because switching like letters does not result in a new permutation.

Our aim is to find an expression for P, where P is the number of permutations of the letters in the word WOOLLOOMOOLOO. First notice that the word has

1 letter W, 1 letter M, 3 letter Ls, 8 letter Os

Replace the three identical Ls with L_1, L_2 and L_3. These three letters can be arranged in 3! different ways. Therefore, by the multiplication principle, there are now

$$P \cdot 3!$$

permutations. Likewise, replace the eight identical Os with $O_1, O_2, \ldots, O_8$. These eight letters can be arranged in 8! different ways. Therefore there are now

$$P \cdot 3! \cdot 8!$$

permutations.

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On the other hand, notice that the 13 letters are now distinct, so there are 13! permutations of these letters. Therefore

$$P \cdot 3! \cdot 8! = 13! \quad \text{and so} \quad P = \frac{13!}{3!\,8!}$$

We can easily generalise this procedure to give the following result.

Permutations of like objects

The number of permutations of n objects of which n_1 are alike, n_2 are alike, ... and n_r are alike is given by

$$\frac{n!}{n_1!\,n_2!\,\cdots\,n_r!}$$

Example 16

a Find the number of permutations of the letters in the word RIFFRAFF.

b There are four identical knives, three identical forks and two identical spoons in a drawer. They are taken out of the drawer and lined up in a row. How many ways can this be done?

Solution

a $\dfrac{8!}{4!\,2!} = 840$

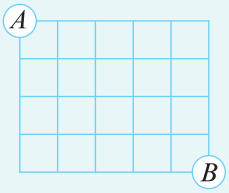
Explanation: There are 8 letters of which 4 are alike and 2 are alike.

b $\dfrac{9!}{4!\,3!\,2!} = 1260$

Explanation: There are 9 items of which 4 are alike, 3 are alike and 2 are alike.

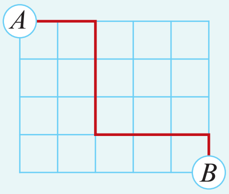
Example 17

The grid shown consists of unit squares. By travelling only right (R) or down (D) along the grid lines, how many paths are there from point A to point B?



Solution

Each path from A to B can be described by a sequence of four Ds and five Rs in some order. For example, the path shown can be described by the sequence RRDDDRRRD.



There are

$$\frac{9!}{4!\,5!} = 126$$

permutations of these letters, since there are 9 letters of which 4 are alike and 5 are alike.

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Section summary

- Switching like objects does not give a new arrangement.
- The number of permutations of n objects of which n_1 are alike, n_2 are alike, ... and n_r are alike is given by

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}$$

Exercise 5D

Skillsheet
Example 16

1 Ying has four identical 20 cent pieces and three identical 10 cent pieces. How many ways can she arrange these coins in a row?

2 How many ways can the letters in the word MISSISSIPPI be arranged?

3 Find the number of permutations of the letters in the word WARRNAMBOOL.

4 Using five 9s and three 7s, how many eight-digit numbers can be made?

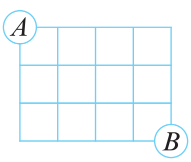
5 Using three As, four Bs and five Cs, how many sequences of 12 letters can be made?

6 How many ways can two red, two black and four blue flags be arranged in a row:

a without restriction
b if the first flag is red
c if the first and last flags are blue
d if every alternate flag is blue
e if the two red flags are adjacent?

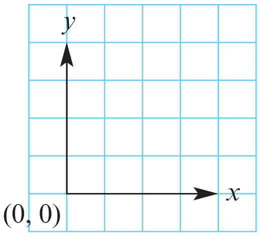
Example 17

7 The grid shown consists of unit squares. By travelling only right (R) or down (D) along the grid lines, how many paths are there from point A to point B?



8 The grid shown consists of unit squares. By travelling only along the grid lines, how many paths are there:

a of length 6 from $(0, 0)$ to the point $(2, 4)$
b of length $m + n$ from $(0, 0)$ to the point (m, n), where m and n are natural numbers?



9 Consider a deck of 52 playing cards.

a How many ways can the deck be arranged? Express your answer in the form $a!$.
b If two identical decks are combined, how many ways can the cards be arranged? Express your answer in the form $\frac{a!}{(b!)^c}$.
c If n identical decks are combined, find an expression for the number of ways that the cards can be arranged.

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10 An ant starts at position (0, 0) and walks north, east, south or west, one unit at a time. How many different paths of length 8 units finish at (0, 0)?

11 Jessica is about to walk up a flight of 10 stairs. She can take either one or two stairs at a time. How many different ways can she walk up the flight of stairs?

5E Combinations

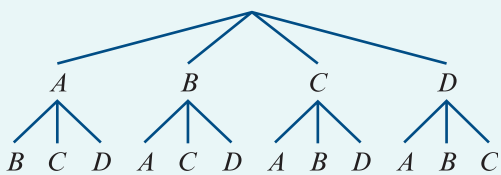
We have seen that a permutation is an ordered arrangement of objects. In contrast, a **combination** is a selection made regardless of order. We use the notation ${}^nP_r$ to denote the number of permutations of n distinct objects taken r at a time. Similarly, we use the notation ${}^nC_r$ to denote the number of combinations of n distinct objects taken r at a time.

Example 18

How many ways can two letters be chosen from the set $\{A, B, C, D\}$?

Solution

The tree diagram below shows the ways that the first and second choices can be made.



This gives 12 arrangements. But there are only six selections, since

$\{A, B\}$ is the same as $\{B, A\}$, $\{A, C\}$ is the same as $\{C, A\}$, $\{A, D\}$ is the same as $\{D, A\}$,
$\{B, C\}$ is the same as $\{C, B\}$, $\{B, D\}$ is the same as $\{D, B\}$ $\{C, D\}$ is the same as $\{D, C\}$

Suppose we want to count the number of ways that three students can be chosen from a group of seven. Let's label the students with the letters $\{A, B, C, D, E, F, G\}$. One such combination might be BDE. Note that this combination corresponds to 3! permutations:

$$BDE, \quad BED, \quad DBE, \quad DEB, \quad EBD, \quad EDB$$

In fact, each combination of three items corresponds to 3! permutations, and so there are 3! times as many permutations as combinations. Therefore

$$^7P_3 = 3! \times {}^7C_3 \quad \text{and so} \quad {}^7C_3 = \frac{^7P_3}{3!}$$

Since we have already established that ${}^7P_3 = \dfrac{7!}{(7-3)!}$, we obtain

$$^7C_3 = \frac{7!}{3!\,(7-3)!}$$

This argument generalises easily so that we can establish a formula for ${}^nC_r$.

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Number of combinations

The number of combinations of n objects taken r at a time is given by the formula

$$^nC_r = \frac{n!}{r!\,(n-r)!}$$

Example 19

a A pizza can have three toppings chosen from nine options. How many different pizzas can be made?

b How many subsets of $\{1, 2, 3, \dots, 20\}$ have exactly two elements?

Solution

a Three objects are to be chosen from nine options. This can be done in $^9C_3$ ways, and

$$\begin{aligned} ^9C_3 &= \frac{9!}{3!\,(9-3)!} \\ &= \frac{9!}{3!\,6!} \\ &= \frac{9 \cdot 8 \cdot 7 \cdot 6!}{3! \cdot 6!} \\ &= \frac{9 \cdot 8 \cdot 7}{3 \cdot 2} \\ &= 84 \end{aligned}$$

b Two objects are to be chosen from 20 options. This can be done in $^{20}C_2$ ways, and

$$\begin{aligned} ^{20}C_2 &= \frac{20!}{2!\,(20-2)!} \\ &= \frac{20!}{2!\,18!} \\ &= \frac{20 \cdot 19 \cdot 18!}{2! \cdot 18!} \\ &= \frac{20 \cdot 19}{2 \cdot 1} \\ &= 190 \end{aligned}$$

Example 20

Using your calculator, find how many ways 10 students can be selected from a class of 20 students.

Using the TI-Nspire

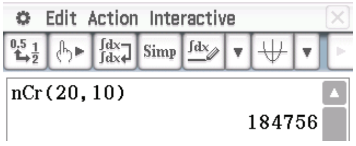
- To evaluate $^{20}C_{10}$, use menu > **Probability** > **Combinations** as shown.

Note: Alternatively, you can simply type ncr(20, 10). The command is not case sensitive.

Using the Casio ClassPad

To evaluate $^{20}C_{10}$:

- In Main $\sqrt{\alpha}$, select nCr from the Advance keyboard.
- In the brackets, enter the numbers 20 and 10, separated by a comma. Then tap EXE.



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Example 21

Consider a group of six students. In how many ways can a group of:

a two students be selected

b four students be selected?

Solution

a
$$^6C_2 = \frac{6!}{2!\,(6-2)!} = \frac{6!}{2!\,4!} = \frac{6 \cdot 5 \cdot 4!}{2! \cdot 4!} = \frac{6 \cdot 5}{2 \cdot 1} = 15$$

b
$$^6C_4 = \frac{6!}{4!\,(6-4)!} = \frac{6!}{4!\,2!} = \frac{6 \cdot 5 \cdot 4!}{4! \cdot 2!} = \frac{6 \cdot 5}{2 \cdot 1} = 15$$

The fact that parts **a** and **b** of the previous example have the same answer is not a coincidence. Choosing two students out of six is the same as *not choosing* the other four students out of six. Therefore $^6C_2 = {}^6C_4$.

More generally:

$$^nC_r = {}^nC_{n-r}$$

Quick calculations

In some instances, you can avoid unnecessary calculations by noting that:

- $^nC_0 = 1$, since there is only one way to select no objects from n objects
- $^nC_n = 1$, since there is only one way to select n objects from n objects
- $^nC_1 = n$, since there are n ways to select one object from n objects
- $^nC_{n-1} = n$, since this corresponds to the number of ways of not selecting one object from n objects.

Example 22

a Six points lie on a circle. How many triangles can you make using these points as the vertices?

b Each of the 20 people at a party shakes hands with every other person. How many handshakes take place?

Solution	*Explanation*
a $^6C_3 = 20$	This is the same as asking how many ways three vertices can be chosen out of six.
b $^{20}C_2 = 190$	This is the same as asking how many ways two people can be chosen to shake hands out of 20 people.

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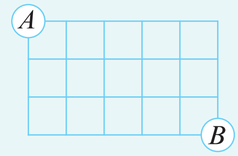
Example 23

The grid shown consists of unit squares. By travelling only right (R) or down (D) along the grid lines, how many paths are there from point A to point B?



Solution

Each path from A to B can be described by a sequence of three Ds and five Rs in some order. Therefore, the number of paths is equal to the number of ways of selecting three of the eight boxes below to be filled with the three Ds. (The rest will be Rs.) This can be done in ${}^8C_3 = 56$ ways.

Alternative notation

We will consistently use the notation ${}^nC_r$ to denote the number of ways of selecting r objects from n objects, regardless of order. However, it is also common to denote this number by $\binom{n}{r}$.

For example:

$$\binom{6}{4} = \frac{6!}{4!\,2!} = 15$$

Section summary

- A **combination** is a selection made regardless of order.
- The number of combinations of n objects taken r at a time is given by

$$ {}^nC_r = \frac{n!}{r!\,(n-r)!} $$

Exercise 5E

1 Evaluate ${}^5C_r$ for $r = 0, 1, 2, 3, 4, 5$.

2 Evaluate each of the following without the use of your calculator:

a ${}^7C_1$ **b** ${}^6C_5$ **c** ${}^{12}C_{10}$
d ${}^8C_5$ **e** ${}^{100}C_{99}$ **f** ${}^{1000}C_{998}$

3 Simplify each of the following:

a ${}^nC_1$ **b** ${}^nC_2$ **c** ${}^nC_{n-1}$
d ${}^{n+1}C_1$ **e** ${}^{n+2}C_n$ **f** ${}^{n+1}C_{n-1}$

ISBN 978-1-316-63608-4

Example 19 **4** A playlist contains ten of Nandi's favourite songs. How many ways can he:

a arrange three songs in a list **b** select three songs for a list?

5 How many ways can five cards be selected from a deck of 52 playing cards?

6 How many subsets of $\{1, 2, 3, \dots, 10\}$ contain exactly:

a 1 element **b** 2 elements
c 8 elements **d** 9 elements?

7 A lottery consists of drawing seven balls out of a barrel of balls numbered from 1 to 45. How many ways can this be done if their order does not matter?

Example 22 **8** Eight points lie on a circle. How many triangles can you make using these points as the vertices?

9 **a** In a hockey tournament, each of the 10 teams plays every other team once. How many games take place?

b In another tournament, each team plays every other team once and 120 games take place. How many teams competed?

10 At a party, every person shakes hands with every other person. Altogether there are 105 handshakes. How many people are at the party?

11 Prove that ${}^nC_r = {}^nC_{n-r}$.

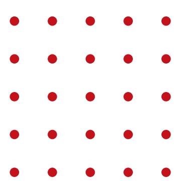
12 Explain why the number of diagonals in a regular polygon with n sides is ${}^nC_2 - n$.

13 Ten students are divided into two teams of five. Explain why the number of ways of doing this is $\dfrac{{}^{10}C_5}{2}$.

14 Twelve students are to be divided into two teams of six. In how many ways can this be done? (Hint: First complete the previous question.)

15 Using the formula for ${}^nC_r$, prove that ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$, where $1 \le r < n$.

16 Consider the 5×5 grid shown.



a How many ways can three dots be chosen?

b How many ways can three dots be chosen so that they lie on a straight line?

c How many ways can three dots be chosen so that they are the vertices of a triangle? (Hint: Use parts **a** and **b**.)

ISBN 978-1-316-63608-4

Cambridge University Press

5F Combinations with restrictions

► Combinations including specific items

In some problems, we want to find the number of combinations that include specific items. This reduces both the number of items we have to select and the number of items from which we are selecting.

Example 24

a Grace belongs to a group of eight workers. How many ways can a team of four workers be selected if Grace must be on the team?

b A hand of cards consists of five cards drawn from a deck of 52 playing cards. How many hands contain both the queen and the king of hearts?

Solution	*Explanation*
a ${}^{7}C_3 = 35$	Grace must be in the selection. Therefore three more workers are to be selected from the remaining seven workers.
b ${}^{50}C_3 = 19\,600$	The queen and king of hearts must be in the selection. So three more cards are to be selected from the remaining 50 cards.

In some other problems, it can be more efficient to count the selections that we don't want.

Example 25

Four students are to be chosen from a group of eight students for the school tennis team. Two members of the group, Sam and Tess, do not get along and cannot both be on the team. How many ways can the team be selected?

Solution

There are ${}^{8}C_4$ ways of selecting four students from eight. We then subtract the number of combinations that include both Sam and Tess. If Sam and Tess are on the team, then we can select two more students from the six that remain in ${}^{6}C_2$ ways. This gives

$${}^{8}C_4 - {}^{6}C_2 = 55$$

► Combinations from multiple groups

Sometimes we are required to make multiple selections from separate groups. In this case, the multiplication principle dictates that we simply multiply the number of ways of performing each task.

Example 26

From seven women and four men in a workplace, how many groups of five can be chosen:

a without restriction

b containing three women and two men

c containing at least one man

d containing at most one man?

ISBN 978-1-316-63608-4

Solution

a There are 11 people in total, from which we must select five. This gives

$$^{11}C_5 = 462$$

b There are $^7C_3$ ways of selecting three women from seven. There are $^4C_2$ ways of selecting two men from four. We then use the multiplication principle to give

$$^7C_3 \cdot {}^4C_2 = 210$$

c Method 1

If you select at least one man, then you select 1, 2, 3 or 4 men and fill the remaining positions with women. We use the multiplication and addition principles to give

$$^4C_1 \cdot {}^7C_4 + {}^4C_2 \cdot {}^7C_3 + {}^4C_3 \cdot {}^7C_2 + {}^4C_4 \cdot {}^7C_1 = 441$$

Method 2

It is more efficient to consider all selections of 5 people from 11 and then subtract the number of combinations containing all women. This gives

$$^{11}C_5 - {}^7C_5 = 441$$

d If there is at most one man, then either there are no men or there is one man. If there are no men, then there are $^7C_5$ ways of selecting all women. If there is one man, then there are $^4C_1$ ways of selecting one man and $^7C_4$ ways of selecting four women. This gives

$$^7C_5 + {}^4C_1 \cdot {}^7C_4 = 161$$

Permutations and combinations combined

In the following example, we first select the items and then arrange them.

Example 27

a How many arrangements of the letters in the word DUPLICATE can be made that have two vowels and three consonants?

b A president, vice-president, secretary and treasurer are to be chosen from a group containing seven women and six men. How many ways can this be done if exactly two women are chosen?

Solution	*Explanation*
a $^4C_2 \cdot {}^5C_3 \cdot 5! = 7200$	There are $^4C_2$ ways of selecting 2 of 4 vowels and $^5C_3$ ways of selecting 3 of 5 consonants. Once chosen, the 5 letters can be arranged in 5! ways.
b $^7C_2 \cdot {}^6C_2 \cdot 4! = 7560$	There are $^7C_2$ ways of selecting 2 of 7 women and $^6C_2$ ways of selecting 2 of 6 men. Once chosen, the 4 people can be arranged into the positions in 4! ways.

ISBN 978-1-316-63608-4

Section summary

- If a selection must include specific items, then this reduces both the number of items that we have to select and the number of items that we select from.
- If we are required to make multiple selections from separate groups, then we multiply the number of ways of performing each task.
- Some problems will require us to select and then arrange objects.

Exercise 5F

Skillsheet
Example 24

1 Jane and Jenny belong to a class of 20 students. How many ways can you select a group of four students from the class if both Jane and Jenny are to be included?

2 How many subsets of $\{1, 2, 3, \dots, 10\}$ have exactly five elements and contain the number 5?

3 Five cards are dealt from a deck of 52 playing cards. How many hands contain the jack, queen and king of hearts?

Example 25

4 Six students are to be chosen from a group of 10 students for the school basketball team. Two members of the group, Rachel and Nethra, do not get along and cannot both be on the team. How many ways can the team be selected?

Example 26

5 From eight girls and five boys, a team of seven is selected for a mixed netball team. How many ways can this be done if:

- **a** there are no restrictions
- **b** there are four girls and three boys on the team
- **c** there must be at least three boys and three girls on the team
- **d** there are at least two boys on the team?

6 There are 10 student leaders at a secondary school. Four are needed for a fundraising committee and three are needed for a social committee. How many ways can the students be selected if they can serve on:

- **a** both committees
- **b** at most one committee?

7 There are 18 students in a class. Seven are required for a basketball team and eight are required for a netball team. How many ways can the teams be selected if students can play in:

- **a** both teams
- **b** at most one team?

ISBN 978-1-316-63608-4

8 From 10 Labor senators and 10 Liberal senators, a committee of five is formed. How many ways can this be done if:

a there are no restrictions
b there are at least two senators from each political party
c there is at least one Labor senator?

9 Consider the set of numbers $\{1, 2, 3, 4, 5, 6, 7\}$.

a How many subsets have exactly five elements?
b How many five-element subsets contain the numbers 2 and 3?
c How many five-element subsets do not contain both 2 and 3?

10 Four letters are selected from the English alphabet. How many of these selections will contain exactly two vowels?

11 A seven-card hand is dealt from a deck of 52 playing cards. How many distinct hands contain:

a four hearts and three spades
b exactly two hearts and three spades?

12 A committee of five people is chosen from four doctors, four dentists and three physiotherapists. How many ways can this be done if the committee contains:

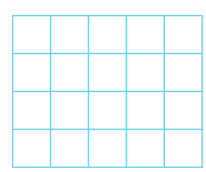
a exactly three doctors and one dentist
b exactly two doctors?

Example 27 **13** There are four girls and five boys. Two of each are chosen and then arranged on a bench. How many ways can this be done?

14 A president, vice-president, secretary and treasurer are to be chosen from a group containing six women and five men. How many ways can this be done if exactly two women must be chosen?

15 Using five letters from the word TRAMPOLINE, how many arrangements contain two vowels and three consonants?

16 How many rectangles are there in the grid shown on the right?
Hint: Every rectangle is determined by a choice of two vertical and two horizontal lines.



17 Five cards are dealt from a deck of 52 playing cards. A full house is a hand that contains 3 cards of one rank and 2 cards of another rank (for example, 3 kings and 2 sevens). How many ways can a full house be dealt?

ISBN 978-1-316-63608-4

5G Pascal's triangle

The diagram below consists of the binomial coefficients ${}^nC_r$ for $0 \le n \le 5$. They form the first 6 rows of **Pascal's triangle**, named after the seventeenth century French mathematician Blaise Pascal, one of the founders of probability theory.

Interestingly, the triangle was well known to Chinese and Indian mathematicians many centuries earlier.

$n = 0$:	${}^0C_0$	1
$n = 1$:	${}^1C_0 \quad {}^1C_1$	1 1
$n = 2$:	${}^2C_0 \quad {}^2C_1 \quad {}^2C_2$	1 2 1
$n = 3$:	${}^3C_0 \quad {}^3C_1 \quad {}^3C_2 \quad {}^3C_3$	1 3 3 1
$n = 4$:	${}^4C_0 \quad {}^4C_1 \quad {}^4C_2 \quad {}^4C_3 \quad {}^4C_4$	1 4 6 4 1
$n = 5$:	${}^5C_0 \quad {}^5C_1 \quad {}^5C_2 \quad {}^5C_3 \quad {}^5C_4 \quad {}^5C_5$	1 5 10 10 5 1

▶ Pascal's rule

Pascal's triangle has many remarkable properties. Most importantly:

> Each entry in Pascal's triangle is the sum of the two entries immediately above.

Pascal's triangle has this property because of the following identity.

> **Pascal's rule**
>
> $${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r \qquad \text{where } 1 \le r < n$$

Proof In Question 15 of Exercise 5E, you are asked to prove Pascal's rule using the formula for ${}^nC_r$. However, there is a much nicer argument.

The number of subsets of $\{1, 2, \dots, n\}$ containing exactly r elements is ${}^nC_r$. Each of these subsets can be put into one of two groups:

1. those that contain n
2. those that do not contain n.

If the subset contains n, then each of the remaining $r - 1$ elements must be chosen from $\{1, 2, \dots, n - 1\}$. Therefore the first group contains ${}^{n-1}C_{r-1}$ subsets.

If the subset does not contain n, then we still have to choose r elements from $\{1, 2, \dots, n - 1\}$. Therefore the second group contains ${}^{n-1}C_r$ subsets.

The two groups together contain all ${}^nC_r$ subsets and so

$${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

which establishes Pascal's rule.

ISBN 978-1-316-63608-4

Example 28

Given that ${}^{17}C_2 = 136$ and ${}^{17}C_3 = 680$, evaluate ${}^{18}C_3$.

Solution

$$\begin{aligned} {}^{18}C_3 &= {}^{17}C_2 + {}^{17}C_3 \\ &= 136 + 680 \\ &= 816 \end{aligned}$$

Explanation

We let $n = 18$ and $r = 3$ in Pascal's rule:

$${}^{n}C_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

Example 29

Write down the $n = 6$ row of Pascal's triangle and then write down the value of ${}^{6}C_3$.

Solution

$n = 6$: 1 6 15 **20** 15 6 1

${}^{6}C_3 = 20$

Explanation

Each entry in the $n = 6$ row is the sum of the two entries immediately above.

Note that ${}^{6}C_3$ is the fourth entry in the row, since the first entry corresponds to ${}^{6}C_0$.

► Subsets of a set

Suppose your friend says to you: 'I have five books that I no longer need, take any that you want.' How many different selections are possible?

We will look at two solutions to this problem.

Solution 1

You could select none of the books (${}^{5}C_0$ ways), or one out of five (${}^{5}C_1$ ways), or two out of five (${}^{5}C_2$ ways), and so on. This gives the answer

$${}^{5}C_0 + {}^{5}C_1 + {}^{5}C_2 + {}^{5}C_3 + {}^{5}C_4 + {}^{5}C_5 = 32$$

Note that this is simply the sum of the entries in the $n = 5$ row of Pascal's triangle.

Solution 2

For each of the five books we have two options: either accept or reject the book. Using the multiplication principle, we obtain the answer

$$2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$$

There are two important conclusions that we can draw from this example.

1. The sum of the entries in row n of Pascal's triangle is 2^n. That is,

$${}^{n}C_0 + {}^{n}C_1 + \cdots + {}^{n}C_{n-1} + {}^{n}C_n = 2^n$$

2. A set of size n has 2^n subsets, including the empty set and the set itself.

ISBN 978-1-316-63608-4

Example 30

a Your friend offers you any of six books that she no longer wants. How many selections are possible assuming that you take at least one book?

b How many subsets of $\{1, 2, 3, \dots, 10\}$ have at least two elements?

Solution

a $2^6 - 1 = 63$

Explanation

There are 2^6 subsets of a set of size 6. We subtract 1 because we discard the empty set of no books.

Solution

b $2^{10} - {}^{10}C_1 - {}^{10}C_0$

$= 2^{10} - 10 - 1$

$= 1013$

Explanation

There are 2^{10} subsets of a set of size 10. There are ${}^{10}C_1$ subsets containing 1 element and ${}^{10}C_0$ subsets containing 0 elements.

Section summary

- The values of ${}^nC_r$ can be arranged to give Pascal's triangle.
- Each entry in Pascal's triangle is the sum of the two entries immediately above.
- Pascal's rule: ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$
- The sum of the entries in row n of Pascal's triangle is 2^n. That is,

$${}^nC_0 + {}^nC_1 + \cdots + {}^nC_{n-1} + {}^nC_n = 2^n$$

- A set of size n has 2^n subsets, including the empty set and the set itself.

Exercise 5G

Example 28

1 Evaluate ${}^7C_2$, ${}^6C_2$ and ${}^6C_1$, and verify that the first is the sum of the other two.

Example 29

2 Write down the $n = 7$ row of Pascal's triangle. Use your answer to write down the values of ${}^7C_2$ and ${}^7C_4$.

3 Write down the $n = 8$ row of Pascal's triangle. Use your answer to write down the values of ${}^8C_4$ and ${}^8C_6$.

Example 30

4 Your friend offers you any of six different DVDs that he no longer wants. How many different selections are possible?

5 How many subsets does the set $\{A, B, C, D, E\}$ have?

6 How many subsets does the set $\{1, 2, 3, \dots, 10\}$ have?

7 How many subsets of $\{1, 2, 3, 4, 5, 6\}$ have at least one element?

8 How many subsets of $\{1, 2, 3, \dots, 8\}$ have at least two elements?

9 How many subsets of $\{1, 2, 3, \dots, 10\}$ contain the numbers 9 and 10?

ISBN 978-1-316-63608-4

10 You have one 5 cent, one 10 cent, one 20 cent and one 50 cent piece. How many different sums of money can you make assuming that at least one coin is used?

11 Let's call a set **selfish** if it contains its size as an element. For example, the set $\{1, 2, 3\}$ is selfish because the set has size 3 and the number 3 belongs to the set.

a How many subsets of $\{1, 2, 3, \dots, 8\}$ are selfish?

b How many subsets of $\{1, 2, 3, \dots, 8\}$ have the property that both the subset and its complement are selfish?

5H The pigeonhole principle

The pigeonhole principle is an intuitively obvious counting technique which can be used to prove some remarkably counterintuitive results. It gets its name from the following simple observation: If $n + 1$ pigeons are placed into n holes, then some hole contains at least two pigeons. Obviously, in most instances we will not be working with pigeons, so we will recast the principle as follows.

Pigeonhole principle

If $n + 1$ or more objects are placed into n holes, then some hole contains at least two objects.

Proof Suppose that each of the n holes contains at most one object. Then the total number of objects is at most n, which is a contradiction.

We are now in a position to prove a remarkable fact: There are at least two people in Australia with the same number of hairs on their head. The explanation is simple. No one has more than 1 million hairs on their head, so let's make 1 million holes labelled with the numbers from 1 to 1 million. We now put each of the 24 million Australians into the hole corresponding to the number of hairs on their head. Clearly, some hole contains at least two people, and all the people in that hole will have the same number of hairs on their head.

Example 31

You have thirteen red, ten blue and eight green socks. How many socks need to be selected at random to ensure that you have a matching pair?

Solution

Label three holes with the colours red, blue and green.

R	B	G

Selecting just three socks is clearly not sufficient, as you might pick one sock of each colour. Select four socks and place each sock into the hole corresponding to the colour of the sock. As there are four socks and three holes, the pigeonhole principle guarantees that some hole contains at least two socks. This is the required pair.

ISBN 978-1-316-63608-4

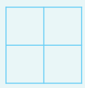
Example 32

a Show that for any five points chosen inside a 2×2 square, at least two of them will be no more than $\sqrt{2}$ units apart.

b Seven football teams play 22 games of football. Show that some pair of teams play each other at least twice.

Solution

a Split the 2×2 square into four unit squares.



Now we have four squares and five points. By the pigeonhole principle, some square contains at least two points. The distance between any two of these points cannot exceed the length of the square's diagonal, $\sqrt{1^2 + 1^2} = \sqrt{2}$.

b There are ${}^7C_2 = 21$ ways that two teams can be chosen to compete from seven. There are 22 games of football, and so some pair of teams play each other at least twice.

The generalised pigeonhole principle

Suppose that 13 pigeons are placed into four holes. By the pigeonhole principle, there is some hole with at least two pigeons. In fact, some hole must contain at least four pigeons. The reason is simple: If each of the four holes contained no more than three pigeons, then there would be no more than 12 pigeons.

This observation generalises as follows.

Generalised pigeonhole principle

If at least $mn + 1$ objects are placed into n holes, then some hole contains at least $m + 1$ objects.

Proof Again, let's suppose that the statement is false. Then each of the n holes contains no more than m objects. However, this means that there are no more than mn objects, which is a contradiction.

Example 33

Sixteen natural numbers are written on a whiteboard. Prove that at least four numbers will leave the same remainder when divided by 5.

Solution

We label five holes with each of the possible remainders on division by 5.

There are 16 numbers to be placed into five holes. Since $16 = 3 \times 5 + 1$, there is some hole with at least four numbers, each of which leaves the same remainder when divided by 5.

ISBN 978-1-316-63608-4

Cambridge University Press

► Pigeons in multiple holes

In some instances, objects can be placed into more than one hole.

Example 34

Seven people sit at a round table with 10 chairs. Show that there are three consecutive chairs that are occupied.

Solution

Number the chairs from 1 to 10. There are 10 groups of three consecutive chairs:

$$\{1, 2, 3\}, \quad \{2, 3, 4\}, \quad \{3, 4, 5\}, \quad \{4, 5, 6\}, \quad \{5, 6, 7\},$$
$$\{6, 7, 8\}, \quad \{7, 8, 9\}, \quad \{8, 9, 10\}, \quad \{9, 10, 1\}, \quad \{10, 1, 2\}$$

Each of the seven people will belong to three of these groups, and so 21 people have to be allocated to 10 groups. Since $21 = 2 \times 10 + 1$, the generalised pigeonhole principle guarantees that some group must contain three people.

Section summary

- Pigeonhole principle
  If $n + 1$ or more objects are placed into n holes, then some hole contains at least two objects.
- Generalised pigeonhole principle
  If at least $mn + 1$ objects are placed into n holes, then some hole contains at least $m + 1$ objects.

Exercise 5H

Example 31

1 You have twelve red, eight blue and seven green socks. How many socks need to be selected at random to ensure that you have a matching pair?

2 A sentence contains 27 English words. Show that there are at least two words that begin with the same letter.

3 Show that in any collection of five natural numbers, at least two will leave the same remainder when divided by 4.

4 How many cards need to be dealt from a deck of 52 playing cards to be certain that you will obtain at least two cards of the same:

a colour **b** suit **c** rank?

5 Eleven points on the number line are located somewhere between 0 and 1. Show that there are at least two points no more than 0.1 apart.

ISBN 978-1-316-63608-4

Example 32

6 An equilateral triangle has side length 2 units. Choose any five points inside the triangle. Prove that there are at least two points that are no more than 1 unit apart.

7 Thirteen points are located inside a rectangle of length 6 and width 8. Show that there are at least two points that are no more than $2\sqrt{2}$ units apart.

8 The **digital sum** of a natural number is defined to be the sum of its digits. For example, the digital sum of 123 is $1 + 2 + 3 = 6$.

a Nineteen two-digit numbers are selected. Prove that at least two of them have the same digital sum.

b Suppose that 82 three-digit numbers are selected. Prove that at least four of them have the same digital sum.

Example 33

9 Whenever Eva writes down 13 integers, she notices that at least four of them leave the same remainder when divided by 4. Explain why this is always the case.

10 Twenty-nine games of football are played among eight teams. Prove that there is some pair of teams who play each other more than once.

11 A teacher instructs each member of her class to write down a different whole number between 1 and 49. She says that there will be at least one pair of students such that the sum of their two numbers is 50. How many students must be in her class?

Example 34

12 There are 10 students seated at a round table with 14 chairs. Show that there are three consecutive chairs that are occupied.

13 There are four points on a circle. Show that three of these points lie on a half-circle.
Hint: Pick any one of the four points and draw a diameter through that point.

14 There are 35 players on a football team and each player has a different number chosen from 1 to 99. Prove that there are at least four pairs of players whose numbers have the same sum.

15 Seven boys and five girls sit evenly spaced at a round table. Prove that some pair of boys are sitting opposite each other.

16 There are n guests at a party and some of these guests shake hands when they meet. Use the pigeonhole principle to show that there is a pair of guests who shake hands with the same number of people.
Hint: Place the n guests into holes labelled from 0 to $n - 1$, corresponding to the number of hands that they shake. Why must either the first or the last hole be empty?

ISBN 978-1-316-63608-4

5I The inclusion–exclusion principle

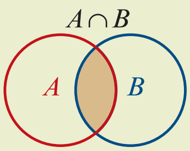
► Basic set theory

A set is any collection of objects where order is not important. The set with no elements is called the **empty set** and is denoted by ∅. We say that set B is a **subset** of set A if each element of B is also in A. In this case, we can write $B \subseteq A$. Note that $\emptyset \subseteq A$ and $A \subseteq A$.

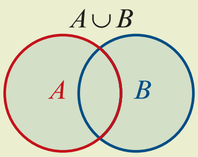
If A is a finite set, then the number of elements in A will be denoted by $|A|$.

Given any two sets A and B we define two important sets:

1 The **intersection** of sets A and B is denoted by $A \cap B$ and consists of elements belonging to A *and* B.



2 The **union** of sets A and B is denoted by $A \cup B$ and consists of elements belonging to A *or* B.



Note: It is important to realise that $A \cup B$ includes elements belonging to A and B.

Example 35

Consider the three sets of numbers $A = \{2, 3\}$, $B = \{1, 2, 3, 4\}$ and $C = \{3, 4, 5\}$.

a Find $B \cap C$.
b Find $A \cup C$.
c Find $A \cap B \cap C$.
d Find $A \cup B \cup C$.
e Find $|A|$.
f List all the subsets of C.

Solution

a $B \cap C = \{3, 4\}$
b $A \cup C = \{2, 3, 4, 5\}$
c $A \cap B \cap C = \{3\}$
d $A \cup B \cup C = \{1, 2, 3, 4, 5\}$
e $|A| = 2$
f $\emptyset$, $\{3\}$, $\{4\}$, $\{5\}$, $\{3, 4\}$, $\{3, 5\}$, $\{4, 5\}$, $\{3, 4, 5\}$

Earlier in the chapter we encountered the addition principle. This principle can be concisely expressed using set notation.

Addition principle

If A and B are two finite sets of objects such that $A \cap B = \emptyset$, then

$$|A \cup B| = |A| + |B|$$

Our aim is to extend this rule for instances where $A \cap B \neq \emptyset$.

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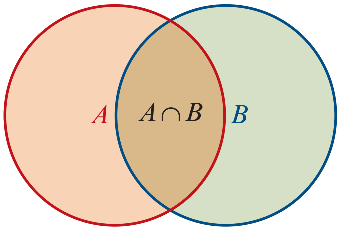
► Two sets

To count the number of elements in the set $A \cup B$, we first add (include) $|A|$ and $|B|$. However, this counts the elements in $A \cap B$ twice, and so we subtract (exclude) $|A \cap B|$.



Inclusion–exclusion principle for two sets

If A and B are two finite sets of objects, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example 36

Each of the 25 students in a Year 11 class studies Physics or Chemistry. Of these students, 15 study Physics and 18 study Chemistry. How many students study both subjects?

Solution

$$|P \cup C| = |P| + |C| - |P \cap C|$$
$$25 = 15 + 18 - |P \cap C|$$
$$25 = 33 - |P \cap C|$$
$$\therefore \quad |P \cap C| = 8$$

Explanation

Let P and C be the sets of students who study Physics and Chemistry respectively.

Since each student studies Physics or Chemistry, we know that $|P \cup C| = 25$.

Example 37

A bag contains 100 balls labelled with the numbers from 1 to 100. How many ways can a ball be chosen that is a multiple of 2 or 5?

Solution

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 50 + 20 - 10$$
$$= 60$$

Explanation

Within the set of numbers $\{1, 2, 3, \dots, 100\}$, let A be the set of multiples of 2 and let B be the set of multiples of 5.

Then $A \cap B$ consists of numbers that are multiples of both 2 and 5, that is, multiples of 10.

Therefore $|A| = 50$, $|B| = 20$ and $|A \cap B| = 10$. We then use the inclusion–exclusion principle.

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Example 38

A hand of five cards is dealt from a deck of 52 cards. How many hands contain exactly:

a two clubs
b three spades
c two clubs and three spades
d two clubs or three spades?

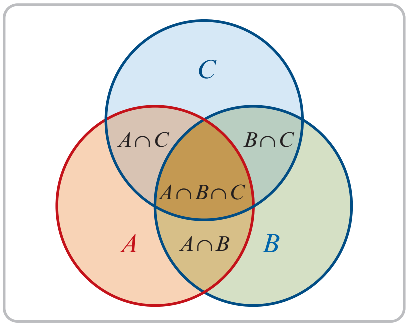
Solution	*Explanation*
a $^{13}C_2 \cdot {}^{39}C_3 = 712\,842$	There are $^{13}C_2$ ways of choosing 2 clubs from 13 and $^{39}C_3$ ways of choosing 3 more cards from the 39 non-clubs.
b $^{13}C_3 \cdot {}^{39}C_2 = 211\,926$	There are $^{13}C_3$ ways of choosing 3 spades from 13 and $^{39}C_2$ ways of choosing 2 more cards from the 39 non-spades.
c $^{13}C_2 \cdot {}^{13}C_3 = 22\,308$	There are $^{13}C_2$ ways of choosing 2 clubs from 13 and $^{13}C_3$ ways of choosing 3 spades from 13.
d $\lvert A \cup B\rvert$ $= \lvert A\rvert + \lvert B\rvert - \lvert A \cap B\rvert$ $= 712\,842 + 211\,926 - 22\,308$ $= 902\,460$	We let A be the set of all hands with 2 clubs and let B be the set of all hands with 3 spades. Then $A \cap B$ is the set of all hands with 2 clubs and 3 spades. We use the inclusion–exclusion principle to find $\lvert A \cup B\rvert$.

▶ Three sets

For three sets A, B and C, the formula for $|A \cup B \cup C|$ is slightly harder to establish.

We first add $|A|$, $|B|$ and $|C|$. However, we have counted the elements in $A \cap B$, $A \cap C$ and $B \cap C$ twice, and the elements in $A \cap B \cap C$ three times.

Therefore we subtract $|A \cap B|$, $|A \cap C|$ and $|B \cap C|$ to compensate. But then the elements in $A \cap B \cap C$ will have been excluded once too often, and so we add $|A \cap B \cap C|$.



Inclusion–exclusion principle for three sets

If A, B and C are three finite sets of objects, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

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Example 39

How many integers from 1 to 140 inclusive are not divisible by 2, 5 or 7?

Solution

Let A, B and C be the sets of all integers from 1 to 140 that are divisible by 2, 5 and 7 respectively. We then have

A	multiples of 2	$\lvert A\rvert = 140 \div 2 = 70$
B	multiples of 5	$\lvert B\rvert = 140 \div 5 = 28$
C	multiples of 7	$\lvert C\rvert = 140 \div 7 = 20$
$A \cap B$	multiples of 10	$\lvert A \cap B\rvert = 140 \div 10 = 14$
$A \cap C$	multiples of 14	$\lvert A \cap C\rvert = 140 \div 14 = 10$
$B \cap C$	multiples of 35	$\lvert B \cap C\rvert = 140 \div 35 = 4$
$A \cap B \cap C$	multiples of 70	$\lvert A \cap B \cap C\rvert = 140 \div 70 = 2$

We use the inclusion–exclusion principle to give

$$\begin{aligned}|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 70 + 28 + 20 - 14 - 10 - 4 + 2 \\ &= 92\end{aligned}$$

Therefore the number of integers not divisible by 2, 5 or 7 is $140 - 92 = 48$.

Section summary

- The inclusion–exclusion principle extends the addition principle to instances where the two sets have objects in common.
- The principle works by ensuring that objects belonging to multiple sets are not counted more than once.
- The inclusion–exclusion principles for two sets and three sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Exercise 5I

Example 35

1 Consider the three sets of numbers $A = \{4, 5, 6\}$, $B = \{1, 2, 3, 4, 5\}$ and $C = \{1, 3, 4, 6\}$.

a Find $B \cap C$. **b** Find $A \cup C$. **c** Find $A \cap B \cap C$.

d Find $A \cup B \cup C$. **e** Find $|A|$. **f** List all the subsets of A.

Example 36

2 In an athletics team, each athlete competes in track or field events. There are 25 athletes who compete in track events, 23 who compete in field events and 12 who compete in both track and field events. How many athletes are in the team?

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3 Fifty patients at a medical clinic are being treated for a disease using two types of medication, A and B. There are 25 patients using medication A and 29 patients using medication B. How many patients are using both types of medication?

Example 37

4 How many integers from 1 to 630 inclusive are multiples of 7 or 9?

5 Consider the integers from 1 to 96 inclusive. How many of these are:

a divisible by 2 or 3

b not divisible by 2 or 3?

Example 38

6 How many five-letter arrangements of the word ALIEN:

a begin with a vowel

b end with a vowel

c begin and end with a vowel

d begin or end with a vowel?

7 **a** How many of the integers from 1 to 100 inclusive are perfect squares or perfect cubes?

b How many of the integers from 1 to 1000 inclusive are perfect squares or perfect cubes?

Example 39

8 How many of the integers from 1 to 120 inclusive are multiples of 2, 3 or 5?

9 How many of the integers from 1 to 220 inclusive are not divisible by 2, 5 or 11?

10 There are 90 Year 11 students at a secondary school and each of them must study at least one of Biology, Physics or Chemistry. There are 36 students who study Biology, 42 who study Physics and 40 who study Chemistry. Moreover, 9 study Biology and Physics, 8 study Biology and Chemistry and 7 study Physics and Chemistry. How many students study all three subjects?

11 A group of six students is selected from four students in Year 10, five in Year 11 and four in Year 12. How many selections have exactly:

a two Year 10 students

b two Year 11 students

c two Year 10 and two Year 11 students

d two Year 10 or two Year 11 students?

12 A hand of five cards is dealt from a deck of 52 cards. How many hands contain exactly one heart or exactly two diamonds?

13 Find the sum of all the integers from 1 to 100 inclusive that are divisible by 2 or 3.

Hint: Use the formula $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

14 There are seventy Year 11 students at a school and each of them must study at least one of three languages. Thirty are studying French, forty-five are studying Chinese, thirty are studying German and fifteen are studying all three languages. How many students are studying exactly two languages?

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Review

Chapter summary

AS
Nrich

- The addition and multiplication principles provide efficient methods for counting the number of ways of performing multiple tasks.
- The number of **permutations** (or arrangements) of n objects taken r at a time is given by

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

- The number of **combinations** (or selections) of n objects taken r at a time is given by

$$^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

- When permutations or combinations involve restrictions, we deal with them first.
- The values of $^{n}C_{r}$ can be arranged to give **Pascal's triangle**, where each entry is the sum of the two entries immediately above.
- The sum of the entries in row n of Pascal's triangle is 2^n. That is,

$$^{n}C_{0} + {}^{n}C_{1} + \cdots + {}^{n}C_{n-1} + {}^{n}C_{n} = 2^n$$

- A set of size n has 2^n subsets.
- The **pigeonhole principle** is used to show that some pair or group of objects have the same property.
- The **inclusion–exclusion principle** allows us to count the number of elements in a union of sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Short-answer questions

1 Evaluate:

a $^{6}C_{3}$ **b** $^{20}C_{2}$ **c** $^{300}C_{1}$ **d** $^{100}C_{98}$

2 Find the value of n if $^{n}C_{2} = 55$.

3 How many three-digit numbers can be formed using the digits 1, 2 and 3 if the digits:

a can be repeated **b** cannot be repeated?

4 How many ways can six students be arranged on a bench seat with space for three?

5 How many ways can three students be allocated to five vacant desks?

6 There are four Year 11 and three Year 12 students in a school debating club. How many ways can a team of four be selected if two are chosen from each year level?

7 There are three boys and four girls in a group. How many ways can three children be selected if at least one of them is a boy?

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8 On a ship's mast are two identical red and three identical black flags that can be arranged to send messages to nearby ships. How many different arrangements using all five flags are possible?

9 There are 53 English words written on a page. How many are guaranteed to share the same first letter?

10 Each of the twenty students in a class plays netball or basketball. Twelve play basketball and four play both sports. How many students play netball?

11 Six people are to be seated in a row. Calculate the number of ways this can be done so that two particular people, A and B, always have exactly one person between them.

Multiple-choice questions

1 Bao plans to study six subjects in Year 12. He has already chosen three subjects and for the remaining three he plans to choose one of four languages, one of three mathematics subjects and one of four science subjects. How many ways can he select his remaining subjects?

A 6 **B** 11 **C** 48 **D** 165 **E** 990

2 There are three flights directly from Melbourne to Brisbane. There are also two flights from Melbourne to Sydney and then four choices of connecting flight from Sydney to Brisbane. How many different paths are there from Melbourne to Brisbane?

A 9 **B** 11 **C** 18 **D** 20 **E** 24

3 In how many ways can 10 people be arranged in a queue at the bank?

A $10!$ **B** 10^{10} **C** 2^{10} **D** ${}^{10}C_2$ **E** ${}^{10}C_1$

4 How many three-digit numbers can be formed using the digits 1, 2, 3, 4, 5 and 6 at most once?

A ${}^{6}C_3$ **B** $3!$ **C** $6!$ **D** $6 \times 5 \times 4$ **E** $6 + 5 + 4$

5 How many permutations of the word UTOPIA begin and end with a vowel?

A 90 **B** 288 **C** 384 **D** 720 **E** 4320

6 How many ways can four identical red flags and three identical blue flags be arranged in a row?

A 4×3 **B** $\dfrac{7!}{4! \times 3!}$ **C** $7! \times 3! \times 4!$ **D** $4! \times 3!$ **E** $2 \times 3! \times 4!$

7 How many ways can three DVDs be chosen from a collection of nine different DVDs?

A $3!$ **B** $9 \times 8 \times 7$ **C** ${}^{9}C_3$ **D** $\dfrac{9!}{3!}$ **E** 3×9

8 The number of subsets of $\{A, B, C, D, E, F\}$ with at least one element is

A ${}^{6}C_2$ **B** ${}^{6}C_2 - 1$ **C** $2^5 - 1$ **D** $2^6 - 1$ **E** 2^6

ISBN 978-1-316-63608-4

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Review

9 A class consists of nine girls and eight boys. How many ways can a group of two boys and two girls be chosen?

A $\dfrac{17!}{2!\,2!}$ **B** $^{17}C_4$ **C** $^9C_2 \cdot {}^8C_2$ **D** $\dfrac{17!}{9!\,8!}$ **E** $9 \times 8 \times 8 \times 7$

10 There are six blue balls and five red balls in a bag. How many balls need to be selected at random before you are certain that three will have the same colour?

A 3 **B** 4 **C** 5 **D** 7 **E** 11

11 Each of the 30 students in a class studies French, German or Chinese. Of these students, 15 study French, 17 study German and 15 study Chinese. There are 15 students that study more than one subject. How many students study all three subjects?

A 2 **B** 3 **C** 4 **D** 5 **E** 6

Extended-response questions

1 A six-digit number is formed using the digits 1, 2, 3, 4, 5 and 6 without repetition. How many ways can this be done if:

a the first digit is 5
b the first digit is even
c even and odd digits alternate
d the even digits are kept together?

2 Three letters from the word AUNTIE are arranged in a row. How many ways can this be done if:

a the first letter is E
b the first letter is a vowel
c the letter E is used?

3 A student leadership team consists of four boys and six girls. A group of four students is required to organise a social function. How many ways can the group be selected:

a without restriction
b if the school captain is included
c if there are two boys
d if there is at least one boy?

4 Consider the eight letters N, N, J, J, T, T, T, T. How many ways can all eight letters be arranged if:

a there is no restriction
b the first and last letters are both N
c the two Js are adjacent
d no two Ts are adjacent?

5 A pizza restaurant offers the following toppings: onion, capsicum, mushroom, olives, ham and pineapple.

a How many different kinds of pizza can be ordered with:

i three different toppings
ii three different toppings including ham
iii any number of toppings (between none and all six)?

b Another pizza restaurant boasts that they can make more than 200 varieties of pizza. What is the smallest number of toppings that they could use?

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6 In how many ways can a group of four people be chosen from five married couples if:

a there is no restriction

b any two women and two men are chosen

c any two married couples are chosen

d a husband and wife cannot both be selected?

7 The name David Smith has initials DS.

a How many different two-letter initials are possible?

b How many different two-letter initials contain at least one vowel?

c Given 50 000 people, how many of them can be guaranteed to share the same two-letter initials?

8 Consider the integers from 1 to 96 inclusive. Let sets A and B consist of those integers that are multiples of 6 and 8 respectively.

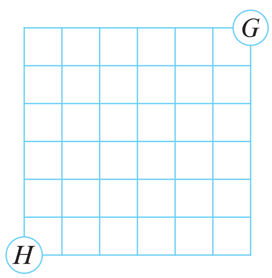
a What is the lowest common multiple of 6 and 8?

b How many integers belong to $A \cap B$?

c How many integers from 1 to 96 are divisible by 6 or 8?

d An integer from 1 to 96 is chosen at random. What is the probability that it is not divisible by 6 or 8?

9 Every morning, Milly walks from her home $H(0, 0)$ to the gym $G(6, 6)$ along city streets that are laid out in a square grid as shown. She always takes a path of shortest distance.



a How many paths are there from H to G?

b Show that there is some path that she takes at least twice in the course of three years.

c On her way to the gym, she often purchases a coffee at a cafe located at point $C(2, 2)$. How many paths are there from:

i H to C **ii** C to G **iii** H to C to G?

d A new cafe opens up at point $B(4, 4)$. How many paths can Milly take, assuming that she buys coffee at either cafe?

Hint: You will need to use the inclusion–exclusion principle here.

10 A box contains 400 balls, each of which is blue, red, green, yellow or orange. The ratio of blue to red to green balls is 1 : 4 : 2. The ratio of green to yellow to orange balls is 1 : 3 : 6. What is the smallest number of balls that must be drawn to ensure that at least 50 balls of one colour are selected?

ISBN 978-1-316-63608-4

Chapter 6

6 Number and proof

Objectives

- To understand and use various methods of proof, including:
  - **direct proof**
  - **proof by contrapositive**
  - **proof by contradiction**.
- To write down the **negation** of a statement.
- To write and prove **converse** statements.
- To understand when mathematical statements are **equivalent**.
- To use the symbols for **implication** ($\Rightarrow$) and **equivalence** ($\Leftrightarrow$).
- To understand and use the quantifiers **'for all'** and **'there exists'**.
- To disprove statements using **counterexamples**.
- To understand and use the **principle of mathematical induction**.

A **mathematical proof** is an argument that demonstrates the absolute truth of a statement.

It is certainty that makes mathematics different from other sciences. In science, a theory is never proved true. Instead, one aims to prove that a theory is not true. And if such evidence is hard to come by, then this increases the likelihood that a theory is correct, but never provides a guarantee. The possibility of absolute certainty is reserved for mathematics alone.

When writing a proof you should always aim for three things:

- correctness
- clarity
- simplicity.

Perhaps the following proof of Pythagoras' theorem exemplifies these three aims.

ISBN 978-1-316-63608-4

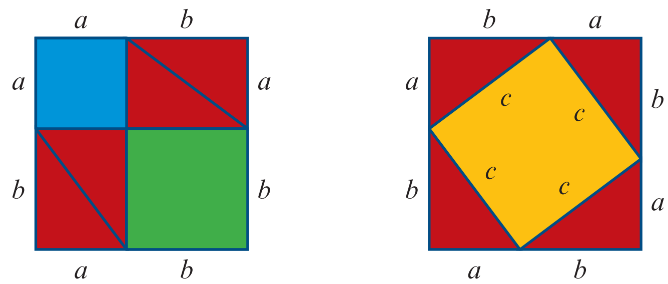
Cambridge University Press

Pythagoras' theorem

Take any triangle with side lengths a, b and c. If the angle between a and b is 90°, then

$$a^2 + b^2 = c^2$$

Proof Consider the two squares shown below.



The two squares each have the same total area. So subtracting four red triangles from each figure will leave the same area. Therefore $a^2 + b^2 = c^2$.

The ideas introduced in this chapter will be used in proofs throughout the rest of this book. We consider these ideas from a different perspective in Chapter 23, which takes a more formal approach to the study of logic.

6A Direct proof

▶ Conditional statements

Consider the following sentence:

Statement	If	it is raining	then	the grass is wet.

This is called a **conditional statement** and has the form:

Statement	If	P is true	then	Q is true.

This can be abbreviated as

$$P \Rightarrow Q$$

which is read 'P **implies** Q'. We call P the **hypothesis** and Q the **conclusion**.

Not all conditional statements will be true. For example, switching the hypothesis and the conclusion above gives:

Statement	If	the grass is wet	then	it is raining.

Anyone who has seen dewy grass on a cloudless day knows this to be false. In this chapter we will learn how to prove (and disprove) mathematical statements.

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Direct proof

To give a **direct proof** of a conditional statement $P \Rightarrow Q$, we assume that the hypothesis P is true, and then show that the conclusion Q follows.

Example 1

Prove the following statements:

a If a is odd and b is even, then $a + b$ is odd.

b If a is odd and b is odd, then ab is odd.

Solution

a Assume that a is odd and b is even.

Since a is odd, we have $a = 2m + 1$ for some $m \in \mathbb{Z}$. Since b is even, we have $b = 2n$ for some $n \in \mathbb{Z}$. Therefore

$$\begin{aligned} a + b &= (2m + 1) + 2n \\ &= 2m + 2n + 1 \\ &= 2(m + n) + 1 \\ &= 2k + 1 \qquad \text{where } k = m + n \in \mathbb{Z} \end{aligned}$$

Hence $a + b$ is odd.

Note: We must use two different pronumerals m and n here, because these two numbers may be different.

b Assume that both a and b are odd. Then $a = 2m + 1$ and $b = 2n + 1$ for some $m, n \in \mathbb{Z}$. Therefore

$$\begin{aligned} ab &= (2m + 1)(2n + 1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \\ &= 2k + 1 \qquad \text{where } k = 2mn + m + n \in \mathbb{Z} \end{aligned}$$

Hence ab is odd.

Example 2

Let $p, q \in \mathbb{Z}$ such that p is divisible by 5 and q is divisible by 3. Prove that pq is divisible by 15.

Solution

Since p is divisible by 5, we have $p = 5m$ for some $m \in \mathbb{Z}$. Since q is divisible by 3, we have $q = 3n$ for some $n \in \mathbb{Z}$. Thus

$$\begin{aligned} pq &= (5m)(3n) \\ &= 15mn \end{aligned}$$

and so pq is divisible by 15.

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Example 3

Let x and y be positive real numbers. Prove that if $x > y$, then $x^2 > y^2$.

Solution

Assume that $x > y$. Then $x - y > 0$.

Since x and y are positive, we also know that $x + y > 0$.

Therefore

$$x^2 - y^2 = \overbrace{(x-y)}^{\text{positive}}\overbrace{(x+y)}^{\text{positive}} > 0$$

Hence $x^2 > y^2$.

Explanation

When trying to prove that $x^2 > y^2$, it is easier to first prove that $x^2 - y^2 > 0$.

Also, note that the product of two positive numbers is positive.

Example 4

Let x and y be any two positive real numbers. Prove that

$$\frac{x+y}{2} \geq \sqrt{xy}$$

Solution

A **false proof** might begin with the statement that we are trying to prove.

$$\begin{aligned}
& \frac{x+y}{2} \geq \sqrt{xy} \\
\Rightarrow \quad & x + y \geq 2\sqrt{xy} \\
\Rightarrow \quad & (x+y)^2 \geq 4xy \qquad \text{(using Example 3)} \\
\Rightarrow \quad & x^2 + 2xy + y^2 \geq 4xy \\
\Rightarrow \quad & x^2 - 2xy + y^2 \geq 0 \\
\Rightarrow \quad & (x-y)^2 \geq 0
\end{aligned}$$

Although it is true that $(x - y)^2 \geq 0$, the argument is faulty. We cannot prove that the result is true by assuming that the result is true! However, the above work is not a waste of time. We can correct the proof by reversing the order of the steps shown above.

Note: In the corrected proof, we need to use the fact that $a > b$ implies $\sqrt{a} > \sqrt{b}$ for all positive numbers a and b. This is shown in Question 8 of Exercise 6B.

Breaking a proof into cases

Sometimes it helps to break a problem up into different cases.

Example 5

Every person on an island is either a knight or a knave. Knights always tell the truth, and knaves always lie. Alice and Bob are residents on the island. Alice says: ‘We are both knaves.’ What are Alice and Bob?

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Solution

We will prove that Alice is a knave and Bob is a knight.

Case 1

Suppose Alice is a knight.

$\Rightarrow$ Alice is telling the truth.

$\Rightarrow$ Alice and Bob are both knaves.

$\Rightarrow$ Alice is a knave and a knight.

This is impossible.

Case 2

Suppose Alice is a knave.

$\Rightarrow$ Alice is not telling the truth.

$\Rightarrow$ Alice and Bob are not both knaves.

$\Rightarrow$ Bob is a knight.

Therefore we conclude that Alice must be a knave and Bob must be a knight.

Section summary

- A **mathematical proof** establishes the truth of a statement.
- A **conditional statement** has the form: If P is true, then Q is true.
  This can be abbreviated as $P \Rightarrow Q$, which is read 'P **implies** Q'.
- To give a **direct proof** of a conditional statement $P \Rightarrow Q$, we assume that P is true and show that Q follows.

Exercise 6A

Skillsheet

Example 1

1 Assume that m is even and n is even. Prove that:

a $m + n$ is even b mn is even.

2 Assume that m is odd and n is odd. Prove that $m + n$ is even.

3 Assume that m is even and n is odd. Prove that mn is even.

Example 2

4 Suppose that m is divisible by 3 and n is divisible by 7. Prove that:

a mn is divisible by 21 b m^2n is divisible by 63.

5 Suppose that m and n are perfect squares. Show that mn is a perfect square.

6 Let m and n be integers. Prove that $(m + n)^2 - (m - n)^2$ is divisible by 4.

7 Suppose that n is an even integer. Prove that $n^2 - 6n + 5$ is odd.

8 Suppose that n is an odd integer. Prove that $n^2 + 8n + 3$ is even.

9 Let $n \in \mathbb{Z}$. Prove that $5n^2 + 3n + 7$ is odd.

Hint: Consider the cases when n is odd and n is even.

Example 3

10 Let x and y be positive real numbers. Show that if $x > y$, then $x^4 > y^4$.

Example 4

11 Let $x, y \in \mathbb{R}$. Show that $x^2 + y^2 \geq 2xy$.

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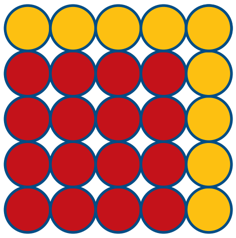
Example 5 **12** Every person on an island is either a knight or a knave. Knights always tell the truth, and knaves always lie. Alice and Bob are residents on the island. Determine whether Alice and Bob are knights or knaves in each of the following separate instances:

a Alice says: 'We are both knaves.'

b Alice says: 'We are both of the same kind.' Bob says: 'We are of a different kind.'

c Alice says: 'Bob is a knave.' Bob says: 'Neither of us is a knave.'

13 The diagram shows that 9 can be written as the difference of two squares: $9 = 5^2 - 4^2$.



a Draw another diagram to show that 11 can be written as the difference of two squares.

b Prove that every odd number can be written as the difference of two squares.

c Hence, express 101 as the difference of two squares.

14 **a** Consider the numbers $\frac{9}{10}$ and $\frac{10}{11}$. Which is larger?

b Let n be a natural number. Prove that $\frac{n}{n+1} > \frac{n-1}{n}$.

15 **a** Prove that

$$\frac{1}{10} - \frac{1}{11} < \frac{1}{100}$$

b Let $n > 0$. Prove that

$$\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n^2}$$

16 Let $a, b \in \mathbb{R}$. Prove that $\frac{a^2 + b^2}{2} \geq \left(\frac{a+b}{2}\right)^2$.

17 **a** Expand $(x - y)(x^2 + xy + y^2)$.

b Prove that $x^2 + yx + y^2 \geq 0$ for all $x, y \in \mathbb{R}$.

Hint: Complete the square by thinking of y as a constant.

c Hence, prove that if $x \geq y$, then $x^3 \geq y^3$.

18 Sally travels from home to work at a speed of 12 km/h and immediately returns home at a speed of 24 km/h.

a Show that her average speed is 16 km/h.

b Now suppose that Sally travels to work at a speed of a km/h and immediately returns home at a speed of b km/h. Show that her average speed is $\frac{2ab}{a+b}$ km/h.

c Let a and b be any two positive real numbers. Prove that

$$\frac{a+b}{2} \geq \frac{2ab}{a+b}$$

Note: This proves that Sally's average speed for the whole journey can be no greater than the average of her speeds for the two individual legs of the journey.

ISBN 978-1-316-63608-4

6B Proof by contrapositive

The negation of a statement

To **negate** a statement P we write its very opposite, which we call '**not** P'. For example, consider the following four statements and their negations.

P		not P	
The sky is green.	(false)	The sky is not green.	(true)
$1 + 1 = 2$	(true)	$1 + 1 \neq 2$	(false)
All prime numbers are odd.	(false)	There exists an even prime number.	(true)
All triangles have three sides.	(true)	Some triangle does not have three sides.	(false)

Notice that negation turns a true statement into a false statement, and a false statement into a true statement.

Example 6

Write down each statement and its negation. Which of the statement and its negation is true and which is false?

a $2 > 1$

b 5 is divisible by 3

c The sum of any two odd numbers is even.

d There are two primes whose product is 12.

Solution

a P: $2 > 1$ (true)
not P: $2 \leq 1$ (false)

b P: 5 is divisible by 3 (false)
not P: 5 is not divisible by 3 (true)

c P: The sum of any two odd numbers is even. (true)
not P: There are two odd numbers whose sum is odd. (false)

d P: There are two primes whose product is 12. (false)
not P: There are no two primes whose product is 12. (true)

De Morgan's laws

Negating statements that involve 'and' and 'or' requires the use of De Morgan's laws.

De Morgan's laws

not (P *and* Q) is the same as (not P) *or* (not Q)

not (P *or* Q) is the same as (not P) *and* (not Q)

ISBN 978-1-316-63608-4

Example 7

Write down each statement and its negation. Which of the statement and its negation is true and which is false?

a 6 is divisible by 2 and 3

b 10 is divisible by 2 or 7

Solution

a P: 6 is divisible by 2 and 6 is divisible by 3 (true)

not P: 6 is not divisible by 2 or 6 is not divisible by 3 (false)

b P: 10 is divisible by 2 or 10 is divisible by 7 (true)

not P: 10 is not divisible by 2 and 10 is not divisible by 7 (false)

Example 8

Every person on an island is either a knight or a knave. Knights always tell the truth, and knaves always lie. Alice and Bob are residents on the island. Alice says: ‘I am a knave or Bob is a knight.’ What are Alice and Bob?

Solution

We will prove that Alice is a knight and Bob is a knight.

Case 1

Suppose Alice is a knave.

$\Rightarrow$ Alice is not telling the truth.

$\Rightarrow$ Alice is a knight AND Bob is a knave.

$\Rightarrow$ Alice is a knight and a knave.

This is impossible.

Case 2

Suppose Alice is a knight.

$\Rightarrow$ Alice is telling the truth.

$\Rightarrow$ Alice is a knave OR Bob is a knight.

$\Rightarrow$ Bob is a knight.

Therefore we conclude that Alice must be a knight and Bob must be a knight.

Proof by contrapositive

Consider this statement:

Statement	If	it is the end of term	then	the students are happy.

By switching the hypothesis and the conclusion and negating both, we obtain the **contrapositive** statement:

Contrapositive	If	the students are *not* happy	then	it is *not* the end of term.

Note that the original statement and its contrapositive are logically equivalent:

- If the original statement is true, then the contrapositive is true.
- If the original statement is false, then the contrapositive is false.

This means that to prove a conditional statement, we can instead prove its contrapositive. This is helpful, as it is often easier to prove the contrapositive than the original statement.

ISBN 978-1-316-63608-4

- The **contrapositive** of $P \Rightarrow Q$ is the statement (not Q) $\Rightarrow$ (not P).
- To prove $P \Rightarrow Q$, we can prove the contrapositive instead.

Example 9

Let $n \in \mathbb{Z}$ and consider this statement: If n^2 is even, then n is even.

a Write down the contrapositive. **b** Prove the contrapositive.

Solution

a If n is odd, then n^2 is odd.

b Assume that n is odd. Then $n = 2m + 1$ for some $m \in \mathbb{Z}$. Squaring n gives

$$\begin{aligned} n^2 &= (2m+1)^2 \\ &= 4m^2 + 4m + 1 \\ &= 2(2m^2 + 2m) + 1 \\ &= 2k + 1 \qquad \text{where } k = 2m^2 + 2m \in \mathbb{Z} \end{aligned}$$

Therefore n^2 is odd.

Note: Although we proved the contrapositive, remember that we have actually proved that if n^2 is even, then n is even.

Example 10

Let $n \in \mathbb{Z}$ and consider this statement: If $n^2 + 4n + 1$ is even, then n is odd.

a Write down the contrapositive. **b** Prove the contrapositive.

Solution

a If n is even, then $n^2 + 4n + 1$ is odd.

b Assume that n is even. Then $n = 2m$ for some $m \in \mathbb{Z}$. Therefore

$$\begin{aligned} n^2 + 4n + 1 &= (2m)^2 + 4(2m) + 1 \\ &= 4m^2 + 8m + 1 \\ &= 2(2m^2 + 4m) + 1 \\ &= 2k + 1 \qquad \text{where } k = 2m^2 + 4m \in \mathbb{Z} \end{aligned}$$

Hence $n^2 + 4n + 1$ is odd.

Example 11

Let x and y be positive real numbers and consider this statement: If $x < y$, then $\sqrt{x} < \sqrt{y}$.

a Write down the contrapositive. **b** Prove the contrapositive.

Solution

a If $\sqrt{x} \geq \sqrt{y}$, then $x \geq y$.

b Assume that $\sqrt{x} \geq \sqrt{y}$. Then $x \geq y$ by Example 3, since $\sqrt{x}$ and $\sqrt{y}$ are positive.

ISBN 978-1-316-63608-4

Section summary

- To **negate** a statement we write its opposite.
- For a statement $P \Rightarrow Q$, the **contrapositive** is the statement (not Q) $\Rightarrow$ (not P). That is, we switch the hypothesis and the conclusion and negate both.
- A statement and its contrapositive are logically equivalent.
- Proving the contrapositive of a statement may be easier than giving a direct proof.

Exercise 6B

Skillshet
Example 6

1 Write down each statement and its negation. Which of the statement and its negation is true and which is false?

a $1 > 0$

b 4 is divisible by 8

c Each pair of primes has an even sum.

d Some rectangle has four sides of equal length.

Example 7

2 Write down each statement and its negation. Which of the statement and its negation is true and which is false?

a 14 is divisible by 7 and 2

b 12 is divisible by 3 or 4

c 15 is divisible by 3 and 6

d 10 is divisible by 2 or 3

Example 8

3 Every person on an island is either a knight or a knave. Knights always tell the truth, and knaves always lie. Alice and Bob are residents on the island. Alice says: 'I am a knave and Bob is a knight.' What are Alice and Bob?

Example 9

4 Write down the contrapositive version of each of these statements:

a If it is raining, then there are clouds in the sky.

b If you are smiling, then you are happy.

c If $x = 1$, then $2x = 2$.

d If $x > y$, then $x^5 > y^5$.

e Let $n \in \mathbb{Z}$. If n^2 is odd, then n is odd.

f Let $m, n \in \mathbb{Z}$. If m and n are odd, then mn is odd.

g Let $m, n \in \mathbb{Z}$. If $m + n$ is even, then m and n are either both even or both odd.

Example 10

5 Let $m, n \in \mathbb{Z}$. For each of the following statements, write down and prove the contrapositive statement:

a If $3n + 5$ is even, then n is odd.

b If n^2 is odd, then n is odd.

c If $n^2 - 8n + 3$ is even, then n is odd.

d If n^2 is not divisible by 3, then n is not divisible by 3.

e If $n^3 + 1$ is even, then n is odd.

f If mn is not divisible by 3, then m is not divisible by 3 and n is not divisible by 3.

g If $m + n$ is odd, then $m \neq n$.

ISBN 978-1-316-63608-4

6 Let $x, y \in \mathbb{R}$. For each of the following statements, write down and prove the contrapositive statement:

a If $x^2 + 3x < 0$, then $x < 0$.

b If $x^3 - x > 0$, then $x > -1$.

c If $x + y \geq 2$, then $x \geq 1$ or $y \geq 1$.

d If $2x + 3y \geq 12$, then $x \geq 3$ or $y \geq 2$.

7 Let $m, n \in \mathbb{Z}$ and consider this statement: If mn and $m + n$ are even, then m and n are even.

a Write down the contrapositive.

b Prove the contrapositive. You will have to consider cases.

Example 11

8 Let x and y be positive real numbers.

a Prove that

$$\sqrt{x} - \sqrt{y} = \frac{x - y}{\sqrt{x} + \sqrt{y}}$$

b Hence, prove that if $x > y$, then $\sqrt{x} > \sqrt{y}$.

c Give a simpler proof by considering the contrapositive.

6C Proof by contradiction

There are various instances when we want to prove mathematically that something cannot be done. To do this, we assume that it can be done, and then show that something goes horribly wrong. Let's first look at a familiar example from geometry.

Example 12

An angle is called **reflex** if it exceeds 180°. Prove that no quadrilateral has more than one reflex angle.

Solution

If there is more than one reflex angle, then the angle sum must exceed $2 \times 180^\circ = 360^\circ$. This contradicts the fact that the angle sum of any quadrilateral is exactly 360°. Therefore there cannot be more than one reflex angle.

The example above is a demonstration of a **proof by contradiction**. The basic outline of a proof by contradiction is:

1. Assume that the statement we want to prove is false.
2. Show that this assumption leads to mathematical nonsense.
3. Conclude that we were wrong to assume that the statement is false.
4. Conclude that the statement must be true.

ISBN 978-1-316-63608-4

Example 13

A **Pythagorean triple** consists of three natural numbers (a, b, c) satisfying

$$a^2 + b^2 = c^2$$

Show that if (a, b, c) is a Pythagorean triple, then a, b and c cannot all be odd numbers.

Solution

This will be a proof by contradiction.

Let (a, b, c) be a Pythagorean triple. Then $a^2 + b^2 = c^2$.

Suppose that a, b and c are all odd numbers.

$\Rightarrow$ a^2, b^2 and c^2 are all odd numbers.

$\Rightarrow$ $a^2 + b^2$ is even and c^2 is odd.

Since $a^2 + b^2 = c^2$, this gives a contradiction.

Therefore a, b and c cannot all be odd numbers.

Possibly the most well-known proof by contradiction is the following.

Theorem

$\sqrt{2}$ is irrational.

Proof This will be a proof by contradiction.

Suppose that $\sqrt{2}$ is rational. Then

$$\sqrt{2} = \frac{p}{q} \quad \text{where } p, q \in \mathbb{Z}$$

We can assume that p and q have no common factors (or else they could be cancelled). Then, squaring both sides and rearranging gives

$$\begin{aligned}
& p^2 = 2q^2 && (1) \\
\Rightarrow\ & p^2 \text{ is divisible by } 2 && \\
\Rightarrow\ & p \text{ is divisible by } 2 && \text{(by Example 9)} \\
\Rightarrow\ & p = 2n \text{ for some } n \in \mathbb{Z} && \\
\Rightarrow\ & (2n)^2 = 2q^2 && \text{(substituting into (1))} \\
\Rightarrow\ & q^2 = 2n^2 && \\
\Rightarrow\ & q^2 \text{ is divisible by } 2 && \\
\Rightarrow\ & q \text{ is divisible by } 2 && \text{(by Example 9)}
\end{aligned}$$

Therefore both p and q are divisible by 2, which contradicts the fact that they have no common factors.

Hence $\sqrt{2}$ is irrational.

ISBN 978-1-316-63608-4

Example 14

Suppose x satisfies $5^x = 2$. Show that x is irrational.

Solution

Suppose that x is rational. Since x must be positive, we can write $x = \frac{m}{n}$ where $m, n \in \mathbb{N}$. Therefore

$$\begin{aligned} 5^x = 2 \quad &\Rightarrow \quad 5^{\frac{m}{n}} = 2 \\ &\Rightarrow \quad \left(5^{\frac{m}{n}}\right)^n = 2^n \qquad \text{(raise both sides to the power } n\text{)} \\ &\Rightarrow \quad 5^m = 2^n \end{aligned}$$

The left-hand side of this equation is odd and the right-hand side is even. This gives a contradiction, and so x is not rational.

We finish on a remarkable result, which is attributed to Euclid some 2300 years ago.

Theorem

There are infinitely many prime numbers.

Proof This is a proof by contradiction, so we will suppose that there are only finitely many primes. This means that we can create a list that contains *every* prime number:

$$2,\ 3,\ 5,\ 7,\ \dots,\ p$$

where p is the largest prime number.

Now for the trick. We a create a new number N by multiplying each number in the list and then adding 1:

$$N = 2 \times 3 \times 5 \times 7 \times \cdots \times p + 1$$

The number N is not divisible by any of the primes $2, 3, 5, 7, \dots, p$, since it leaves a remainder of 1 when divided by any of these numbers.

However, every natural number greater than 1 is divisible by a prime number. (This is proved in Question 13 of Exercise 6F.) Therefore N is divisible by some prime number q. But this prime number q is not in the list $2, 3, 5, 7, \dots, p$, contradicting the fact that our list contains every prime number.

Hence there are infinitely many prime numbers.

Section summary

- A **proof by contradiction** is used to prove that something cannot be done.
- These proofs always follow the same basic structure:
  1. Assume that the statement we want to prove is false.
  2. Show that this assumption leads to mathematical nonsense.
  3. Conclude that we were wrong to assume that the statement is false.
  4. Conclude that the statement must be true.

ISBN 978-1-316-63608-4

Exercise 6C

Skillsheet

Example 12

1 Prove that every triangle has some interior angle with a magnitude of at least $60°$.

2 Prove that there is no smallest positive rational number.

3 Let p be a prime number. Show that $\sqrt{p}$ is not an integer.

Example 14

4 Suppose that $3^x = 2$. Prove that x is irrational.

5 Prove that $\log_2 5$ is irrational.

6 Suppose that $x > 0$ is irrational. Prove that $\sqrt{x}$ is also irrational.

7 Suppose that a is rational and b is irrational. Prove that $a + b$ is irrational.

8 Suppose that $c^2 - b^2 = 4$. Prove that b and c cannot both be natural numbers.

9 Let a, b and c be real numbers with $a \neq 0$. Prove by contradiction that there is only one solution to the equation $ax + b = c$.

10 **a** Prove that all primes $p > 2$ are odd.

b Hence, prove that there are no two primes whose sum is 1001.

11 **a** Prove that there are no integers a and b for which $42a + 7b = 1$.

Hint: The left-hand side is divisible by 7.

b Prove that there are no integers a and b for which $15a + 21b = 2$.

12 **a** Prove that if n^2 is divisible by 3, then n is divisible by 3.

Hint: Prove the contrapositive by considering two cases.

b Hence, prove that $\sqrt{3}$ is irrational.

13 **a** Prove that if n^3 is divisible by 2, then n is divisible by 2.

Hint: Prove the contrapositive.

b Hence, prove that $\sqrt[3]{2}$ is irrational.

14 Prove that if $a, b \in \mathbb{Z}$, then $a^2 - 4b - 2 \neq 0$.

15 **a** Let $a, b, n \in \mathbb{N}$. Prove that if $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

b Hence, show that 97 is a prime number.

16 **a** Let m be an integer. Prove that m^2 is divisible by 4 or leaves a remainder of 1.

Hint: Suppose that $m = 4n + r$ and consider m^2 for $r = 0, 1, 2, 3$.

b Let $a, b, c \in \mathbb{Z}$. Prove by contradiction: If $a^2 + b^2 = c^2$, then a is even or b is even.

17 **a** Let $a, b, c, d \in \mathbb{Z}$. Prove that if $a + b\sqrt{2} = c + d\sqrt{2}$, then $a = c$ and $b = d$.

b Hence, find $c, d \in \mathbb{Z}$ if $\sqrt{3 + 2\sqrt{2}} = c + d\sqrt{2}$. Hint: Square both sides.

18 Let $a, b, c \in \mathbb{Z}$. Prove that if a, b and c are all odd, then the equation $ax^2 + bx + c = 0$ cannot have a rational solution.

ISBN 978-1-316-63608-4

6D Equivalent statements

► The converse of a statement

At the beginning of this chapter, we proved Pythagoras' theorem. Consider any triangle with side lengths a, b and c.

Statement	If	the angle between a and b is 90°	then	$a^2 + b^2 = c^2$.

By switching the hypothesis and the conclusion, we obtain the **converse** statement:

Converse	If	$a^2 + b^2 = c^2$	then	the angle between a and b is 90°.

In Chapter 7 we will prove that this is also a true statement.

When we switch the hypothesis and the conclusion of a conditional statement, $P \Rightarrow Q$, we obtain the **converse** statement, $Q \Rightarrow P$.

Note: The converse of a true statement may not be true. For example:

Statement	If it is raining, then there are clouds in the sky.	(true)
Converse	If there are clouds in the sky, then it is raining.	(false)

Example 15

Let x and y be positive real numbers. Consider the statement: If $x < y$, then $x^2 < y^2$.

a Write down the converse of this statement.

b Prove the converse.

Solution

a If $x^2 < y^2$, then $x < y$.

b Assume that $x^2 < y^2$. Then, since both x and y are positive,

$$\begin{aligned} x^2 - y^2 &< 0 && \text{(subtract } y^2\text{)} \\ \Rightarrow \quad (x-y)(x+y) &< 0 && \text{(factorising)} \\ \Rightarrow \quad x - y &< 0 && \text{(divide both sides by } x + y > 0\text{)} \\ \Rightarrow \quad x &< y \end{aligned}$$

as required.

Example 16

Let m and n be integers. Consider the statement: If m and n are even, then $m + n$ is even.

a Write down the converse of this statement.

b Show that the converse is not true.

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Solution

a If $m + n$ is even, then m is even and n is even.

b Clearly $1 + 3 = 4$ is even, although 1 and 3 are not.

Equivalent statements

Now consider the following two statements:

P: your heart is beating

Q: you are alive

Notice that both $P \Rightarrow Q$ and its converse $Q \Rightarrow P$ are true statements. In this case, we say that P and Q are **equivalent** statements and we write

$$P \Leftrightarrow Q$$

We will also say that P is true **if and only if** Q is true. So in the above example, we can say

Your heart is beating	if and only if	you are alive.

To prove that two statements P and Q are equivalent, you have to prove two things:

$$P \Rightarrow Q \quad \text{and} \quad Q \Rightarrow P$$

Example 17

Let $n \in \mathbb{Z}$. Prove that n is even if and only if $n + 1$ is odd.

Solution

($\Rightarrow$) Assume that n is even. Then $n = 2m$ for some $m \in \mathbb{Z}$.
Therefore $n + 1 = 2m + 1$, and so $n + 1$ is odd.

($\Leftarrow$) Assume that $n + 1$ is odd. Then $n + 1 = 2m + 1$ for some $m \in \mathbb{Z}$.
Subtracting 1 from both sides gives $n = 2m$. Therefore n is even.

Note: To prove that $P \Leftrightarrow Q$, we have to show that $P \Rightarrow Q$ and $P \Leftarrow Q$. When we are about to prove $P \Rightarrow Q$, we write ($\Rightarrow$). When we are about to prove $P \Leftarrow Q$, we write ($\Leftarrow$).

Section summary

- For a statement $P \Rightarrow Q$, the **converse** is the statement $Q \Rightarrow P$.
  That is, we switch the hypothesis and the conclusion.
- If $P \Rightarrow Q$ is true and $Q \Rightarrow P$ is true, then we say that P is **equivalent** to Q, or that P is true **if and only if** Q is true.
- If P and Q are equivalent, we write $P \Leftrightarrow Q$.

ISBN 978-1-316-63608-4

Exercise 6D

Skillsheet
Example 15

1 Write down and prove the converse of each of these statements:

a Let $x \in \mathbb{R}$. If $2x + 3 = 5$, then $x = 1$.

b Let $n \in \mathbb{Z}$. If n is odd, then $n - 3$ is even.

c Let $m \in \mathbb{Z}$. If $m^2 + 2m + 1$ is even, then m is odd.

d Let $n \in \mathbb{Z}$. If n^2 is divisible by 5, then n is divisible by 5.

Example 16

2 Let m and n be integers. Consider the statement: If m and n are even, then mn is a multiple of 4.

a Write down the converse of this statement.

b Show that the converse is not true.

3 Which of these pairs of statements are equivalent?

a P: Vivian is in China.
Q: Vivian is in Asia.

b P: $2x = 4$
Q: $x = 2$

c P: $x > 0$ and $y > 0$
Q: $xy > 0$

d P: m is even or n is even, where $m, n \in \mathbb{Z}$
Q: mn is even, where $m, n \in \mathbb{Z}$

Example 17

4 Let n be an integer. Prove that $n + 1$ is odd if and only if $n + 2$ is even.

5 Let $n \in \mathbb{N}$. Prove that $n^2 - 4$ is a prime number if and only if $n = 3$.

6 Let n be an integer. Prove that n^3 is even if and only if n is even.

7 Let n be an integer. Prove that n is odd if and only if $n = 4k \pm 1$ for some $k \in \mathbb{Z}$.

8 Let $x, y \in \mathbb{R}$. Prove that $(x + y)^2 = x^2 + y^2$ if and only if $x = 0$ or $y = 0$.

9 Let m and n be integers.

a By expanding the right-hand side, prove that $m^3 - n^3 = (m - n)(m^2 + mn + n^2)$.

b Hence, prove that $m - n$ is even if and only if $m^3 - n^3$ is even.

10 Prove that an integer is divisible by 4 if and only if the number formed by its last two digits is divisible by 4. Hint: 100 is divisible by 4.

ISBN 978-1-316-63608-4

6E Disproving statements

▶ Quantification using 'for all' and 'there exists'

For all

Universal quantification claims that a property holds for *all* members of a given set. For example, consider this statement:

Statement	For all natural numbers n, we have $2n \geq n + 1$.

To prove that this statement is true, we need to give a general argument that applies to every natural number n.

There exists

Existential quantification claims that a property holds for *at least one* member of a given set. For example, consider this statement:

Statement	There exists an integer m such that $m^2 = 25$.

To prove that this statement is true, we just need to give an example: $5 \in \mathbb{Z}$ with $5^2 = 25$.

Example 18

Rewrite each statement using either 'for all' or 'there exists':

a Some real numbers are irrational.

b Every integer that is divisible by 4 is also divisible by 2.

Solution

a There exists $x \in \mathbb{R}$ such that $x \notin \mathbb{Q}$.

b For all $m \in \mathbb{Z}$, if m is divisible by 4, then m is divisible by 2.

Negating 'for all' and 'there exists'

To negate a statement involving a quantifier, we interchange 'for all' with 'there exists' and then negate the rest of the statement.

Example 19

Write down the negation of each of the following statements:

a For all natural numbers n, we have $2n \geq n + 1$.

b There exists an integer m such that $m^2 = 4$ and $m^3 = -8$.

Solution

a There exists a natural number n such that $2n < n + 1$.

b For all integers m, we have $m^2 \neq 4$ or $m^3 \neq -8$.

Note: For part **b**, we used one of De Morgan's laws.

ISBN 978-1-316-63608-4

Notation

The words 'for all' can be abbreviated using the *turned A* symbol, $\forall$. The words 'there exists' can be abbreviated using the *turned E* symbol, $\exists$. For example:

- 'For all natural numbers n, we have $2n \geq n + 1$' can be written as $(\forall n \in \mathbb{N})\ 2n \geq n + 1$.
- 'There exists an integer m such that $m^2 = 25$' can be written as $(\exists m \in \mathbb{Z})\ m^2 = 25$.

Despite the ability of these new symbols to make certain sentences more concise, we do not believe that they make written sentences clearer. Therefore we have avoided using them in this chapter.

▶ Counterexamples

Consider the quadratic function $f(n) = n^2 - n + 11$. Notice how $f(n)$ is a prime number for small natural numbers n:

n	1	2	3	4	5	6	7	8	9	10
$f(n)$	11	13	17	23	31	41	53	67	83	101

From this, we might be led to believe that the following statement is true:

Statement	For all natural numbers n, the number $f(n)$ is prime.

We call this a **universal statement**, because it asserts the truth of a statement without exception. So to disprove a universal statement, we need only show that it is not true in some particular instance. For our example, we need to find $n \in \mathbb{N}$ such that $f(n)$ is not prime. Luckily, we do not have to look very hard.

Example 20

Let $f(n) = n^2 - n + 11$. Disprove this statement: For all $n \in \mathbb{N}$, the number $f(n)$ is prime.

Solution

When $n = 11$, we obtain

$$f(11) = 11^2 - 11 + 11 = 11^2$$

Therefore $f(11)$ is not prime.

To disprove a statement of the form $P \Rightarrow Q$, we simply need to give one example for which P is true and Q is not true. Such an example is called a **counterexample**.

Example 21

Find a counterexample to disprove this statement: For all $x, y \in \mathbb{R}$, if $x > y$, then $x^2 > y^2$.

Solution

Let $x = 1$ and $y = -2$. Clearly $1 > -2$, but $1^2 = 1 \leq 4 = (-2)^2$.

ISBN 978-1-316-63608-4

► Disproving existence statements

Consider this statement:

Statement	There exists $n \in \mathbb{N}$ such that $n^2 + 3n + 2$ is a prime number.

We call this an **existence statement**, because it claims the existence of an object possessing a particular property. To show that such a statement is false, we prove that its negation is true:

Negation	For all $n \in \mathbb{N}$, the number $n^2 + 3n + 2$ is not a prime number.

This is easy to prove, as

$$n^2 + 3n + 2 = (n + 1)(n + 2)$$

is clearly a composite number for each $n \in \mathbb{N}$.

Example 22

Disprove this statement: There exists $n \in \mathbb{N}$ such that $n^2 + 13n + 42$ is a prime number.

Solution

We need to prove that, for all $n \in \mathbb{N}$, the number $n^2 + 13n + 42$ is not prime.

This is true, since

$$n^2 + 13n + 42 = (n + 6)(n + 7)$$

is clearly a composite number for each $n \in \mathbb{N}$.

Example 23

Show that this statement is false: There exists some real number x such that $x^2 = -1$.

Solution

We have to prove that the negation is true: For *all* real numbers x, we have $x^2 \neq -1$.

This is easy to prove since, for any real number x, we have $x^2 \geq 0$ and so $x^2 \neq -1$.

Section summary

- A **universal statement** claims that a property holds for all members of a given set. Such a statement can be written using the quantifier **'for all'**.
- An **existence statement** claims that a property holds for some member of a given set. Such a statement can be written using the quantifier **'there exists'**.
- A universal statement of the form $P \Rightarrow Q$ can be disproved by giving one example of an instance when P is true but Q is not.
- Such an example is called a **counterexample**.
- To disprove an existence statement, we prove that its negation is true.

ISBN 978-1-316-63608-4

Exercise 6E

Skillsheet

Example 18

1 Which of the following are universal statements ('for all') and which are existence statements ('there exists')?

a For each $n \in \mathbb{N}$, the number $5n^2 + 3n + 7$ is odd.

b There is an even prime number.

c Every natural number greater than 1 has a prime factorisation.

d All triangles have three sides.

e Some natural numbers are primes.

f At least one real number x satisfies the equation $x^2 - x - 1 = 0$.

g Any positive real number has a square root.

h The angle sum of a triangle is 180°.

2 Which of the following statements are true and which are false?

a There exists a real number x such that $x^2 = 2$.

b There exists a real number x such that $x^2 < 0$.

c For all natural numbers n, the number $2n - 1$ is odd.

d There exists $n \in \mathbb{N}$ such that $2n$ is odd.

e For all $x \in \mathbb{R}$, we have $x^3 \geq 0$.

Example 19

3 Write down the negation of each of the following statements:

a For every natural number n, the number $2n^2 - 4n + 31$ is prime.

b For all $x \in \mathbb{R}$, we have $x^2 > x$.

c There exists $x \in \mathbb{R}$ such that $2 + x^2 = 1 - x^2$.

d For all $x, y \in \mathbb{R}$, we have $(x + y)^2 = x^2 + y^2$.

e There exist $x, y \in \mathbb{R}$ such that $x < y$ and $x^2 > y^2$.

4 Prove that each of the following statements is false by finding a counterexample:

Example 20

a For every natural number n, the number $2n^2 - 4n + 31$ is prime.

b If $x, y \in \mathbb{R}$, then $(x + y)^2 = x^2 + y^2$.

Example 21

c For all $x \in \mathbb{R}$, we have $x^2 > x$.

d Let $n \in \mathbb{Z}$. If $n^3 - n$ is even, then n is even.

e If $m, n \in \mathbb{N}$, then $m + n \leq mn$.

f Let $m, n \in \mathbb{Z}$. If 6 divides mn, then 6 divides m or 6 divides n.

5 Show that each of the following existence statements is false:

Example 22

a There exists $n \in \mathbb{N}$ such that $9n^2 - 1$ is a prime number.

b There exists $n \in \mathbb{N}$ such that $n^2 + 5n + 6$ is a prime number.

Example 23

c There exists $x \in \mathbb{R}$ such that $2 + x^2 = 1 - x^2$.

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6 Provide a counterexample to disprove each of the following statements.
Hint: $\sqrt{2}$ might come in handy.

a If a is irrational and b is irrational, then ab is irrational.
b If a is irrational and b is irrational, then $a + b$ is irrational.
c If a is irrational and b is irrational, then $\frac{a}{b}$ is irrational.

7 Let $a \in \mathbb{Z}$.

a Prove that if a is divisible by 4, then a^2 is divisible by 4.
b Prove that the converse is not true.

8 Let $a, b \in \mathbb{Z}$.

a Prove that if $a - b$ is divisible by 3, then $a^2 - b^2$ is divisible by 3.
b Prove that the converse is not true.

9 Prove that each of the following statements is false:

a There exist real numbers a and b such that $a^2 - 2ab + b^2 = -1$.
b There exists some real number x such that $x^2 - 4x + 5 = \frac{3}{4}$.

10 The numbers $\{1, 2, \ldots, 8\}$ can be paired so that the sum of each pair is a square number:

$$1 + 8 = 9, \quad 2 + 7 = 9, \quad 3 + 6 = 9, \quad 4 + 5 = 9$$

a Prove that you can also do this with the numbers $\{1, 2, \ldots, 16\}$.
b Prove that you cannot do this with the numbers $\{1, 2, \ldots, 12\}$.

11 Let $f(n) = an^2 + bn + c$ be a quadratic function, where a, b, c are natural numbers and $c \geq 2$. Show that there is an $n \in \mathbb{N}$ such that $f(n)$ is not a prime number.

6F Mathematical induction

Consider the sum of the first n odd numbers:

$$\begin{aligned} 1 &= 1 = 1^2 \\ 1 + 3 &= 4 = 2^2 \\ 1 + 3 + 5 &= 9 = 3^2 \\ 1 + 3 + 5 + 7 &= 16 = 4^2 \end{aligned}$$

From this limited number of examples, we could make the following proposition $P(n)$ about the number n: the sum of the first n odd numbers is n^2. Since the nth odd number is $2n - 1$, we can write this proposition as

$$P(n): \quad 1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

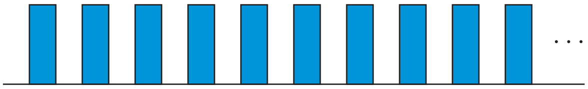
However, we have to be careful here: Just because something looks true does not mean that it is true. In this section, we will learn how to prove statements like the one above.

ISBN 978-1-316-63608-4

The principle of mathematical induction

Imagine a row of dominoes extending infinitely to the right. Each of these dominoes can be knocked over provided two conditions are met:

1 The first domino is knocked over.
2 Each domino is sufficiently close to the next domino.



This scenario provides an accurate physical model of the following proof technique.

Principle of mathematical induction

Let $P(n)$ be some proposition about the natural number n.

We can prove that $P(n)$ is true for every natural number n as follows:

a Show that $P(1)$ is true.
b Show that, for every natural number k, if $P(k)$ is true, then $P(k + 1)$ is true.

The idea is simple: Condition **a** tells us that $P(1)$ is true. But then condition **b** means that $P(2)$ will also be true. However, if $P(2)$ is true, then condition **b** also guarantees that $P(3)$ is true, and so on. This process continues indefinitely, and so $P(n)$ is true for all $n \in \mathbb{N}$.

$$P(1) \text{ is true} \quad \Rightarrow \quad P(2) \text{ is true} \quad \Rightarrow \quad P(3) \text{ is true} \quad \Rightarrow \quad \cdots$$

Let's see how mathematical induction is used in practice.

Example 24

Prove that

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

for all $n \in \mathbb{N}$.

Solution

For each natural number n, let $P(n)$ be the proposition:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

Step 1 $P(1)$ is the proposition $1 = 1^2$, that is, $1 = 1$. Therefore $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

Step 3 We now have to prove that $P(k + 1)$ is true, that is,

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2$$

Notice that we have written the last and the second-last term in the summation. This is so we can easily see how to use our assumption that $P(k)$ is true.

ISBN 978-1-316-63608-4

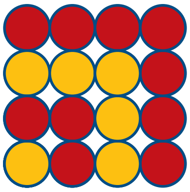
We have

$$\begin{aligned}\text{LHS of } P(k+1) &= 1 + 3 + 5 + \cdots + (2k-1) + (2k+1)\\ &= k^2 + (2k+1) && \text{(using } P(k)\text{)}\\ &= (k+1)^2\\ &= \text{RHS of } P(k+1)\end{aligned}$$

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number k.

By the principle of mathematical induction, it follows that $P(n)$ is true for every natural number n.

While mathematical induction is good for proving that formulas are true, it rarely indicates why they should be true in the first place. The formula $1 + 3 + 5 + \cdots + (2n-1) = n^2$ can be discovered in the diagram shown on the right.



Example 25

Prove by induction that $7^n - 4$ is divisible by 3 for all $n \in \mathbb{N}$.

Solution

For each natural number n, let $P(n)$ be the proposition:

$7^n - 4$ is divisible by 3

Step 1 $P(1)$ is the proposition $7^1 - 4 = 3$ is divisible by 3. Clearly, $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$7^k - 4 = 3m$$

for some $m \in \mathbb{Z}$.

Step 3 We now have to prove that $P(k+1)$ is true, that is, $7^{k+1} - 4$ is divisible by 3.

We have

$$\begin{aligned}7^{k+1} - 4 &= 7 \times 7^k - 4\\ &= 7(3m+4) - 4 && \text{(using } P(k)\text{)}\\ &= 21m + 28 - 4\\ &= 21m + 24\\ &= 3(7m+8)\end{aligned}$$

Therefore $7^{k+1} - 4$ is divisible by 3.

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number k.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$, by the principle of mathematical induction.

ISBN 978-1-316-63608-4

Cambridge University Press

Proving inequalities

Induction can be used to prove certain inequalities.

For example, consider this table of values:

n	1	2	3	4	5
3^n	3	9	27	81	243
3×2^n	6	12	24	48	96

From the table, it certainly looks as though

$$3^n > 3 \times 2^n \quad \text{for all } n \geq 3$$

We will prove this formally using induction; this time starting with the proposition $P(3)$ instead of $P(1)$.

Example 26

Prove that

$$3^n > 3 \times 2^n$$

for every natural number $n \geq 3$.

Solution

For each natural number $n \geq 3$, let $P(n)$ be the proposition:

$$3^n > 3 \times 2^n$$

Step 1 $P(3)$ is the proposition $3^3 > 3 \times 2^3$, that is, $27 > 24$. Therefore $P(3)$ is true.

Step 2 Let k be a natural number with $k \geq 3$, and assume $P(k)$ is true. That is,

$$3^k > 3 \times 2^k$$

Step 3 We now have to prove that $P(k+1)$ is true, that is,

$$3^{k+1} > 3 \times 2^{k+1}$$

We have

$$\begin{aligned} \text{LHS of } P(k+1) &= 3^{k+1} \\ &= 3 \times 3^k \\ &> 3 \times 3 \times 2^k \qquad \text{(using } P(k)) \\ &> 3 \times 2 \times 2^k \\ &= 3 \times 2^{k+1} \\ &= \text{RHS of } P(k+1) \end{aligned}$$

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number $k \geq 3$.

By the principle of mathematical induction, it follows that $P(n)$ is true for every natural number $n \geq 3$.

ISBN 978-1-316-63608-4

▶ Applications to sequences

Sequences are studied in Mathematical Methods Year 11. A **sequence** is a list of numbers, with order being important. An example is the sequence of odd numbers:

$$1, 3, 5, 7, 9, \dots$$

The nth term of a sequence is denoted by t_n. So, for the sequence of odd numbers, we have $t_1 = 1$, $t_2 = 3$ and $t_3 = 5$. In general, we have $t_n = 2n - 1$ for all $n \in \mathbb{N}$.

Some sequences can be defined by a rule that enables each subsequent term to be found from previous terms. We can define the sequence of odd numbers by $t_1 = 1$ and $t_{n+1} = t_n + 2$. This type of rule is called a **recurrence relation**.

Induction proofs are frequently used in the study of sequences. As an example, we will consider the sequence defined by the recurrence relation

$$t_1 = 11 \quad \text{and} \quad t_{n+1} = 10t_n - 9$$

The first five terms of this sequence are listed in the following table.

n	1	2	3	4	5
t_n	11	101	1001	10 001	100 001

Notice that each of these terms is one more than a power of 10. Let's see if we can prove that this is true for *every* term in the sequence.

Example 27

Given $t_1 = 11$ and $t_{n+1} = 10t_n - 9$, prove that $t_n = 10^n + 1$.

Solution

For each natural number n, let $P(n)$ be the proposition: $t_n = 10^n + 1$.

Step 1 Since $t_1 = 11$ and $10^1 + 1 = 11$, it follows that $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$t_k = 10^k + 1$$

Step 3 We now have to prove that $P(k+1)$ is true, that is,

$$t_{k+1} = 10^{k+1} + 1$$

We have

$$\begin{aligned}
\text{LHS of } P(k+1) &= t_{k+1} \\
&= 10t_k - 9 \\
&= 10 \times (10^k + 1) - 9 \qquad (\text{using } P(k)) \\
&= 10^{k+1} + 1 \\
&= \text{RHS of } P(k+1)
\end{aligned}$$

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for each $k \in \mathbb{N}$.

By the principle of mathematical induction, it follows that $P(n)$ is true for all $n \in \mathbb{N}$.

ISBN 978-1-316-63608-4

► Tower of Hanoi

You have three pegs and a collection of n discs of different sizes. Initially, all the discs are stacked in size order on the left-hand peg. Discs can be moved one at a time from one peg to any other peg, provided that a larger disc never rests on a smaller one. The aim of the puzzle is to transfer all the discs to another peg using the smallest possible number of moves.

Example 28

Let a_n be the minimum number of moves needed to solve the Tower of Hanoi with n discs.

a Find a formula for a_{n+1} in terms of a_n.

b Evaluate a_n for $n = 1, 2, 3, 4, 5$. Guess a formula for a_n in terms of n.

c Confirm your formula for a_n using mathematical induction.

d If $n = 20$, how many days are needed to transfer all the discs to another peg, assuming that one disc can be moved per second?

Solution

a Suppose there are $n + 1$ discs on the left-hand peg.

If we want to be able to move the largest disc to the right-hand peg, then first we must transfer the other n discs to the centre peg. This takes a minimum of a_n moves.

It takes 1 move to transfer the largest disc to the right-hand peg. Now we can complete the puzzle by transferring the n discs on the centre peg to the right-hand peg. This takes a minimum of a_n moves.

Hence the minimum number of moves required to transfer all the discs is

$$\begin{aligned} a_{n+1} &= a_n + 1 + a_n \\ &= 2a_n + 1 \end{aligned}$$

b We have $a_1 = 1$, since one disc can be moved in one move. Using the recurrence relation from part **a**, we find that

$$a_2 = 2a_1 + 1 = 2 \times 1 + 1 = 3$$
$$a_3 = 2a_2 + 1 = 2 \times 3 + 1 = 7$$

Continuing in this way, we obtain the following table.

n	1	2	3	4	5
a_n	1	3	7	15	31

It seems as though every term is one less than a power of 2. We guess that

$$a_n = 2^n - 1$$

ISBN 978-1-316-63608-4

c For each natural number n, let $P(n)$ be the proposition:

$$a_n = 2^n - 1$$

Step 1 The minimum number of moves required to solve the Tower of Hanoi puzzle with one disc is 1. Since $2^1 - 1 = 1$, it follows that $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$a_k = 2^k - 1$$

Step 3 We now wish to prove that $P(k + 1)$ is true, that is,

$$a_{k+1} = 2^{k+1} - 1$$

We have

$$\begin{aligned} \text{LHS of } P(k+1) &= a_{k+1} \\ &= 2a_k + 1 && \text{(using part } \mathbf{a}) \\ &= 2 \times (2^k - 1) + 1 && \text{(using } P(k)) \\ &= 2^{k+1} - 2 + 1 \\ &= 2^{k+1} - 1 \\ &= \text{RHS of } P(k+1) \end{aligned}$$

We have proved that if $P(k)$ is true, then $P(k + 1)$ is true, for every natural number k.

By the principle of mathematical induction, it follows that $P(n)$ is true for all $n \in \mathbb{N}$. Hence we have shown that $a_n = 2^n - 1$ for all $n \in \mathbb{N}$.

d A puzzle with 20 discs requires a minimum of $2^{20} - 1$ seconds.

Since there are $60 \times 60 \times 24 = 86\,400$ seconds in a day, it will take

$$\frac{2^{20} - 1}{86\,400} \approx 12.14 \text{ days}$$

to complete the puzzle.

Section summary

The basic outline of a proof by mathematical induction is:

0 Define the proposition $P(n)$ for $n \in \mathbb{N}$.

1 Show that $P(1)$ is true.

2 Assume that $P(k)$ is true for some $k \in \mathbb{N}$.

3 Show that $P(k + 1)$ is true.

4 Conclude that $P(n)$ is true for all $n \in \mathbb{N}$.

ISBN 978-1-316-63608-4

Exercise 6F

Skillsheet
Example 24

1 Prove each of the following by mathematical induction:

a $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$

b $1 + x + x^2 + \cdots + x^n = \dfrac{1 - x^{n+1}}{1 - x}$, where $x \neq 1$

c $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

d $1 \cdot 2 + 2 \cdot 3 + \cdots + n \cdot (n+1) = \dfrac{n(n+1)(n+2)}{3}$

e $\dfrac{1}{1 \times 3} + \dfrac{1}{3 \times 5} + \cdots + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$

f $\left(1 - \dfrac{1}{2^2}\right)\left(1 - \dfrac{1}{3^2}\right)\cdots\left(1 - \dfrac{1}{n^2}\right) = \dfrac{n+1}{2n}$, for $n \geq 2$

Example 25

2 Prove each of the following divisibility statements by mathematical induction:

a $11^n - 1$ is divisible by 10 for all $n \in \mathbb{N}$

b $3^{2n} + 7$ is divisible by 8 for all $n \in \mathbb{N}$

c $7^n - 3^n$ is divisible by 4 for all $n \in \mathbb{N}$

d $5^n + 6 \times 7^n + 1$ is divisible by 4 for all $n \in \mathbb{N}$

Example 26

3 Prove each of the following inequalities by mathematical induction:

a $4^n > 10 \times 2^n$ for all integers $n \geq 4$

b $3^n > 5 \times 2^n$ for all integers $n \geq 5$

c $2^n > 2n$ for all integers $n \geq 3$

d $n! > 2^n$ for all integers $n \geq 4$

Example 27

4 Prove each of the following statements by mathematical induction:

a If $a_{n+1} = 2a_n - 1$ and $a_1 = 3$, then $a_n = 2^n + 1$.

b If $a_{n+1} = 5a_n + 4$ and $a_1 = 4$, then $a_n = 5^n - 1$.

c If $a_{n+1} = 2a_n - n + 1$ and $a_1 = 3$, then $a_n = 2^n + n$.

5 Prove that 3^n is odd for every $n \in \mathbb{N}$.

6 **a** Prove by mathematical induction that $n^2 - n$ is even for all $n \in \mathbb{N}$.

b Find an easier proof by factorising $n^2 - n$.

7 **a** Prove by mathematical induction that $n^3 - n$ is divisible by 6 for all $n \in \mathbb{N}$.

b Find an easier proof by factorising $n^3 - n$.

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8 Consider the sequence defined by $a_{n+1} = 10a_n + 9$ where $a_1 = 9$.

a Find a_n for $n = 1, 2, 3, 4, 5$.

b Guess a formula for a_n in terms of n.

c Confirm that your formula is valid by using mathematical induction.

9 The Fibonacci numbers are defined by $f_1 = 1$, $f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$.

a Find f_n for $n = 1, 2, \dots, 10$.

b Prove that $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$.

c Evaluate $f_1 + f_3 + \cdots + f_{2n-1}$ for $n = 1, 2, 3, 4$.

d Try to find a formula for the above expression.

e Confirm that your formula works using mathematical induction.

f Using induction, prove that every third Fibonacci number, f_{3n}, is even.

10 Prove that $4^n + 5^n$ is divisible by 9 for all odd integers n.

11 Prove by induction that, for all $n \in \mathbb{N}$, every set of numbers S with exactly n elements has a largest element.

12 Standing around a circle, there are n friends and n thieves. You begin with no money, but as you go around the circle clockwise, each friend will give you \$1 and each thief will steal \$1. Prove that no matter where the friends and thieves are placed, it is possible to walk once around the circle without going into debt, provided you start at the correct point.

13 Prove by induction that every natural number $n \geq 2$ is divisible by some prime number.
Hint: Let $P(n)$ be the statement that every integer j such that $2 \leq j \leq n$ is divisible by some prime number.

14 If n straight lines are drawn across a sheet of paper, they will divide the paper into regions. Show that it is always possible to colour each region black or white, so that no two adjacent regions have the same colour.

ISBN 978-1-316-63608-4

Review

Chapter summary

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Nrich

- A **conditional statement** has the form: If P is true, then Q is true.
  This can be abbreviated as $P \Rightarrow Q$, which is read 'P implies Q'.
- To give a **direct proof** of a conditional statement $P \Rightarrow Q$, we assume that P is true and show that Q follows.
- The **converse** of $P \Rightarrow Q$ is $Q \Rightarrow P$.
- Statements P and Q are **equivalent** if $P \Rightarrow Q$ and $Q \Rightarrow P$. We write $P \Leftrightarrow Q$.
- The **contrapositive** of $P \Rightarrow Q$ is (not Q) $\Rightarrow$ (not P).
- Proving the contrapositive of a statement may be easier than giving a direct proof.
- A **proof by contradiction** begins by assuming the negation of what is to be proved.
- A **universal statement** claims that a property holds for all members of a given set. Such a statement can be written using the quantifier **'for all'**.
- An **existence statement** claims that a property holds for some member of a given set. Such a statement can be written using the quantifier **'there exists'**.
- **Counterexamples** can be used to demonstrate that a universal statement is false.
- **Mathematical induction** is used to prove that a statement is true for all natural numbers.

Short-answer questions

1 For each of the following statements, if the statement is true, then prove it, and otherwise give a counterexample to show that it is false:

a The sum of any three consecutive integers is divisible by 3.

b The sum of any four consecutive integers is divisible by 4.

2 Assume that n is even. Prove that $n^2 - 3n + 1$ is odd.

3 Let $n \in \mathbb{Z}$. Consider the statement: If n^3 is even, then n is even.

a Write down the contrapositive of this statement.

b Prove the contrapositive.

c Hence, prove by contradiction that $\sqrt[3]{6}$ is irrational.

4 a Show that one of three consecutive integers is always divisible by 3.

b Hence, prove that $n^3 + 2n^2 + 2n$ is divisible by 3 for all $n \in \mathbb{Z}$.

5 a Suppose that both m and n are divisible by d. Prove that $m - n$ is divisible by d.

b Hence, prove that the highest common factor of two consecutive integers is 1.

c Find the highest common factor of 1002 and 999.

6 A student claims that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$, for all $x \geq 0$ and $y \geq 0$.

a Using a counterexample, prove that the equation is not always true.

b Prove that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ if and only if $x = 0$ or $y = 0$.

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Review

7 Let $n \in \mathbb{Z}$. Prove that $n^2 + 3n + 4$ is even.
Hint: Consider the cases when n is odd and n is even.

8 Suppose that a, b, c and d are positive integers.

a Provide a counterexample to disprove the equation

$$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$$

b Now suppose that $\frac{c}{d} > \frac{a}{b}$. Prove that

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

9 Prove by mathematical induction that:

a $6^n + 4$ is divisible by 10 for all $n \in \mathbb{N}$

b $1^2 + 3^2 + \cdots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$ for all $n \in \mathbb{N}$

Multiple-choice questions

1 If m is even and n is odd, then which of these statements is true?

A $m + 2n$ is odd **B** $m + n$ is even **C** $m \times n$ is odd
D $m^2 - n^2$ is even **E** $m - 3n$ is odd

2 If m is divisible by 6 and n is divisible by 15, then which of these statements might be false?

A $m \times n$ is divisible by 90 **B** $m \times n$ is divisible by 30 **C** $m \times n$ is divisible by 15
D $m + n$ is divisible by 3 **E** $m + n$ is divisible by 15

3 The contrapositive of $P \Rightarrow Q$ is

A $Q \Rightarrow P$ **B** (not P) $\Rightarrow$ (not Q) **C** (not Q) $\Rightarrow$ (not P)
D $Q \Leftrightarrow P$ **E** (not P) $\Leftrightarrow$ (not Q)

4 The converse of $P \Rightarrow Q$ is

A (not Q) $\Rightarrow$ (not P) **B** $Q \Rightarrow P$ **C** $Q \Leftrightarrow Q$
D (not P) $\Leftrightarrow$ (not Q) **E** (not Q) $\Leftrightarrow$ (not P)

5 The number of pairs of integers (m, n) that satisfy $m + n = mn$ is

A 0 **B** 1 **C** 2 **D** 3 **E** 4

6 If a, b and c are any real numbers with $a > b$, the statement that must be true is

A $\frac{1}{a} > \frac{1}{b}$ **B** $\frac{1}{a} < \frac{1}{b}$ **C** $ac > bc$ **D** $a + c > b + c$ **E** $a^2 > b^2$

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7 If $n = (m-1)(m-2)(m-3)$ where m is an integer, then n will not always be divisible by

A 1 **B** 2 **C** 3 **D** 5 **E** 6

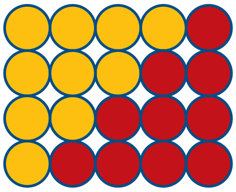
8 Let $m, n \in \mathbb{Z}$. Which of the following statements is false?

A n is even if and only if $n + 1$ is odd
B $m + n$ is odd if and only if $m - n$ is odd
C $m + n$ is even if and only if m and n are even
D m and n are odd if and only if mn is odd
E mn is even if and only if m is even or n is even

Extended-response questions

1 **a** Use the diagram on the right to deduce the equation

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \qquad (1)$$



b Using equation (1), prove that the sum $1 + 2 + \cdots + 99$ is divisible by 99.

c Using equation (1), prove that if n is odd, then the sum of any n consecutive odd natural numbers is divisible by n.

d With the help of equation (1) and mathematical induction, prove that

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2 \quad \text{for all } n \in \mathbb{N}$$

2 Define $n! = n \times (n-1) \times \cdots \times 2 \times 1$.

a Prove that $10! + 2, 10! + 3, \dots, 10! + 10$ are nine consecutive composite numbers.
Hint: The first number is divisible by 2.

b Find a sequence of ten consecutive composite numbers.

3 We call (a, b, c) a Pythagorean triple if a, b, c are natural numbers such that $a^2 + b^2 = c^2$.

a Let $n \in \mathbb{N}$. Prove that if (a, b, c) is a Pythagorean triple, then so is (na, nb, nc).

b Prove that there is only one Pythagorean triple (a, b, c) of consecutive natural numbers.

c Prove that there is no Pythagorean triple (a, b, c) containing the numbers 1 or 2.

4 Let a be an integer that is not divisible by 3. We know that $a = 3k + 1$ or $a = 3k + 2$, for some $k \in \mathbb{Z}$.

a Show that a^2 must leave a remainder of 1 when divided by 3.

b Hence, prove that if (a, b, c) is any Pythagorean triple, then a or b is divisible by 3.

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5 **a** Prove by mathematical induction that $n^2 + n$ is even for all $n \in \mathbb{N}$.

b Find an easier proof by factorising $n^2 + n$.

c Hence, prove that if n is odd, then there exists an integer k such that $n^2 = 8k + 1$.

6 Let $n \in \mathbb{Z}$ and consider the statement: If n is divisible by 8, then n^2 is divisible by 8.

a Prove the statement.

b Write down the converse of the statement.

c If the converse is true, prove it. Otherwise, give a counterexample.

7 Goldbach's conjecture is that every even integer greater than 2 can be expressed as the sum of two primes. To date, no one has been able to prove this, although it has been verified for all integers less than 4×10^{18}.

a Express 100 and 102 as the sum of two prime numbers.

b Prove that 101 cannot be written as the sum of two prime numbers.

c Express 101 as the sum of three prime numbers.

d Assuming that Goldbach's conjecture is true, prove that every odd integer greater than 5 can be written as the sum of three prime numbers.

8 **a** Simplify the expression $\dfrac{1}{n-1} - \dfrac{1}{n}$.

b Hence, show that

$$\frac{1}{2 \times 1} + \frac{1}{3 \times 2} + \frac{1}{4 \times 3} + \cdots + \frac{1}{n(n-1)} = 1 - \frac{1}{n}$$

c Give another proof of the above equation using mathematical induction.

d Using the above equation, prove that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 \quad \text{for all } n \in \mathbb{N}$$

9 **a** Let $x \geq 0$ and $y \geq 0$. Prove that

$$\frac{x+y}{2} \geq \sqrt{xy}$$

by substituting $x = a^2$ and $y = b^2$ into $\dfrac{x+y}{2} - \sqrt{xy}$.

b Using the above inequality, or otherwise, prove each of the following:

i If $a > 0$, then $a + \dfrac{1}{a} \geq 2$.

ii If a, b and c are positive real numbers, then $(a+b)(b+c)(c+a) \geq 8abc$.

iii If a, b and c are positive real numbers, then $a^2 + b^2 + c^2 \geq ab + bc + ca$.

c Take any rectangle of length x and width y. Prove that a square with the same perimeter has an area greater than or equal to that of the rectangle.

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Review

10 Exactly one of the following three people is lying. Who is the liar?

- Jay says: 'Kaye is lying.'
- Kaye say: 'Elle is lying.'
- Elle says: 'I am not lying.'

11 There are four sentences written below. Which of them is true?

- Exactly one of these statements is false.
- Exactly two of these statements are false.
- Exactly three of these statements are false.
- Exactly four of these statements are false.

12 We will say that a set of numbers can be **split** if it can be divided into two groups so that no two numbers appear in the same group as their sum. For example, the set $\{1, 2, 3, 4, 5, 6\}$ can be split into the two groups $\{1, 2, 4\}$ and $\{3, 5, 6\}$.

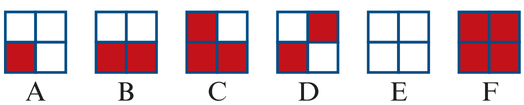
a Prove that the set $\{1, 2, \dots, 8\}$ can be split.

b Hence, explain why the set $\{1, 2, \dots, n\}$ can be split, where $1 \le n \le 8$.

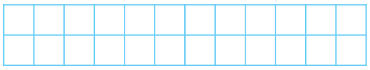
c Prove that it is impossible to split the set $\{1, 2, \dots, 9\}$.

d Hence, prove that it is impossible to split the set $\{1, 2, \dots, n\}$, where $n \ge 9$.

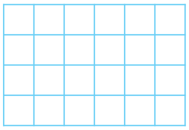
13 Consider the set of six 2×2 square tiles shown below.



a Tile the 2×12 grid shown using all six tiles, so that neighbouring squares have matching colours along the boundaries between tiles. Tiles can be rotated.



b Prove that there are only four ways to tile the 4×6 grid shown using all six tiles, so that neighbouring squares have matching colours along the boundaries between tiles. Tiles can be rotated.



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Chapter 7

7 Geometry in the plane and proof

Objectives

- To consider necessary and sufficient conditions for two lines to be **parallel**.
- To determine the **angle sum** of a polygon.
- To define **congruence** of two figures.
- To determine when two triangles are congruent.
- To write geometric proofs.
- To use **Pythagoras' theorem** and its converse.
- To apply **transformations** that are expansions from the origin.
- To define **similarity** of two figures.
- To determine when two triangles are similar.
- To determine and apply **similarity factors** for areas and volumes.
- To investigate properties of the **golden ratio**.

There are three main reasons for the study of geometry at school.

The first reason is that the properties of figures in two and three dimensions are helpful in other areas of mathematics. The second reason is that the subject provides a good setting to show how a large body of results may be deduced from a small number of assumptions. The third reason is that it gives you, the student, the opportunity to practise writing coherent, logical mathematical arguments.

In this chapter and the next, we use some of the proof techniques introduced in the previous chapter. Review of geometry from Years 9 and 10 is included, but in such a way that you can see the building of the results.

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7A Points, lines and angles

In this section we do not pretend to be fully rigorous, but aim to make you aware that assumptions are being made and that we base the proofs of the results on these assumptions. The assumptions do seem obvious to us, but there are ways of making the study of geometry even more rigorous. However, whatever we do, we will need to accept a set of results as our starting point.

▶ Points, lines and planes

We begin with a few basic concepts. No formal definitions are given.

Point In geometry, a point is used to indicate position.

Line In the physical world, we may illustrate the idea of a line as a tightly stretched wire or a fold in a piece of paper. A line has no width and is infinite in length.

Plane A plane has no thickness and it extends infinitely in all directions.

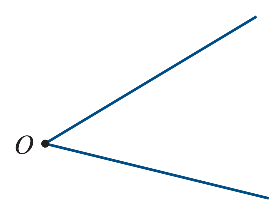
We make the following assumptions about points and lines:

- Given a point and a line, the point may or may not lie on the line.
- Two distinct points are contained in exactly one line.
- Two distinct lines do not have more than one point in common.

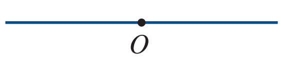
▶ Angles

A **ray** is a portion of a line consisting of a point O and all the points on one side of O.

An **angle** is the figure formed by two distinct rays which have a common endpoint O. The common endpoint is called the **vertex** of the angle.



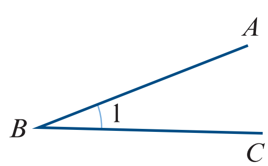
- If the two rays are part of one straight line, the angle is called a **straight angle** and measures 180°.



- A **right angle** is an angle of 90°.
- An **acute angle** is an angle which is less than 90°.
- An **obtuse angle** is an angle which is greater than 90° and less than 180°.
- **Supplementary angles** are two angles whose sum is 180°.
- **Complementary angles** are two angles whose sum is 90°.

Naming angles

The convention for naming an angle is to fully describe the rays of the angle and the endpoint where the rays meet.



The marked angle is denoted by $\angle ABC$.

When there is no chance of ambiguity, it can be written as $\angle B$.

Sometimes an angle can simply be numbered as shown, and in a proof we refer to the angle as $\angle 1$.

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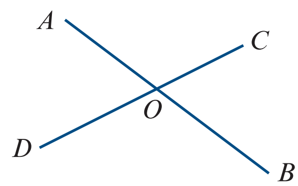
The important thing is that the writing of your argument must be clear and unambiguous. With complicated diagrams, the $\angle ABC$ notation is safest.

Theorem

If two straight lines intersect, then the opposite angles are equal in pairs.

Such angles are said to be **vertically opposite**.

Proof using angle names



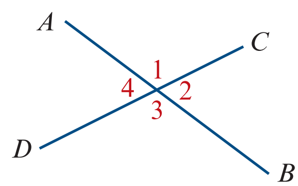
$\angle AOC$ and $\angle COB$ are supplementary.
That is, $\angle AOC + \angle COB = 180^\circ$.

Also, $\angle COB$ and $\angle BOD$ are supplementary.
That is, $\angle COB + \angle BOD = 180^\circ$.

Hence $\angle AOC = \angle BOD$.

The proof can also be presented with the labelling technique.

Proof using number labels



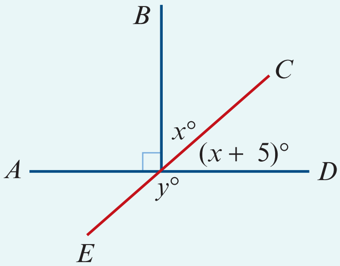
$\angle 1 + \angle 2 = 180^\circ$ (supplementary angles)

$\angle 2 + \angle 3 = 180^\circ$ (supplementary angles)

$\therefore \quad \angle 1 = \angle 3$

Example 1

Find the values of x and y in the diagram.



Solution

$x + (x + 5) = 90$ (complementary angles)

$2x = 85$

$\therefore \quad x = 42.5$

$y + (x + 5) = 180$ (supplementary angles)

$y + 47.5 = 180$

$\therefore \quad y = 132.5$

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▶ Parallel lines

Given two distinct lines ℓ_1 and ℓ_2 in the plane, either the lines intersect in a single point or the lines have no point in common. In the latter case, the lines are said to be **parallel**. We can write this as $\ell_1 \parallel \ell_2$.

Here is another important assumption.

Playfair's axiom

Given any point P not on a line ℓ, there is only one line through P parallel to ℓ.

From this we have the following results for three distinct lines ℓ_1, ℓ_2 and ℓ_3 in the plane:

- If $\ell_1 \parallel \ell_2$ and $\ell_2 \parallel \ell_3$, then $\ell_1 \parallel \ell_3$.
- If $\ell_1 \parallel \ell_2$ and ℓ_3 intersects ℓ_1, then ℓ_3 also intersects ℓ_2.

We prove the first of these and leave the other as an exercise. The proof is by contradiction.

Proof Let ℓ_1, ℓ_2 and ℓ_3 be three distinct lines in the plane such that $\ell_1 \parallel \ell_2$ and $\ell_2 \parallel \ell_3$. Now suppose that ℓ_1 is not parallel to ℓ_3. Then ℓ_1 and ℓ_3 meet at a point P. But by Playfair's axiom, there is only one line parallel to ℓ_2 passing through P. Therefore $\ell_1 = \ell_3$. But this gives a contradiction, as ℓ_1 and ℓ_3 are distinct by assumption.

Corresponding, alternate and co-interior angles

The following types of pairs of angles play an important role in considering parallel lines.

In the diagram, the lines ℓ_1 and ℓ_2 are crossed by a **transversal** ℓ_3.

Corresponding angles:

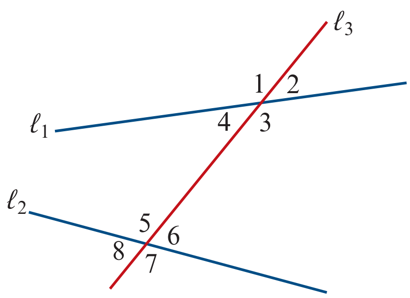
- Angles 1 and 5
- Angles 2 and 6
- Angles 3 and 7
- Angles 4 and 8

Alternate angles:

- Angles 3 and 5
- Angles 4 and 6

Co-interior angles:

- Angles 3 and 6
- Angles 4 and 5



The following result is easy to prove, and you should complete it as an exercise.

Theorem

When two lines are crossed by a transversal, any one of the following three conditions implies the other two:

- a pair of alternate angles are equal
- a pair of corresponding angles are equal
- a pair of co-interior angles are supplementary.

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The next result is important as it gives us the ability to establish properties of the angles associated with parallel lines crossed by a transversal, and it also gives us an easily applied method for proving that two lines are parallel.

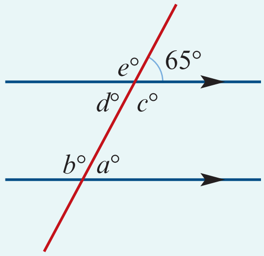
Theorem

- If two parallel lines are crossed by a transversal, then alternate angles are equal.
- Conversely, if two lines crossed by a transversal form an equal pair of alternate angles, then the two lines are parallel.

Example 2

Find the values of the pronumerals.

Note: The arrows indicate that the two lines are parallel.



Solution

$a = 65$ (corresponding)

$d = 65$ (alternate with a)

$b = 115$ (co-interior with d)

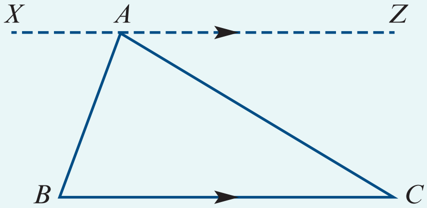
$e = 115$ (corresponding with b)

$c = 115$ (vertically opposite e)

Example 3

For $\triangle ABC$ shown in the diagram, the line XAZ is drawn through vertex A parallel to BC.

Use this construction to prove that the sum of the interior angles of a triangle is a straight angle (180°).



Solution

$\angle ABC = \angle XAB$ (alternate angles)

$\angle ACB = \angle ZAC$ (alternate angles)

$\angle XAB + \angle ZAC + \angle BAC$ is a straight angle.

Therefore $\angle ABC + \angle ACB + \angle BAC = 180°$.

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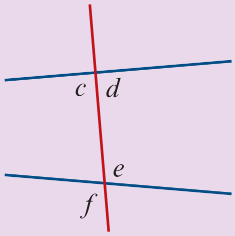
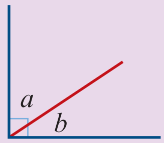
Section summary

- Pairs of angles
  - complementary (a and b)
  - supplementary (c and d)
  - vertically opposite (e and f)
  - alternate (c and e)
  - corresponding (c and f)
  - co-interior (d and e)





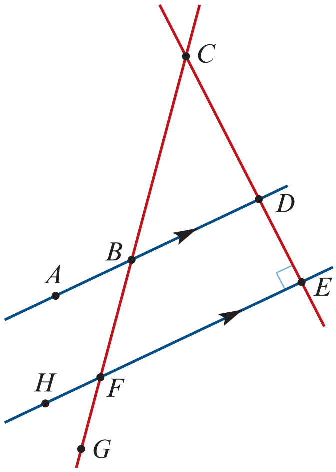
- Parallel lines

  If two parallel lines are crossed by a transversal, then:
  - alternate angles are equal
  - corresponding angles are equal
  - co-interior angles are supplementary.

  If two lines crossed by a transversal form an equal pair of alternate angles, then the two lines are parallel.

Exercise 7A

1 Consider the diagram shown.



a State whether each of the following angles is acute, obtuse, right or straight:

i $\angle ABC$ **ii** $\angle HFE$ **iii** $\angle CBD$ **iv** $\angle FED$

b State which angle is:

i corresponding to $\angle ABC$

ii alternate to $\angle ABF$

iii vertically opposite $\angle BFE$

iv co-interior to $\angle DBF$

c State which angles are:

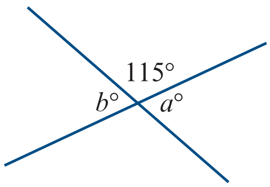
i complementary to $\angle BCD$

ii supplementary to $\angle CBD$

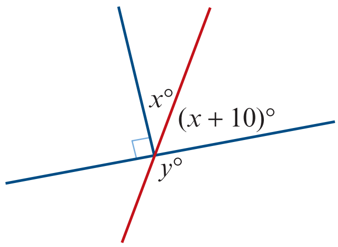
Example 1, 2

2 Calculate the values of the unknowns for each of the following. Give reasons.

a



b



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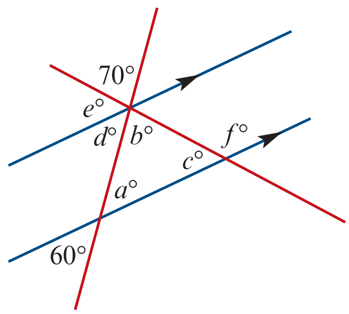
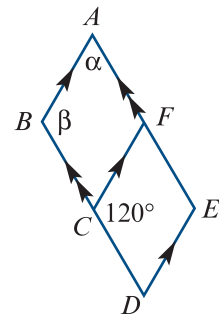
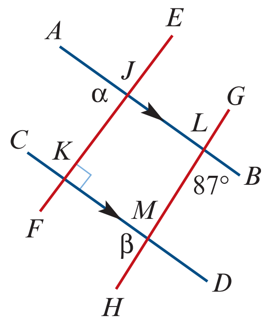
c



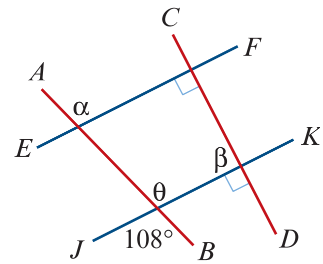
d



e

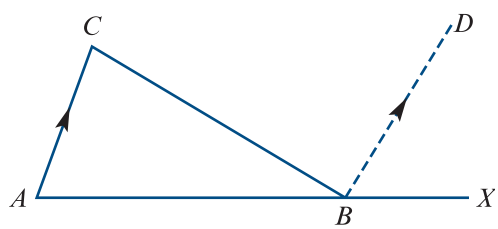


f

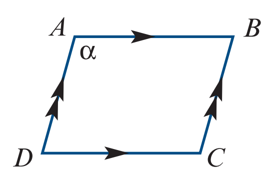


Example 3

3 Side AB of $\triangle ABC$ is extended to point X and line BD is drawn parallel to side AC.
Prove that the sum of two interior angles of a triangle is equal to the opposite exterior angle.
Hint: Using the diagram, this means showing that $\angle CAB + \angle ACB = \angle CBX$.



4 Recall that a parallelogram is a quadrilateral whose opposite sides are parallel. A parallelogram $ABCD$ is shown on the right. Let $\angle A = \alpha$.

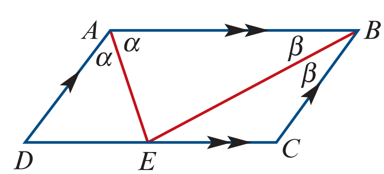


a Find the sizes of $\angle B$ and $\angle D$ in terms of α.

b Hence find the size of $\angle C$ in terms of α.

5 Prove the converse of the result in Question 4. That is, prove that if the opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram.

6 Prove that AE is perpendicular to EB.



7 The lines PQ and RS are parallel. A transversal meets PQ at X and RS at Y. Lines XA and YB are bisectors of the angles PXY and XYS. Prove that XA is parallel to YB.

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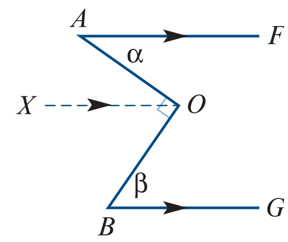
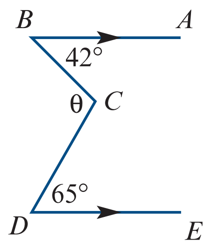
8 For the diagram on the right, show that $\alpha + \beta = 90°$.

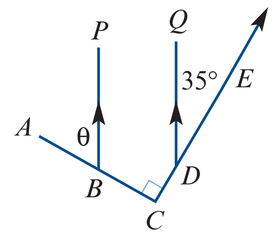


9 For each of the following, use a construction line to find the angle marked θ:

a



b



7B Triangles and polygons

We first define polygons.

A **line segment** AB is a portion of a line consisting of two distinct points A and B and all the points between them.

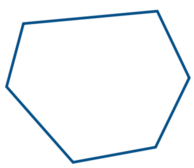
If distinct points $A_1, A_2, \dots, A_n$ in the plane are connected in order by the line segments $A_1A_2, A_2A_3, \dots, A_nA_1$, then the figure formed is a **polygon**. The points $A_1, A_2, \dots, A_n$ are the vertices of the polygon, and the line segments $A_1A_2, A_2A_3, \dots, A_nA_1$ are its sides.

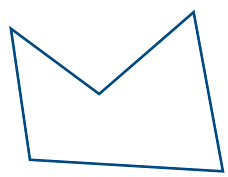
Types of polygons

A **simple polygon** is a polygon such that no two sides have a point in common except a vertex.

A **convex polygon** is a polygon that contains each line segment connecting any pair of points on its boundary.

For example, the left-hand figure is convex, while the right-hand figure is not.


A convex polygon


A non-convex polygon

Note: In this chapter we will always assume that the polygons being considered are convex.

A **regular polygon** is a polygon in which all the angles are equal and all the sides are equal.

Names of polygons

- triangle (3 sides)
- quadrilateral (4 sides)
- pentagon (5 sides)
- hexagon (6 sides)
- heptagon (7 sides)
- octagon (8 sides)
- nonagon (9 sides)
- decagon (10 sides)
- dodecagon (12 sides)

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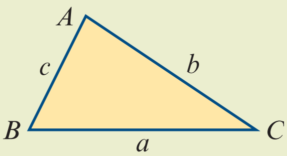
► Triangles

A **triangle** is a figure formed by three line segments determined by a set of three points not on one line. If the three points are A, B and C, then the figure is called triangle ABC and commonly written $\triangle ABC$. The points A, B and C are called the vertices of the triangle.

Triangle inequality

An important property of a triangle is that any side is shorter than the sum of the other two.

In $\triangle ABC$: $a < b + c$, $b < c + a$ and $c < a + b$.



Note: For $\triangle ABC$ labelled as shown, we have $c < b < a$ if and only if $\angle C < \angle B < \angle A$.

The following two results have been proved in Example 3 and in Question 3 of Exercise 7A.

Angles of a triangle

- The sum of the three interior angles of a triangle is 180°.
- The sum of two interior angles of a triangle is equal to the opposite exterior angle.

Classification of triangles

Equilateral triangle	a triangle in which all three sides are equal
Isosceles triangle	a triangle in which two sides are equal
Scalene triangle	a triangle in which all three sides are unequal

Important lines in a triangle

Median A **median** of a triangle is a line segment from a vertex to the midpoint of the opposite side.

Altitude An **altitude** of a triangle is a line segment from a vertex to the opposite side (possibly extended) which forms a right angle where it meets the opposite side.

Example 4

The sides of a triangle are $6 - x$, $4x + 1$ and $2x + 3$. Find the value of x for which the triangle is isosceles, and show that if it is isosceles, then it is equilateral.

Solution

$$6 - x = 4x + 1$$
$$\Rightarrow \quad 5x = 5$$
$$\Rightarrow \quad x = 1$$

When $x = 1$, we have $6 - x = 5$, $4x + 1 = 5$ and $2x + 3 = 5$. Hence the triangle is equilateral with each side of length 5 units.

Explanation

We want to show that if any two side lengths are equal, then the third length is the same.

It is enough to show that the three lines $y = 6 - x$, $y = 4x + 1$ and $y = 2x + 3$ intersect in a common point.

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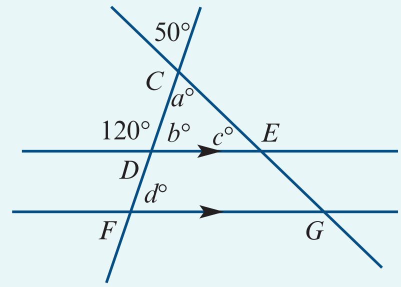
Example 5

Find the values of a, b, c and d, giving reasons.



Solution

$a = 50$ (vertically oposite angles)

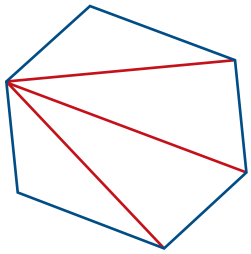
$b = 60$ (supplementary angles)

$c = 180 - (50 + 60) = 70$ (angle sum of a triangle)

$d = 60$ (corresponding angles $DE \parallel FG$)

▶ Angle sum of a polygon

If a polygon has n sides, then we can draw $n - 3$ diagonals from a vertex. In this way, we can divide the polygon into $n - 2$ triangles, each with an angle sum of 180°.



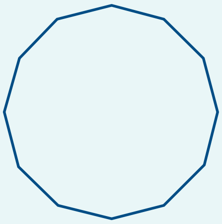
We have drawn a hexagon to illustrate this, but we could have used any polygon.

Angle sum of a polygon

- The sum of the interior angles of an n-sided polygon is $(n - 2)180°$.
- Each interior angle of a regular n-sided polygon has size $\dfrac{(n-2)}{n}180°$.

Example 6

A regular dodecagon is shown to the right.



a Find the sum of the interior angles of a dodecagon.

b Find the size of each interior angle of a regular dodecagon.

Solution

a The angle sum of a polygon with n sides is $(n - 2)180°$.
Therefore the angle sum of a dodecagon is 1800°.

b Each of the interior angles is $\dfrac{1800}{12} = 150°$.

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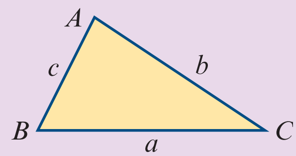
Section summary

- Polygons
  - The sum of the interior angles of an n-sided polygon is $(n-2)180°$.
  - In a **regular polygon**, all the angles are equal and all the sides are equal.
    Each interior angle of a regular n-sided polygon has size $\dfrac{(n-2)}{n}180°$.
- Triangles
  - An **equilateral triangle** is a triangle in which all three sides are equal.
  - An **isosceles triangle** is a triangle in which two sides are equal.
  - A **scalene triangle** is a triangle in which all three sides are unequal.
  - The sum of the three interior angles of a triangle is 180°.
  - The sum of two interior angles of a triangle is equal to the opposite exterior angle.

  In $\triangle ABC$:
  - $a < b + c$, $b < c + a$ and $c < a + b$
  - $c < b < a$ if and only if $\angle C < \angle B < \angle A$



Exercise 7B

1 Is it possible for a triangle to have sides of lengths:

a 12 cm, 9 cm, 20 cm
b 24 cm, 24 cm, 40 cm
c 5 cm, 5 cm, 5 cm
d 12 cm, 9 cm, 2 cm?

2 Describe each of the triangles in Question 1.

3 If a triangle has sides 10 cm and 20 cm, what can be said about the third side?

4 The sides of a triangle are $2n - 1$, $n + 5$ and $3n - 8$.

a Find the value(s) of n for which the triangle is isosceles.
b Is there a value of n which makes the triangle equilateral?

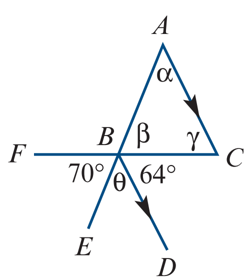
Example 4

5 The sides of a triangle are $2n - 1$, $n + 7$ and $3n - 9$. Prove that if the triangle is isosceles, then it is equilateral.

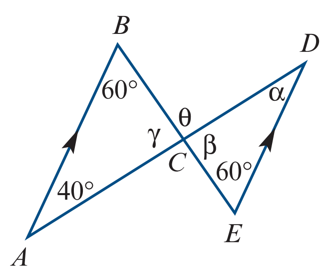
Example 5

6 Calculate the value of the unknowns for each of the following. Give reasons.

a



b



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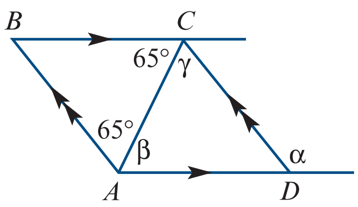
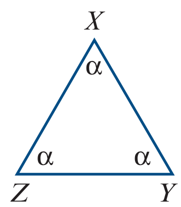
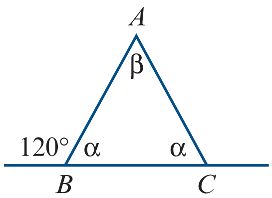
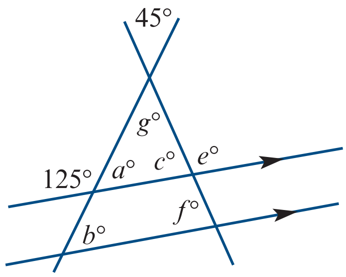
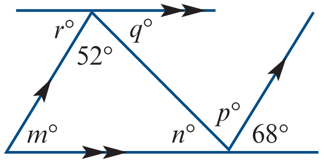
c 

d 

e 

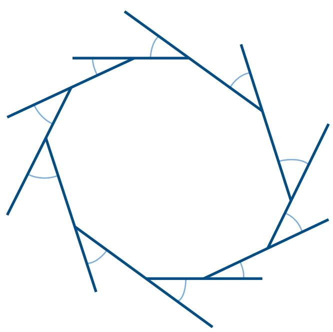
f 

g 

Example 6 **7** Find the interior-angle sum and the size of each angle of a regular polygon with:

a 6 sides **b** 12 sides **c** 20 sides

8 In the decagon shown on the right, each side has been extended to form an exterior angle.



a Explain why the sum of the interior angles plus the sum of the exterior angles is 1800°.

b Hence find the sum of the decagon's 10 exterior angles.

9 Prove that the sum of the exterior angles of any polygon is 360°.

10 If the sum of the interior angles of a polygon is four times the sum of the exterior angles, how many sides does the polygon have?

11 Assume that the sum of the interior angles of a polygon is k times the sum of the exterior angles (where $k \in \mathbb{N}$). Prove that the polygon has $2(k + 1)$ sides.

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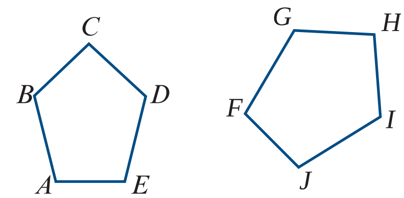
7C Congruence and proofs

Two plane figures are called **congruent** if one figure can be moved on top of the other figure, by a sequence of translations, rotations and reflections, so that they coincide exactly.

Congruent figures have exactly the same shape and size. For example, the two figures shown are congruent. We can write:

pentagon $ABCDE \equiv$ pentagon $FGHIJ$



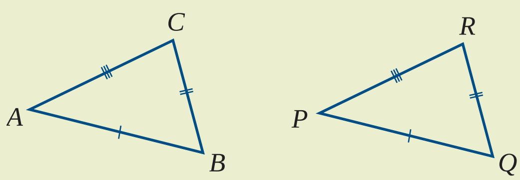
When two figures are congruent, we can find a transformation that pairs up every part of one figure with the corresponding part of the other, so that:

- paired angles have the same size
- paired line segments have the same length
- paired regions have the same area.

Congruent triangles

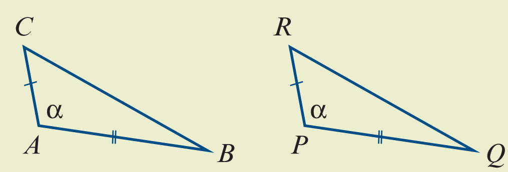
There are four standard tests for two triangles to be congruent.

- The SSS congruence test
  If the three sides of one triangle are respectively equal to the three sides of another triangle, then the two triangles are congruent.



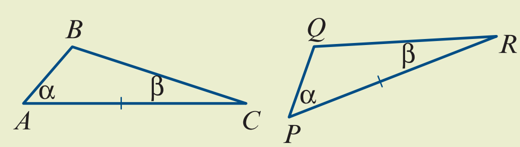
$\triangle ABC \equiv \triangle PQR$

- The SAS congruence test
  If two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another triangle, then the two triangles are congruent.



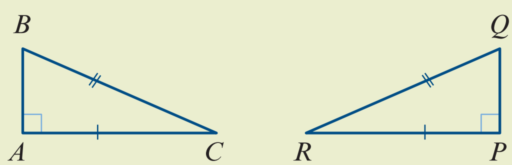
$\triangle ABC \equiv \triangle PQR$

- The AAS congruence test
  If two angles and one side of one triangle are respectively equal to two angles and the matching side of another triangle, then the two triangles are congruent.



$\triangle ABC \equiv \triangle PQR$

- The RHS congruence test
  If the hypotenuse and one side of one right-angled triangle are respectively equal to the hypotenuse and one side of another right-angled triangle, then the two triangles are congruent.



$\triangle ABC \equiv \triangle PQR$

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Classification of quadrilaterals

Trapezium	a quadrilateral with at least one pair of opposite sides parallel
Parallelogram	a quadrilateral with both pairs of opposite sides parallel
Rhombus	a parallelogram with a pair of adjacent sides equal
Rectangle	a quadrilateral in which all angles are right angles
Square	a quadrilateral that is both a rectangle and a rhombus

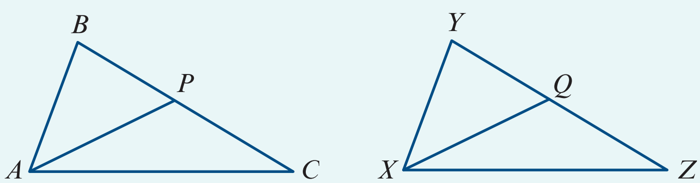
Proofs using congruence

Example 7

Let $\triangle ABC$ and $\triangle XYZ$ be such that $\angle BAC = \angle YXZ$, $AB = XY$ and $AC = XZ$.

If P and Q are the midpoints of BC and YZ respectively, prove that $AP = XQ$.

Solution



From the given conditions, we have $\triangle ABC \equiv \triangle XYZ$ (SAS).

Therefore $\angle ABP = \angle XYQ$ and $BC = YZ$.

Thus $BP = YQ$, as P and Q are the midpoints of BC and YZ respectively.

Hence $\triangle ABP \equiv \triangle XYQ$ (SAS) and so $AP = XQ$.

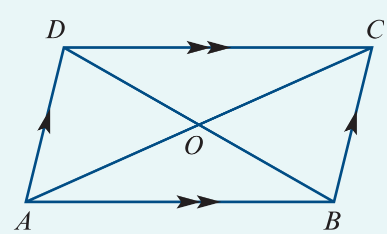
Example 8

a Prove that, in a parallelogram, the diagonals bisect each other.

b Prove that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

Solution

a Note that opposite sides of a parallelogram are equal. (See Question 8 of Exercise 7C.)



In triangles DOC and BOA:

$\angle ODC = \angle OBA$	(alternate angles $CD \parallel AB$)
$\angle OCD = \angle OAB$	(alternate angles $CD \parallel AB$)
$\angle AOB = \angle DOC$	(vertically opposite)
$AB = CD$	(opposite sides of parallelogram are equal)
$\triangle DOC \equiv \triangle BOA$	(AAS)

Hence $AO = OC$ and $DO = OB$.

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b

$OD = OB$ (diagonals bisect each other)

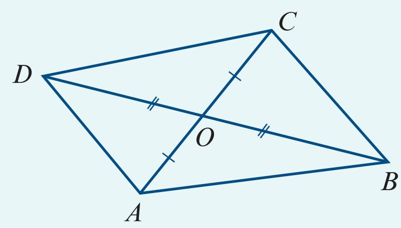
$OA = OC$ (diagonals bisect each other)

$\angle AOB = \angle DOC$ (vertically opposite)

$\angle DOA = \angle COB$ (vertically opposite)

$\triangle DOC \equiv \triangle BOA$ (SAS)

$\triangle DOA \equiv \triangle BOC$ (SAS)



Therefore $\angle ODC = \angle OBA$ and so $CD \parallel AB$, since alternate angles are equal. Similarly, we have $AD \parallel BC$. Hence $ABCD$ is a parallelogram.

Example 9

Prove that the triangle formed by joining the midpoints of the three sides of an isosceles triangle (with the midpoints as the vertices of the new triangle) is also isosceles.

Solution

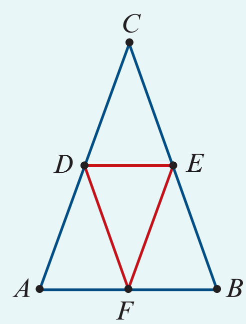
Assume $\triangle ABC$ is isosceles with $CA = CB$ and $\angle CAB = \angle CBA$. (See Question 3 of Exercise 7C.)

Then we have $DA = EB$, where D and E are the midpoints of CA and CB respectively.

We also have $AF = BF$, where F is the midpoint of AB.

Therefore $\triangle DAF \equiv \triangle EBF$ (SAS).

Hence $DF = EF$ and so $\triangle DEF$ is isosceles.



Section summary

- **Congruent figures** have exactly the same shape and size.
- If triangle ABC is congruent to triangle XYZ, this can be written as $\triangle ABC \equiv \triangle XYZ$.
- Two triangles are congruent provided any one of the following four conditions holds:
  - **SSS** the three sides of one triangle are equal to the three sides of the other triangle
  - **SAS** two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle
  - **AAS** two angles and one side of one triangle are equal to two angles and the matching side of the other triangle
  - **RHS** the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and one side of another right-angled triangle.

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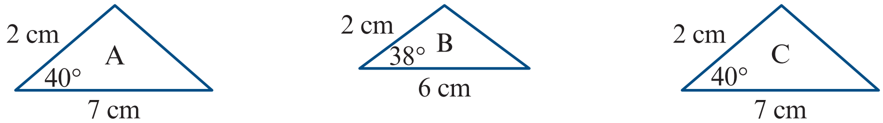
Exercise 7C

1 In each part, find pairs of congruent triangles. State the congruence tests used.

a



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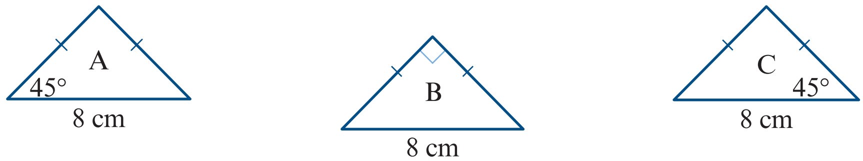

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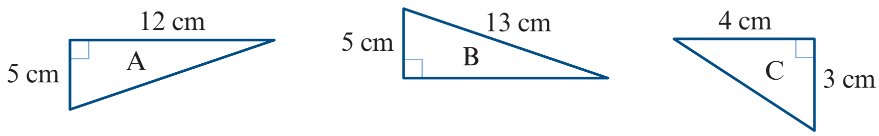
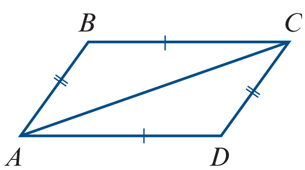


2 Name the congruent triangles and state the congruence test used:

a



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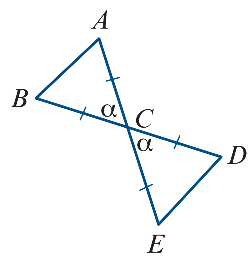

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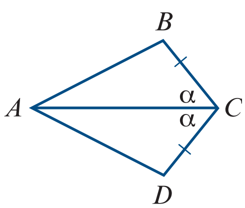

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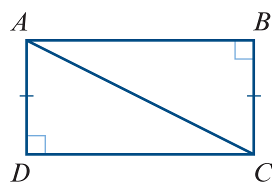

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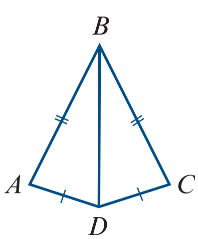

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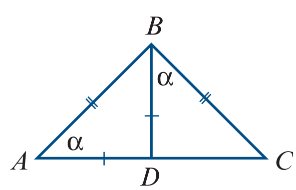

Example 7

3 Prove that if $\triangle ABC$ is isosceles with $AB = AC$, then $\angle ABC = \angle ACB$.

4 Prove that if $\triangle ABC$ is such that $\angle ABC = \angle ACB$, then $\triangle ABC$ is isosceles. (This is the converse of Question 3.)

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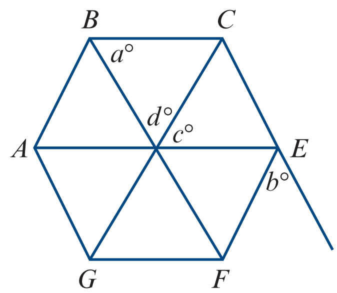
5 For the quadrilateral shown, prove that $AB \parallel CD$.

6 $ABCEFG$ is a regular hexagon.

a Find the values of a, b, c and d.

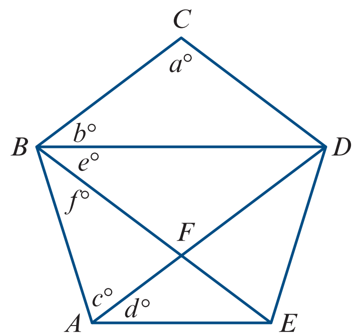
b Prove that $AE \parallel BC$ and $CG \parallel BA$.



7 $ABCDE$ is a regular pentagon.

a Find the values of a, b, c, d, e and f.

b Prove that $AE \parallel BD$ and $BE \parallel CD$.



Example 8

8 Proofs involving parallelograms Prove each of the following:

a In a parallelogram, opposite sides are equal and opposite angles are equal.

b If each side of a quadrilateral is equal to the opposite side, then the quadrilateral is a parallelogram.

c If each angle of a quadrilateral is equal to the opposite angle, then the quadrilateral is a parallelogram.

d If one side of a quadrilateral is equal and parallel to the opposite side, then the quadrilateral is a parallelogram.

9 Let $ABCD$ be a parallelogram and let P and Q be the midpoints of AB and DC respectively. Prove that $APCQ$ is a parallelogram.

10 Let $PQRS$ be a parallelogram whose diagonals meet at O. Let X, Y, Z and W be the midpoints of PO, QO, RO and SO respectively. Prove that $XYZW$ is a parallelogram.

11 Proofs involving rhombuses Prove each of the following:

a The diagonals of a rhombus bisect each other at right angles.

b The diagonals of a rhombus bisect the vertex angles through which they pass.

c If the diagonals of a quadrilateral bisect each other at right angles, then the quadrilateral is a rhombus.

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12 Proofs involving rectangles Prove each of the following:

a The diagonals of a rectangle are equal and bisect each other.

b A parallelogram with one right angle is a rectangle.

c If the diagonals of a quadrilateral are equal and bisect each other, then the quadrilateral is a rectangle.

Example 9 **13** $ABCDE$ is a pentagon in which all the sides are equal and diagonal AC is equal to diagonal AD. Prove that $\angle ABC = \angle AED$.

14 $\triangle ABC$ is equilateral and its sides are extended to points X, Y and Z so that AY, BZ and CX are all equal in length to the sides of $\triangle ABC$. Prove that $\triangle XYZ$ is also equilateral.

15 $ABCD$ is a quadrilateral in which $AB = BC$ and $AD = DC$. The diagonal BD is extended to a point K. Prove that $AK = CK$.

16 Prove that if the angle C of a triangle ABC is equal to the sum of the other two angles, then the length of side AB is equal to twice the length of the line segment joining C with the midpoint of AB.

17 Prove that if NO is the base of isosceles triangle MNO and if the perpendicular from N to MO meets MO at A, then angle ANO is equal to half of angle NMO.

18 If a median of a triangle is drawn, prove that the perpendiculars from the other vertices upon this median are equal. (The median may be extended.)

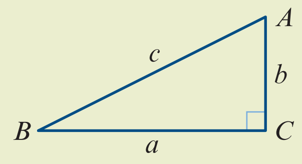
7D Pythagoras' theorem

Pythagoras' theorem

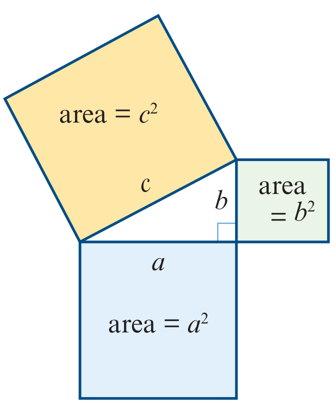
Let ABC be a triangle with side lengths a, b and c.

If $\angle C$ is a right angle, then

$$a^2 + b^2 = c^2$$



Pythagoras' theorem can be illustrated by the diagram shown here. The sum of the areas of the two smaller squares is equal to the area of the square on the longest side (hypotenuse).



There are many different proofs of Pythagoras' theorem. One was given at the start of Chapter 6. Here we give another proof, due to James A. Garfield, the 20th President of the United States.

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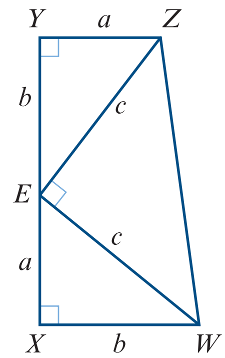
Proof The proof is based on the diagram shown on the right.



Area of trapezium $XYZW = \frac{1}{2}(a+b)(a+b)$

Area of $\triangle EWX$ + area of $\triangle EYZ$ + area of $\triangle EWZ$

$$= \tfrac{1}{2}ab + \tfrac{1}{2}ab + \tfrac{1}{2}c^2$$
$$= ab + \tfrac{1}{2}c^2$$

Thus $\quad \frac{1}{2}(a+b)(a+b) = ab + \frac{1}{2}c^2$

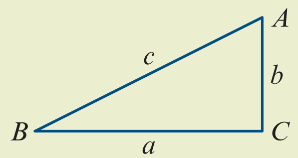
$$a^2 + 2ab + b^2 = 2ab + c^2$$

Hence $\quad a^2 + b^2 = c^2$

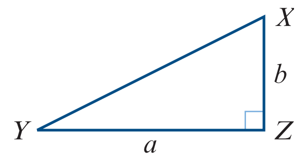
Converse of Pythagoras' theorem

Let ABC be a triangle with side lengths a, b and c.

If $a^2 + b^2 = c^2$, then $\angle C$ is a right angle.



Proof Assume $\triangle ABC$ has side lengths $a = BC$, $b = CA$ and $c = AB$ such that $a^2 + b^2 = c^2$.



Construct a second triangle $\triangle XYZ$ with $YZ = a$ and $ZX = b$ such that $\angle XZY$ is a right angle.

By Pythagoras' theorem, the length of the hypotenuse of $\triangle XYZ$ is

$$\sqrt{a^2 + b^2} = \sqrt{c^2}$$
$$= c$$

Therefore $\triangle ABC \equiv \triangle XYZ$ (SSS).

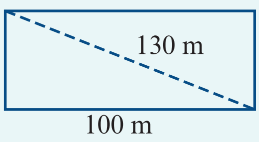
Hence $\angle C$ is a right angle.

Example 10

The diagonal of a soccer field is 130 m and the length of the long side of the field is 100 m. Find the length of the short side, correct to the nearest centimetre.

Solution

Let x be the length of the short side. Then



$$x^2 + 100^2 = 130^2$$
$$x^2 = 130^2 - 100^2$$
$$\therefore \quad x = \sqrt{6900}$$

Correct to the nearest centimetre, the length of the short side is 83.07 m.

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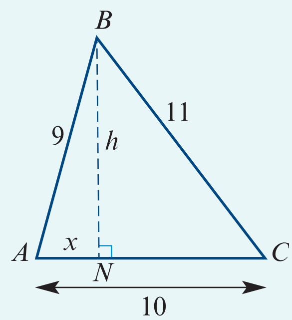
Example 11

Consider $\triangle ABC$ with $AB = 9$ cm, $BC = 11$ cm and $AC = 10$ cm. Find the length of the altitude of $\triangle ABC$ on AC.

Solution



Let BN be the altitude on AC as shown, with $BN = h$ cm.

Let $AN = x$ cm. Then $CN = (10 - x)$ cm.

In $\triangle ABN$: $\quad x^2 + h^2 = 81 \quad (1)$

In $\triangle CBN$: $\quad (10 - x)^2 + h^2 = 121 \quad (2)$

Expanding in equation (2) gives

$$100 - 20x + x^2 + h^2 = 121$$

Substituting for $x^2 + h^2$ from (1) gives

$$100 - 20x + 81 = 121$$

$$\therefore \quad x = 3$$

Substituting in (1), we have

$$9 + h^2 = 81$$

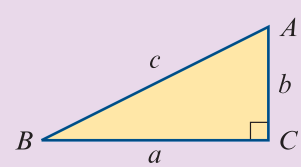
$$h^2 = 72$$

$$\therefore \quad h = 6\sqrt{2}$$

The length of altitude BN is $6\sqrt{2}$ cm.

Section summary

Pythagoras' theorem and its converse



Let ABC be a triangle with side lengths a, b and c.

- If $\angle C$ is a right angle, then $a^2 + b^2 = c^2$.
- If $a^2 + b^2 = c^2$, then $\angle C$ is a right angle.

Exercise 7D

1. An 18 m ladder is 7 m away from the bottom of a vertical wall. How far up the wall does it reach?

Example 10

2. Find the length of the diagonal of a rectangle with dimensions 40 m by 9 m.

3. In a circle of centre O, a chord AB is of length 4 cm. The radius of the circle is 14 cm. Find the distance of the chord from O.

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4 A square has an area of 169 cm^2. What is the length of the diagonal?

5 Find the area of a square with a diagonal of length:

a 10 cm

b 8 cm

6 $ABCD$ is a square of side length 2 cm. If E is a point on AB extended and $CA = CE$, find the length of DE.

7 In a square of side length 2 cm, the midpoints of each side are joined to form a new square. Find the area of the new square.

Example 11

8 Consider $\triangle ABC$ with $AB = 7$ cm, $BC = 6$ cm and $AC = 5$ cm. Find the length of AN, the altitude on BC.

9 Which of the following are the three side lengths of a right-angled triangle?

a 5 cm, 6 cm, 7 cm

b 3.9 cm, 3.6 cm, 1.5 cm

c 2.4 cm, 2.4 cm, 4 cm

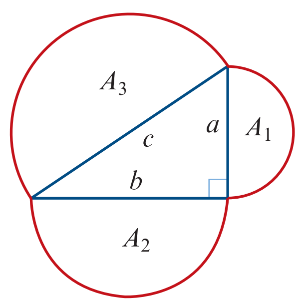
d 82 cm, 18 cm, 80 cm

10 Prove that a triangle with sides lengths $x^2 - 1$, $2x$ and $x^2 + 1$ is a right-angled triangle.

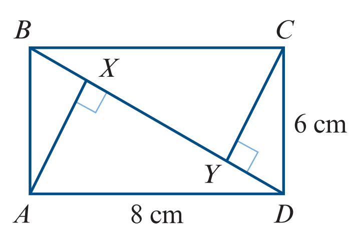
11 Consider $\triangle ABC$ such that $AB = 20$ cm, $AC = 15$ cm and the altitude AN has length 12 cm. Prove that $\triangle ABC$ is a right-angled triangle.

12 Find the length of an altitude in an equilateral triangle with side length 16 cm.

13 Three semicircles are drawn on the sides of this right-angled triangle. Let A_1, A_2 and A_3 be the areas of these semicircles. Prove that $A_3 = A_1 + A_2$.



14 Rectangle $ABCD$ has $CD = 6$ cm and $AD = 8$ cm. Line segments CY and AX are drawn such that points X and Y lie on BD and $\angle AXD = \angle CYD = 90°$. Find the length of XY.



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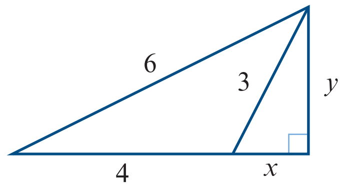
15 Find the values of x and y.



16 If P is a point in rectangle $ABCD$ such that $PA = 3$ cm, $PB = 4$ cm and $PC = 5$ cm, find the length of PD.

17 Let AQ be an altitude of $\triangle ABC$, where Q lies between B and C. Let P be the midpoint of BC. Prove that $AB^2 + AC^2 = 2PB^2 + 2AP^2$.

18 For a parallelogram $ABCD$, prove that $2AB^2 + 2BC^2 = AC^2 + BD^2$.

7E Ratios

This section is revision of work of previous years.

Example 12

Divide 300 in the ratio 3 : 2.

Solution

$$\begin{aligned} \text{one part} &= 300 \div 5 = 60 \\ \therefore \quad \text{two parts} &= 60 \times 2 = 120 \\ \therefore \quad \text{three parts} &= 60 \times 3 = 180 \end{aligned}$$

Example 13

Divide 3000 in the ratio 3 : 2 : 1.

Solution

$$\begin{aligned} \text{one part} &= 3000 \div 6 = 500 \\ \therefore \quad \text{two parts} &= 500 \times 2 = 1000 \\ \therefore \quad \text{three parts} &= 500 \times 3 = 1500 \end{aligned}$$

Exercise 7E

Skillsheet

Example 12

Example 13

1 Divide 9000 in the ratio 2 : 7.

2 Divide 15 000 in the ratio 2 : 2 : 1.

3 Given that $x : 6 = 9 : 15$, find x.

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4 The ratio of the numbers of orange flowers to pink flowers in a garden is 6 : 11. There are 144 orange flowers. How many pink flowers are there?

5 Given that $15 : 2 = x : 3$, find x.

6 The angles of a triangle are in the ratio 6 : 5 : 7. Find the sizes of the three angles.

7 Three men X, Y and Z share an amount of money in the ratio 2 : 3 : 7. If Y receives \$2 more than X, how much does Z receive?

8 An alloy consists of copper, zinc and tin in the ratio 1 : 3 : 4 (by weight). If there is 10 g of copper in the alloy, find the weights of zinc and tin.

9 The ratio of red beads to white beads to green beads in a bag is 7 : 2 : 1. If there are 56 red beads, how many white beads and how many green beads are there?

10 On a map, the length of a road is represented by 45 mm. If the scale is 1 : 125 000, find the actual length of the road.

11 Five thousand two hundred dollars was divided between a mother and daughter in the ratio 8 : 5. Find the difference between the sums they received.

12 Points A, B, C and D are placed in that order on a line so that $AB = 2BC = CD$. Express BD as a fraction of AD.

13 If the radius of a circle is increased by two units, find the ratio of the new circumference to the new diameter.

14 In a class of 30 students, the ratio of boys to girls is 2 : 3. If six boys join the class, find the new ratio of boys to girls in the class.

15 If $a : b = 3 : 4$ and $a : (b + c) = 2 : 5$, find the ratio $a : c$.

16 The scale of a map is 1 : 250 000. Find the distance, in kilometres, between two towns which are 3.5 cm apart on the map.

17 Prove that if $\dfrac{a - c}{b - d} = \dfrac{c}{d}$, then $\dfrac{a}{b} = \dfrac{c}{d}$.

18 Prove that if $\dfrac{a}{x} = \dfrac{b}{y} = \dfrac{c}{z} = \dfrac{2}{3}$, then $\dfrac{a + b + c}{x + y + z} = \dfrac{2}{3}$.

19 Prove that if $\dfrac{x}{y} = \dfrac{m}{n}$, then $\dfrac{x + y}{x - y} = \dfrac{m + n}{m - n}$.

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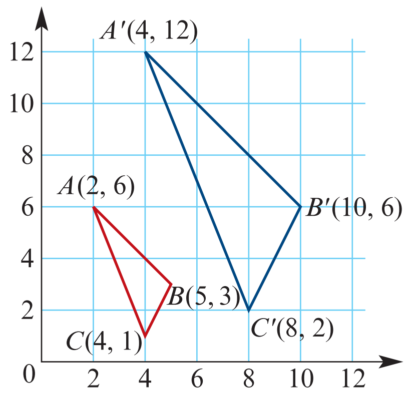
7F An introduction to similarity

The two triangles ABC and $A'B'C'$ shown in the diagram are similar.



Note: $OA' = 2OA$, $OB' = 2OB$, $OC' = 2OC$

Triangle $A'B'C'$ can be considered as the image of triangle ABC under a mapping of the plane in which the coordinates are multiplied by 2.

This mapping is called an **expansion** from the origin of factor 2. From now on we will call this factor the **similarity factor**.

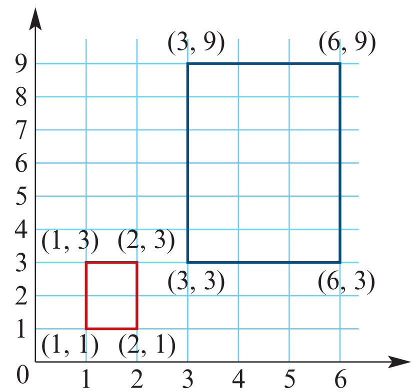
The rule for this mapping can be written in transformation notation as $(x, y) \to (2x, 2y)$.

There is also a mapping from $\triangle A'B'C'$ to $\triangle ABC$, which is an expansion from the origin of factor $\frac{1}{2}$. The rule for this is $(x, y) \to (\frac{1}{2}x, \frac{1}{2}y)$.

Two figures are called **similar** if we can enlarge one figure so that its enlargement is congruent to the other figure.

- Matching lengths of similar figures are in the same ratio.
- Matching angles of similar figures are equal.

For example, the rectangle with side lengths 1 and 2 is similar to the rectangle with side lengths 3 and 6.



Here the similarity factor is 3 and the rule for the mapping is $(x, y) \to (3x, 3y)$.

Notes:

- Any two circles are similar.
- Any two squares are similar.
- Any two equilateral triangles are similar.

Similar triangles

If triangle ABC is similar to triangle $A'B'C'$, we can write this as

$$\triangle ABC \sim \triangle A'B'C'$$

The triangles are named so that angles of equal magnitude hold the same position. That is, A matches to A', B matches to B' and C matches to C'. So we have

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA} = k$$

where k is the **similarity factor**.

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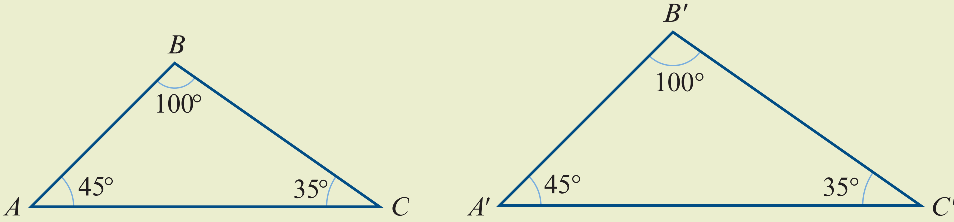
There are four standard tests for two triangles to be similar.

- **The AAA similarity test**

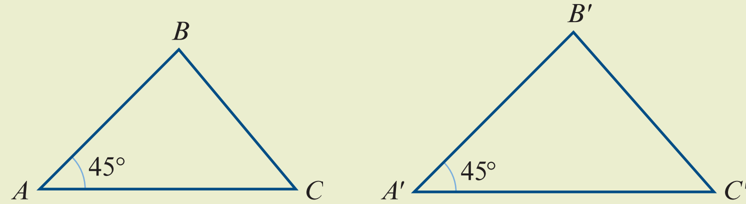
  If two angles of one triangle are respectively equal to two angles of another triangle, then the two triangles are similar.



- **The SAS similarity test**

  If the ratios of two pairs of matching sides are equal and the included angles are equal, then the two triangles are similar.

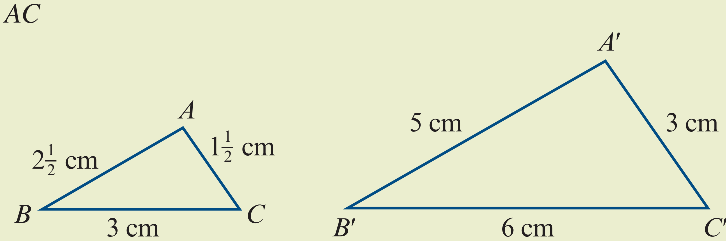
$$\frac{A'B'}{AB} = \frac{A'C'}{AC}$$



- **The SSS similarity test**

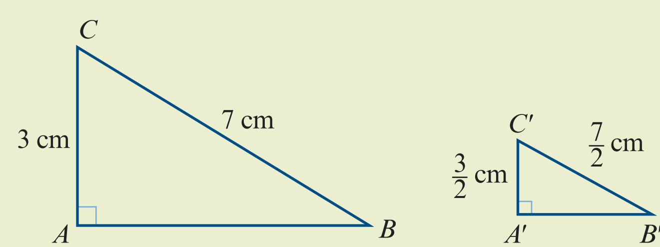
  If the sides of one triangle can be matched up with the sides of another triangle so that the ratio of matching lengths is constant, then the two triangles are similar.

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{A'C'}{AC}$$



- **The RHS similarity test**

  If the ratio of the hypotenuses of two right-angled triangles equals the ratio of another pair of sides, then the two triangles are similar.



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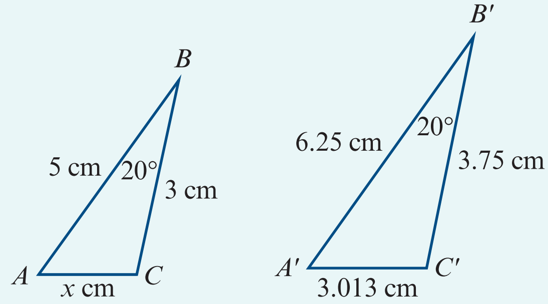
Example 14

a Give the reason for triangle ABC being similar to triangle $A'B'C'$.

b Find the value of x.



Solution

a Triangle ABC is similar to triangle $A'B'C'$ by SAS, since

$$\frac{5}{6.25} = 0.8 = \frac{3}{3.75}$$

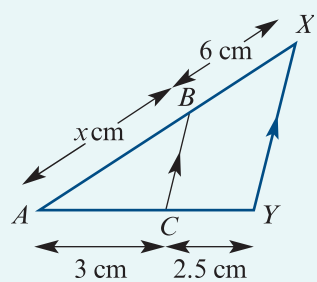
and $\angle ABC = 20^\circ = \angle A'B'C$

b
$$\frac{x}{3.013} = \frac{5}{6.25}$$
$$\therefore \quad x = \frac{5}{6.25} \times 3.013$$
$$= 2.4104$$

Example 15

a Give the reason for triangle ABC being similar to triangle AXY.

b Find the value of x.



Solution

a Corresponding angles are of equal magnitude (AAA).

b
$$\frac{AB}{AX} = \frac{AC}{AY}$$
$$\frac{x}{x+6} = \frac{3}{5.5}$$
$$5.5x = 3(x+6)$$
$$2.5x = 18$$
$$\therefore \quad x = 7.2$$

Section summary

- Two figures are **similar** if we can enlarge one figure so that its enlargement is congruent to the other figure.
  - Matching lengths of similar figures are in the same ratio.
  - Matching angles of similar figures are equal.

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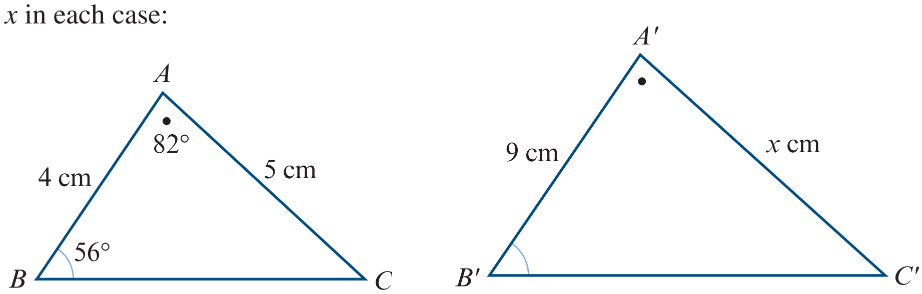
- Two triangles are similar provided any one of the following four conditions holds:
  - **AAA** two angles of one triangle are equal to two angles of the other triangle
  - **SAS** the ratios of two pairs of matching sides are equal and the included angles are equal
  - **SSS** the sides of one triangle can be matched up with the sides of the other triangle so that the ratio of matching lengths is constant
  - **RHS** the ratio of the hypotenuses of two right-angled triangles equals the ratio of another pair of sides.

Exercise 7F

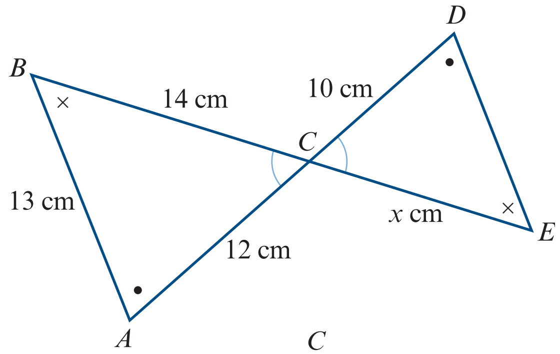
Example 14

1 Give reasons why each of the following pairs of triangles are similar and find the value of x in each case:

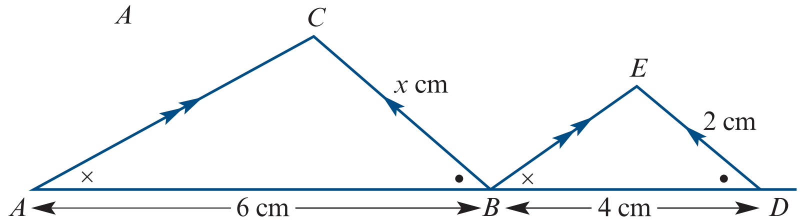
a



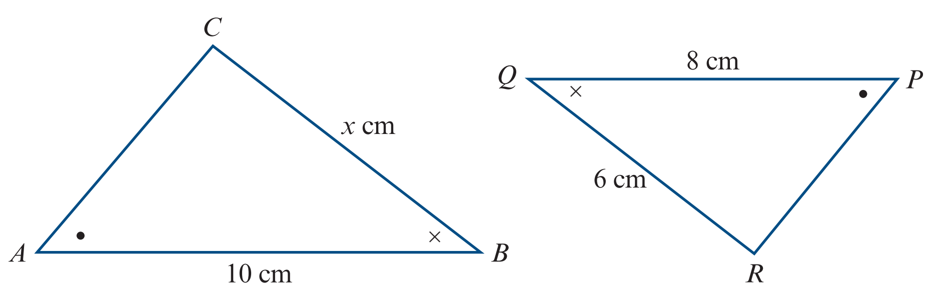
b



c



d



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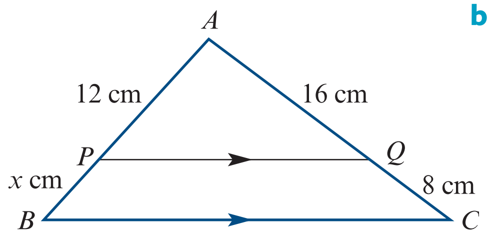
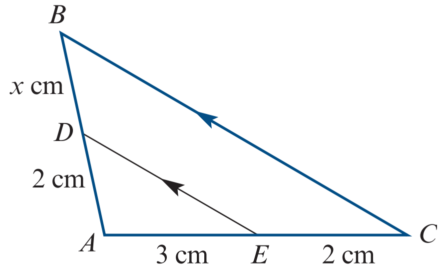
Example 15

2 Give reasons why each of the following pairs of triangles are similar and find the value of x in each case:

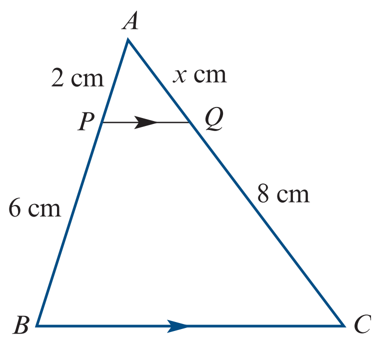
a



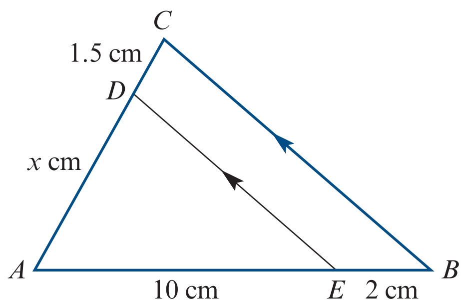
b



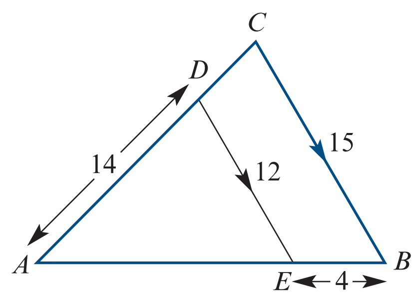
c



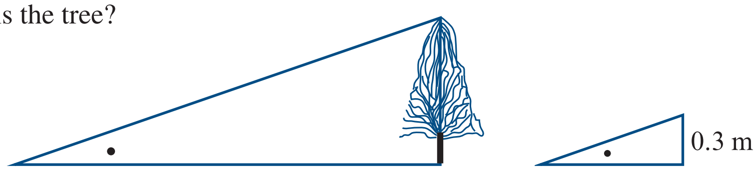
d



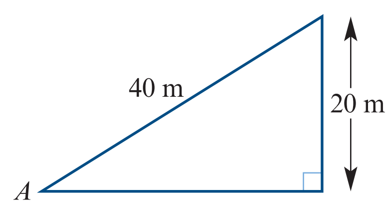
3 Given that $AD = 14$, $DE = 12$, $BC = 15$ and $EB = 4$, find AC, AE and AB.



4 A tree casts a shadow of 33 m and at the same time a stick 30 cm long casts a shadow of 224 cm. How tall is the tree?



5 A 20 m high neon sign is supported by a 40 m steel cable as shown. An ant crawls along the cable starting at A. How high is the ant when it is 15 m from A?



6 A hill has a gradient of 1 in 20, i.e. for every 20 m horizontally there is a 1 m increase in height. If you go 300 m horizontally, how high up will you be?

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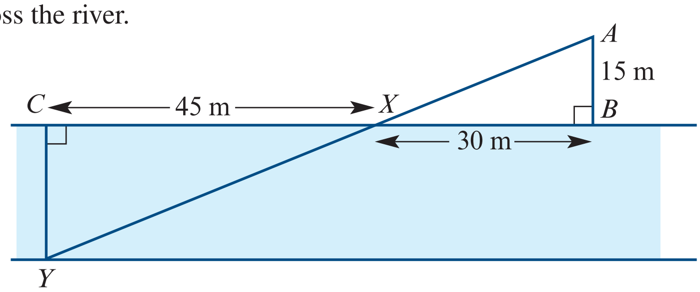
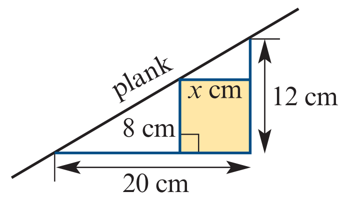
7 A man stands at A and looks at point Y across the river. He gets a friend to place a stone at X so that the three points A, X and Y are collinear (that is, they all lie on a single line). He then measures AB, BX and XC to be 15 m, 30 m and 45 m respectively. Find CY, the distance across the river.

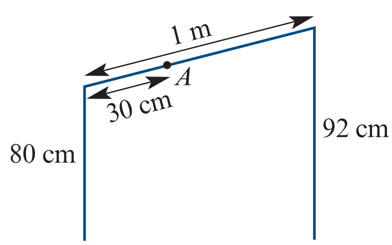


8 Find the height, h m, of a tree that casts a shadow 32 m long at the same time that a vertical straight stick 2 m long casts a shadow 6.2 m long.

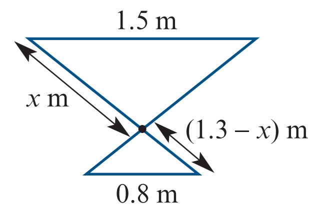
9 A plank is placed straight up stairs that are 20 cm wide and 12 cm deep. Find x, where x cm is the width of the widest rectangular box of height 8 cm that can be placed on a stair under the plank.



10 The sloping edge of a technical drawing table is 1 m from front to back. Calculate the height above the ground of a point A, which is 30 cm from the front edge.



11 Two similar rods 1.3 m long have to be hinged together to support a table 1.5 m wide. The rods have been fixed to the floor 0.8 m apart. Find the position of the hinge by finding the value of x.



12 A man whose eyes are 1.7 m from the ground, when standing 3.5 m in front of a wall 3 m high, can just see the top of a tower that is 100 m away from the wall. Find the height of the tower.

13 A man is 8 m up a 10 m ladder, the top of which leans against a vertical wall and touches it at a height of 9 m above the ground. Find the height of the man above the ground.

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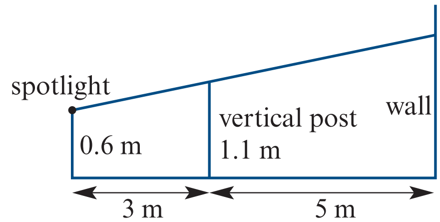
14 A spotlight is at a height of 0.6 m above ground level. A vertical post 1.1 m high stands 3 m away and 5 m further away there is a vertical wall. How high up the wall does the shadow reach?



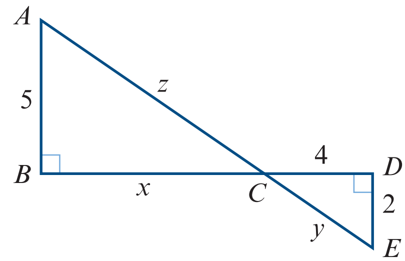
15 Consider the diagram on the right.

a Prove that $\triangle ABC \sim \triangle EDC$.

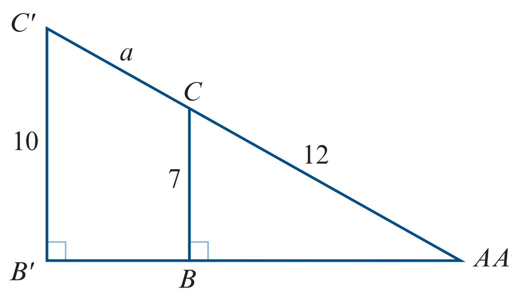
b Find x.

c Use Pythagoras' theorem to find y and z.

d Verify that $y : z = ED : AB$.

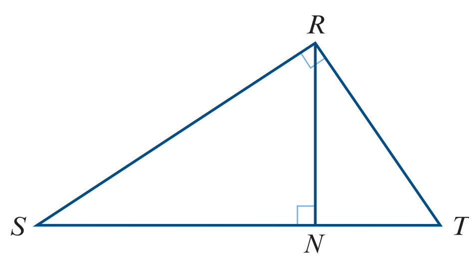


16 Find a.



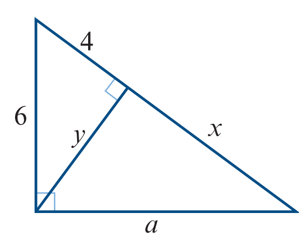
17 A man who is 1.8 m tall casts a shadow of 0.76 m in length. If at the same time a telephone pole casts a 3 m shadow, find the height of the pole.

18 In the diagram shown, $RT = 4$ cm and $ST = 10$ cm. Find the length NT.



19 ABC is a triangular frame with $AB = 14$ m, $BC = 10$ m and $CA = 7$ m. A point P on AB, 1.5 m from A, is linked by a rod to a point Q on AC, 3 m from A. Calculate the length PQ.

20 Using this diagram, find a, x and y.



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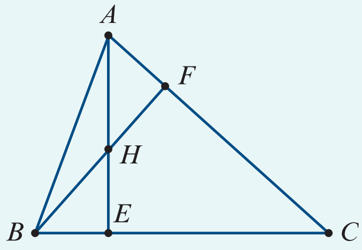
7G Proofs involving similarity

Example 16

The altitudes AE and BF of $\triangle ABC$ intersect at H.
Prove that

$$\frac{AE}{BF} = \frac{AC}{BC}$$



Solution

$\angle CEA = \angle CFB$ (AE and BF are altitudes)

$\angle ACE = \angle BCF$ (common)

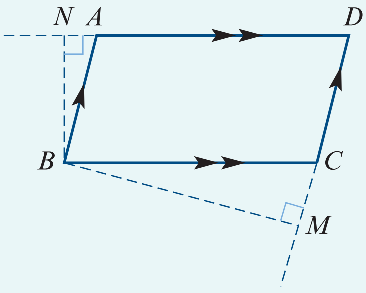
$\therefore$ $\triangle CAE \sim \triangle CBF$ (AAA)

$\therefore$ $\dfrac{AE}{BF} = \dfrac{AC}{BC}$

Example 17

$ABCD$ is a parallelogram with $\angle ABC$ acute.
BM is perpendicular to DC extended, and BN is perpendicular to DA extended. Prove that

$$DC \times CM = DA \times AN$$



Solution

$\angle BCM = \angle ABC$ (alternate angles $AB \parallel CD$)

$\angle BAN = \angle ABC$ (alternate angles $BC \parallel AD$)

$\therefore$ $\angle BCM = \angle BAN$

$\angle BNA = \angle BMC = 90^\circ$ (given)

$\therefore$ $\triangle BCM \sim \triangle BAN$

Hence $\dfrac{CM}{AN} = \dfrac{BC}{AB}$

But $AB = CD$ and $BC = AD$, giving

$$\frac{CM}{AN} = \frac{AD}{CD}$$

Hence $DC \times CM = DA \times AN$.

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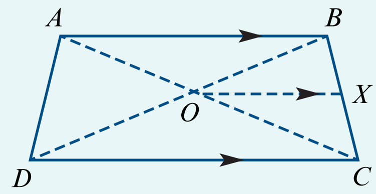
Example 18

$ABCD$ is a trapezium with diagonals intersecting at O. A line through O, parallel to the base CD, meets BC at X. Prove that $BX \times DC = XC \times AB$.

Solution



$\triangle ABC \sim \triangle OXC$ (AAA)

$\triangle DCB \sim \triangle OXB$ (AAA)

Thus $\dfrac{BX}{BC} = \dfrac{OX}{DC}$ (1)

and $\dfrac{XC}{BC} = \dfrac{OX}{AB}$ (2)

Divide (1) by (2):

$$\frac{BX}{XC} = \frac{AB}{DC}$$

$\therefore$ $BX \times DC = XC \times AB$

Exercise 7G

Skillsheet
Example 16

1 Let M be the midpoint of a line segment AB. Assume that AXB and MYB are equilateral triangles on opposite sides of AB and that XY cuts AB at Z. Prove that $\triangle AXZ \sim \triangle BYZ$ and hence prove that $AZ = 2ZB$.

Example 17

2 $ABCD$ is a rectangle. Assume that P, Q and R are points on AB, BC and CD respectively such that $\angle PQR$ is a right angle. Prove that $BQ \times QC = PB \times CR$.

3 **a** AC is a diagonal of a regular pentagon $ABCDE$. Find the sizes of $\angle BAC$ and $\angle CAE$.

b AC, AD and BD are diagonals of a regular pentagon $ABCDE$, with AC and BD meeting at X. Prove that $(AB)^2 = BX \times BD$.

4 $\triangle ABC$ has a right angle at A, and AD is the altitude to BC.

a Prove that $AD \times BC = AB \times AC$.

b Prove that $(DA)^2 = DB \times DC$.

c Prove that $(BA)^2 = BD \times BC$.

Example 18

5 $ABCD$ is a trapezium with AB one of the parallel sides. The diagonals meet at O. OX is the perpendicular from O to AB, and XO extended meets CD at Y.

Prove that $\dfrac{OX}{OY} = \dfrac{OA}{OC} = \dfrac{AB}{CD}$.

6 P is the point on side AB of $\triangle ABC$ such that $AP : AB = 1 : 3$, and Q is the point on BC such that $CQ : CB = 1 : 3$. The line segments AQ and CP intersect at X. Prove that $AX : AQ = 3 : 5$.

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7 P and Q are points on sides AB and AC respectively of $\triangle ABC$ such that $PQ \parallel BC$. The median AD meets PQ at M. Prove that $PM = MQ$.

8 $ABCD$ is a straight line and $AB = BC = CD$. An equilateral triangle $\triangle BCP$ is drawn with base BC. Prove that $(AP)^2 = AB \times AD$.

9 $ABCD$ is a quadrilateral such that $\angle BAD = \angle DBC$ and $\dfrac{DA}{AB} = \dfrac{DB}{BC}$. Prove that DB bisects $\angle ADC$.

10 $\triangle ABC$ has a right angle at C. The bisector of $\angle BCA$ meets AB at D, and DE is the perpendicular from D to AC. Prove that $\dfrac{1}{BC} + \dfrac{1}{AC} = \dfrac{1}{DE}$.

11 Proportions in a right-angled triangle

a Prove that, for a right-angled triangle, the altitude on its hypotenuse forms two triangles which are similar to the original triangle, and hence to each other.

b Prove Pythagoras' theorem by using part **a** (or by using similar triangles directly).

7H Areas, volumes and similarity

In this section we look at the areas of similar shapes and the volumes of similar solids.

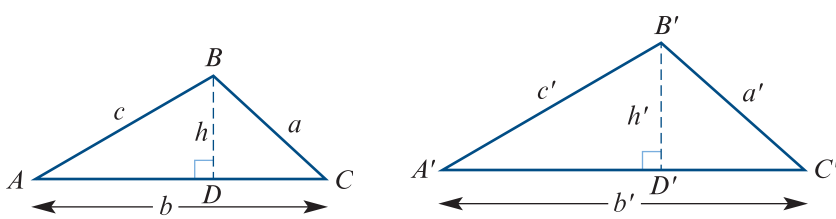
▶ Similarity and area

If two shapes are similar and the similarity factor is k (that is, if for any length AB of one shape, the corresponding length $A'B'$ of the similar shape is kAB), then

area of similar shape $= k^2 \times$ area of original shape

For example, if triangles ABC and $A'B'C'$ are similar with $A'B' = kAB$, then

area of $\triangle A'B'C' = k^2 \times$ area of $\triangle ABC$



This can be shown by observing that, since $\triangle ABC \sim \triangle A'B'C'$, we have

$$
\begin{aligned}
\text{area of } \triangle A'B'C' &= \frac{1}{2} b'h' \\
&= \frac{1}{2}(kb)(kh) \\
&= k^2\left(\frac{1}{2} bh\right) \\
&= k^2 \times \text{area of } \triangle ABC
\end{aligned}
$$

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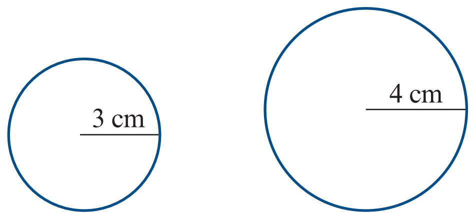
Here are some more examples of similar shapes and the ratio of their areas.

Similar circles

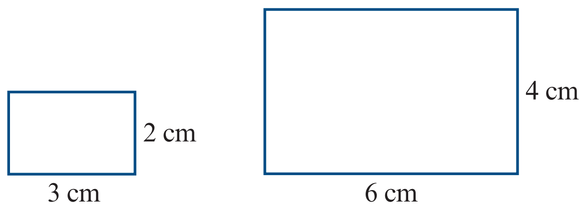


Area = 9π cm^2 Area = 16π cm^2

Similarity factor $= \frac{4}{3}$

Ratio of areas $= \frac{16\pi}{9\pi} = \left(\frac{4}{3}\right)^2$

Similar rectangles

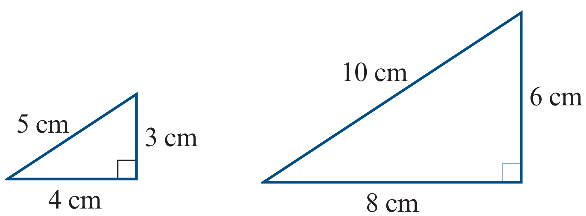


Area = 6 cm^2 Area = 24 cm^2

Similarity factor = 2

Ratio of areas $= \frac{24}{6} = 4 = 2^2$

Similar triangles



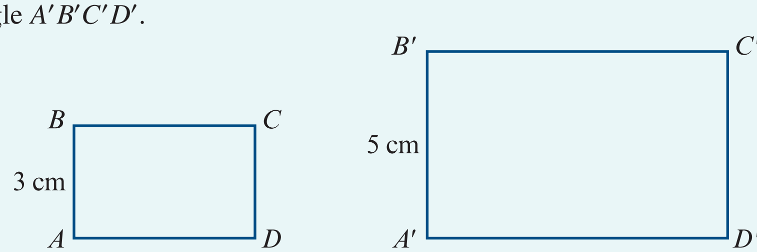
Area = 6 cm^2 Area = 24 cm^2

Similarity factor = 2

Ratio of areas $= \frac{24}{6} = 4 = 2^2$

Example 19

The two rectangles shown below are similar. The area of rectangle $ABCD$ is 20 cm^2. Find the area of rectangle $A'B'C'D'$.



Solution

The ratio of their side lengths is $\frac{A'B'}{AB} = \frac{5}{3}$.

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The ratio of their areas is $\dfrac{\text{Area of } A'B'C'D'}{\text{Area of } ABCD} = \left(\dfrac{5}{3}\right)^2 = \dfrac{25}{9}$.

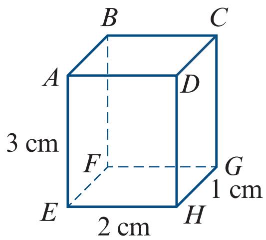
$\therefore$ Area of $A'B'C'D' = \dfrac{25}{9} \times 20$

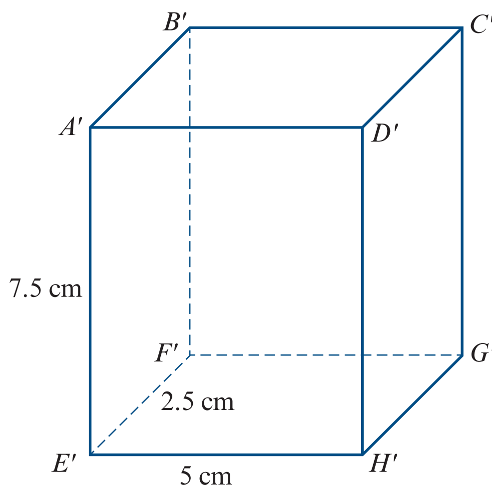
$= 55\frac{5}{9}$ cm^2

Similarity and volume

Two solids are considered to be similar if they have the same shape and the ratios of their corresponding linear dimensions are equal.

For example, the two cuboids $ABCDEFGH$ and $A'B'C'D'E'F'G'H'$ shown are similar, with similarity factor 2.5.





If two solids are similar and the similarity factor is k, then

volume of similar solid = k^3 × volume of original solid

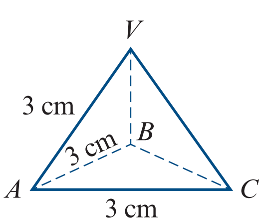
For example, for the two cuboids shown, we have

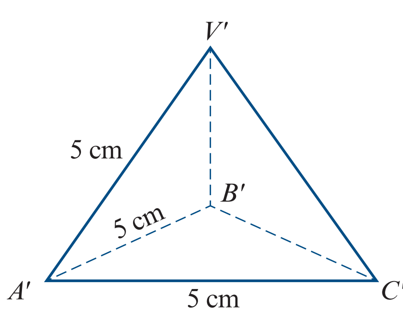
Volume of $ABCDEFGH = 2 \times 1 \times 3 = 6$ cm^3

Volume of $A'B'C'D'E'F'G'H' = 5 \times 2.5 \times 7.5 = 93.75$ cm^3

$\therefore$ Ratio of volumes $= \dfrac{93.75}{6} = 15.625 = 2.5^3$

Here is another example:





Similarity factor $= \dfrac{5}{3}$

Ratio of volumes $= \left(\dfrac{5}{3}\right)^3$

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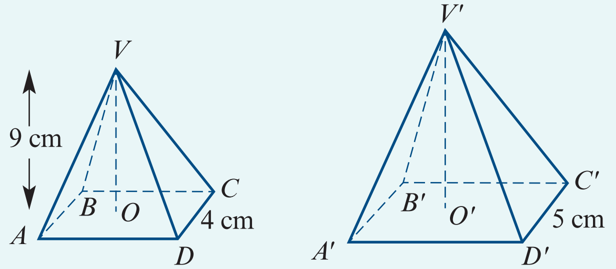
Example 20

The two square pyramids are similar and $VO = 9$ cm.

a Find the ratio of the lengths of their bases, and hence find the height $V'O'$ of pyramid $V'A'B'C'D'$.



b The volume of $VABCD$ is 48 cm^3. Find the ratio of their volumes, and hence find the volume of $V'A'B'C'D'$.

Solution

a The ratio of the length of their bases is

$$\frac{C'D'}{CD} = \frac{5}{4}$$

$$\therefore \quad V'O' = \frac{5}{4} \times 9 = 11.25 \text{ cm}$$

b The ratio of their volumes is

$$\frac{\text{Volume of } V'A'B'C'D'}{\text{Volume of } VABCD} = \left(\frac{5}{4}\right)^3 = \frac{125}{64}$$

$$\therefore \quad \text{Volume of } V'A'B'C'D' = \frac{125}{64} \times 48 = 93.75 \text{ cm}^3$$

Section summary

- If two shapes are similar and the similarity factor is k (that is, if for any length AB of one shape, the corresponding length $A'B'$ of the similar shape is kAB), then

  area of similar shape $= k^2 \times$ area of original shape

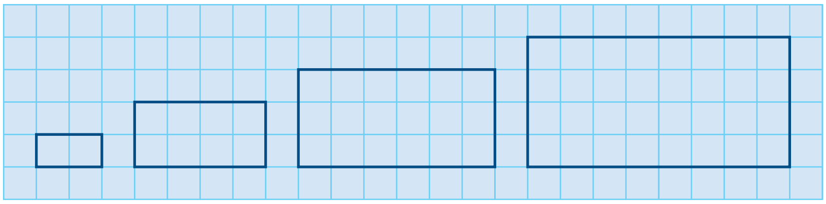
- If two solids are similar and the similarity factor is k, then

  volume of similar solid $= k^3 \times$ volume of original solid

Exercise 7H

Skillsheet

1 These four rectangles are similar:



a Write down the ratio of the lengths of their bases.

b By counting rectangles, write down the ratio of their areas.

c Is there a relationship between these two ratios?

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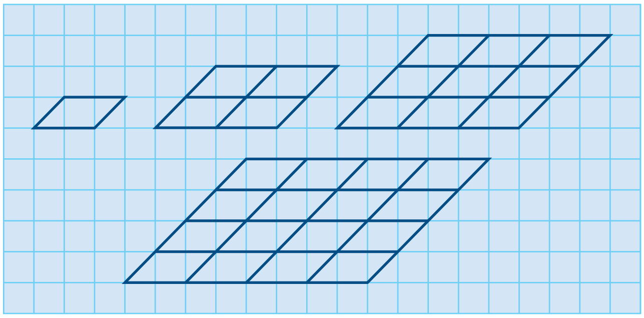
2 These four parallelograms are similar:



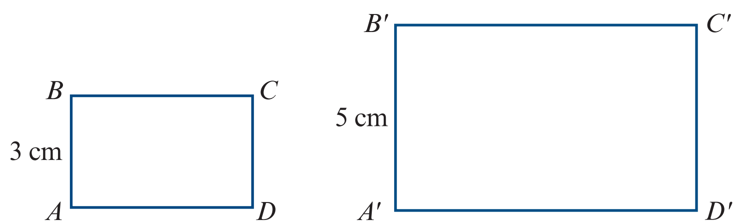
a Write down the ratio of the lengths of their bases.

b By counting parallelograms, write down the ratio of their areas.

c Is there a relationship between these two ratios?

Example 19

3 The two rectangles shown are similar. The area of rectangle $ABCD$ is 7 cm^2. Find the area of rectangle $A'B'C'D'$.



4 Triangle ABC is similar to triangle XYZ with

$$\frac{XY}{AB} = \frac{YZ}{BC} = \frac{ZX}{CA} = 2.1$$

The area of triangle XYZ is 20 cm^2. Find the area of triangle ABC.

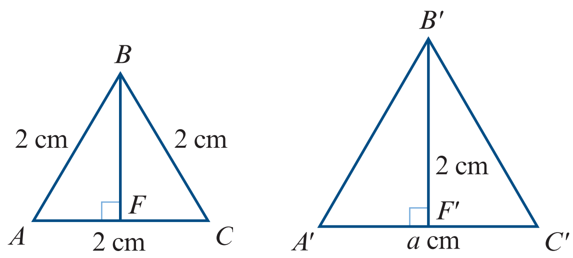
5 Triangles ABC and $A'B'C'$ are equilateral triangles.

a Find the length of BF.

b Find a.

c Find the ratio

$$\frac{\text{Area of } \triangle A'B'C'}{\text{Area of } \triangle ABC}$$



6 The areas of two similar triangles are 16 and 25. What is the ratio of a pair of corresponding sides?

7 The areas of two similar triangles are 144 and 81. If the base of the larger triangle is 30, what is the corresponding base of the smaller triangle?

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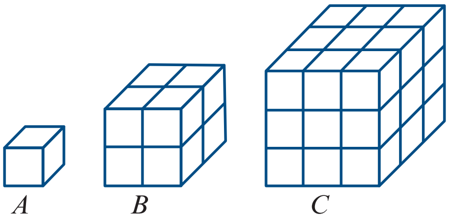
8 These three solids are similar.

a Write down the ratio of the lengths of the bases.

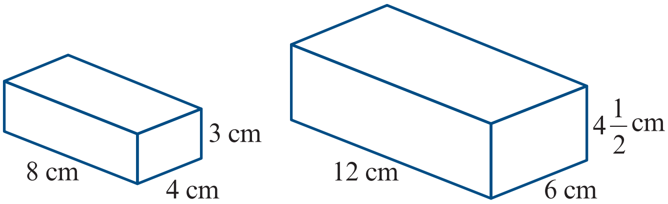
b Write down the ratio of the lengths of the heights.

c By counting cuboids equal in shape and size to cuboid A, write down the ratio of the volumes.

d Is there a relationship between the answers to **a**, **b** and **c**?



9 These are two similar rectangular blocks.



a Write down the ratio of their:

i longest edges **ii** depths **iii** heights.

b By counting cubes of side length 1 cm, write down the ratio of their volumes.

c Is there any relationship between the ratios in **a** and **b**?

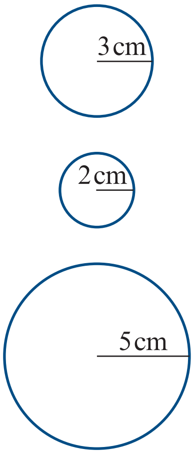
Example 20 **10** These three solids are spheres.

a Write down the ratio of the radii of the three spheres.

b The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.

Express the volume of each sphere as a multiple of π.
Hence write down the ratio of their volumes.

c Is there any relationship between the ratios found in **a** and **b**?



In each of Questions 11–20, the objects are mathematically similar.

11 The sides of two cubes are in the ratio 2 : 1. What is the ratio of their volumes?

12 The radii of two spheres are in the ratio 3 : 4. What is the ratio of their volumes?

13 Two regular tetrahedrons have volumes in the ratio 8 : 27. What is the ratio of their sides?

14 Two right cones have volumes in the ratio 64 : 27. What is the ratio of:

a their heights **b** their base radii?

15 Two similar bottles are such that one is twice as high as the other. What is the ratio of:

a their surface areas **b** their capacities?

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16 Each linear dimension of a model car is $\frac{1}{10}$ of the corresponding car dimension. Find the ratio of:

a the areas of their windscreens
b the capacities of their boots
c the widths of the cars
d the number of wheels they have.

17 Three similar jugs have heights 8 cm, 12 cm and 16 cm. If the smallest jug holds $\frac{1}{2}$ litre, find the capacities of the other two.

18 Three similar drinking glasses have heights 7.5 cm, 9 cm and 10.5 cm. If the tallest glass holds 343 millilitres, find the capacities of the other two.

19 A toy manufacturer produces model cars which are similar in every way to the actual cars. If the ratio of the door area of the model to the door area of the car is 1 : 2500, find:

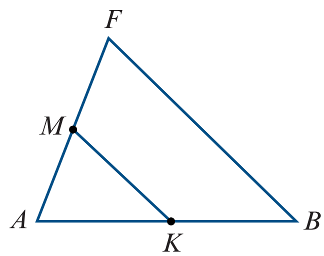
a the ratio of their lengths
b the ratio of the capacities of their petrol tanks
c the width of the model, if the actual car is 150 cm wide
d the area of the rear window of the actual car if the area of the rear window of the model is 3 cm^2.

20 The ratio of the areas of two similar labels on two similar jars of coffee is 144 : 169. Find the ratio of:

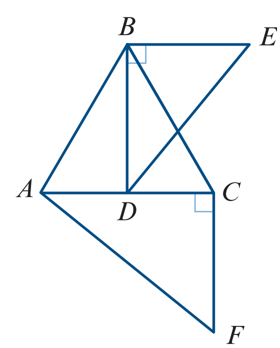
a the heights of the two jars
b their capacities.

21 **a** In the figure, if M is the midpoint of AF and K is the midpoint of AB, then how many times larger is the area of $\triangle ABF$ than the area of $\triangle AKM$?

b If the area of $\triangle ABF$ is 15, find the area of $\triangle AKM$.



22 In the diagram, $\triangle ABC$ is equilateral, $\angle BDE = \angle CAF$ and D is the midpoint of AC. Find the ratio of the area of $\triangle BDE$ to the area of $\triangle ACF$.



23 The areas of two similar triangles are 144 cm^2 and 81 cm^2. If the length of one side of the first triangle is 6 cm, what is the length of the corresponding side of the second?

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7I The golden ratio

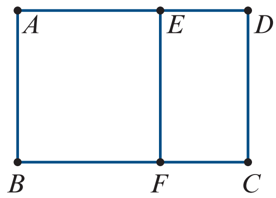
The golden ratio is the irrational number

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.618\,033\,988\ldots$$

This number is mentioned in the books of Euclid, where it is used in the construction of a regular pentagon. (See Extended-response question 2.) The golden ratio also arises naturally in connection with the sequence of Fibonacci numbers.

▶ Golden rectangles

A **golden rectangle** is a rectangle that can be cut up into a square and a rectangle that is similar to the original one.



Let $ABCD$ be a rectangle with $AB < BC$.

Then there is a point E on AD and a point F on BC such that $ABFE$ is a square. We say that $ABCD$ is a golden rectangle if $FCDE$ is similar to $ABCD$.

Theorem

All golden rectangles are similar, with ratio of length to width given by

$$\frac{1+\sqrt{5}}{2} : 1$$

Proof Assume that $ABCD$, as shown in the diagram above, is a golden rectangle. Let $AD = \ell$ and $CD = w$. Then $ED = \ell - w$.

As the two rectangles are similar, we have

$$\frac{AD}{CD} = \frac{CD}{ED} = k$$

where k is the ratio that we want to find. Thus

$$\frac{\ell}{w} = \frac{w}{\ell - w} = k$$

and therefore

$$\ell^2 - \ell w = w^2$$

Substitute $\ell = kw$:

$$(kw)^2 - kw^2 = w^2$$

$$k^2 - k - 1 = 0$$

Using the quadratic formula gives $k = \dfrac{1+\sqrt{5}}{2}$ since $k > 0$.

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The **golden ratio** is denoted by φ and is given by

$$\varphi = \frac{1 + \sqrt{5}}{2}$$

Alternatively, the golden ratio can be defined as the unique positive number φ such that

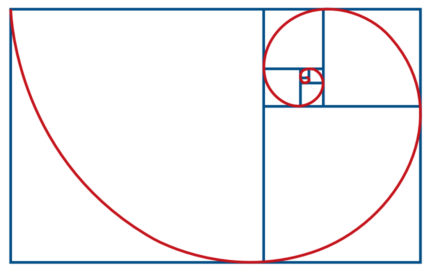
$$\varphi^2 - \varphi - 1 = 0$$

Note: A rectangle of length φ and width 1 is indeed a golden rectangle, since $\varphi^2 - \varphi - 1 = 0$ implies that $\dfrac{\varphi}{1} = \dfrac{1}{\varphi - 1}$.

A sequence of golden rectangles

Starting from any golden rectangle, we can remove a square to form another golden rectangle. Therefore we can remove another square to form yet another golden rectangle.

Continuing in this way, we can create the spiral shown.



The golden ratio and the geometric mean

You are familiar with the **arithmetic mean** of two numbers a and b, defined as $\dfrac{a + b}{2}$.

The **geometric mean** of two positive numbers a and b is defined as $\sqrt{ab}$.

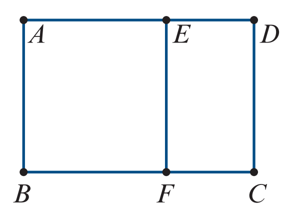
Note that the arithmetic mean of numbers a and b is the unique number c such that

$$c - a = b - c$$

Similarly, we can see that the geometric mean of positive numbers a and b is the unique positive number c such that

$$\frac{c}{a} = \frac{b}{c}$$

Again, consider a rectangle $ABCD$ with $AB < BC$.



Choose points E and F as shown such that $ABFE$ is a square. Then $CD = AE$.

Therefore $ABCD$ is a golden rectangle if and only if

$$\frac{AD}{AE} = \frac{AE}{ED}$$

This is precisely the requirement that AE is the geometric mean of AD and ED.

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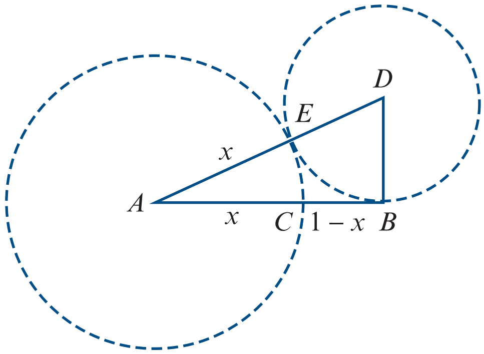
Construction of the golden ratio

We can construct the golden ratio as follows.



1. Start with a line segment AB of unit length.
2. Draw BD perpendicular to AB with length $\frac{AB}{2} = \frac{1}{2}$.
3. Draw line segment AD.
4. Draw a circle with centre D and radius DB, cutting AD at E.
5. Draw a circle with centre A and radius AE, cutting AB at C.
6. Use Pythagoras' theorem to show that $x = AE = \frac{\sqrt{5}-1}{2}$. Then $\frac{AB}{AC} = \frac{1}{x} = \varphi$.

Irrationality of the golden ratio

One way to prove that the golden ratio is irrational is first to prove that $\sqrt{5}$ is irrational. However, we can give a more direct proof as follows.

Theorem

The golden ratio φ is irrational.

Proof Suppose that the golden ratio is rational. Then we can write

$$\varphi = \frac{m}{n} \qquad \text{for some } m, n \in \mathbb{N}$$

We can assume that m and n have no common factors, and hence the numerator m is as small as possible. Note that $\varphi > 1$ and so $m > n$.

Since $\varphi^2 - \varphi - 1 = 0$, we have

$$\begin{aligned}\varphi &= \frac{1}{\varphi - 1}\\ &= \frac{1}{\frac{m}{n} - 1}\\ &= \frac{n}{m-n}\end{aligned}$$

As $m > n$, we have now expressed φ as a fraction with a numerator smaller than m. But this contradicts our initial assumption. Hence φ is irrational.

Section summary

- The **golden ratio** is $\varphi = \frac{1+\sqrt{5}}{2}$.
- All golden rectangles are similar, with the ratio of length to width $\varphi : 1$.
- The golden ratio is the unique positive number φ such that $\varphi^2 - \varphi - 1 = 0$.

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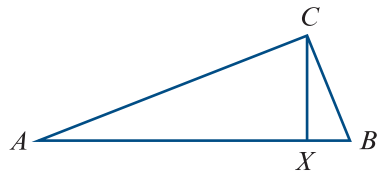
Exercise 7I

1 For the golden ratio φ show that:

a $\varphi - 1 = \dfrac{1}{\varphi}$ **b** $\varphi^3 = 2\varphi + 1$ **c** $2 - \varphi = (\varphi - 1)^2 = \dfrac{1}{\varphi^2}$

2 ABC is a right-angled triangle with the right angle at C, and CX is the altitude of the triangle from C.



a Prove that $\dfrac{AX}{CX} = \dfrac{CX}{XB}$.

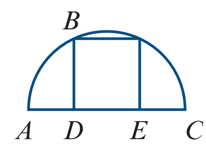
Note: This shows that the length CX is the geometric mean of lengths AX and XB.

b Find CX if:

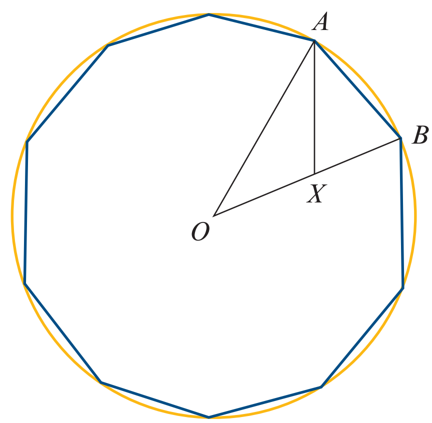
i $AX = 2$ and $XB = 8$ **ii** $AX = 1$ and $XB = 10$.

3 A square is inscribed in a semicircle as shown. Prove that

$$\frac{AD}{BD} = \frac{BD}{CD} = \varphi - 1$$



4 A regular decagon is inscribed in a circle with unit radius as shown.



a Find the magnitude of angle:

i AOB **ii** OAB

b The line AX bisects angle OAB. Prove that:

i triangle AXB is isosceles

ii triangle AXO is isosceles

iii triangle AOB is similar to triangle BXA.

c Find the length of AB, correct to two decimal places.

5 Calculate $\varphi^0, \varphi^1, \varphi^2, \varphi^3, \varphi^4$ and $\varphi^{-1}, \varphi^{-2}, \varphi^{-3}, \varphi^{-4}$. Show that each power of φ is equal to the sum of the two powers before it. That is, show that $\varphi^{n+1} = \varphi^n + \varphi^{n-1}$.

6 The Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$ is defined by $t_1 = t_2 = 1$ and $t_{n+1} = t_{n-1} + t_n$. Consider the sequence

$$\frac{t_2}{t_1}, \frac{t_3}{t_2}, \frac{t_4}{t_3}, \frac{t_5}{t_4}, \ldots$$

Show numerically that, as n gets very large, the ratio $\dfrac{t_{n+1}}{t_n}$ approaches φ.

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Chapter summary

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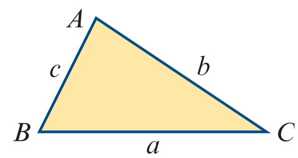
Parallel lines

- If two parallel lines are crossed by a transversal, then:
  - alternate angles are equal
  - corresponding angles are equal
  - co-interior angles are supplementary.
- If two lines crossed by a transversal form an equal pair of alternate angles, then the two lines are parallel.

Polygons

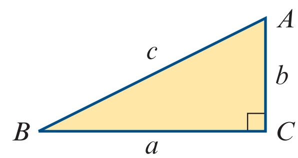
- The sum of the interior angles of an n-sided polygon is $(n - 2)180°$.
- A **regular polygon** is a polygon in which all angles are equal and all sides are equal.
- Triangle inequality

  In $\triangle ABC$: $a < b + c$, $b < c + a$ and $c < a + b$.



- Pythagoras' theorem and its converse

  Let ABC be a triangle with side lengths a, b and c.
  - If $\angle C$ is a right angle, then $a^2 + b^2 = c^2$.
  - If $a^2 + b^2 = c^2$, then $\angle C$ is a right angle.



- Classification of quadrilaterals:
  - A **trapezium** is a quadrilateral with at least one pair of opposite sides parallel.
  - A **parallelogram** is a quadrilateral with both pairs of opposite sides parallel.
  - A **rhombus** is a parallelogram with a pair of adjacent sides equal.
  - A **rectangle** is a parallelogram in which one angle is a right angle.
  - A **square** is a rectangle with a pair of adjacent sides equal.

Congruence

- **Congruent figures** have exactly the same shape and size.
- If triangle ABC is congruent to triangle XYZ, this can be written as $\triangle ABC \equiv \triangle XYZ$.
- Two triangles are congruent provided any one of the following four conditions holds:

  SSS the three sides of one triangle are equal to the three sides of the other triangle

  SAS two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle

  AAS two angles and one side of one triangle are equal to two angles and the matching side of the other triangle

  RHS the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and one side of another right-angled triangle.

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Review

Similarity

- Two figures are **similar** if we can enlarge one figure so that its enlargement is congruent to the other figure.
  - Matching lengths of similar figures are in the same ratio.
  - Matching angles of similar figures are equal.
- If triangle ABC is similar to triangle XYZ, this can be written as $\triangle ABC \sim \triangle XYZ$.
- Two triangles are similar provided any one of the following four conditions holds:

  AAA two angles of one triangle are equal to two angles of the other triangle

  SAS the ratios of two pairs of matching sides are equal and the included angles are equal

  SSS the sides of one triangle can be matched up with the sides of the other triangle so that the ratio of matching lengths is constant

  RHS the ratio of the hypotenuses of two right-angled triangles equals the ratio of another pair of sides.
- If two shapes are similar and the similarity factor is k (that is, if for any length AB of one shape, the corresponding length $A'B'$ of the similar shape is kAB), then

  $$\text{area of similar shape} = k^2 \times \text{area of original shape}$$
- If two solids are similar and the similarity factor is k, then

  $$\text{volume of similar solid} = k^3 \times \text{volume of original solid}$$

Short-answer questions

1 $ABCD$ is a rhombus with $AB = 16$ cm. The midpoints of its sides are joined to form a quadrilateral.

a Describe the quadrilateral formed.

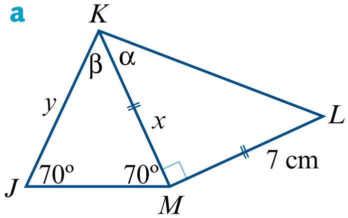
b What is the length of the diagonal of this quadrilateral?

2 Prove that a triangle with sides $x^2 - y^2$, $x^2 + y^2$ and $2xy$ is a right-angled triangle.

3 Find the side length of a rhombus whose diagonals are 6 cm and 10 cm.

4 Find the values of the unknowns (x, y, α, β, γ and θ) for each of the following:

a



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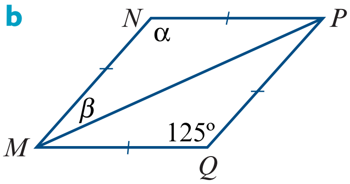

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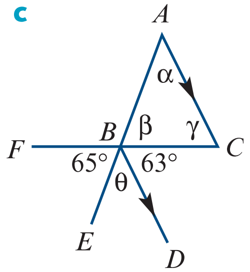

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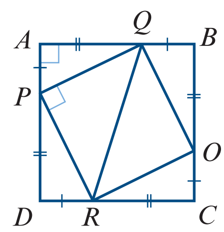
Review

5 A 25 m pole is leaning against a vertical wall with the foot of the pole 20 m from the wall. If its foot slips a further 4 m from the wall, find the distance that the top of the pole has slipped down the wall.

6 **a** Prove that $\triangle PAQ \equiv \triangle QBO$.

b Prove that $\triangle PQR \equiv \triangle ORQ$.



7 Let XYZ be a triangle with a point P on XY and a point Q on XZ such that PQ is parallel to YZ.

a Show that the two triangles XYZ and XPQ are similar.

b If $XY = 36$ cm, $XZ = 30$ cm and $XP = 24$ cm, find:

i XQ **ii** QZ

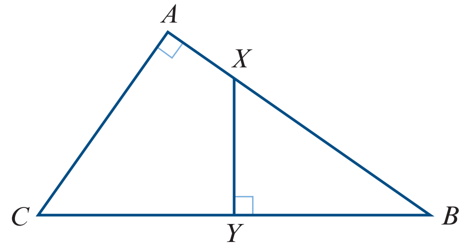
c Write down the values of $XP : PY$ and $PQ : YZ$.

8 Triangles ABC and DEF are similar. If the area of triangle ABC is 12.5 cm^2, the area of triangle DEF is 4.5 cm^2 and $AB = 5$ cm, find:

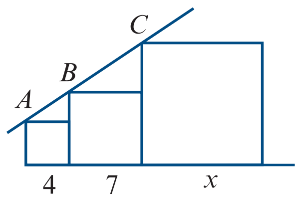
a the length of DE **b** the value of $AC : DF$ **c** the value of $EF : BC$.

9 If a 1 m stake casts a shadow 2.3 m long, find the height (in metres) of a tree which casts a shadow 21 m long.

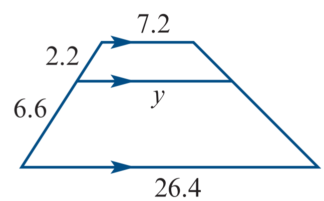
10 ABC is a right-angled triangle with $AB = 4$ and $AC = 3$. If the triangle is folded along the line XY, then vertex C coincides with vertex B. Find the length of XY.



11 Points A, B and C lie on a straight line. The squares are adjacent and have side lengths 4, 7 and x. Find the value of x.



12 Find the value of y in the diagram on the right.



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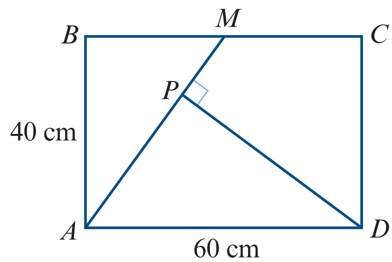
13 An alloy is produced by mixing metal X with metal Y in the ratio of 5 : 3 by volume. The mass of 1 cm^3 of metal X is $\frac{8}{5}$ g and the mass of 1 cm^3 of metal Y is $\frac{4}{3}$ g. Calculate:

- **a** the mass of a solid cube of alloy with edge length 4 cm
- **b** the ratio by mass, in the form n : 1, of metal X to metal Y in the alloy
- **c** the volume, to the nearest cm^3, of a cubic block of alloy with a mass of 1.5 kg
- **d** the length, in mm, of the edge of this cubic block.

14 $ABCD$ is a rectangle in which $AB = 40$ cm and $AD = 60$ cm. The midpoint of BC is M, and DP is perpendicular to AM.



- **a** Prove that the triangles BMA and PAD are similar.
- **b** Calculate the ratio of the areas of the triangles BMA and PAD.
- **c** Calculate the length of PD.

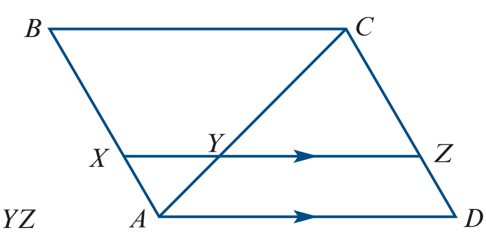
15 A sculptor is commissioned to create a bronze statue 2 m high. He begins by making a clay model 30 cm high.

- **a** Express, in simplest form, the ratio of the height of the completed bronze statue to the height of the clay model.
- **b** If the surface area of the model is 360 cm^2, find the surface area of the statue.
- **c** If the volume of the model is 1000 cm^3, find the volume of the statue.

16 The radius of a spherical soap bubble increases by 1%. Find, correct to the nearest whole number, the percentage increase in:

- **a** its surface area
- **b** its volume.

17 AC is the diagonal of a rhombus $ABCD$. The line XYZ is parallel to AD, and $AX = 3$ cm and $AB = 9$ cm. Find:



- **a** $\dfrac{XY}{BC}$
- **b** $\dfrac{AY}{AC}$
- **c** $\dfrac{CY}{AC}$
- **d** $\dfrac{YZ}{AD}$
- **e** $\dfrac{\text{area } \triangle AXY}{\text{area } \triangle ABC}$
- **f** $\dfrac{\text{area } \triangle CYZ}{\text{area } \triangle ACD}$

18 AB and DC are parallel sides of a trapezium and $DC = 3AB$. The diagonals AC and DB intersect at O. Prove that $AO = \frac{1}{4}AC$.

19 Triangles ABC and PQR are similar. The medians AX and PY are drawn, where X is the midpoint of BC and Y is the midpoint of QR. Prove that:

- **a** triangles ABX and PQY are similar
- **b** $\dfrac{AX}{PY} = \dfrac{BC}{QR}$

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Review

Multiple-choice questions

1 One angle of a triangle is twice the size of the second angle, and the third angle is $66°$. The smallest angle is

A $66°$ **B** $53°$ **C** $38°$ **D** $24°$ **E** $22°$

2 Three of the angles of a pentagon are right angles. The other two angles are of equal size $x°$. The angle $x°$ is equal to

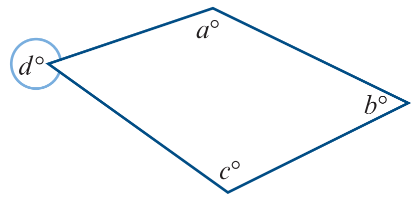
A $45°$ **B** $135°$ **C** $120°$ **D** $150°$ **E** $108°$

3 d is equal to

A $360 - (a + b + c)$ **B** $a + b + c$

C $b - a + c$ **D** $a - b + c$

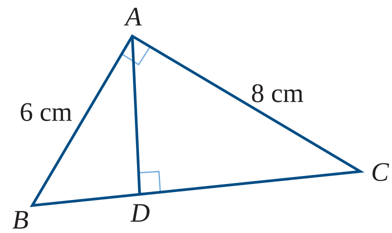
E $a + b - c$



4 In the figure, $\angle BAC$ is a right angle and AD is perpendicular to BC. If $AB = 6$ cm and $AC = 8$ cm, then the length of AD is

A 4 cm **B** $\frac{24}{5}$ cm **C** $\frac{17}{3}$ cm

D $\frac{13}{2}$ cm **E** 7 cm



5 Two sides of a triangle have lengths 14 and 18. Which of the following *cannot* be the length of the third side?

A 2 **B** 6 **C** 7 **D** 28 **E** 30

6 If $5 : 3 = 7 : x$, then x is equal to

A 12 **B** $\frac{35}{3}$ **C** 5 **D** $\frac{21}{5}$ **E** $\frac{5}{21}$

7 Brass is composed of a mixture of copper and zinc. If the ratio of copper to zinc is $85 : 15$, then the amount of copper in 400 kg of brass is

A 60 kg **B** 340 kg **C** 360 kg **D** 380 kg **E** 150 kg

8 If the total cost of P articles is Q dollars, then the cost of R articles of the same type is

A PQR **B** $\frac{P}{QR}$ **C** $\frac{PQ}{R}$ **D** $\frac{QR}{P}$ **E** $\frac{R}{PQ}$

9 A car is 3.2 m long. The length of a model of the car at the scale $1 : 100$ is

A 0.032 cm **B** 0.32 cm **C** 3.2 cm **D** 320 cm **E** 32 cm

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10 An athlete runs 75 m in 9 s. If she were to maintain the same average speed for 100 m, then her time for 100 m would be

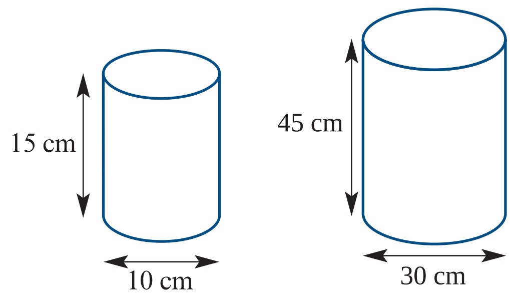
A 11.6 s **B** 12.0 s **C** 11.8 s **D** 12.2 s **E** 12.4 s

11 If 50 is divided into three parts in the ratio 1 : 3 : 6, then the largest part is

A 5 **B** 15 **C** $\frac{50}{3}$ **D** 30 **E** 3

12 Two similar cylinders are shown. The ratio of the volume of the smaller cylinder to the volume of the larger cylinder is

A 1 : 3 **B** 1 : 9
C 1 : 27 **D** 1 : 5
E 2 : 9

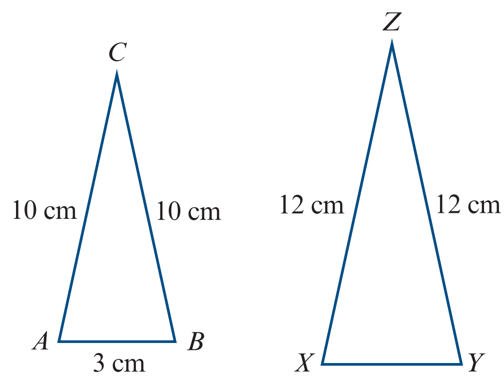


13 The radius of sphere A is $\frac{4}{5}$ times the radius of sphere B. Hence, the ratio of the volume of sphere A to the volume of sphere B is

A 16 : 25 **B** 4 : 5 **C** 5 : 4 **D** 25 : 16 **E** 64 : 125

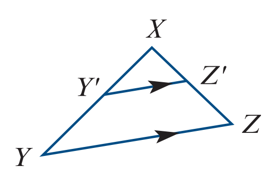
14 Triangles ABC and XYZ are similar isosceles triangles. The length of XY is

A 4 cm **B** 5 cm
C 4.2 cm **D** 2.5 cm
E 3.6 cm



15 YZ is parallel to $Y'Z'$ and $Y'Y = \frac{1}{3}YX$. The area of triangle XYZ is 60 cm^2. The area of triangle $XY'Z'$ is

A 20 cm^2 **B** 30 cm^2 **C** $\frac{20}{9}$ cm^2
D $\frac{20}{3}$ cm^2 **E** $\frac{80}{3}$ cm^2



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Review

Extended-response questions

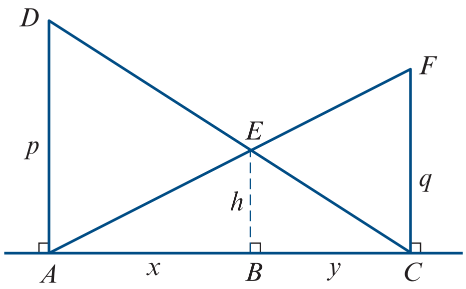
1 **a** In this diagram, which other triangle is similar to $\triangle DAC$?

b Explain why $\frac{h}{p} = \frac{y}{x + y}$.

c Use another pair of similar triangles to write down an expression for $\frac{h}{q}$ in terms of x and y.

d Explain why $h \cdot \left(\frac{1}{p} + \frac{1}{q}\right) = 1$.

e Calculate h when $p = 4$ and $q = 5$.



2 $ABCDE$ is a regular pentagon whose sides are each 1 unit long. Each diagonal is of length d units. In a regular pentagon, each diagonal is parallel to one of the sides of the pentagon.

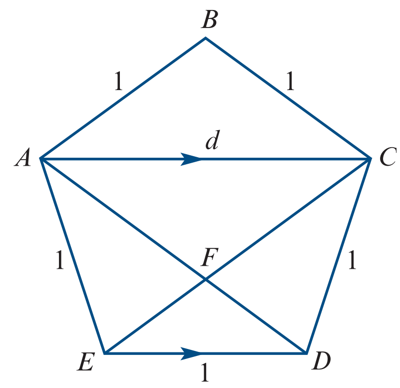
a What kind of shape is $ABCF$ and what is the length of CF?

b Explain why the length of EF is $d - 1$.

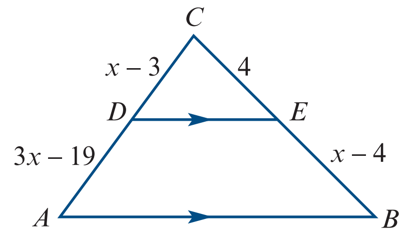
c Which triangle is similar to $\triangle EFD$?

d Use the pair of similar triangles to write an equation for d and show that the equation can be rewritten as $d^2 - d - 1 = 0$.

e Find d.



3 Place conditions upon x such that DE is parallel to AB given that $CD = x - 3$, $DA = 3x - 19$, $CE = 4$ and $EB = x - 4$.



4 **a** If BR, CS and DT are perpendicular to BD, name the pairs of similar triangles.

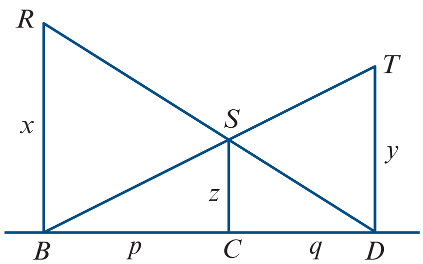
b Which of the following is correct?

$$\frac{z}{y} = \frac{p}{q} \quad \text{or} \quad \frac{z}{y} = \frac{p}{p+q}$$

c Which of the following is correct?

$$\frac{z}{x} = \frac{q}{p} \quad \text{or} \quad \frac{z}{x} = \frac{q}{p+q}$$

d Show that $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$.



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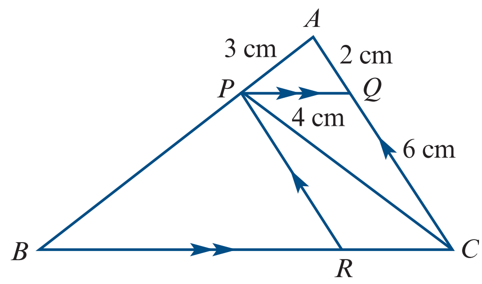
5 In the diagram, PQ is parallel to BC and PR is parallel to AC, with $AQ = 2$ cm, $QC = 6$ cm, $AP = 3$ cm and $PQ = 4$ cm.



a Calculate:

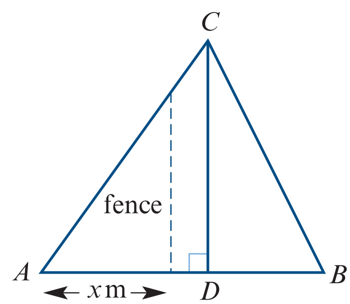
- **i** PB
- **ii** BR
- **iii** $\dfrac{\text{area } \triangle APQ}{\text{area } \triangle ABC}$
- **iv** $\dfrac{\text{area } \triangle BPR}{\text{area } \triangle ABC}$

b If the area of triangle APQ is a cm^2, express in terms of a:

- **i** area $\triangle ABC$
- **ii** area $\triangle CPQ$

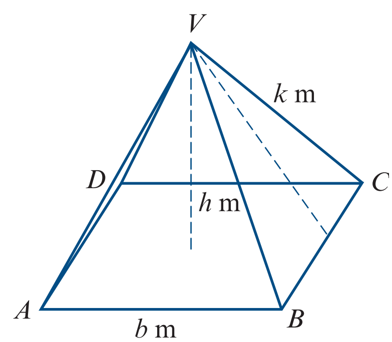
6 Construct a triangle ABC such that $BC = 10$ cm, $AC = 9$ cm and $AB = 6$ cm. Find a point D on AB and a point E on AC such that DE is parallel to BC and the area of $\triangle ADE$ is one-ninth the area of $\triangle ABC$.

7 A triangular lot has boundaries of lengths $AB = 130$ m, $BC = 40\sqrt{10}$ m and $CA = 150$ m. The length of CD is 120 m. A fence is to be erected which runs at right angles to AB. If the lot is to be divided into two equal areas, find x.



8 The Greek historian Herodotus wrote that the proportions of the great pyramid at Giza in Egypt were chosen so that the area of a square, for which the side lengths are equal to the height of the great pyramid, is equal to the area of one of the triangular faces.

Let h m be the height of the pyramid, let k m be the altitude of one of the face triangles, and let b m be the length of a side of the square base.



Show that Herodotus' definition gives $k : \dfrac{b}{2} = \varphi : 1$, where φ is the golden ratio.

Review

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Chapter 8

8 Circle geometry

Objectives

- To establish the following results and use them to solve problems:
  - The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.
  - Angles in the same segment of a circle are equal.
  - A tangent to a circle is perpendicular to the radius drawn from the point of contact.
  - The two tangents drawn from an external point to a circle are the same length.
  - The angle between a tangent and a chord drawn from the point of contact is equal to any angle in the alternate segment.
  - A quadrilateral is cyclic if and only if the sum of each pair of opposite angles is 180°.
  - If AB and CD are two chords of a circle that cut at a point P, then $PA \cdot PB = PC \cdot PD$.

The two basic figures of geometry in the plane are the triangle and the circle. We have considered triangles and their properties in Chapter 7, and we will use the results of that chapter in establishing results involving circles.

A **circle** is the set of all points in the plane at a fixed distance r from a point O. Circles with the same radius are congruent to each other (and are said to be equal circles). We have seen in the previous chapter that all circles are similar to each other.

You may have come across the Cartesian equation of the circle in Mathematical Methods Year 11. For example, the circle with radius 1 and centre the origin has equation $x^2 + y^2 = 1$. In this chapter we take a different approach to the study of circles.

The theorems and related results in this chapter can be investigated through a geometry package such as GeoGebra or Cabri Geometry.

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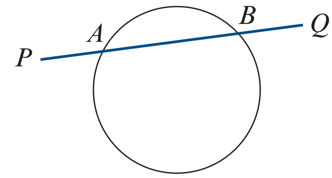
8A Angle properties of the circle



A line segment joining two points on a circle is called a **chord**. A line that cuts a circle at two distinct points is called a **secant**.

For example, in the diagram, the line PQ is a secant and the line segment AB is a chord.

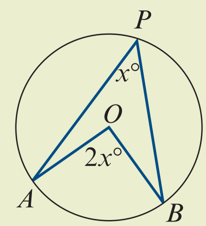
Suppose that we have a line segment or an arc AB and a point P not on AB. Then $\angle APB$ is the angle **subtended** by AB at the point P.

You should prove the following two results. The first proof uses the SSS congruence test and the second uses the SAS congruence test.

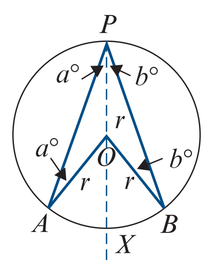
- Equal chords of a circle subtend equal angles at the centre.
- If two chords subtend equal angles at the centre, then the chords are equal.

Theorem 1

The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.



Proof Join points P and O and extend the line through O, as shown in the diagram on the right.



Note that $AO = BO = PO = r$, the radius of the circle. Therefore triangles PAO and PBO are isosceles.

Let $\angle APO = \angle PAO = a°$ and $\angle BPO = \angle PBO = b°$.

Then angle AOX is $2a°$ (exterior angle of a triangle) and angle BOX is $2b°$ (exterior angle of a triangle).

Hence $\angle AOB = 2a° + 2b° = 2(a + b)° = 2\angle APB$.

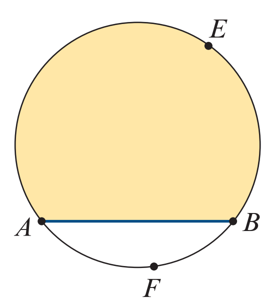
Note: In this proof, the centre O and point P are on the same side of chord AB. Slight variations of this proof can be used for other cases. The result is always true.

Converse of Theorem 1 Let A and B be points on a circle, centre O, and let P be a point on the same side of AB as O. If the angle APB is half the angle AOB, then P lies on the circle.

Segments

A **segment** of a circle is the part of the plane bounded by an arc and its chord. For example, in the diagram:

- Arc AEB and chord AB define a **major segment** (shaded).
- Arc AFB and chord AB define a **minor segment** (unshaded).



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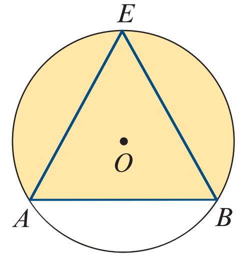
Angles in a segment

$\angle AEB$ is said to be an angle in segment AEB.



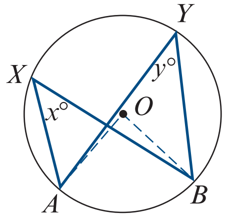
Theorem 2: Angles in the same segment

Angles in the same segment of a circle are equal.

Proof Let $\angle AXB = x^\circ$ and $\angle AYB = y^\circ$.

Then, by Theorem 1, $\angle AOB = 2x^\circ = 2y^\circ$.

Therefore $x = y$.

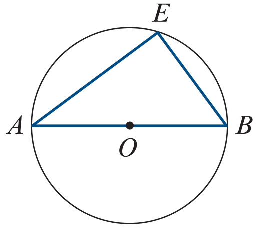


Note: A converse of this result is proved later in this section.

Theorem 3: Angle subtended by a diameter

The angle subtended by a diameter at the circumference is equal to a right angle (90°).

Proof The angle subtended at the centre is 180°, and so the result follows from Theorem 1.



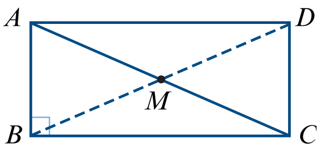
We can give a straightforward proof of a converse of this result.

Converse of Theorem 3 The circle whose diameter is the hypotenuse of a right-angled triangle passes through all three vertices of the triangle.

Proof In the diagram, triangle ABC has a right angle at B. Let M be the midpoint of the hypotenuse AC.

We need to prove that $MA = MB = MC$.

Complete the rectangle $ABCD$.



In Question 12 of Exercise 7C, you proved that the diagonals of a rectangle are equal and bisect each other.

Hence $AC = BD$ and M is also the midpoint of BD. It follows that $MA = MB = MC$.

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Cyclic polygons

- A set of points is said to be **concyclic** if they all lie on a common circle.
- A polygon is said to be **inscribed in a circle** if all its vertices lie on the circle. This implies that no part of the polygon lies outside the circle.
- A quadrilateral that can be inscribed in a circle is called a **cyclic quadrilateral**.

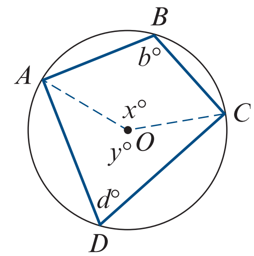
Theorem 4

The opposite angles of a quadrilateral inscribed in a circle sum to 180°.

That is, the opposite angles of a cyclic quadrilateral are supplementary.

Proof In the diagram, the quadrilateral $ABCD$ is inscribed in a circle with centre O.

By Theorem 1, we have $x = 2d$ and $y = 2b$.



Now $x + y = 360$

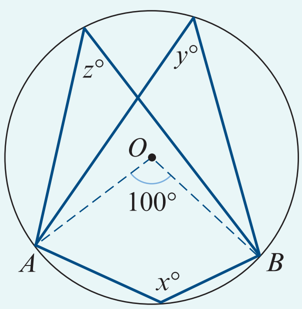
and so $2b + 2d = 360$

Hence $b + d = 180$

Converse of Theorem 4 If opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

Example 1

Find the value of each of the pronumerals in the diagram, where O is the centre of the circle and $\angle AOB = 100°$.



Solution

Theorem 1 gives $y = z = 50$.

The value of x can be found by observing either of the following:

1 Reflex angle AOB is 260°.
Therefore $x = 130$ (by Theorem 1).

2 We have $x + y = 180$ (by Theorem 4).
Therefore $x = 180 - 50 = 130$.

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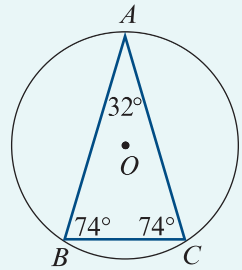
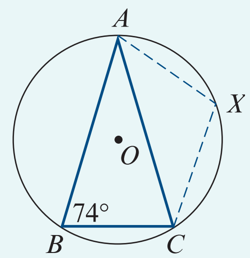
Example 2

An isosceles triangle is inscribed in a circle as shown. Find the angles in the three minor segments of the circle cut off by the sides of this triangle.



Solution

To find $\angle AXC$, form the cyclic quadrilateral $AXCB$. Then $\angle AXC$ and $\angle ABC$ are supplementary. Therefore $\angle AXC = 106°$, and so all angles in the minor segment formed by AC have magnitude 106°.



Similarly, it can be shown that all angles in the minor segment formed by AB have magnitude 106°, and that all angles in the minor segment formed by BC have magnitude 148°.

Example 3

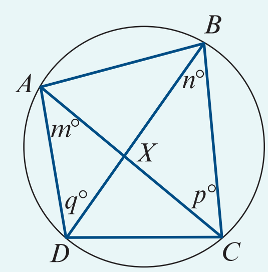
A, B, C and D are points on a circle. The diagonals of quadrilateral $ABCD$ meet at X. Prove that triangles ADX and BCX are similar.

Solution

$\angle DAC$ and $\angle DBC$ are in the same segment. Therefore $m = n$.

$\angle ADB$ and $\angle ACB$ are in the same segment. Therefore $q = p$.

Hence triangles ADX and BCX are similar (AAA).



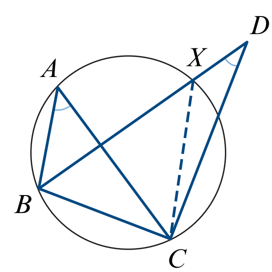
▶ The converse theorems

We only prove a converse of Theorem 2 here, but the proofs of the converses of Theorems 1 and 4 use similar techniques. Try them for yourself.

Converse of Theorem 2 If a line segment subtends equal angles at two points on the same side of the line segment, then the two points and the endpoints of the line segment are concyclic.

Proof A circle is drawn through points A, B and C. (This can be done with any three non-collinear points.)

Assume that $\angle BAC = \angle BDC$ and that D lies outside the circle. (There is another case to consider when D is inside, but the proof is similar. If D lies on the circle, then we are finished.)



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Let X be the point of intersection of line BD with the circle. Then, by Theorem 2, $\angle BAC = \angle BXC$ and so $\angle BDC = \angle BXC$. But this is impossible. (You can use the equality of the angle sums of $\triangle BXC$ and $\triangle BDC$ to show this.)

Hence D lies on the same circle as A, B and C.

Section summary

- Equal chords of a circle subtend equal angles at the centre.
- If two chords subtend equal angles at the centre, then the chords are equal.
- Theorem 1 The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.
- Theorem 2 Angles in the same segment of a circle are equal.
- Theorem 3 The angle subtended by a diameter at the circumference is 90°.
- Theorem 4 Opposite angles of a cyclic quadrilateral sum to 180°.

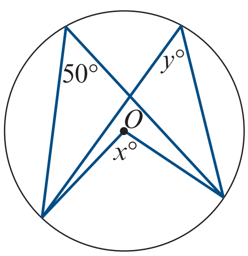
Exercise 8A

Skillsheet

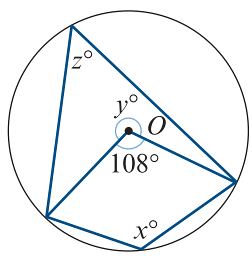
1 Find the values of the pronumerals for each of the following, where O denotes the centre of the given circle:

Example 1

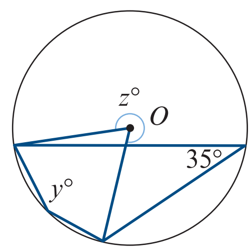
a



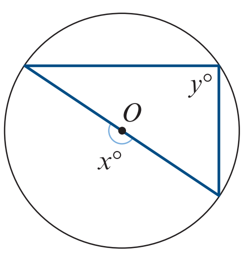
b



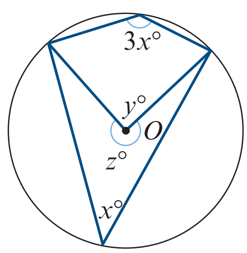
c



d

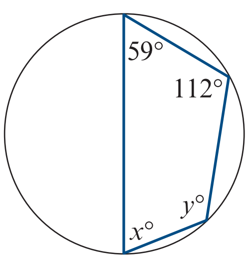


e

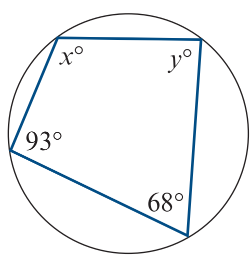


2 Find the values of the pronumerals for each of the following:

a



b



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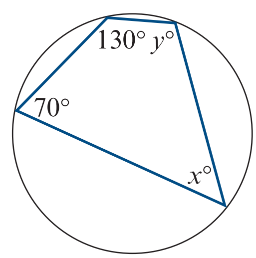

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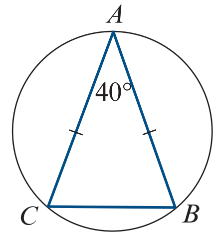
Example 2 **3** An isosceles triangle ABC is inscribed in a circle. (Inscribed means that all the vertices of the triangle lie on the circle.) What are the angles in the three minor segments cut off by the sides of this triangle?



4 $ABCDE$ is a pentagon inscribed in a circle. If $AE = DE$, $\angle BDC = 20°$, $\angle CAD = 28°$ and $\angle ABD = 70°$, find all the interior angles of the pentagon.

Example 3 **5** Prove that if two opposite sides of a cyclic quadrilateral are equal, then the other two sides are parallel.

6 $ABCD$ is a parallelogram. The circle through the points A, B and C cuts CD (extended if necessary) at E. Prove that $AE = AD$.

7 $ABCD$ is a cyclic quadrilateral and O is the centre of the circle through A, B, C and D. If $\angle AOC = 120°$, find the magnitude of $\angle ADC$.

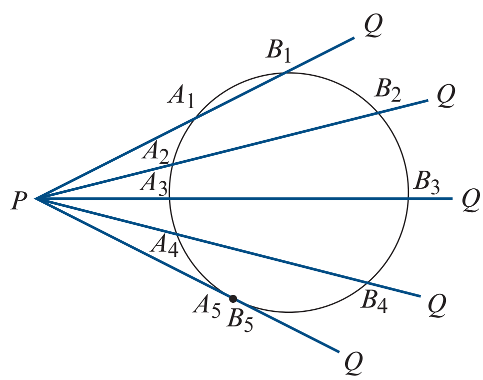
8 $PQRS$ is a cyclic quadrilateral with $\angle SQR = 36°$, $\angle PSQ = 64°$ and $\angle RSQ = 42°$. Find the interior angles of the quadrilateral.

9 Prove that if a parallelogram can be inscribed in a circle, then it must be a rectangle.

10 Prove that the bisectors of the four interior angles of a quadrilateral form a cyclic quadrilateral.

8B Tangents

Consider a point P outside a circle, as shown in the diagram. By rotating the secant PQ, with P as the pivot point, we obtain a sequence of pairs of points on the circle. As PQ moves towards the edge of the circle, the pairs of points become closer together, until they eventually coincide.



When PQ is in this final position (i.e. when the intersection points A and B coincide), it is called a **tangent** to the circle.

A tangent touches the circle at only one point, and this point is called the **point of contact**.

The **length of a tangent** from a point P outside the circle is the distance between P and the point of contact.

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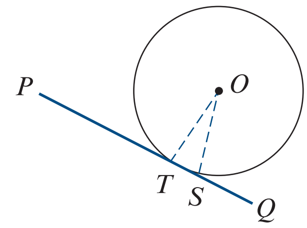
Theorem 5: Tangent is perpendicular to radius

A tangent to a circle is perpendicular to the radius drawn from the point of contact.

Proof This will be a proof by contradiction.

Let T be the point of contact of tangent PQ and suppose that $\angle OTP$ is not a right angle.

Let S be the point on PQ, not T, such that OSP is a right angle. Then triangle OST has a right angle at S.



Therefore $OT > OS$, as OT is the hypotenuse of triangle OST.

This implies that S is inside the circle, as OT is a radius.

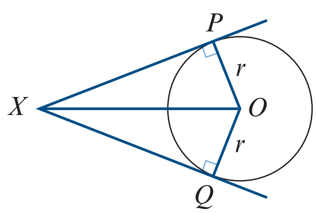
Thus the line through T and S must cut the circle again. But PQ is a tangent, and so this is a contradiction. Hence we have shown that $\angle OTP$ is a right angle.

Theorem 6: Two tangents from the same point

The two tangents drawn from an external point to a circle are the same length.

Proof We can see that $\triangle XPO$ is congruent to $\triangle XQO$ using the RHS test, as $\angle XPO = \angle XQO = 90°$, the side XO is common and $OP = OQ$ (radii).

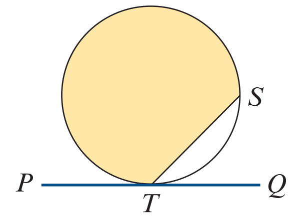
Therefore $XP = XQ$.



The alternate segment theorem

In the diagram:

- The shaded segment is called the **alternate segment** in relation to $\angle STQ$.
- The unshaded segment is alternate to $\angle STP$.



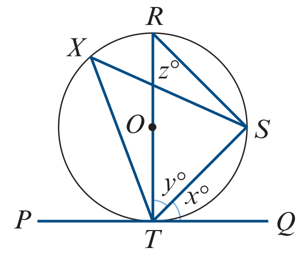
Theorem 7: Alternate segment theorem

The angle between a tangent and a chord drawn from the point of contact is equal to any angle in the alternate segment.

Proof Let $\angle STQ = x°$, $\angle RTS = y°$ and $\angle TRS = z°$, where RT is a diameter.

Then $\angle RST = 90°$ (Theorem 3, angle subtended by a diameter), and therefore $y + z = 90$.

Also $\angle RTQ = 90°$ (Theorem 5, tangent is perpendicular to radius), and therefore $x + y = 90$.



Thus $x = z$. But $\angle TXS$ is in the same segment as $\angle TRS$ and so $\angle TXS = x°$.

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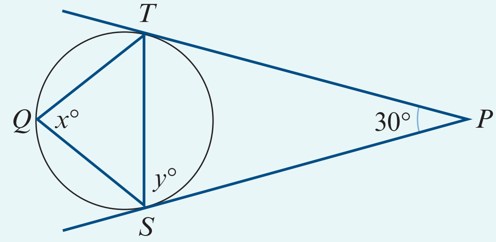
Example 4

Find the magnitudes of the angles x and y in the diagram.



Solution

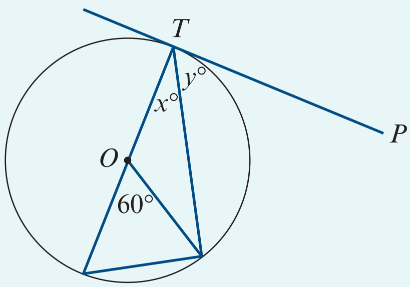
Triangle PST is isosceles (Theorem 6, two tangents from the same point).

Therefore $\angle PST = \angle PTS$ and so $y = 75$.

The alternate segment theorem gives $x = y = 75$.

Example 5

Find the values of x and y, where PT is tangent to the circle centre O.

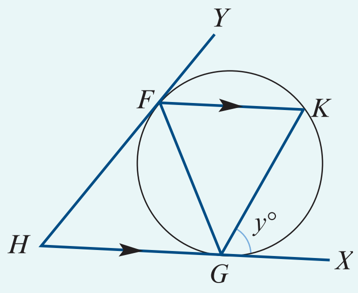


Solution

$x = 30$ as the angle at the circumference is half the angle subtended at the centre, and so $y = 60$ as $\angle OTP$ is a right angle.

Example 6

The tangents to a circle at F and G meet at H. A chord FK is drawn parallel to HG. Prove that triangle FGK is isosceles.



Solution

Let $\angle XGK = y°$.

Then $\angle GFK = y°$ (alternate segment theorem) and $\angle GKF = y°$ (alternate angles).

Therefore triangle FGK is isosceles with $FG = KG$.

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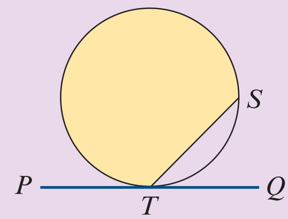
Section summary

- A tangent to a circle is perpendicular to the radius drawn from the point of contact.
- The two tangents drawn from an external point to a circle are the same length.
- In the diagram, the **alternate segment** to $\angle STQ$ is shaded, and the alternate segment to $\angle STP$ is unshaded.
- Alternate segment theorem The angle between a tangent and a chord drawn from the point of contact is equal to any angle in the alternate segment.



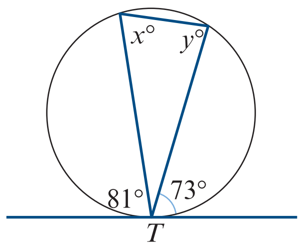
Exercise 8B

Skillsheet

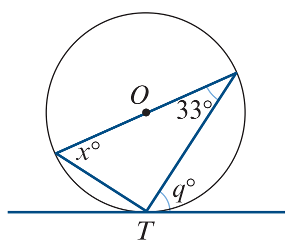
1 Find the values of the pronumerals for each of the following, where T is the point of contact of the tangent and O is the centre of the circle:

Example 4

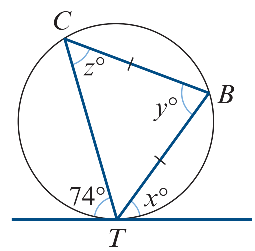
a



b

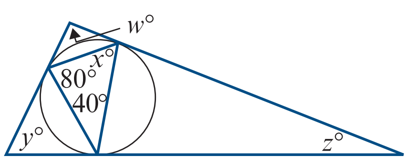


c

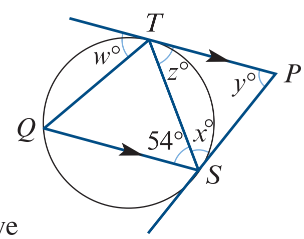


$BC = BT$

d



e



Note: In the diagram for part **e**, the two tangents from P have points of contact at S and T, and TP is parallel to QS.

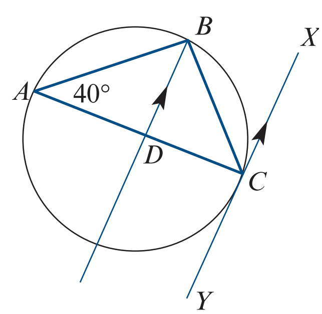
Example 5

2 A triangle ABC is inscribed in a circle, and the tangent to the circle at C is parallel to the bisector of angle ABC.

a Find the magnitude of $\angle BCX$.

b Find the magnitude of $\angle CBD$, where D is the point of intersection of the bisector of angle ABC with AC.

c Find the magnitude of $\angle ABC$.



3 Assume that AB and AC are two tangents to a circle, touching the circle at B and C, and that $\angle BAC = 116°$. Find the magnitudes of the angles in the two segments into which BC divides the circle.

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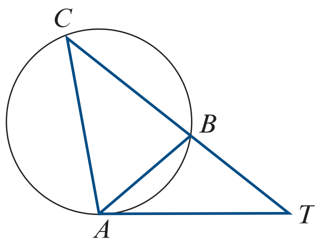
4 AT is a tangent at A and TBC is a secant to the circle. Given that $\angle CTA = 30°$ and $\angle CAT = 110°$, find the magnitude of angles ACB, ABC and BAT.



Example 6 **5** From a point A outside a circle, a secant ABC is drawn cutting the circle at B and C, and a tangent AD touching it at D. A chord DE is drawn equal in length to chord DB. Prove that triangles ABD and CDE are similar.

6 Assume that AB is a chord of a circle and that CT, the tangent at C, is parallel to AB. Prove that $CA = CB$.

7 Through a point T, a tangent TA and a secant TPQ are drawn to a circle APQ. The chord AB is drawn parallel to PQ. Prove that the triangles PAT and BAQ are similar.

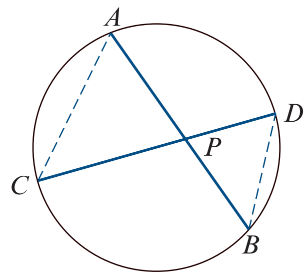
8 PQ is a diameter of a circle and AB is a perpendicular chord cutting it at N. Prove that PN is equal in length to the perpendicular from P onto the tangent at A.

8C Chords in circles

Theorem 8

If AB and CD are two chords of a circle that cut at a point P (which may be inside or outside the circle), then $PA \cdot PB = PC \cdot PD$.

Proof Case 1: The intersection point P is inside the circle.



Consider triangles APC and DPB:

$\angle APC = \angle DPB$ (vertically opposite)

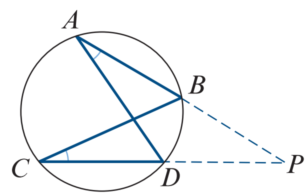
$\angle CAB = \angle BDC$ (angles in the same segment)

Thus triangle APC is similar to triangle DPB. This gives

$$\frac{PA}{PD} = \frac{PC}{PB}$$

$$\therefore \quad PA \cdot PB = PC \cdot PD$$

Case 2: The intersection point P is outside the circle.



Show that triangle APD is similar to triangle CPB. This gives

$$\frac{PA}{PC} = \frac{PD}{PB}$$

$$\therefore \quad PA \cdot PB = PC \cdot PD$$

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A converse of Theorem 8 is:

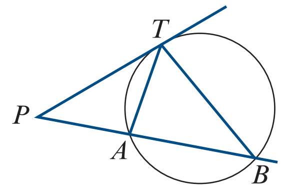
> If line segments AB and CD intersect at a point M and $AM \cdot BM = CM \cdot DM$, then the points A, B, C and D are concyclic.

This is proved in Extended-response question 2.

Theorem 9

If P is a point outside a circle and T, A, B are points on the circle such that PT is a tangent and PAB is a secant, then $PT^2 = PA \cdot PB$.

Proof Consider triangles PAT and PTB:



$\angle ATP = \angle TBA$ (alternate segment theorem)

$\angle PAT = \angle PTB$ (angle sum of a triangle)

Therefore triangle PAT is similar to triangle PTB.

This gives

$$\frac{PA}{PT} = \frac{PT}{PB}$$

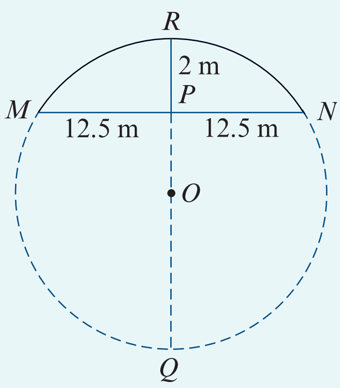
$$\therefore \quad PT^2 = PA \cdot PB$$

Example 7

The arch of a bridge is to be in the form of an arc of a circle. The span of the bridge is to be 25 m and the height in the middle 2 m. Find the radius of the circle.

Solution

Let r be the radius of the circle. Then $PQ = 2r - 2$.



Use Theorem 8 with the chords RQ and MN:

$$RP \cdot PQ = MP \cdot PN$$

Therefore

$$2PQ = 12.5^2$$

$$PQ = \frac{12.5^2}{2}$$

$$2r - 2 = \frac{12.5^2}{2} \qquad \text{as } PQ = 2r - 2$$

$$\therefore \quad r = \frac{1}{2}\left(\frac{12.5^2}{2} + 2\right)$$

$$= \frac{641}{16} \text{ m}$$

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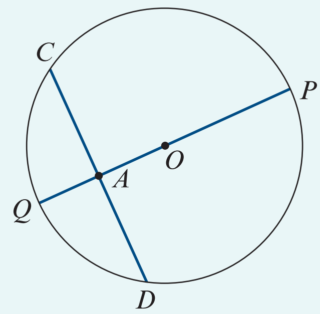
Example 8

Let A be any point inside a circle with radius r and centre O. Show that, if CD is a chord through A, then $CA \cdot AD = r^2 - OA^2$.

Solution

Let PQ be a diameter through A as shown.



By Theorem 8:

$$CA \cdot AD = QA \cdot AP$$

Since $QA = r - OA$ and $AP = r + OA$, this gives

$$CA \cdot AD = (r - OA)(r + OA)$$
$$= r^2 - OA^2$$

Section summary

- **Theorem 8** If AB and CD are two chords of a circle that cut at a point P (which may be inside or outside the circle), then $PA \cdot PB = PC \cdot PD$.
- **Theorem 9** If P is a point outside a circle and T, A, B are points on the circle such that PT is a tangent and PAB is a secant, then $PT^2 = PA \cdot PB$.

Exercise 8C

Skillsheet
Example 7

1 Two chords AB and CD intersect at a point P within a circle.

a Given that $AP = 5$ cm, $PB = 4$ cm, $CP = 2$ cm, find PD.

b Given that $AP = 4$ cm, $CP = 3$ cm, $PD = 8$ cm, find PB.

2 If AB is a chord and P is a point on AB such that $AP = 8$ cm, $PB = 5$ cm and P is 3 cm from the centre of the circle, find the radius.

3 If AB is a chord of a circle with centre O and P is a point on AB such that $BP = 4PA$, $OP = 5$ cm and the radius of the circle is 7 cm, find AB.

Example 8

4 Two circles intersect at points A and B. From any point P on the line AB, tangents PQ and PR are drawn to the circles. Prove that $PQ = PR$.

5 PQ is a variable chord of the smaller of two fixed concentric circles, and PQ extended meets the circumference of the larger circle at R. Prove that the product $RP \cdot RQ$ is constant for all positions and lengths of PQ.

6 ABC is an isosceles triangle with $AB = AC$. A line through A meets BC at D and the circumcircle of the triangle at E. Prove that $AB^2 = AD \cdot AE$.

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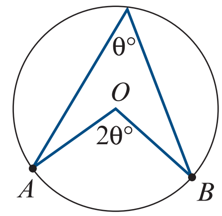
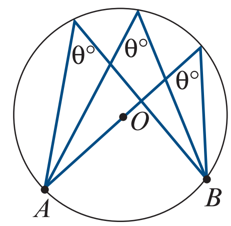
Chapter summary

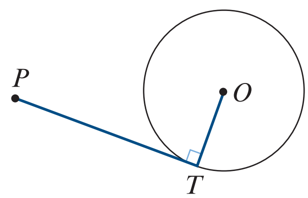
- The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.



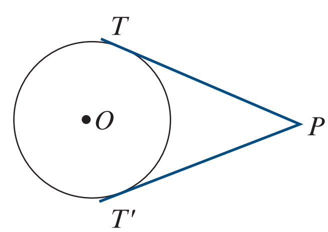
- Angles in the same segment of a circle are equal.



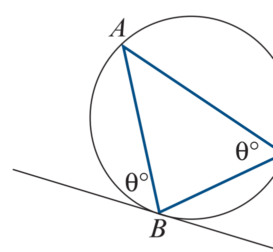
- A tangent to a circle is perpendicular to the radius drawn from the point of contact.



- The two tangents drawn from an external point are the same length, i.e. $PT = PT'$.



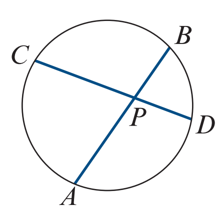
- The angle between a tangent and a chord drawn from the point of contact is equal to any angle in the alternate segment.

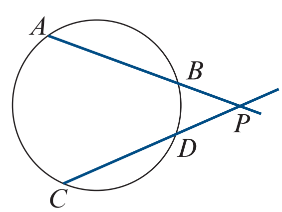


- A quadrilateral is cyclic if and only if the sum of each pair of opposite angles is 180°.

- If AB and CD are two chords of a circle that cut at a point P, then

$$PA \cdot PB = PC \cdot PD$$





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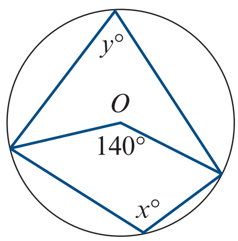
Short-answer questions

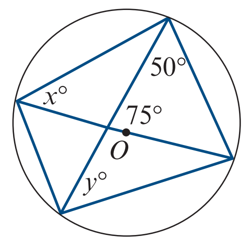
1 $\triangle ABC$ has $\angle A = 36°$ and $\angle C = 90°$. M is the midpoint of AB and CN is the altitude on AB. Find the size of $\angle MCN$.

2 Find the values of the pronumerals in each of the following:

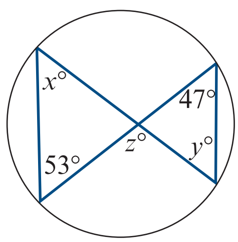
a



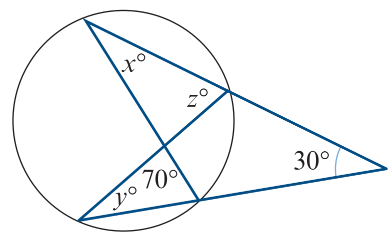
b



c



d



3 Let OP be a radius of a circle with centre O. A chord BA is drawn parallel to OP. The lines OA and BP intersect at C. Prove that:

a $\angle CAB = 2\angle CBA$

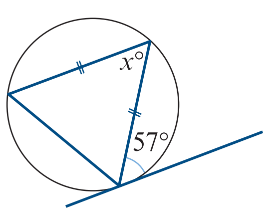
b $\angle PCA = 3\angle PBA$

4 A chord AB of a circle, centre O, is extended to C. The straight line bisecting $\angle OAB$ meets the circle at E. Prove that EB bisects $\angle OBC$.

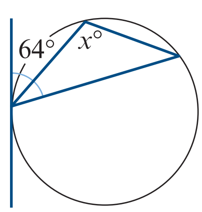
5 Two circles intersect at A and B. The tangent at B to one circle meets the second again at D, and a straight line through A meets the first circle at P and the second at Q. Prove that BP is parallel to DQ.

6 Find the values of the pronumerals for each of the following:

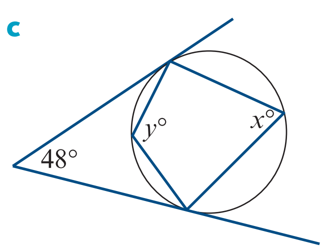
a



b



c



7 Two circles intersect at M and N. The tangent to the first circle at M meets the second circle at P, while the tangent to the second at N meets the first at Q. Prove that $MN^2 = NP \cdot QM$.

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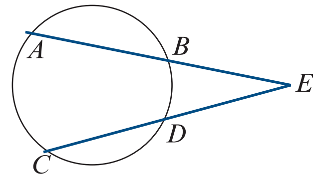
Review

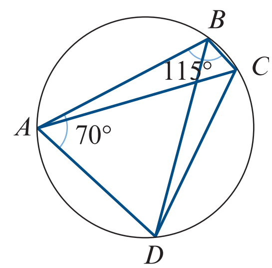
8 If $AB = 10$ cm, $BE = 5$ cm and $CE = 25$ cm, find DE.



Multiple-choice questions

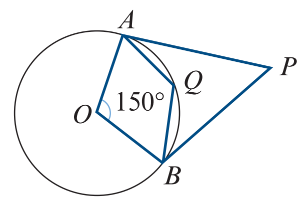
1 In the diagram, the points A, B, C and D lie on a circle, $\angle ABC = 115°$, $\angle BAD = 70°$ and $AB = AD$. The magnitude of $\angle ACD$ is

A 45° **B** 55° **C** 40° **D** 70° **E** 50°



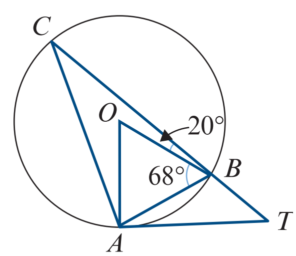
2 In the diagram, PA and PB are tangents to the circle centre O. Given that Q is a point on the minor arc AB and that $\angle AOB = 150°$, the magnitudes of $\angle APB$ and $\angle AQB$ are

A $\angle APB = 30°$ and $\angle AQB = 105°$
B $\angle APB = 40°$ and $\angle AQB = 110°$
C $\angle APB = 25°$ and $\angle AQB = 105°$
D $\angle APB = 30°$ and $\angle AQB = 110°$
E $\angle APB = 25°$ and $\angle AQB = 100°$



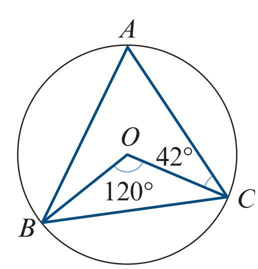
3 A circle with centre O passes through A, B and C. The line AT is the tangent to the circle at A, and CBT is a straight line. Given that $\angle ABO = 68°$ and $\angle OBC = 20°$, the magnitude of $\angle ATB$ is

A 60° **B** 64° **C** 65° **D** 70° **E** 66°



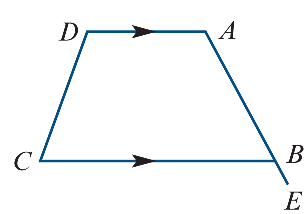
4 In the diagram, the points A, B and C lie on a circle centre O. If $\angle BOC = 120°$ and $\angle ACO = 42°$, then the magnitude of $\angle ABO$ is

A 18° **B** 20° **C** 22° **D** 24° **E** 26°



5 $ABCD$ is a cyclic quadrilateral with AD parallel to BC, and $\angle DCB = 65°$. The magnitude of $\angle CBE$ is

A 100° **B** 110° **C** 115° **D** 120° **E** 122°



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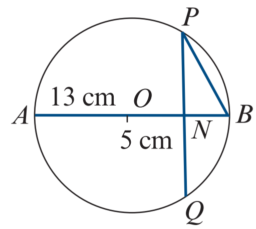
6 A chord AB of a circle subtends an angle of 50° at a point on the circumference of the circle. The acute angle between the tangents at A and B has magnitude

A 80° **B** 65° **C** 75° **D** 85° **E** 82°

7 Chords AB and CD of a circle intersect at P. If $AP = 12$ cm, $PB = 6$ cm and $CP = 2$ cm, then the length of PD in centimetres is

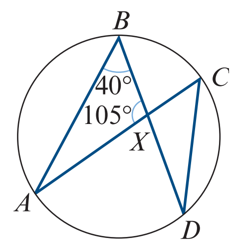
A 12 **B** 24 **C** 36 **D** 48 **E** 56

8 In the diagram, AB is the diameter of a circle with centre O and radius 13 cm. The chord PQ is perpendicular to AB, and N is the point of intersection of AB and PQ, with $ON = 5$ cm. The length of chord PB, in centimetres, is



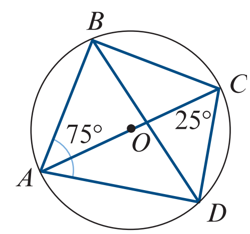
A 12 **B** $4\sqrt{13}$ **C** $2\sqrt{13}$
D 14 **E** 8

9 A, B, C and D are points on a circle, with $\angle ABD = 40°$ and $\angle AXB = 105°$. The magnitude of $\angle XDC$ is



A 35° **B** 40° **C** 45° **D** 50° **E** 55°

10 A, B, C and D are points on a circle, centre O, such that AC is a diameter of the circle. If $\angle BAD = 75°$ and $\angle ACD = 25°$, then the magnitude of $\angle BDC$ is

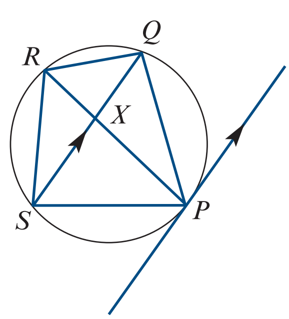


A 10° **B** 15° **C** 20° **D** 25° **E** 30°

Extended-response questions

1 The diagonals PR and QS of a cyclic quadrilateral $PQRS$ intersect at X. The tangent at P is parallel to QS.
Prove that:

a $PQ = PS$
b PR bisects $\angle QRS$



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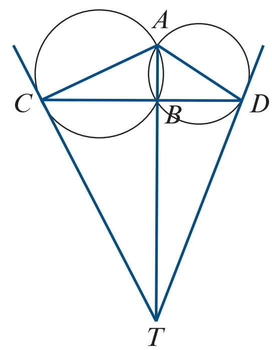
2 *If line segments AB and CD intersect at a point M and* $AM \cdot BM = CM \cdot DM$*, then the points A, B, C and D are concyclic.*

To prove this claim, show that:

a $\dfrac{AM}{CM} = \dfrac{DM}{BM}$ **b** $\triangle AMC \sim \triangle DMB$

c $\angle CAM = \angle BDM$ **d** $ABCD$ is cyclic

3 Two circles intersect at A and B. The tangents at C and D intersect at T on the extension of AB. Prove that, if CBD is a straight line, then:

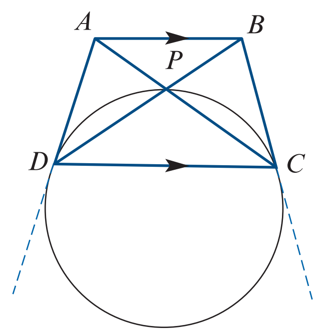


a $TCAD$ is a cyclic quadrilateral

b $\angle TAC = \angle TAD$

c $TC = TD$.

4 $ABCD$ is a trapezium in which AB is parallel to DC and the diagonals meet at P. The circle through D, P and C touches AD and BC at D and C respectively.
Prove that:

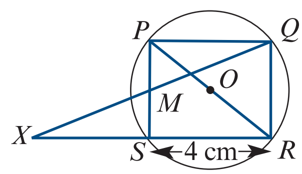


a $\angle BAC = \angle ADB$

b the circle through A, P and D touches BA at A

c $ABCD$ is a cyclic quadrilateral.

5 $PQRS$ is a square of side length 4 cm inscribed in a circle with centre O. The midpoint of the side PS is M. The line segments QM and RS are extended to meet at X.



a Prove that:

i $\triangle XPR$ is isosceles

ii PX is the tangent to the circle at P.

b Calculate the area of trapezium $PQRX$.

6 **a** An isosceles triangle ABC, with $AB = AC$, is inscribed in a circle. A chord AD intersects BC at E. Prove that

$$AB^2 - AE^2 = BE \cdot CE$$

b Diameter AB of a circle with centre O is extended to C, and from C a line is drawn tangent to the circle at P. The line PT is drawn perpendicular to AB at T. Prove that

$$CA \cdot CB - TA \cdot TB = CT^2$$

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9 Revision of Chapters 5–8

9A Short-answer questions

1 How many ways can four different books be arranged on a shelf?

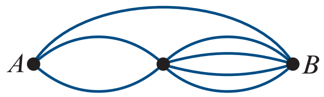
2 How many ways can three teachers and three students be arranged in a row if a teacher must be at the start of the row?

3 How many different three-digit numbers can be formed using the digits 1, 3, 5, 7 and 9:

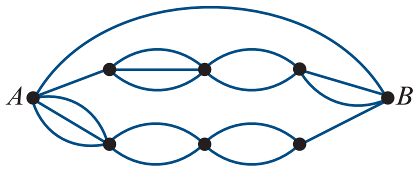
a as many times as you would like **b** at most once?

4 Travelling from left to right, how many paths are there from point A to point B in each of the following diagrams?

a



b



5 Evaluate each of the following:

a $4!$ **b** $\dfrac{6!}{4!}$ **c** $\dfrac{8!}{6!\,2!}$ **d** ${}^{10}C_2$

6 How many ways can five children be arranged on a bench with space for:

a four children **b** five children?

7 A bookshelf has three different mathematics books and two different physics books. How many ways can these books be arranged:

a without restriction

b if the mathematics books are kept together?

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8 Using the digits 0, 1, 2, 3 and 4 without repetition, how many five-digit numbers can you form:

a without restriction
b that are divisible by 10
c that are greater than 20 000
d that are even?

9 Asha has three identical 20 cent pieces and two identical 10 cent pieces. How many ways can she arrange these coins in a row?

10 How many ways can you select:

a three children from a group of six
b two letters from the alphabet
c four numbers from the set $\{1, 2, \ldots, 10\}$
d three sides of an octagon?

11 Consider the set of numbers $X = \{1, 2, \ldots, 8\}$.

a How many subsets of X have exactly two elements?
b How many subsets of X have exactly three elements, one of which is the number 8?
c Find the total number of subsets of X.

12 How many ways can you select three boys and two girls from a group of five boys and four girls?

13 There are four Labor and five Liberal parliamentarians, from which four are to be selected to form a committee. If the committee must include at least one member from each party, how many ways can this be done?

14 There are 10 blue, 11 green and 12 red balls in a bag. How many balls must be chosen at random to be sure that at least three will have the same colour?

15 How many different natural numbers from 1 to 99 inclusive must be chosen at random to be sure there will be at least one pair of numbers that sums to 100?

16 How many integers from 1 to 120 inclusive are divisible by 2 or 3?

17 Suppose that n is odd. Prove that $n^2 + n$ is even.

18 Prove that if m and n are consecutive integers, then $n^2 - m^2 = n + m$.

19 Let $n \in \mathbb{Z}$. Consider the statement: If $5n + 3$ is even, then n is odd.

a Write down the converse statement.
b Prove the converse.
c Write down the contrapositive statement.
d Prove the contrapositive.

20 Suppose the number x is irrational. Prove by contradiction that $x + 1$ is also irrational.

21 Prove by contradiction that 6 cannot be written as the difference of two perfect squares.

22 Let $n \in \mathbb{Z}$. Prove that $3n + 1$ is even if and only if n is odd.

23 Prove that each of the following statements is false by finding a counterexample:

a The sum of two prime numbers cannot be a prime number.
b For all $x \in \mathbb{R}$, we have $x^3 > x^2$.

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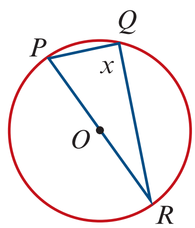
24 Show that this statement is false: There exists $n \in \mathbb{N}$ such that $25n^2 - 9$ is a prime number.

25 Prove by mathematical induction that:

a $2 + 4 + \cdots + 2n = n(n+1)$ **b** $11^n - 6$ is divisible by 5, for all $n \in \mathbb{N}$

26 Find the values of x and y in each of the following diagrams, giving reasons.

a



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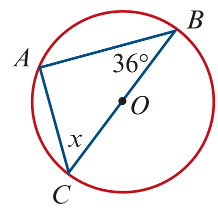

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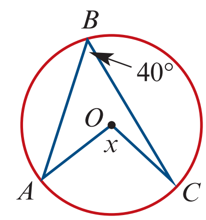

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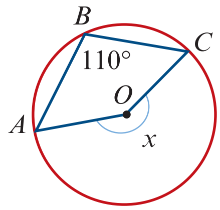

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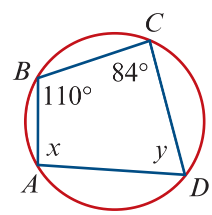

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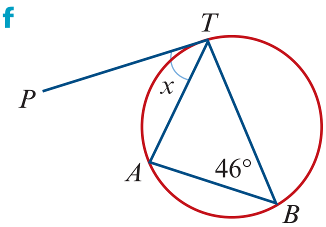
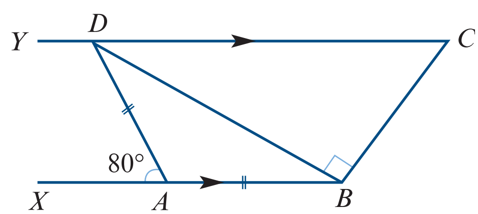


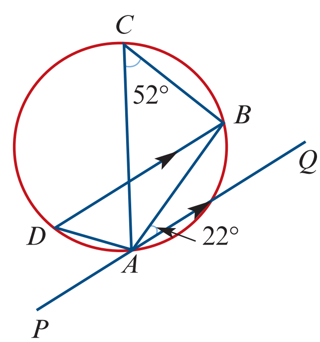
27 In the diagram, XB and YC are parallel. Given that $AD = AB$, $\angle XAD = 80°$ and $\angle DBC = 90°$, find:

a $\angle ABD$ **b** $\angle BDY$ **c** $\angle BCD$



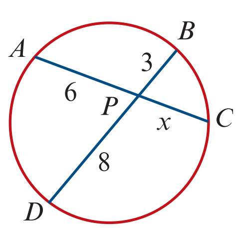
28 In the diagram, AC is a diameter of the circle, $PQ \parallel DB$, $\angle BCA = 52°$ and $\angle BAQ = 22°$. Find:

a $\angle CAB$ **b** $\angle PAD$ **c** $\angle CBD$



29 In each of the following, find the value of x:

a



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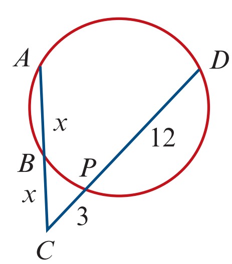

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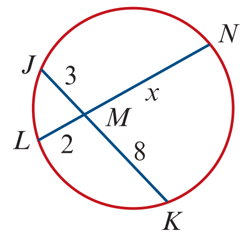

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30 In quadrilateral $ABCD$, $AB = 9$ cm, $BC = 6$ cm, $CD = 3$ cm, $DA = 6$ cm and $\angle CDA = 90°$. Prove that $\angle ACB$ is a right angle.

31 PN is an altitude of $\triangle PQR$, which has a right angle at P. Given that $PN = 60$ cm and $PR = 65$ cm, find:

a PQ **b** QN **c** NR

32 $ABCD$ is a square and X is a point outside the square such that $\triangle AXB$ is equilateral. Find the size of $\angle ACX$.

33 Prove that the sum of the interior angles of an 11-sided polygon is three times that of a pentagon.

34 $ABCD$ is a square and the bisector of $\angle ACD$ cuts BD at Q. Prove that $BQ = CD$.

35 The sides of a triangle are 28 cm, 45 cm and 54 cm. Is it an obtuse angle or not?
Hint: Consider the right-angled triangle with shorter sides 28 cm and 45 cm.

9B Multiple-choice questions

1 How many ways can five people be arranged in a line?

A $5!$ **B** 2^5 **C** 5^5 **D** ${}^5C_1$ **E** $5 + 4 + 3 + 2 + 1$

2 How many arrangements of the word HOBART begin with a vowel?

A 24 **B** 48 **C** 240 **D** 120 **E** 720

3 How many four-digit numbers can be formed using the digits 1, 2, 3, 4 and 5 at most once?

A ${}^5C_4$ **B** $5 + 4 + 3 + 2$ **C** $5 \times 4 \times 3 \times 2$ **D** $4!$ **E** 5^4

4 The number of arrangements of the digits in the number 111222 is

A $\dfrac{6!}{3! \times 3!}$ **B** $6!$ **C** $\dfrac{6!}{3!}$ **D** $3! \times 3! \times 3!$ **E** $3! \times 3!$

5 Sam has n identical 10 cent pieces and n identical 20 cent pieces. How many ways can these coins be arranged in a row?

A $n! \times n!$ **B** $\dfrac{(2n)!}{(n!)^2}$ **C** $\dfrac{(n!)^2}{(2n)!}$ **D** $(2n)!$ **E** $2n!$

6 There are 10 flavours of ice-cream at a shop. Mark will select three flavours for his cone, one of which must be chocolate. The total number of different selections is

A ${}^{10}C_3$ **B** ${}^{10}C_2$ **C** ${}^9C_3$ **D** ${}^9C_2$ **E** ${}^8C_2$

7 There are four Labor and five Liberal parliamentarians, from which two of each are to be selected to form a committee. How many ways can this be done?

A ${}^9C_2$ **B** ${}^9C_4$ **C** ${}^9C_2 \times {}^9C_2$ **D** ${}^4C_2 \times {}^5C_2$ **E** ${}^4C_2 + {}^5C_2$

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8 From 10 friends, you can invite any number of them to the movies. Assuming that you invite at least one friend, how many different selections can you make?

A 2^9 **B** $2^9 - 1$ **C** 2^{10} **D** $2^{10} - 1$ **E** ${}^{10}C_1$

9 An untidy kitchen drawer has a jumbled collection of eight knives, six forks and ten spoons. What is the smallest number of items that must be randomly chosen to ensure that at least four items of the same type are selected?

A 10 **B** 11 **C** 12 **D** 13 **E** 14

10 Whenever n integers are written on a whiteboard, at least six of them leave the same remainder when divided by 3. What is the smallest possible value of n?

A 3 **B** 4 **C** 7 **D** 15 **E** 16

11 How many integers from 1 to 60 inclusive are multiples of 2 or 5?

A 32 **B** 36 **C** 40 **D** 44 **E** 50

12 Suppose that both m and n are odd. Which of the following statements is false?

A $m + n$ is even **B** $m - n$ is even **C** $3m + 5n$ is even
D $2m + n$ is odd **E** $mn + 1$ is odd

13 Suppose that m is divisible by 4 and n is divisible by 12. Which of the following statements might be false?

A $m \times n$ is divisible by 3 **B** $m \times n$ is divisible by 48 **C** $m + n$ is divisible by 4
D $m^2 n$ is divisible by 48 **E** n is divisible by m

14 Let m and n be integers. Which of the following statements is always true?

A If mn is even, then m is even.
B The number $m + n$ is even if and only if both m and n are even.
C If $m + n$ is odd, then mn is odd.
D If mn is odd, then $m + n$ is even.
E If $m + n$ is even, then $m - n$ is odd.

15 Consider the statement: If n is even, then $n + 3$ is odd. The converse of this statement is

A If $n + 3$ is even, then n is even. **B** If n is odd, then $n + 3$ is even.
C If $n + 3$ is odd, then n is even. **D** If $n + 3$ is odd, then n is odd.
E If $n + 3$ is even, then n is odd.

16 Assume that a and b are positive real numbers with $a > b$. Which of the following might be false?

A $\dfrac{1}{a - b} > 0$ **B** $\dfrac{a}{b} - \dfrac{b}{a} > 0$ **C** $a + b > 2b$ **D** $a + 3 > b + 2$ **E** $2a > 3b$

17 The number of pairs of integers (m, n) that satisfy $mn - n = 12$ is

A 2 **B** 3 **C** 4 **D** 6 **E** 12

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18 Suppose that n is a positive integer. For how many values of n is the number $9n^2 - 4$ a prime?

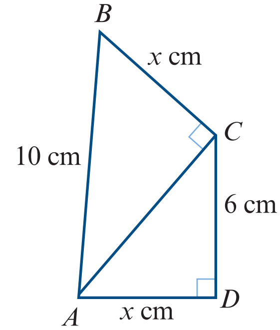
A 0 **B** 1 **C** 2 **D** 3 **E** 4

19 If a, b, c and d are consecutive integers, then which of the following statements may be false?

A $a + b + c + d$ is divisible by 2
B $a + b + c + d$ is divisible by 4
C $a \times b \times c \times d$ is divisible by 3
D $a \times b \times c \times d$ is divisible by 8
E $a \times b \times c \times d$ is divisible by 24

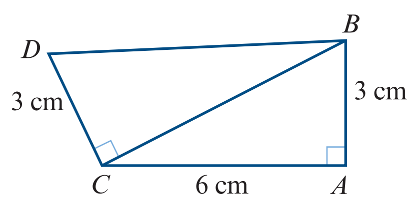
20 In this diagram, angles ACB and ADC are right angles. If BC and AD each have a length of x cm, then x is equal to

A $2\sqrt{17}$ **B** 4 **C** 5
D $4\sqrt{2}$ **E** $5\sqrt{2}$



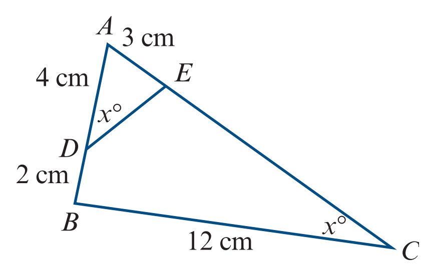
21 In this figure, the length of DB, in centimetres, is

A 6 **B** 9 **C** $3\sqrt{5}$
D $3\sqrt{6}$ **E** $3\sqrt{7}$



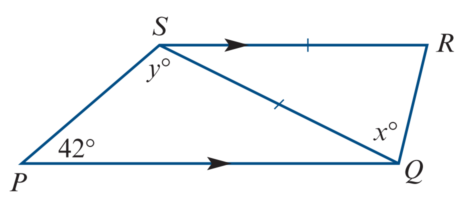
22 D and E are points on AB and AC, respectively. $AD = 4$ cm, $DB = 2$ cm, $AE = 3$ cm and $BC = 12$ cm. If $\angle ADE = \angle ACB$, then the length DE, in centimetres, is

A 6 **B** $\frac{9}{2}$ **C** 9 **D** 10 **E** 11



23 In this diagram, PQ and SR are parallel and $SR = SQ$. The angles x and y satisfy the equation

A $x = y$ **B** $x + y = 138$
C $2x + y = 42$ **D** $x = y + 42$
E $2x - y = 42$



24 A map is drawn so that a wall 17.1 m long is represented by a line 45 mm long. The scale is

A 1 : 3.8 **B** 1 : 38 **C** 1 : 380 **D** 1 : 3800 **E** 1 : 38 000

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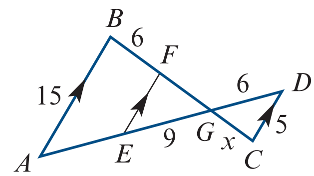
25 In the figure, $AB = 15$, $CD = 5$, $BF = 6$, $GD = 6$ and $EG = 9$. The length x is equal to



A 3 **B** 4 **C** 4.5
D 4.75 **E** 5

26 A model car is 8 cm long and the real car is 3.2 m long. The scale factor is

A 1 : 8 **B** 1 : 32 **C** 1 : 24 **D** 1 : 400 **E** 1 : 40

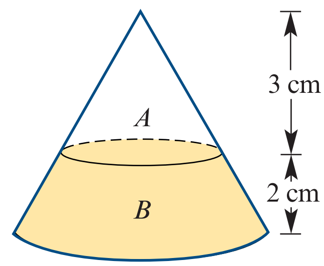
27 A ladder rests against a wall, touching the wall at a height of 5.6 m. The bottom of the ladder is 2 m from the wall. The distance (to the nearest centimetre) that a person of height 1.6 m must be from the wall to just fit under the ladder is

A 1.43 m **B** 0.57 m **C** 1.75 m **D** 0.25 m **E** 1.2 m

28 Let $\triangle ABC$ and $\triangle DEF$ be similar triangles such that $AB = 4$ cm and $DE = 10$ cm. If the area of $\triangle ABC$ is 24 cm^2, then the area of $\triangle DEF$ is

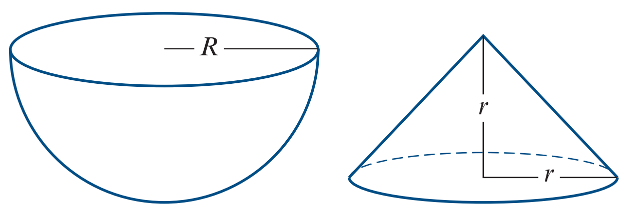
A 60 cm^2 **B** 240 cm^2 **C** 150 cm^2 **D** 96 cm^2 **E** none of these

29 In the figure, the volume of the shaded solid B is 49 cm^3. The volume of A is



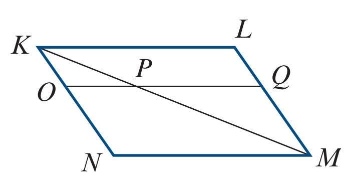
A 19.5 cm^3 **B** 17.3 cm^3 **C** 13.5 cm^3
D 12.5 cm^3 **E** 10.5 cm^3

30 The ratio of the volume of the hemisphere to the volume of the right circular cone is 27 : 4. Let R be the radius of the hemisphere and r the radius of the base of the cone. The ratio $R : r$ is equal to



A 1 : 2 **B** 2 : 3
C $3 : \sqrt{2}$ **D** 27 : 8
E 3 : 2

31 $KLMN$ is a parallelogram and OQ is parallel to KL. If O divides KN in the ratio 1 : 2, then the ratio $\dfrac{\text{area } \triangle KOP}{\text{area } KLMN}$ is equal to



A $\frac{1}{4}$ **B** $\frac{1}{9}$ **C** $\frac{1}{12}$
D $\frac{1}{18}$ **E** $\frac{1}{20}$

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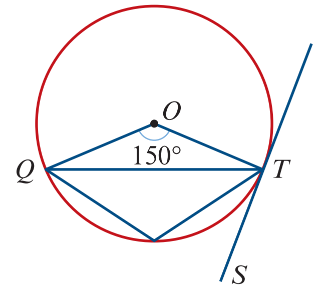
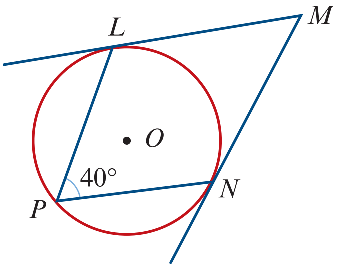
32 ST is a tangent at T to the circle with centre O. If $\angle QOT = 150°$, then the magnitude of $\angle QTS$ is

A 70° **B** 75° **C** 95°
D 105° **E** 150°



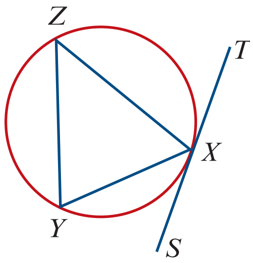
33 ML and MN are tangents to the circle at L and N. The magnitude of angle LMN is

A 80° **B** 90° **C** 100°
D 110° **E** 140°



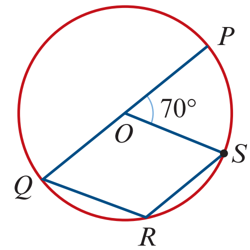
34 TS is a tangent at X and ZX bisects angle TXY. Given these facts, it can be proved that

A $YZ = XT$
B $YZ = XZ$
C $\angle YZX = \angle ZXT$
D $\angle SXY = \angle ZXY$
E $TX^2 = XY \cdot YZ$



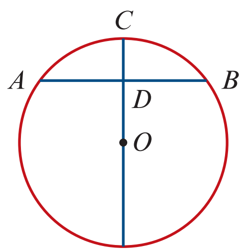
35 POQ is a diameter of the circle centre O. The size of angle QRS is

A 90° **B** 100° **C** 110°
D 125° **E** 160°



36 In the figure, O is the centre of the circle and D is the midpoint of AB. If $AB = 8$ cm and $CD = 2$ cm, the radius of the circle is

A 3 cm **B** 4 cm **C** 5 cm
D 6 cm **E** 7 cm



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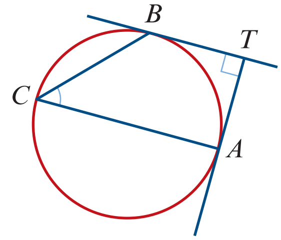
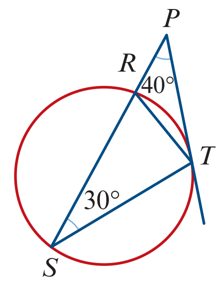
37 In the figure, TA and TB are tangents to the circle.
If TA is perpendicular to TB and TA is perpendicular to AC, then the magnitude of $\angle BCA$ is

A 30° **B** 40° **C** 45°
D 55° **E** 65°



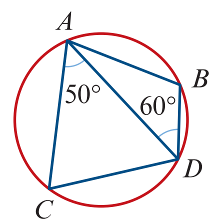
38 R, S and T are three points on the circumference of a circle, with $\angle RST$ equal to 30°. The tangent to the circle at T meets the line SR at P, and $\angle RPT$ is equal to 40°. The magnitude of $\angle RTS$ is

A 70° **B** 80° **C** 90°
D 100° **E** 110°



39 If $AB = AC$, $\angle ADB = 60°$ and $\angle CAD = 50°$, then $\angle ABD$ is equal to

A 80° **B** 90° **C** 100°
D 110° **E** 120°



9C Extended-response questions

1 A five-digit number is formed using the digits 0, 1, 2, 3, 4, 5 and 6 without repetition. How many ways can this be done:

a without restriction
b if the number is divisible by 10
c if the number is odd
d if the number is even?

2 Mike and Sonia belong to a group of eight coworkers. There are three men and five women in this group. A team of four workers is required to complete a project. How many ways can the team be selected:

a without restriction
b if it must contain two men and two women
c if it must contain both Mike and Sonia
d if it must not contain both Mike and Sonia?

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3 A sailing boat has three identical black flags and three identical red flags. The boat can send signals to nearby boats by arranging flags along its mast.

a How many ways can all six flags be arranged in a row?

b How many ways can all six flags be arranged in a row if no two black flags are adjacent?

c Using at least one flag, how many arrangements in a row are possible?

4 Consider the letters in the word BAGGAGE.

a How many arrangements of these letters are there?

b How many arrangements begin and end with a vowel?

c How many arrangements begin and end with a consonant?

d How many arrangements have all vowels together and all consonants together?

5 There are 25 people at a party.

a If every person shakes hands with every other person, what is the total number of handshakes?

b In fact, there are two rival groups at the party, so everyone only shakes hands with every other person in their group. If there are 150 handshakes, how many people are in each of the rival groups?

c At another party, there are 23 guests. Explain why it is not possible for each person to shake hands with exactly three other guests.

6 On a clock's face, twelve points are evenly spaced around a circle.

a How many ways can you select four of these points?

b How many ways can you select two points that are not diametrically opposite?

c For every selection of two points that are not diametrically opposite, you can draw one rectangle on the face that has these two points as vertices. What are the other two vertices?

d How many ways can you select four points that are the vertices of a rectangle?
Hint: Why must you divide the answer to part **b** by 4?

e Four points are randomly selected. What is the probability that the four points are the vertices of a rectangle?

7 Let a, b and c be integers. Suppose you know that $a + b$ is even and $b + c$ is odd.

a Is it possible to work out whether a, b and c are even or odd?

b What if you also know that $a + b + c$ is even?

8 **a** Find all positive integer values of a, b and c such that $a < b < c$ and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$$

b Find all positive integer values of a, b, c and d such that $a < b < c < d$ and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} > 2$$

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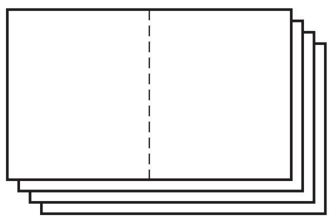
9 Let a, b and c be positive real numbers. Prove that if $b > a$, then $\dfrac{a+c}{b+c} > \dfrac{a}{b}$.

10 **a** Find the smallest value of $n \in \mathbb{N}$ such that $2^n > 10^3$.

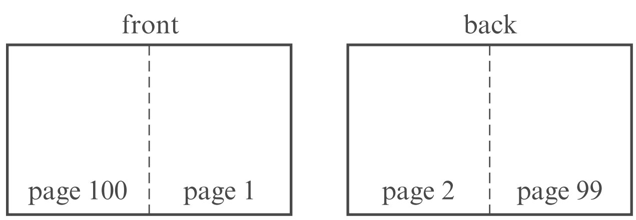
b Hence, prove that 2^{100} has at least 31 digits.

c Hence, explain why some digit in the decimal expansion of 2^{100} occurs at least four times. (Hint: There are 10 different digits: $0, 1, \dots, 9$.)

11 A stack of paper, printed on both sides, is folded in the middle to make a newspaper.



Each sheet contains four pages. The page numbers on the top sheet of Monday's newspaper are 1, 2, 99 and 100.



a What are the page numbers on the bottom sheet of Monday's stack?

b One of the sheets in Monday's newspaper has page numbers 7 and 8. What are its other two page numbers?

c Suppose that a newspaper is made from n sheets of paper. Prove that the sum of the four page numbers on each sheet is a constant.

d Tuesday's newspaper has a sheet whose pages are numbered 11, 12, 33 and 34. How many pages does this newspaper have?

12 Sam has 20 one-dollar coins and seven pockets. He wants to put coins into his pockets so that each pocket contains a different number of coins. (The number 0 is allowed.)

a Prove that this is impossible.

b What is the minimum number of coins Sam would need to do this?

c If Sam had 50 one-dollar coins, find the maximum number of pockets that he could fill, each with a different number of coins.

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13 Take a close look at the following square numbers:

$$15^2 = 225, \quad 25^2 = 625, \quad 35^2 = 1225, \quad 45^2 = 2025, \quad 55^2 = 3025, \quad 65^2 = 4225$$

a Find and describe the pattern that you see in these square numbers.

b Confirm that your pattern works for the number 75.

c Prove that your pattern actually works. (Hint: Each number is of the form $10n + 5$.)

14 Heidi has 10 wooden cubes, with edges of length 1 cm through to 10 cm.

a Using all the cubes, can she build two towers of the same height?

b Now Heidi has n wooden cubes, with edges of length 1 through to n. For what values of n can Heidi use all the cubes to build two towers of the same height?

15 **a** Suppose that a is odd and b is odd. Prove that ab is odd.

b Suppose that a is odd and $n \in \mathbb{N}$. Prove by induction that a^n is odd.

c Hence, prove that if x satisfies $3^x = 2$, then x is irrational.

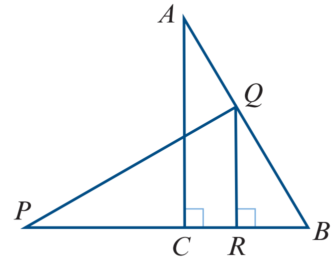
16 **a** If $n^4 + 6n^3 + 11n^2 + 6n + 1 = (an^2 + bn + c)^2$, find the positive values of a, b and c.

b Hence, prove that when 1 is added to the product of four consecutive integers, the result is always a perfect square.

c Hence, write the number $5 \times 6 \times 7 \times 8 + 1$ as a product of prime numbers.

17 Proof of Pythagoras' theorem

In the diagram, the right-angled triangle ABC is congruent to the right-angled triangle PQR. The side PR is on the same line as BC, and the point Q is on AB. Let $AB = c$, $BC = a$ and $AC = b$.



a Prove that $\triangle PBQ$ is similar to $\triangle ABC$.

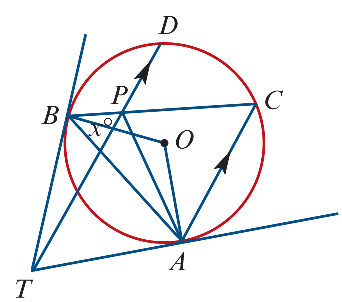
b Find PB in terms of a, b and c.

c Prove that $\triangle QBR$ is similar to $\triangle ABC$.

d Prove that $PB = b + \dfrac{a^2}{b}$.

e Prove that $c^2 = a^2 + b^2$.

18 In the figure, O is the centre of a circle. TD and AC are parallel. TA and TB are tangents to the circle. Let $\angle BPT = x°$.



a Prove that $TBOA$ is a cyclic quadrilateral.

b Find $\angle BCA$, $\angle BOA$, $\angle TAB$ and $\angle TBA$ in terms of x.

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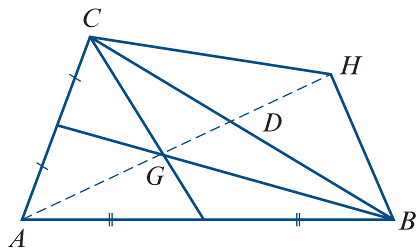
Revision

19 *The medians of a triangle are concurrent.*

In $\triangle ABC$:

- E and F are the midpoints of AC and AB respectively
- BE and CF intersect at G
- the line segment AG is extended to H so that $AG = GH$
- AH intersects BC at D.



a Prove that $\triangle AFG$ is similar to $\triangle ABH$.

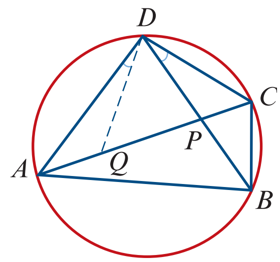
b Hence show $GC \parallel BH$.

c Prove that $GB \parallel CH$.

d Prove that $GBHC$ is a parallelogram and hence $BD = DC$.

20 Ptolemy's theorem: *In a cyclic quadrilateral, the sum of the products of the opposite sides is equal to the product of the diagonals.*

a To prove Ptolemy's theorem, we need to show that $BC \cdot AD + AB \cdot CD = AC \cdot BD$.
Line DQ is drawn such that $\angle BDC = \angle ADQ$.



i Prove that $\triangle ADQ \sim \triangle BDC$.

ii Prove that $\triangle ADB \sim \triangle QDC$.

iii Show that $AQ = \dfrac{BC \cdot AD}{BD}$ and $QC = \dfrac{AB \cdot CD}{BD}$.

iv Complete the proof of Ptolemy's theorem.
(Hint: $AC = AQ + QC$.)

b Prove Pythagoras' theorem by applying Ptolemy's theorem to a rectangle.

c Use Ptolemy's theorem to prove that, in a regular pentagon with side length 1, the length of each diagonal is the golden ratio φ.

Hint: Draw the diagonals and take three sides of the pentagon and one diagonal to form a quadrilateral.

d Given an equilateral triangle inscribed in a circle and a point on the circle, use Ptolemy's theorem to prove that the distance from the point to the most distant vertex of the triangle is the sum of the distances from the point to the two nearer vertices.

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10 Circular functions

Chapter 10

Objectives

- ▶ To review **radian measure**.
- ▶ To review the definitions of the circular functions **sine**, **cosine** and **tangent**.
- ▶ To review the **symmetry properties** of circular functions.
- ▶ To **solve equations** involving circular functions.
- ▶ To **sketch graphs** of circular functions.
- ▶ To apply circular functions to model periodic phenomena.

This chapter reviews and extends the study of the three circular functions – sine, cosine and tangent – introduced in Mathematical Methods Year 11.

An important property of these three functions is that they are periodic. That is, they each repeat their values in regular intervals or periods. In general, a function f is **periodic** if there is a positive constant a such that $f(x + a) = f(x)$. The sine and cosine functions each have period 2π, while the tangent function has period π.

The sine and cosine functions are used to model wave motion, and are therefore central to the application of mathematics to any problem in which periodic motion is involved – from the motion of the tides and ocean waves to sound waves and modern telecommunications.

We will continue our study of circular functions in Chapter 11, where we introduce the reciprocal circular functions – secant, cosecant and cotangent – and we derive and apply several important trigonometric identities.

Note: A more detailed introduction to sine, cosine and tangent as functions is given in Mathematical Methods Year 11 and also in an online chapter for this book, available in the Interactive Textbook.

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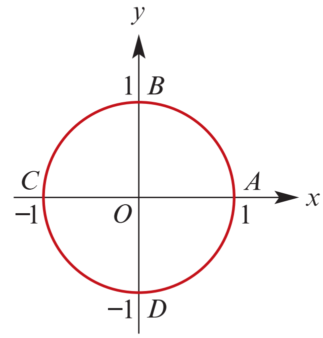
10A Defining the circular functions

▶ Measuring angles in degrees and radians

The diagram shows a **unit circle**, i.e. a circle of radius 1 unit.



$$\text{The circumference of the unit circle} = 2\pi \times 1 = 2\pi \text{ units}$$

Thus, the distance in an anticlockwise direction around the circle from

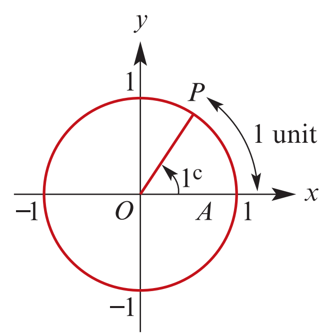
$$A \text{ to } B = \frac{\pi}{2} \text{ units}$$

$$A \text{ to } C = \pi \text{ units}$$

$$A \text{ to } D = \frac{3\pi}{2} \text{ units}$$

Definition of a radian

In moving around the circle a distance of 1 unit from A to P, the angle POA is defined. The measure of this angle is 1 radian.



One **radian** (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit.

Note: Angles formed by moving **anticlockwise** around the unit circle are defined as **positive**; those formed by moving **clockwise** are defined as **negative**.

Converting between degrees and radians

The angle, in radians, swept out in one revolution of a circle is $2\pi^c$.

$$2\pi^c = 360°$$

$$\therefore \quad \pi^c = 180°$$

$$\therefore \quad 1^c = \frac{180°}{\pi} \quad \text{or} \quad 1° = \frac{\pi^c}{180}$$

Example 1

a Convert 135° to radians.

b Convert $\frac{\pi^c}{6}$ to degrees.

Solution

a $135° = \frac{135 \times \pi^c}{180} = \frac{3\pi^c}{4}$

b $\frac{\pi^c}{6} = \frac{\pi}{6} \times \frac{180°}{\pi} = 30°$

Note: Usually the symbol for radians, ^{c}, is omitted. Any angle is assumed to be measured in radians unless indicated otherwise.

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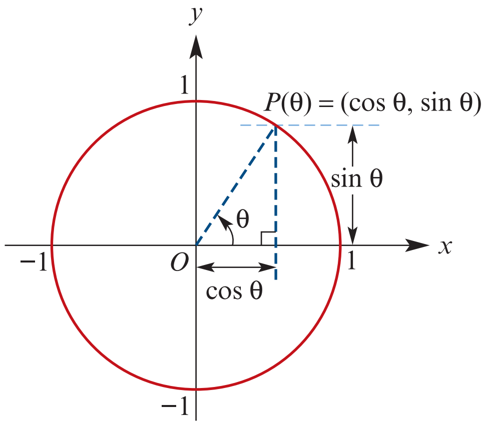
Defining sine and cosine

Let $P(\theta)$ denote the point on the unit circle corresponding to an angle θ. Then:

- $\cos\theta$ is the x-coordinate of $P(\theta)$
- $\sin\theta$ is the y-coordinate of $P(\theta)$



Hence the coordinates of the point $P(\theta)$ are $(\cos\theta, \sin\theta)$.

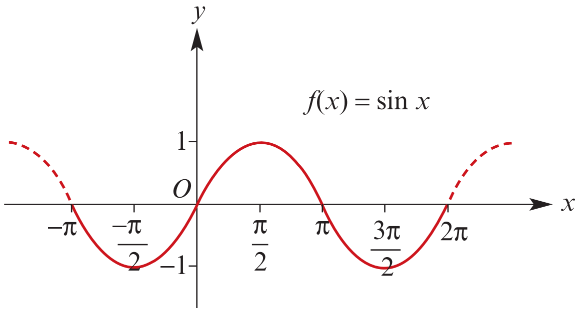
Note: Adding 2π to the angle results in a return to the same point on the unit circle. Thus $\cos(2\pi + \theta) = \cos\theta$ and $\sin(2\pi + \theta) = \sin\theta$.

The graphs of sine and cosine

A sketch of the graph of

$$f\colon \mathbb{R} \to \mathbb{R},\ f(x) = \sin x$$

is shown opposite.

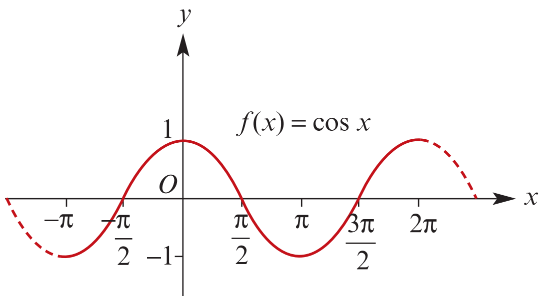


As $\sin(x + 2\pi) = \sin x$ for all $x \in \mathbb{R}$, the sine function is **periodic**. The period is 2π. The amplitude is 1.

A sketch of the graph of

$$f\colon \mathbb{R} \to \mathbb{R},\ f(x) = \cos x$$

is shown opposite.



The period of the cosine function is 2π. The amplitude is 1.

Example 2

Evaluate each of the following:

a $\sin\left(\dfrac{3\pi}{2}\right)$ **b** $\cos\left(-\dfrac{\pi}{2}\right)$ **c** $\cos\left(\dfrac{21\pi}{2}\right)$

Solution	*Explanation*
a $\sin\left(\dfrac{3\pi}{2}\right) = -1$	since $P\left(\dfrac{3\pi}{2}\right)$ has coordinates $(0, -1)$.
b $\cos\left(-\dfrac{\pi}{2}\right) = 0$	since $P\left(-\dfrac{\pi}{2}\right)$ has coordinates $(0, -1)$.
c $\cos\left(\dfrac{21\pi}{2}\right) = \cos\left(10\pi + \dfrac{\pi}{2}\right) = 0$	since $P\left(\dfrac{\pi}{2}\right)$ has coordinates $(0, 1)$.

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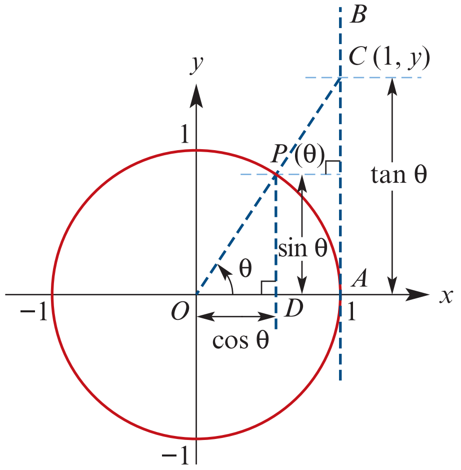
Defining tangent

Again consider the unit circle.

If we draw a tangent to the unit circle at A, then the y-coordinate of C, the point of intersection of the line OP and the tangent, is called **tangent** θ (abbreviated to $\tan\theta$).



By considering the similar triangles OPD and OCA:

$$\frac{\tan\theta}{1} = \frac{\sin\theta}{\cos\theta}$$

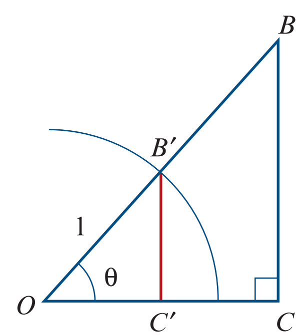
$$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

Note that $\tan\theta$ is undefined when $\cos\theta = 0$. The domain of tan is $\mathbb{R} \setminus \{\, \theta : \cos\theta = 0 \,\}$ and so $\tan\theta$ is undefined when $\theta = \pm\frac{\pi}{2}, \ \pm\frac{3\pi}{2}, \ \pm\frac{5\pi}{2}, \ \dots$

Note: Adding π to the angle does not change the line OP. Thus $\tan(\pi + \theta) = \tan\theta$.

The trigonometric ratios

For a right-angled triangle OBC, we can construct a similar triangle $OB'C'$ that lies in the unit circle.



From the diagram:

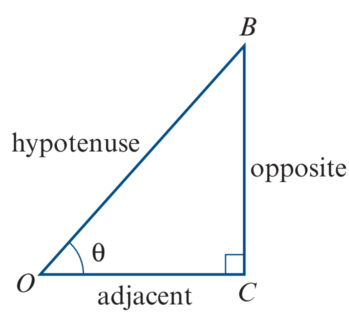
$$B'C' = \sin\theta \qquad \text{and} \qquad OC' = \cos\theta$$

The similarity factor is the length OB, giving

$$BC = OB\sin\theta \qquad \text{and} \qquad OC = OB\cos\theta$$

$$\therefore \quad \frac{BC}{OB} = \sin\theta \qquad \text{and} \qquad \frac{OC}{OB} = \cos\theta$$

This gives the ratio definition of sine and cosine for a right-angled triangle. The naming of sides with respect to an angle θ is as shown.



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

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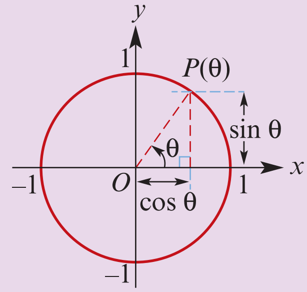
Section summary

- One radian (written 1^c) is the angle formed at the centre of the unit circle by an arc of length 1 unit:

$$1^c = \frac{180^\circ}{\pi} \quad \text{and} \quad 1^\circ = \frac{\pi^c}{180}$$

- The point $P(\theta)$ on the unit circle corresponding to an angle θ has coordinates $(\cos\theta, \sin\theta)$.



Exercise 10A

Example 1a

1 Convert the following angles from degrees to exact values in radians:

a 720° **b** 540° **c** -450° **d** 15° **e** -10° **f** -315°

Example 1b

2 Convert the following angles from radians to degrees:

a $\frac{5\pi}{4}$ **b** $-\frac{2\pi}{3}$ **c** $\frac{7\pi}{12}$ **d** $-\frac{11\pi}{6}$ **e** $\frac{13\pi}{9}$ **f** $-\frac{11\pi}{12}$

Example 2

3 Find the exact value of each of the following:

a $\cos\left(\frac{3\pi}{2}\right)$ **b** $\sin\left(-\frac{\pi}{2}\right)$ **c** $\cos(6\pi)$ **d** $\sin\left(\frac{15\pi}{2}\right)$

4 Find the exact value of each of the following:

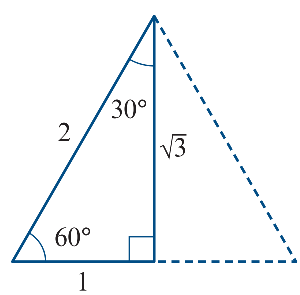
a $\sin(270^\circ)$ **b** $\cos(-540^\circ)$ **c** $\sin(450^\circ)$ **d** $\cos(720^\circ)$

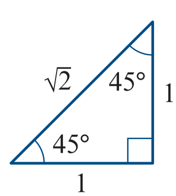
10B Symmetry properties and the Pythagorean identity

▶ Exact values of circular functions

The values in this table for 30°, 45° and 60° can be determined from the two triangles shown.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$ (30°)	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$ (45°)	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$ (60°)	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$ (90°)	1	0	undefined





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▶ Symmetry properties

The coordinate axes divide the unit circle into four quadrants, numbered anticlockwise from the positive direction of the x-axis. Using symmetry, we can determine relationships between the circular functions for angles in different quadrants:

Quadrant 2

By symmetry

$\sin(\pi - \theta) = b = \sin\theta$

$\cos(\pi - \theta) = -a = -\cos\theta$

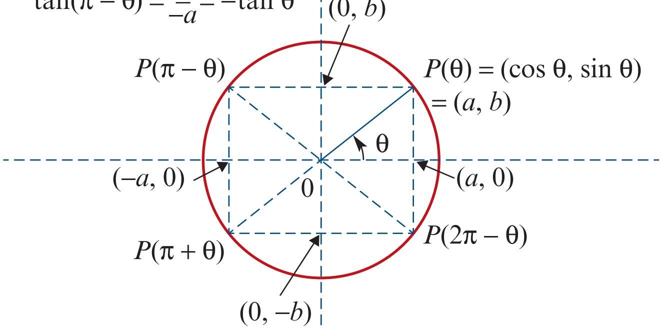
$\tan(\pi - \theta) = \dfrac{b}{-a} = -\tan\theta$

Quadrant 1

Note: $\sin(2\pi + \theta) = \sin\theta$

$\cos(2\pi + \theta) = \cos\theta$

$\tan(2\pi + \theta) = \tan\theta$



Quadrant 3

$\sin(\pi + \theta) = -b = -\sin\theta$

$\cos(\pi + \theta) = -a = -\cos\theta$

$\tan(\pi + \theta) = \dfrac{-b}{-a} = \tan\theta$

Quadrant 4

$\sin(2\pi - \theta) = -b = -\sin\theta$

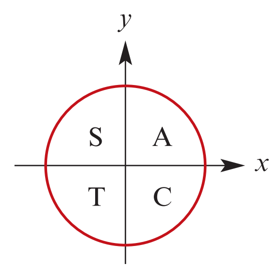
$\cos(2\pi - \theta) = a = \cos\theta$

$\tan(2\pi - \theta) = \dfrac{-b}{a} = -\tan\theta$

Signs of circular functions

Using these symmetry properties, the signs of sin, cos and tan for the four quadrants can be summarised as follows:

1st quadrant	all are positive	(A)
2nd quadrant	sin is positive	(S)
3rd quadrant	tan is positive	(T)
4th quadrant	cos is positive	(C)



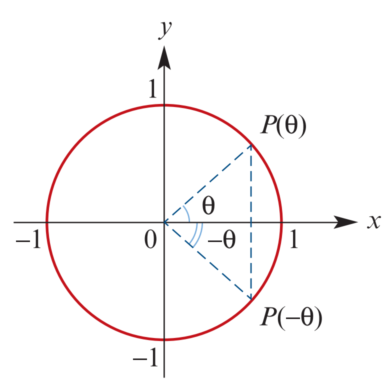
Negative angles

By symmetry:

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$



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Example 3

Find the exact value of:

a $\sin\left(\frac{11\pi}{6}\right)$ **b** $\cos\left(\frac{-5\pi}{4}\right)$

Solution

a
$$\begin{aligned}\sin\left(\frac{11\pi}{6}\right) &= \sin\left(2\pi - \frac{\pi}{6}\right)\\ &= -\sin\left(\frac{\pi}{6}\right)\\ &= -\frac{1}{2}\end{aligned}$$

b
$$\begin{aligned}\cos\left(\frac{-5\pi}{4}\right) &= \cos\left(\frac{5\pi}{4}\right)\\ &= \cos\left(\pi + \frac{\pi}{4}\right)\\ &= -\cos\left(\frac{\pi}{4}\right)\\ &= -\frac{1}{\sqrt{2}}\end{aligned}$$

Example 4

Find the exact value of:

a $\sin 150^\circ$ **b** $\cos(-585^\circ)$

Solution

a
$$\begin{aligned}\sin 150^\circ &= \sin(180^\circ - 150^\circ)\\ &= \sin 30^\circ\\ &= \frac{1}{2}\end{aligned}$$

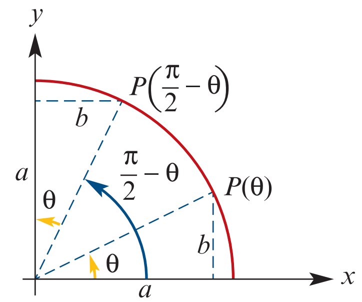
b
$$\begin{aligned}\cos(-585^\circ) &= \cos 585^\circ\\ &= \cos(585^\circ - 360^\circ)\\ &= \cos 225^\circ\\ &= -\cos 45^\circ\\ &= -\frac{1}{\sqrt{2}}\end{aligned}$$

▶ Complementary relationships

From the diagram to the right:

$$\sin\left(\frac{\pi}{2} - \theta\right) = a = \cos\theta$$

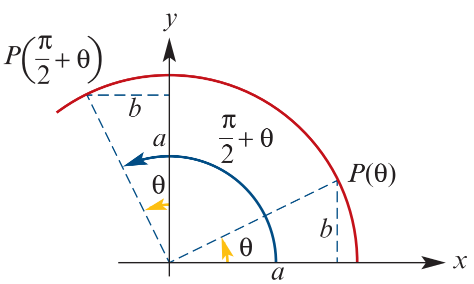
$$\cos\left(\frac{\pi}{2} - \theta\right) = b = \sin\theta$$



From the diagram to the right:

$$\sin\left(\frac{\pi}{2} + \theta\right) = a = \cos\theta$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -b = -\sin\theta$$



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Example 5

If $\sin\theta = 0.4$ and $\cos\alpha = 0.8$, find the value of:

a $\sin\left(\frac{\pi}{2} - \alpha\right)$ **b** $\cos\left(\frac{\pi}{2} + \theta\right)$ **c** $\sin(-\theta)$

Solution

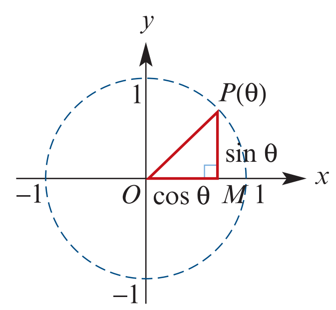
a $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha$
$= 0.8$

b $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$
$= -0.4$

c $\sin(-\theta) = -\sin\theta$
$= -0.4$

▶ The Pythagorean identity

Consider a point, $P(\theta)$, on the unit circle.



By Pythagoras' theorem:

$$OP^2 = OM^2 + MP^2$$
$$\therefore \quad 1 = (\cos\theta)^2 + (\sin\theta)^2$$

Since this is true for all values of θ, it is called an identity.

We can write $(\cos\theta)^2$ and $(\sin\theta)^2$ as $\cos^2\theta$ and $\sin^2\theta$, and therefore we obtain:

Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

Example 6

If $\sin x = 0.3$ and $0 < x < \frac{\pi}{2}$, find:

a $\cos x$ **b** $\tan x$

Solution

a $\cos^2 x + \sin^2 x = 1$

$\cos^2 x + 0.09 = 1$

$\cos^2 x = 0.91$

$\therefore \quad \cos x = \pm\sqrt{0.91}$

Since the point $P(x)$ is in the 1st quadrant, this gives

$$\cos x = \sqrt{0.91} = \sqrt{\frac{91}{100}}$$
$$= \frac{\sqrt{91}}{10}$$

b $\tan x = \frac{\sin x}{\cos x} = \frac{0.3}{\sqrt{0.91}}$

$= \frac{3}{\sqrt{91}}$

$= \frac{3\sqrt{91}}{91}$

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Section summary

- **Exact values**

θ	sin θ	cos θ	tan θ
0	0	1	0
$\frac{\pi}{6}$ (30°)	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$ (45°)	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$ (60°)	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$ (90°)	1	0	undefined

- **Complementary relationships**

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

- **Pythagorean identity**

$$\cos^2\theta + \sin^2\theta = 1$$

Exercise 10B

Example 3

1 Evaluate each of the following:

a $\cos\left(\frac{3\pi}{4}\right)$ **b** $\sin\left(\frac{5\pi}{4}\right)$ **c** $\sin\left(\frac{25\pi}{2}\right)$ **d** $\sin\left(\frac{15\pi}{6}\right)$ **e** $\cos\left(\frac{17\pi}{4}\right)$

f $\sin\left(-\frac{15\pi}{4}\right)$ **g** $\sin(27\pi)$ **h** $\sin\left(-\frac{17\pi}{3}\right)$ **i** $\cos\left(\frac{75\pi}{6}\right)$ **j** $\cos\left(-\frac{15\pi}{6}\right)$

k $\sin\left(-\frac{35\pi}{2}\right)$ **l** $\cos\left(-\frac{45\pi}{6}\right)$ **m** $\cos\left(\frac{16\pi}{3}\right)$ **n** $\sin\left(-\frac{105\pi}{2}\right)$ **o** $\cos(1035\pi)$

Example 4

2 Find the exact value of each of the following:

a $\sin(135°)$ **b** $\cos(-300°)$ **c** $\sin(480°)$

d $\cos(240°)$ **e** $\sin(-225°)$ **f** $\sin(420°)$

Example 5

3 If $\sin x = 0.3$ and $\cos\alpha = 0.6$, find the value of:

a $\cos(-\alpha)$ **b** $\sin\left(\frac{\pi}{2} + \alpha\right)$ **c** $\cos\left(\frac{\pi}{2} - x\right)$ **d** $\sin(-x)$

e $\cos\left(\frac{\pi}{2} + x\right)$ **f** $\sin\left(\frac{\pi}{2} - \alpha\right)$ **g** $\sin\left(\frac{3\pi}{2} + \alpha\right)$ **h** $\cos\left(\frac{3\pi}{2} - x\right)$

Example 6

4 If $\sin x = 0.5$ and $\frac{\pi}{2} < x < \pi$, find $\cos x$ and $\tan x$.

5 If $\cos x = -0.7$ and $\pi < x < \frac{3\pi}{2}$, find $\sin x$ and $\tan x$.

6 If $\sin x = -0.5$ and $\pi < x < \frac{3\pi}{2}$, find $\cos x$ and $\tan x$.

7 If $\sin x = -0.3$ and $\frac{3\pi}{2} < x < 2\pi$, find $\cos x$ and $\tan x$.

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10C Solution of equations involving sine and cosine

If a trigonometric equation has a solution, then it will have a corresponding solution in each 'cycle' of its domain. Such an equation is solved by using the symmetry of the graph to obtain solutions within one 'cycle' of the function. Other solutions may be obtained by adding multiples of the period to these solutions.

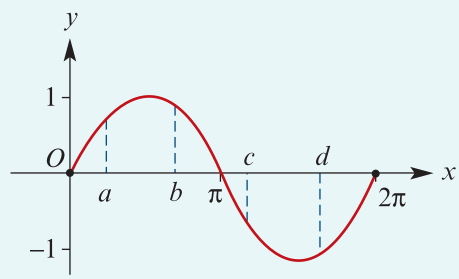
Example 7

The graph of $y = f(x)$ for

$$f : [0, 2\pi] \to \mathbb{R},\ f(x) = \sin x$$

is shown.

For each pronumeral marked on the x-axis, find the other x-value which has the same y-value.



Solution

For $x = a$, the other value is $\pi - a$.

For $x = b$, the other value is $\pi - b$.

For $x = c$, the other value is $2\pi - (c - \pi) = 3\pi - c$.

For $x = d$, the other value is $\pi + (2\pi - d) = 3\pi - d$.

Example 8

Solve the equation $\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$ for $x \in [0, 2\pi]$.

Solution

Let $\theta = 2x + \frac{\pi}{3}$. Note that

$$\begin{aligned} 0 \le x \le 2\pi \quad &\Leftrightarrow \quad 0 \le 2x \le 4\pi \\ &\Leftrightarrow \quad \frac{\pi}{3} \le 2x + \frac{\pi}{3} \le \frac{13\pi}{3} \\ &\Leftrightarrow \quad \frac{\pi}{3} \le \theta \le \frac{13\pi}{3} \end{aligned}$$

To solve $\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$ for $x \in [0, 2\pi]$, we first solve $\sin\theta = \frac{1}{2}$ for $\frac{\pi}{3} \le \theta \le \frac{13\pi}{3}$.

Consider $\sin\theta = \frac{1}{2}$.

$$\therefore \quad \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \text{ or } 2\pi + \frac{\pi}{6} \text{ or } 2\pi + \frac{5\pi}{6} \text{ or } 4\pi + \frac{\pi}{6} \text{ or } 4\pi + \frac{5\pi}{6} \text{ or } \dots$$

The solutions $\frac{\pi}{6}$ and $\frac{29\pi}{6}$ are not required, as they lie outside the restricted domain for θ.

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For $\frac{\pi}{3} \le \theta \le \frac{13\pi}{3}$:

$$\theta = \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{17\pi}{6} \text{ or } \frac{25\pi}{6}$$

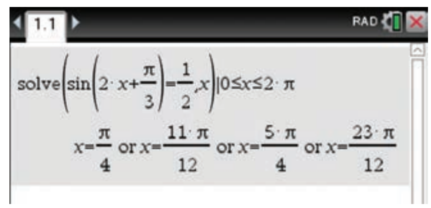
$$\therefore \quad 2x + \frac{2\pi}{6} = \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{17\pi}{6} \text{ or } \frac{25\pi}{6}$$

$$\therefore \quad 2x = \frac{3\pi}{6} \text{ or } \frac{11\pi}{6} \text{ or } \frac{15\pi}{6} \text{ or } \frac{23\pi}{6}$$

$$\therefore \quad x = \frac{\pi}{4} \text{ or } \frac{11\pi}{12} \text{ or } \frac{5\pi}{4} \text{ or } \frac{23\pi}{12}$$

Using the TI-Nspire

- Ensure your calculator is in radian mode. (To change the mode, go to [on] > **Settings** > **Document Settings**.)
- Complete as shown.

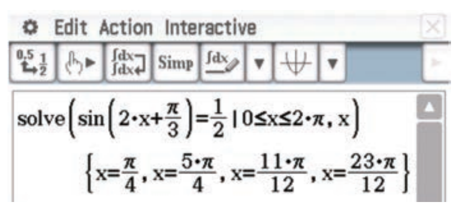


Note: The **Graph** application has its own settings, which are accessed from a **Graph** page using [menu] > **Settings**.

Using the Casio ClassPad

- Open the Main $\sqrt{\alpha}$ application.
- Ensure your calculator is in radian mode (with **Rad** in the status bar).
- Using the [Math1] and [Math3] keyboards, enter and then highlight

$$\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2} \mid 0 \le x \le 2\pi$$



- Select **Interactive** > **Equation/Inequality** > **solve**.

Exercise 10C

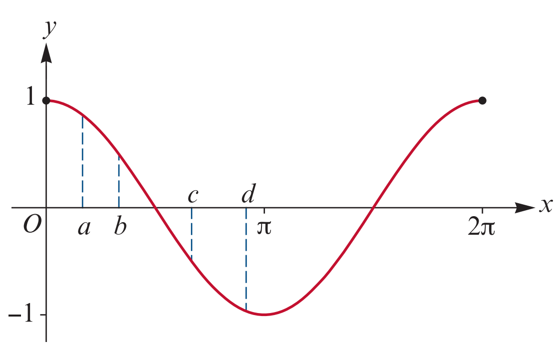
Skillsheet
Example 7

1 The graph of $y = f(x)$ for

$f\colon [0, 2\pi] \to \mathbb{R}, f(x) = \cos x$

is shown.

For each pronumeral marked on the x-axis, find the other x-value which has the same y-value.



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Example 8

2 Solve each of the following for $x \in [0, 2\pi]$:

a $\sin x = -\dfrac{\sqrt{3}}{2}$ **b** $\sin(2x) = -\dfrac{\sqrt{3}}{2}$ **c** $2\cos(2x) = -1$

d $\sin\left(x + \dfrac{\pi}{3}\right) = -\dfrac{1}{2}$ **e** $2\cos\left(2\left(x + \dfrac{\pi}{3}\right)\right) = -1$ **f** $2\sin\left(2x + \dfrac{\pi}{3}\right) = -\sqrt{3}$

3 Solve each of the following for $x \in [-\pi, \pi]$:

a $\sin x + \dfrac{1}{2} = 0$ **b** $\sin(3x) = 0$ **c** $\cos\left(\dfrac{x}{2}\right) = 1$

d $\sin\left(x - \dfrac{\pi}{6}\right) = -\dfrac{1}{2}$ **e** $2\cos\left(2\left(x + \dfrac{\pi}{6}\right)\right) = -1$

10D Transformations of the graphs of sine and cosine

The graphs of functions with rules of the form

$$f(x) = a\sin(n(x + \varepsilon)) + b \qquad \text{and} \qquad f(x) = a\cos(n(x + \varepsilon)) + b$$

can be obtained from the graphs of $y = \sin x$ and $y = \cos x$ by transformations.

Example 9

Sketch the graph of the function

$$h\colon [0, 2\pi] \to \mathbb{R},\ h(x) = 3\cos\left(2x + \frac{\pi}{3}\right) + 1$$

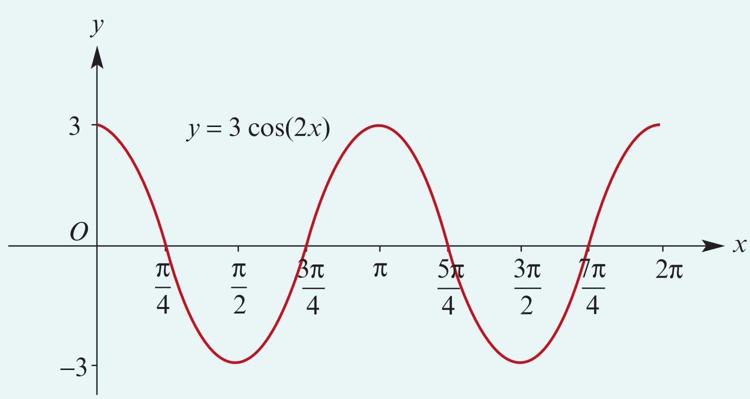
Solution

We can write $h(x) = 3\cos\left(2\left(x + \dfrac{\pi}{6}\right)\right) + 1$.

The graph of $y = h(x)$ is obtained from the graph of $y = \cos x$ by:

- a dilation of factor $\frac{1}{2}$ from the y-axis
- a dilation of factor 3 from the x-axis
- a translation of $\dfrac{\pi}{6}$ units in the negative direction of the x-axis
- a translation of 1 unit in the positive direction of the y-axis.

First apply the two dilations to the graph of $y = \cos x$.



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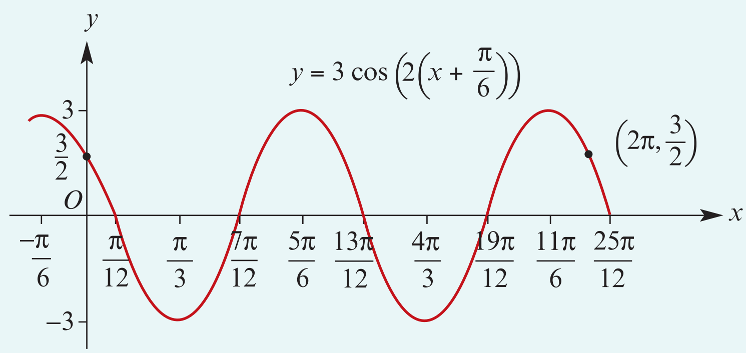
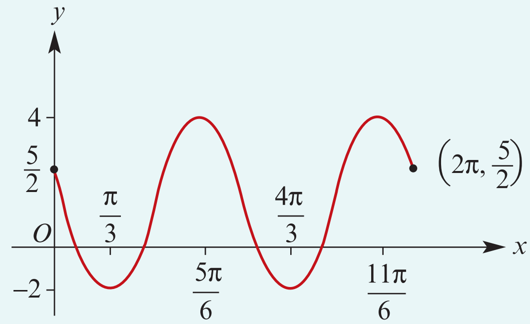
Next apply the translation $\frac{\pi}{6}$ units in the negative direction of the x-axis.



Apply the final translation and restrict the graph to the required domain.



Section summary

For the graphs of $y = a\sin(nx)$ and $y = a\cos(nx)$, where $a > 0$ and $n > 0$:

- Period $= \frac{2\pi}{n}$
- Amplitude $= a$
- Range $= [-a, a]$

Exercise 10D

Skillsheet

Example 9

1 Sketch the graph of each of the following for the stated domain:

a $f(x) = \sin(2x), \ x \in [0, 2\pi]$

b $f(x) = \cos\left(x + \frac{\pi}{3}\right), \ x \in \left[\frac{-\pi}{3}, \pi\right]$

c $f(x) = \cos\left(2\left(x + \frac{\pi}{3}\right)\right), \ x \in [0, \pi]$

d $f(x) = 2\sin(3x) + 1, \ x \in [0, \pi]$

e $f(x) = 2\sin\left(x - \frac{\pi}{4}\right) + \sqrt{3}, \ x \in [0, 2\pi]$

f $f(x) = \cos(2x), \ x \in [-\pi, \pi]$

2 Sketch the graph of each of the following for the stated domain:

a $f(x) = \sin\left(x + \frac{\pi}{6}\right), \ x \in [-\pi, \pi]$

b $f(x) = \sin\left(2\left(x + \frac{\pi}{4}\right)\right), \ x \in [-\pi, \pi]$

c $f(x) = 2\cos\left(\frac{x}{3}\right) + 1, \ x \in [0, 6\pi]$

d $f(x) = 2\cos\left(x - \frac{\pi}{3}\right) + \sqrt{3}, \ x \in [0, 2\pi]$

e $f(x) = \cos(\pi x), \ x \in [-2, 2]$

f $f(x) = \cos\left(\frac{\pi x}{6}\right), \ x \in [-12, 12]$

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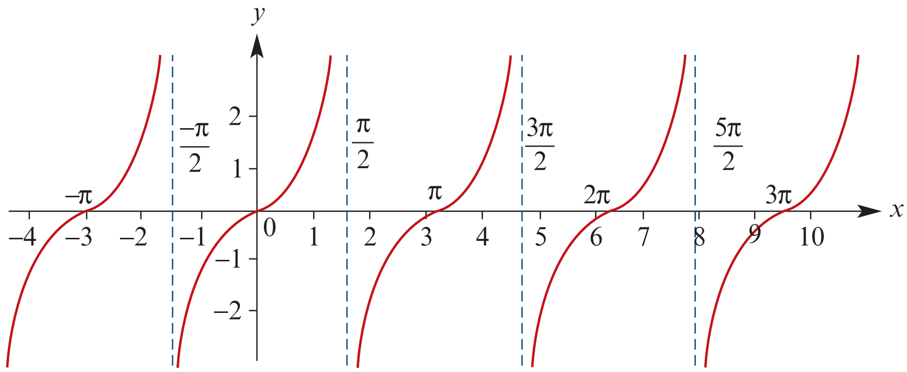
10E The tangent function

Recall that the tangent function is given by $\tan x = \dfrac{\sin x}{\cos x}$ for $\cos x \neq 0$.

The graph of $y = \tan x$ is shown below.



Properties of the tangent function

- The graph repeats itself every π units, i.e. the period of tan is π.
- The range of tan is $\mathbb{R}$.
- The vertical asymptotes have equations $x = \dfrac{(2k+1)\pi}{2}$ where $k \in \mathbb{Z}$.
- The axis intercepts are at $x = k\pi$ where $k \in \mathbb{Z}$.

▶ Symmetry properties of tangent

Using symmetry properties of sine and cosine, we have

$$\tan(\pi - \theta) = \frac{\sin(\pi - \theta)}{\cos(\pi - \theta)} = \frac{\sin\theta}{-\cos\theta} = -\tan\theta$$

Similarly, we obtain:

- $\tan(\pi + \theta) = \tan\theta$
- $\tan(2\pi - \theta) = -\tan\theta$
- $\tan(-\theta) = -\tan\theta$
- $\tan\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{\cos\theta}{\sin\theta}$
- $\tan\left(\dfrac{\pi}{2} + \theta\right) = -\dfrac{\cos\theta}{\sin\theta}$

Note: We will see in Section 11A that $\dfrac{\cos\theta}{\sin\theta}$ can be written as $\cot\theta$.

Example 10

Find the exact value of:

a $\tan\left(\dfrac{4\pi}{3}\right)$

b $\tan 330^\circ$

Solution

a
$$\tan\left(\frac{4\pi}{3}\right) = \tan\left(\pi + \frac{\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

b
$$\tan 330^\circ = \tan(360^\circ - 30^\circ) = -\tan 30^\circ = -\frac{1}{\sqrt{3}}$$

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▶ Solution of equations involving the tangent function

We now consider the solution of equations involving the tangent function, which can be applied to finding the x-axis intercepts for graphs of the tangent function.

The method is similar to that used for solving equations involving sine and cosine, except that only one solution needs to be found and then all other solutions are one period length apart.

Example 11

Solve the equation $3\tan(2x) = \sqrt{3}$ for $x \in (0, 2\pi)$.

Solution

$$3\tan(2x) = \sqrt{3}$$

$$\tan(2x) = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

$$\therefore \quad 2x = \frac{\pi}{6} \text{ or } \frac{7\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{19\pi}{6}$$

$$x = \frac{\pi}{12} \text{ or } \frac{7\pi}{12} \text{ or } \frac{13\pi}{12} \text{ or } \frac{19\pi}{12}$$

Explanation

Since we want solutions for x in $(0, 2\pi)$, we find solutions for $2x$ in $(0, 4\pi)$.

Once we have found one solution for $2x$, we can obtain all other solutions by adding and subtracting multiples of π.

Example 12

Solve the equation $\tan\left(\frac{1}{2}\left(x - \frac{\pi}{4}\right)\right) = -1$ for $x \in [-2\pi, 2\pi]$.

Solution

Let $\theta = \frac{1}{2}\left(x - \frac{\pi}{4}\right)$. Note that

$$-2\pi \le x \le 2\pi \iff -\frac{9\pi}{4} \le x - \frac{\pi}{4} \le \frac{7\pi}{4}$$

$$\iff -\frac{9\pi}{8} \le \frac{1}{2}\left(x - \frac{\pi}{4}\right) \le \frac{7\pi}{8}$$

$$\iff -\frac{9\pi}{8} \le \theta \le \frac{7\pi}{8}$$

We first solve the equation $\tan\theta = -1$ for $-\frac{9\pi}{8} \le \theta \le \frac{7\pi}{8}$:

$$\theta = \frac{-\pi}{4} \text{ or } \frac{3\pi}{4}$$

$$\frac{1}{2}\left(x - \frac{\pi}{4}\right) = \frac{-\pi}{4} \text{ or } \frac{3\pi}{4}$$

$$x - \frac{\pi}{4} = \frac{-\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$\therefore \quad x = \frac{-\pi}{4} \text{ or } \frac{7\pi}{4}$$

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▶ Graphing the tangent function

When graphing a transformation of the tangent function:

- Find the period.
- Find the equations of the asymptotes.
- Find the intercepts with the axes.

Example 13

Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

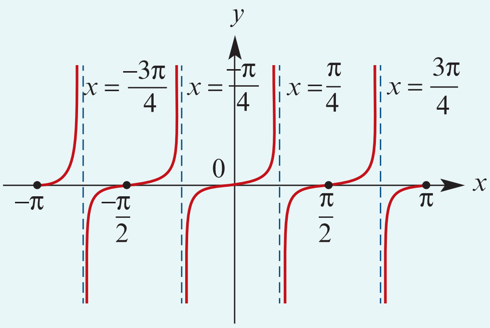
a $y = 3\tan(2x)$ **b** $y = -2\tan(3x)$

Solution

a Period $= \frac{\pi}{n} = \frac{\pi}{2}$

Asymptotes: $x = \frac{(2k+1)\pi}{4}$, $k \in \mathbb{Z}$

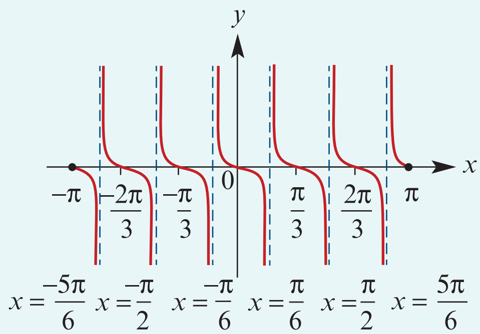
Axis intercepts: $x = \frac{k\pi}{2}$, $k \in \mathbb{Z}$



b Period $= \frac{\pi}{n} = \frac{\pi}{3}$

Asymptotes: $x = \frac{(2k+1)\pi}{6}$, $k \in \mathbb{Z}$

Axis intercepts: $x = \frac{k\pi}{3}$, $k \in \mathbb{Z}$



Section summary

- The tangent function is given by $\tan\theta = \frac{\sin\theta}{\cos\theta}$ for $\cos\theta \neq 0$.
- The graph of $y = \tan x$:
  - The period is π.
  - The vertical asymptotes have equations $x = \frac{(2k+1)\pi}{2}$ where $k \in \mathbb{Z}$.
  - The axis intercepts are at $x = k\pi$ where $k \in \mathbb{Z}$.
- Useful symmetry properties:
  - $\tan(\pi + \theta) = \tan\theta$
  - $\tan(-\theta) = -\tan\theta$

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Exercise 10E

Example 10

1 Find the exact value of each of the following:

a $\tan\left(\frac{5\pi}{4}\right)$ **b** $\tan\left(-\frac{2\pi}{3}\right)$ **c** $\tan\left(-\frac{29\pi}{6}\right)$

2 Find the exact value of each of the following:

a $\tan 240°$ **b** $\tan(-150°)$ **c** $\tan 315°$

3 If $\tan x = \frac{1}{4}$ and $\pi \le x \le \frac{3\pi}{2}$, find the exact value of:

a $\sin x$ **b** $\cos x$ **c** $\tan(-x)$ **d** $\tan(\pi - x)$

4 If $\tan x = -\frac{\sqrt{3}}{2}$ and $\frac{\pi}{2} \le x \le \pi$, find the exact value of:

a $\sin x$ **b** $\cos x$ **c** $\tan(-x)$ **d** $\tan(x - \pi)$

Example 11

5 Solve each of the following equations for x in the stated interval:

a $\tan x = -1,\ x \in (0, 2\pi)$ **b** $\tan x = \sqrt{3},\ x \in (0, 2\pi)$

c $\tan x = \frac{1}{\sqrt{3}},\ x \in (0, 2\pi)$ **d** $\tan(2x) = 1,\ x \in (-\pi, \pi)$

e $\tan(2x) = \sqrt{3},\ x \in (-\pi, \pi)$ **f** $\tan(2x) = -\frac{1}{\sqrt{3}},\ x \in (-\pi, \pi)$

Example 12

6 Solve each of the following equations for x in the stated interval:

a $\tan\left(2\left(x - \frac{\pi}{4}\right)\right) = 1,\ x \in (0, 2\pi)$ **b** $\tan\left(2\left(x - \frac{\pi}{4}\right)\right) = -1,\ x \in (-\pi, \pi)$

c $\tan\left(3\left(x - \frac{\pi}{6}\right)\right) = \sqrt{3},\ x \in (-\pi, \pi)$ **d** $\tan\left(\frac{1}{2}\left(x - \frac{\pi}{6}\right)\right) = -\frac{1}{\sqrt{3}},\ x \in (-\pi, \pi)$

Example 13

7 Sketch the graph of each of the following:

a $y = \tan(2x)$ **b** $y = \tan(3x)$ **c** $y = -\tan(2x)$

d $y = 3\tan x$ **e** $y = \tan\left(\frac{x}{2}\right)$ **f** $y = 2\tan\left(x + \frac{\pi}{4}\right)$

g $y = 3\tan x + 1$ **h** $y = 2\tan\left(x + \frac{\pi}{2}\right) + 1$ **i** $y = 3\tan\left(2\left(x - \frac{\pi}{4}\right)\right) - 2$

8 Sketch the graph of each of the following for the stated domain:

a $y = \tan\left(x + \frac{\pi}{3}\right) + \sqrt{3}$ for $x \in [0, 2\pi]$ **b** $y = \tan\left(\frac{x}{2}\right)$ for $x \in [0, 4\pi]$

c $y = \tan\left(\frac{\pi x}{2}\right)$ for $x \in [0, 4]$

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10F General solution of trigonometric equations

We have seen how to solve equations involving circular functions over a restricted domain. We now consider the general solutions of such equations over the maximal domain for each function.

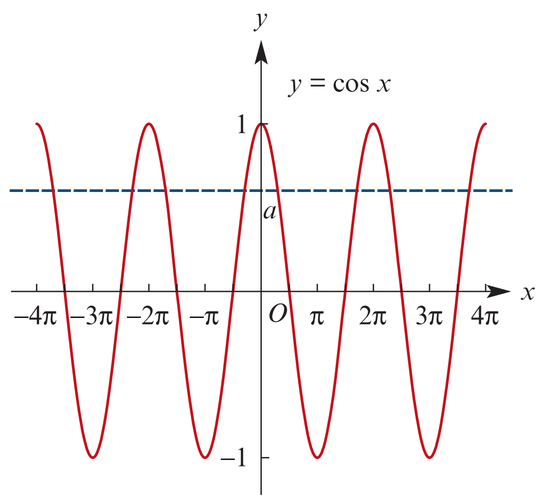
By convention:

- $\cos^{-1}$ has range $[0, \pi]$
- $\sin^{-1}$ has range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $\tan^{-1}$ has range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

For example:

$$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}, \qquad \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}, \qquad \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}, \qquad \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

If an equation involving a circular function has one or more solutions in one 'cycle', then it will have corresponding solutions in each 'cycle' of its domain, i.e. there will be infinitely many solutions.



For example, consider the equation

$$\cos x = a$$

for some fixed $a \in [-1, 1]$. The solution in the interval $[0, \pi]$ is given by

$$x = \cos^{-1}(a)$$

By the symmetry properties of the cosine function, the other solutions are given by

$$-\cos^{-1}(a),\ \pm 2\pi + \cos^{-1}(a),\ \pm 2\pi - \cos^{-1}(a),\ \pm 4\pi + \cos^{-1}(a),\ \pm 4\pi - \cos^{-1}(a),\ \dots$$

In general, we have the following:

- For $a \in [-1, 1]$, the general solution of the equation $\cos x = a$ is

  $$x = 2n\pi \pm \cos^{-1}(a), \qquad \text{where } n \in \mathbb{Z}$$

- For $a \in \mathbb{R}$, the general solution of the equation $\tan x = a$ is

  $$x = n\pi + \tan^{-1}(a), \qquad \text{where } n \in \mathbb{Z}$$

- For $a \in [-1, 1]$, the general solution of the equation $\sin x = a$ is

  $$x = 2n\pi + \sin^{-1}(a) \quad \text{or} \quad x = (2n+1)\pi - \sin^{-1}(a), \qquad \text{where } n \in \mathbb{Z}$$

Note: An alternative and more concise way to express the general solution of $\sin x = a$ is $x = n\pi + (-1)^n \sin^{-1}(a)$, where $n \in \mathbb{Z}$.

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Example 14

Find the general solution of each of the following equations:

a $\cos x = 0.5$ **b** $\sqrt{3}\tan(3x) = 1$ **c** $2\sin x = \sqrt{2}$

Solution

a $\cos x = 0.5$

$$\begin{aligned} x &= 2n\pi \pm \cos^{-1}(0.5) \\ &= 2n\pi \pm \frac{\pi}{3} \\ &= \frac{(6n \pm 1)\pi}{3}, \quad n \in \mathbb{Z} \end{aligned}$$

b $\tan(3x) = \dfrac{1}{\sqrt{3}}$

$$\begin{aligned} 3x &= n\pi + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \\ &= n\pi + \frac{\pi}{6} \\ &= \frac{(6n+1)\pi}{6} \\ x &= \frac{(6n+1)\pi}{18}, \quad n \in \mathbb{Z} \end{aligned}$$

c $\sin x = \dfrac{\sqrt{2}}{2} = \dfrac{1}{\sqrt{2}}$

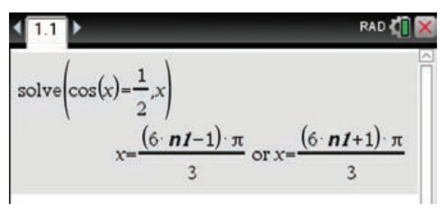
$$\begin{aligned} x &= 2n\pi + \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) \\ &= 2n\pi + \frac{\pi}{4} \\ &= \frac{(8n+1)\pi}{4}, \quad n \in \mathbb{Z} \end{aligned}$$

or

$$\begin{aligned} x &= (2n+1)\pi - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) \\ &= (2n+1)\pi - \frac{\pi}{4} \\ &= \frac{(8n+3)\pi}{4}, \quad n \in \mathbb{Z} \end{aligned}$$

Using the TI-Nspire

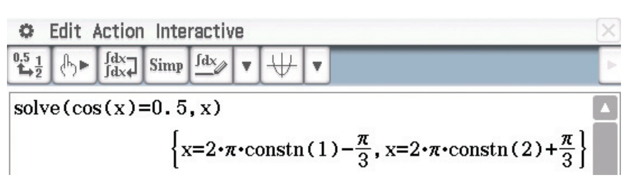
- Make sure the calculator is in radian mode.
- Use **solve** from the **Algebra** menu and complete as shown. Note the use of $\frac{1}{2}$ rather than 0.5 to ensure that the answer is exact.



Using the Casio ClassPad

- Check that the calculator is in radian mode.
- In Main $\sqrt{\alpha}$, enter and highlight the equation $\cos(x) = 0.5$.
- Select **Interactive** > **Equation/Inequality** > **solve**. Then tap (EXE).
- To view the entire solution, rotate the screen by selecting Rotate.

Note: Replace constn(1) and constn(2) with n in the written answer.



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Example 15

Find the first three positive solutions of each of the following equations:

a $\cos x = 0.5$ **b** $\sqrt{3}\tan(3x) = 1$ **c** $2\sin x = \sqrt{2}$

Solution

a The general solution (from Example 14a) is given by $x = \dfrac{(6n \pm 1)\pi}{3}$, $n \in \mathbb{Z}$.

When $n = 0$, $x = \pm\dfrac{\pi}{3}$, and when $n = 1$, $x = \dfrac{5\pi}{3}$ or $x = \dfrac{7\pi}{3}$.

Thus the first three positive solutions of $\cos x = 0.5$ are $x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}, \dfrac{7\pi}{3}$.

b The general solution (from Example 14b) is given by $x = \dfrac{(6n + 1)\pi}{18}$, $n \in \mathbb{Z}$.

When $n = 0$, $x = \dfrac{\pi}{18}$, and when $n = 1$, $x = \dfrac{7\pi}{18}$, and when $n = 2$, $x = \dfrac{13\pi}{18}$.

Thus the first three positive solutions of $\sqrt{3}\tan(3x) = 1$ are $x = \dfrac{\pi}{18}, \dfrac{7\pi}{18}, \dfrac{13\pi}{18}$.

c The general solution (from Example 14c) is $x = \dfrac{(8n + 1)\pi}{4}$ or $x = \dfrac{(8n + 3)\pi}{4}$, $n \in \mathbb{Z}$.

When $n = 0$, $x = \dfrac{\pi}{4}$ or $x = \dfrac{3\pi}{4}$, and when $n = 1$, $x = \dfrac{9\pi}{4}$ or $x = \dfrac{11\pi}{4}$.

Thus the first three positive solutions of $2\sin x = \sqrt{2}$ are $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{9\pi}{4}$.

Example 16

Find the general solution for each of the following:

a $\sin\left(x - \dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$ **b** $\tan\left(2x - \dfrac{\pi}{3}\right) = 1$

Solution

a

$$\sin\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$x - \frac{\pi}{3} = n\pi + (-1)^n \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\therefore \quad x = n\pi + (-1)^n\left(\frac{\pi}{3}\right) + \frac{\pi}{3}, \quad n \in \mathbb{Z}$$

The solutions are $x = \dfrac{(3n + 2)\pi}{3}$ for n even and $x = n\pi$ for n odd.

b

$$\tan\left(2x - \frac{\pi}{3}\right) = 1$$

$$2x - \frac{\pi}{3} = n\pi + \frac{\pi}{4}$$

$$2x = n\pi + \frac{\pi}{4} + \frac{\pi}{3}$$

$$\therefore \quad x = \frac{1}{2}\left(n\pi + \frac{7\pi}{12}\right)$$

$$= \frac{(12n + 7)\pi}{24}, \quad n \in \mathbb{Z}$$

ISBN 978-1-316-63608-4

Section summary

- For $a \in [-1, 1]$, the general solution of the equation $\cos x = a$ is

$$x = 2n\pi \pm \cos^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

- For $a \in \mathbb{R}$, the general solution of the equation $\tan x = a$ is

$$x = n\pi + \tan^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

- For $a \in [-1, 1]$, the general solution of the equation $\sin x = a$ is

$$x = 2n\pi + \sin^{-1}(a) \quad \text{or} \quad x = (2n + 1)\pi - \sin^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

Exercise 10F

Skillsheet

1 Evaluate each of the following for:

i $n = 1$ **ii** $n = 2$ **iii** $n = -2$

a $2n\pi \pm \cos^{-1}(1)$ **b** $2n\pi \pm \cos^{-1}\left(-\frac{1}{2}\right)$

Example 14

2 Find the general solution of each of the following equations:

a $\cos x = \frac{\sqrt{3}}{2}$ **b** $2\sin(3x) = \sqrt{3}$ **c** $\sqrt{3}\tan x = 3$

Example 15

3 Find the first two positive solutions of each of the following equations:

a $\sin x = 0.5$ **b** $2\cos(2x) = \sqrt{3}$ **c** $\sqrt{3}\tan(2x) = -3$

4 Given that a trigonometric equation has general solution $x = n\pi + (-1)^n \sin^{-1}\left(\frac{1}{2}\right)$, where $n \in \mathbb{Z}$, find the solutions of the equation in the interval $[-2\pi, 2\pi]$.

5 Given that a trigonometric equation has general solution $x = 2n\pi \pm \cos^{-1}\left(\frac{1}{2}\right)$, where $n \in \mathbb{Z}$, find the solutions of the equation in the interval $[-\pi, 2\pi]$.

Example 16

6 Find the general solution for each of the following:

a $\cos\left(2\left(x + \frac{\pi}{3}\right)\right) = \frac{1}{2}$ **b** $2\tan\left(2\left(x + \frac{\pi}{4}\right)\right) = 2\sqrt{3}$ **c** $2\sin\left(x + \frac{\pi}{3}\right) = -1$

7 Find the general solution of $2\cos\left(2x + \frac{\pi}{4}\right) = \sqrt{2}$ and hence find all the solutions for x in the interval $(-2\pi, 2\pi)$.

8 Find the general solution of $\sqrt{3}\tan\left(\frac{\pi}{6} - 3x\right) - 1 = 0$ and hence find all the solutions for x in the interval $[-\pi, 0]$.

9 Find the general solution of $2\sin(4\pi x) + \sqrt{3} = 0$ and hence find all the solutions for x in the interval $[-1, 1]$.

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10G Applications of circular functions

A **sinusoidal function** has a rule of the form $y = a\sin(nt + \varepsilon) + b$ or, equivalently, of the form $y = a\cos(nt + \varepsilon) + b$. Such functions can be used to model periodic phenomena.

Example 17

The height, $h(t)$ metres, of the tide above mean sea level at a harbour entrance over one day is given by the rule $h(t) = 3\sin\left(\frac{\pi t}{6}\right)$, where t is the number of hours after midnight.

a Draw the graph of $y = h(t)$ for $0 \le t \le 24$.

b When was high tide?

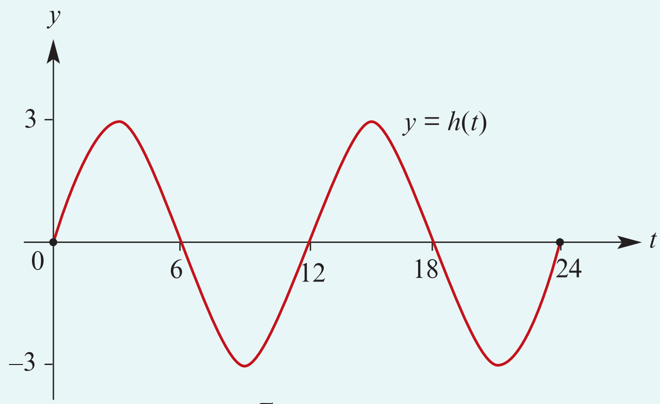
c What was the height of the high tide?

d What was the height of the tide at 8 a.m.?

e A boat can only enter the harbour when the tide is at least 1 metre above mean sea level. When could the boat enter the harbour during this particular day?

Solution

a



Note: Period $= 2\pi \div \frac{\pi}{6} = 12$

b High tide occurs when $h(t) = 3$:

$$3\sin\left(\frac{\pi t}{6}\right) = 3$$

$$\sin\left(\frac{\pi t}{6}\right) = 1$$

$$\frac{\pi t}{6} = \frac{\pi}{2}, \frac{5\pi}{2}$$

$$\therefore \quad t = 3,\ 15$$

i.e. high tide occurs at 03:00 and 15:00 (3 p.m.).

c The high tide has height 3 metres above the mean height.

d $h(8) = 3\sin\left(\frac{8\pi}{6}\right) = 3\sin\left(\frac{4\pi}{3}\right) = 3 \times \frac{-\sqrt{3}}{2} = -\frac{3\sqrt{3}}{2}$

The water is $\frac{3}{2}\sqrt{3}$ metres below the mean height at 8 a.m.

e First consider when $h(t) = 1$:

$$3\sin\left(\frac{\pi t}{6}\right) = 1$$

$$\sin\left(\frac{\pi t}{6}\right) = \frac{1}{3}$$

$$\therefore \quad t = 0.649,\ 5.351,\ 12.649,\ 17.351$$

i.e. the water is at height 1 metre at 00:39, 05:21, 12:39, 17:21.
Thus the boat can enter the harbour between 00:39 and 05:21, and between 12:39 and 17:21.

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Exercise 10G

Example 17

1 The number of hours of daylight at a point on the Arctic Circle is given approximately by $d = 12 - 12\cos\left(\frac{\pi}{6}\left(t + \frac{1}{3}\right)\right)$, where t is the number of months which have elapsed since 1 January.

a **i** Find d on 21 December ($t \approx 11.7$).

ii Find d on 21 June ($t \approx 5.7$).

b When will there be 5 hours of daylight?

2 The depth, $D(t)$ metres, of water at the entrance to a harbour at t hours after midnight on a particular day is given by $D(t) = 12 + 4\sin\left(\frac{\pi t}{6}\right)$, $0 \le t \le 24$.

a Sketch the graph of $D(t)$ for $0 \le t \le 24$.

b Find the values of t for which $D(t) \ge 12$.

c Boats which need a depth of w metres are permitted to enter the harbour only if the depth of the water at the entrance is at least w metres for a continuous period of 1 hour. Find, correct to one decimal place, the largest value of w which satisfies this condition.

3 A particle moves on a straight line, OX, and its distance x metres from O at time t seconds is given by $x = 4 + 3\sin(2\pi t)$.

a Find its greatest distance from O.

b Find its least distance from O.

c Find the times at which it is 7 metres from O for $0 \le t \le 2$.

d Find the times at which it is 1 metre from O for $0 \le t \le 2$.

e Describe the motion of the particle.

4 The temperature, A°C, inside a house at t hours after 4 a.m. is given by the rule $A = 21 - 3\cos\left(\frac{\pi t}{12}\right)$, for $0 \le t \le 24$. The temperature, B°C, outside the house at the same time is given by $B = 22 - 5\cos\left(\frac{\pi t}{12}\right)$, for $0 \le t \le 24$.

a Find the temperature inside the house at 8 a.m.

b Write down an expression for $D = A - B$, the difference between the inside and outside temperatures.

c Sketch the graph of D for $0 \le t \le 24$.

d Determine when the inside temperature is less than the outside temperature.

5 The water level on a beach wall is given by $d(t) = 6 + 4\cos\left(\frac{\pi}{6}t - \frac{\pi}{3}\right)$, where t is the number of hours after midnight and d is the depth of the water in metres.

a What is the earliest time of day at which the water is at its highest?

b When is the water 2 m up the wall?

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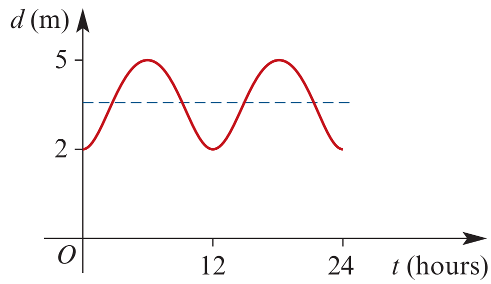
6 The graph shows the distance, $d(t)$, of the tip of the hour hand of a large clock from the ceiling at time t hours.



a The function d is sinusoidal. Find:

i the amplitude

ii the period

iii the rule for $d(t)$

iv the length of the hour hand.

b At what times is the distance less than 3.5 metres from the ceiling?

7 In a tidal river, the time between high tide and low tide is 8 hours. The average depth of water at a point in the river is 4 metres; at high tide the depth is 5 metres.

a Sketch the graph of the depth of water at the point for the time interval from 0 to 24 hours if the relationship between time and depth is sinusoidal and there is a high tide at noon.

b If a boat requires a depth of 4 metres of water in order to sail, at what time before noon can it enter the point in the river and by what time must it leave if it is not to be stranded?

c If a boat requires a depth of 3.5 metres of water in order to sail, at what time before noon can it enter the point in the river and by what time must it leave if it is not to be stranded?

8 The population of a particular species of ant varies with time. The population, $N(t)$, at time t weeks after 1 January 2016 is given by

$$N(t) = 3000\sin\left(\frac{\pi(t-10)}{26}\right) + 4000$$

a For the rule $N(t)$, state:

i the period **ii** the amplitude **iii** the range.

b **i** State the values of $N(0)$ and $N(100)$.

ii Sketch the graph of $y = N(t)$ for $t \in [0, 100]$.

c Find the values of $t \in [0, 100]$ for which the population is:

i 7000 **ii** 1000

d Find $\{ t \in [0, 100] : N(t) > 5500 \}$. That is, find the intervals of time during the first 100 weeks for which the population of ants is greater than 5500.

e A second population of ants also varies with time. The rule for the population, $M(t)$, at time t weeks after 1 January 2016 is of the form

$$M(t) = a\sin\left(\frac{\pi(t-c)}{b}\right) + d$$

where a, b, c and d are positive constants. Find a set of possible values for the constants a, b, c and d given that the population has the following properties:

- the maximum population is 40 000 and occurs at $t = 10$
- the minimum population is 10 000 and occurs at $t = 20$
- the maximum and minimum values do not occur between $t = 10$ and $t = 20$.

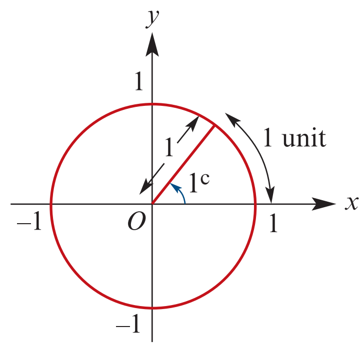
ISBN 978-1-316-63608-4

Chapter summary

- **Definition of a radian**

  One radian (written 1^c) is the angle formed at the centre of the unit circle by an arc of length 1 unit.

  $$1^c = \frac{180^\circ}{\pi} \qquad 1^\circ = \frac{\pi^c}{180}$$



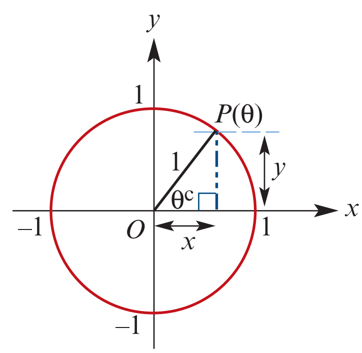
- **Sine and cosine functions**

  x-coordinate of $P(\theta)$ on unit circle:

  $$x = \cos\theta, \qquad \theta \in \mathbb{R}$$

  y-coordinate of $P(\theta)$ on unit circle:

  $$y = \sin\theta, \qquad \theta \in \mathbb{R}$$



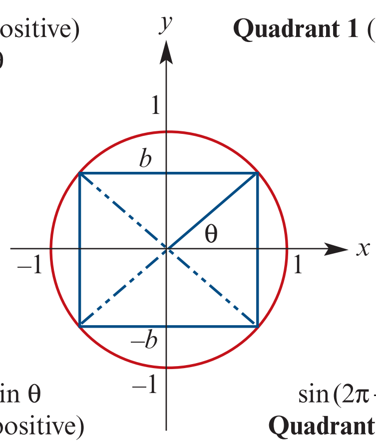
- **Tangent function**

  $$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \text{for } \cos\theta \neq 0$$

- **Symmetry properties of circular functions**

Quadrant 2 (sin is positive)
$\sin(\pi - \theta) = b = \sin\theta$

Quadrant 1 (all functions are positive)
e.g. $\sin\theta = b$



$\sin(\pi + \theta) = -b = -\sin\theta$
Quadrant 3 (tan is positive)

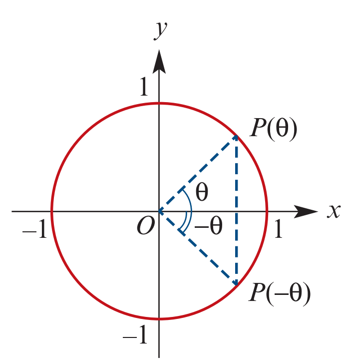
$\sin(2\pi - \theta) = -b = -\sin\theta$
Quadrant 4 (cos is positive)

Negative angles:

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$



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Review

Complementary relationships:

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta, \qquad \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \qquad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

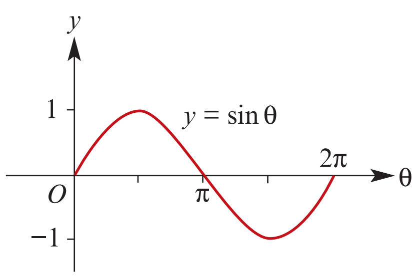
- Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

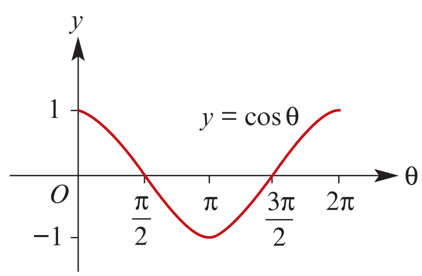
- Exact values of circular functions

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	undefined

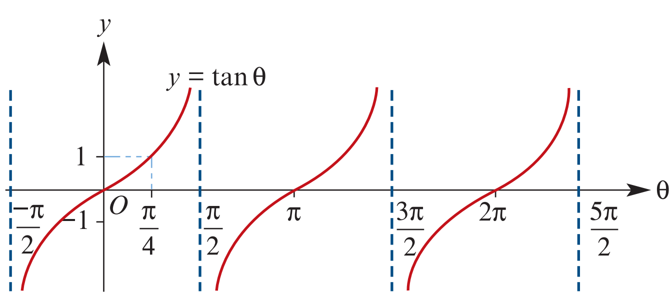
- Graphs of circular functions



Amplitude = 1
Period = 2π



Amplitude = 1
Period = 2π



Amplitude is undefined
Period = π

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- **Transformations of the graphs of circular functions**

For the graphs of $y = a\sin(n(x+\varepsilon)) + b$ and $y = a\cos(n(x+\varepsilon)) + b$, where $a, n \in \mathbb{R}^+$:

- Period $= \dfrac{2\pi}{n}$
- Amplitude $= a$
- Range $= [-a+b, a+b]$

For the graph of $y = a\tan(n(x+\varepsilon)) + b$, where $n \in \mathbb{R}^+$:

- Period $= \dfrac{\pi}{n}$
- Asymptotes: $x = \dfrac{(2k+1)\pi}{2n} - \varepsilon$, where $k \in \mathbb{Z}$

- **General solution of trigonometric equations**

- For $a \in [-1, 1]$, the general solution of the equation $\cos x = a$ is

$$x = 2n\pi \pm \cos^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

- For $a \in \mathbb{R}$, the general solution of the equation $\tan x = a$ is

$$x = n\pi + \tan^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

- For $a \in [-1, 1]$, the general solution of the equation $\sin x = a$ is

$$x = 2n\pi + \sin^{-1}(a) \quad \text{or} \quad x = (2n+1)\pi - \sin^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

Short-answer questions

1 Convert each of the following to radian measure in terms of π:

a 390° **b** 840° **c** 1110° **d** 1065° **e** 165°
f 450° **g** 420° **h** 390° **i** 40°

2 Convert each of the following to degree measure:

a $\dfrac{11\pi}{6}$ **b** $\dfrac{17\pi}{4}$ **c** $\dfrac{9\pi}{4}$ **d** $\dfrac{7\pi}{12}$ **e** $\dfrac{17\pi}{2}$
f $-\dfrac{11\pi}{4}$ **g** $-\dfrac{5\pi}{4}$ **h** $-\dfrac{13\pi}{4}$ **i** $\dfrac{23\pi}{4}$

3 Give the exact value of each of the following:

a $\sin\left(\dfrac{9\pi}{4}\right)$ **b** $\cos\left(\dfrac{-5\pi}{4}\right)$ **c** $\sin\left(\dfrac{3\pi}{2}\right)$ **d** $\cos\left(\dfrac{-3\pi}{2}\right)$
e $\cos\left(\dfrac{11\pi}{6}\right)$ **f** $\sin\left(\dfrac{21\pi}{6}\right)$ **g** $\tan\left(\dfrac{-25\pi}{3}\right)$ **h** $\tan\left(\dfrac{-15\pi}{4}\right)$

4 State the amplitude and period of each of the following:

a $4\sin\left(\dfrac{\theta}{2}\right)$ **b** $-5\sin(6\theta)$ **c** $\dfrac{1}{3}\sin(4\theta)$
d $-2\cos(5x)$ **e** $-7\sin\left(\dfrac{\pi x}{4}\right)$ **f** $\dfrac{2}{3}\sin\left(\dfrac{2\pi x}{3}\right)$

5 Find the maximum and minimum values of the function with rule:

a $3 + 2\sin\theta$ **b** $4 - 5\cos\theta$

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6 Sketch the graph of each of the following (showing one cycle):

a $y = 2\cos(2x)$ **b** $y = -3\sin\left(\frac{x}{3}\right)$ **c** $y = -2\cos(3x)$

d $y = 2\cos\left(\frac{x}{3}\right)$ **e** $y = \cos\left(x - \frac{\pi}{4}\right)$ **f** $y = \cos\left(x + \frac{2\pi}{3}\right)$

g $y = 2\sin\left(x - \frac{5\pi}{6}\right)$ **h** $y = -3\sin\left(x + \frac{\pi}{6}\right)$

7 Solve each of the following equations:

a $\cos\theta = -\frac{\sqrt{3}}{2}$ for $\theta \in [-\pi, \pi]$ **b** $\cos(2\theta) = -\frac{\sqrt{3}}{2}$ for $\theta \in [-\pi, \pi]$

c $\cos\left(\theta - \frac{\pi}{3}\right) = -\frac{1}{2}$ for $\theta \in [0, 2\pi]$ **d** $\cos\left(\theta + \frac{\pi}{3}\right) = -1$ for $\theta \in [0, 2\pi]$

e $\cos\left(\frac{\pi}{3} - \theta\right) = -\frac{1}{2}$ for $\theta \in [0, 2\pi]$

8 Sketch the graph of each of the following for $x \in [-\pi, 2\pi]$:

a $f(x) = 2\cos(2x) + 1$ **b** $f(x) = 1 - 2\cos(2x)$ **c** $f(x) = 3\sin\left(x + \frac{\pi}{3}\right)$

d $f(x) = 2 - \sin\left(x + \frac{\pi}{3}\right)$ **e** $f(x) = 1 - 2\cos(3x)$

9 Solve each of the following for $x \in [0, 2\pi]$:

a $\tan x = -\sqrt{3}$ **b** $\tan\left(3x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$

c $2\tan\left(\frac{x}{2}\right) + 2 = 0$ **d** $3\tan\left(\frac{\pi}{2} + 2x\right) = -3$

10 Sketch the graph of each of the following for $x \in [0, \pi]$, clearly labelling all intercepts with the axes and all asymptotes:

a $f(x) = \tan(2x)$ **b** $f(x) = \tan\left(x - \frac{\pi}{3}\right)$

c $f(x) = 2\tan\left(2x + \frac{\pi}{3}\right)$ **d** $f(x) = 2\tan\left(2x + \frac{\pi}{3}\right) - 2$

11 Find the values of $\theta \in [0, 2\pi]$ for which:

a $\sin^2\theta = \frac{1}{4}$ **b** $\sin(2\theta) = \frac{1}{2}$ **c** $\cos(3\theta) = \frac{\sqrt{3}}{2}$ **d** $\sin^2(2\theta) = 1$

12 Solve the equation $\tan(\theta°) = 2\sin(\theta°)$ for values of $\theta°$ from $0°$ to $360°$.

13 Find the general solution for each of the following:

a $\sin(2x) = -1$ **b** $\cos(3x) = 1$ **c** $\tan x = -1$

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Multiple-choice questions

1 The period of the graph of $y = 2\sin(3x - \pi) + 4$ is

A $\frac{2\pi}{3}$ **B** 2 **C** 3 **D** π **E** 2π

2 The amplitude of the graph of $y = -5\cos(5x) + 3$ is

A -5 **B** -2 **C** 2 **D** 5 **E** 8

3 The number of solutions of $5\sin(2x - \pi) + 2 = 0$ in the interval $[0, 2\pi]$ is

A 1 **B** 2 **C** 3 **D** 4 **E** 8

4 An angle of $\frac{3\pi}{11}$ radians expressed in degrees (correct to two decimal places) is

A 49.00 **B** 154.22 **C** 49.09 **D** 0.01 **E** 0.00

5 The solutions of $2\sin(3x) + \sqrt{2} = 0$ in the interval $\left(\frac{5\pi}{12}, \frac{23\pi}{12}\right)$ are

A $\frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{23\pi}{4}$ **B** 1.83, 3.40, 3.93, 5.50

C $\frac{4\pi}{9}, \frac{5\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{16\pi}{9}, \frac{17\pi}{9}$ **D** $\frac{7\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{4}, \frac{7\pi}{4}$

E none of the above

6 $\cos\left(-\frac{13\pi}{6}\right)$ is equal to

A $-\cos\left(\frac{13\pi}{6}\right)$ **B** $-\frac{\sqrt{3}}{2}$ **C** $\cos\left(-\frac{7\pi}{6}\right)$ **D** $-\frac{1}{2}$ **E** $\sin\left(\frac{2\pi}{3}\right)$

7 $\tan(180 - \theta)^\circ$ is equal to

A $\frac{\sin(90 + \theta)^\circ}{\cos(90 - \theta)^\circ}$ **B** $\frac{\cos(180 - \theta)^\circ}{\sin(180 - \theta)^\circ}$ **C** $\frac{\sin(90 - \theta)^\circ}{\cos(90 + \theta)^\circ}$

D $\frac{\cos(90 - \theta)^\circ}{\sin(90 + \theta)^\circ}$ **E** $\frac{\cos(90 + \theta)^\circ}{\sin(90 - \theta)^\circ}$

8 $\sin\left(\frac{\pi}{2} - x\right)$ is not equal to

A $\cos(2\pi - x)$ **B** $-\sin\left(\frac{3\pi}{2} + x\right)$ **C** $\sin x$

D $\cos(-x)$ **E** $\sin\left(\frac{\pi}{2} + x\right)$

9 Given that $\sin a^\circ = b$, which of the following is equal to b?

A $\sin(90 + a)^\circ$ **B** $\sin(90 - a)^\circ$ **C** $\sin(180 + a)^\circ$ **D** $\sin(180 - a)^\circ$ **E** $\sin(360 - a)^\circ$

10 The period of the graph of $f(x) = 4\sin(3\pi x) - 3\cos(2\pi x)$ is

A 1 **B** 2 **C** 3 **D** 4 **E** $\frac{2}{3}$

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Extended-response questions

1 The depth, D metres, of sea water in a bay t hours after midnight on a particular day may be represented by a function with rule

$$D(t) = a + b\cos\left(\frac{2\pi t}{k}\right)$$

where a, b and k are real numbers. The water is at a maximum depth of 15.4 metres at midnight and noon, and is at a minimum depth of 11.4 metres at 6 a.m. and 6 p.m.

a Find the value of:

i a **ii** b **iii** k

b Find the times when the depth of the water is 13.4 metres.

c Find the values of t for which the depth of the water is less than 14.4 metres.

2 The temperature, T°C, in a small town in the mountains over a day is modelled by the function with rule

$$T = 15 - 8\cos\left(\frac{\pi t}{12} + 6\right), \quad 0 \le t \le 24$$

where t is the time in hours after midnight.

a What is the temperature at midnight, correct to two significant figures?

b What are the maximum and minimum temperatures reached?

c At what times of the day, to the nearest minute, are temperatures warmer than 20°C?

d Sketch the graph for the temperatures over a day.

3 A particle oscillates back and forth, in a straight line, between points A and B about a point O. Its position, $x(t)$ metres, relative to O at time t seconds is given by the rule $x(t) = 3\sin(2\pi t - a)$. The position of the particle when $t = 1$ is $x = -1.5$.

a If $a \in \left[0, \frac{\pi}{2}\right]$, find the value of a.

b Sketch the graph of $x(t)$ against t for $t \in [0, 2]$. Label the maximum and minimum points, the axis intercepts and the endpoints with their coordinates.

c How far from O is point A?

d At what time does the particle first pass through A?

e How long is it before the particle returns to A?

f How long does it take for the particle to go from A to O?

g How far does the particle travel in:

i the first 2 seconds of its motion

ii the first 2.5 seconds of its motion?

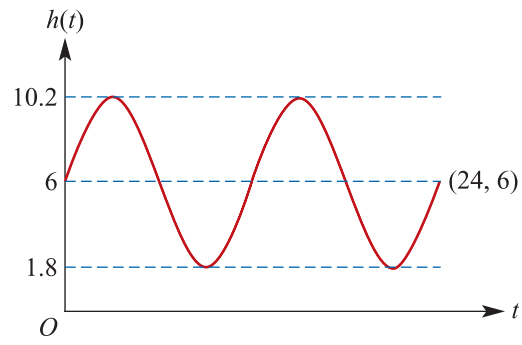
ISBN 978-1-316-63608-4

Review

4 The depth of water, $h(t)$ m, at a particular jetty in a harbour at time t hours after midnight is given by the rule

$$h(t) = p + q\sin\left(\frac{\pi t}{6}\right)$$

for constants p and q. The graph of $h(t)$ against t for $t \in [0, 24]$ is shown. The maximum depth is 10.2 m and the minimum depth is 1.8 m.



a Find the values of p and q.

b At what times during the time interval $[0, 24]$ is the depth of water at a maximum?

c What is the average depth of the water over the time interval $[0, 24]$?

d At what times during the time interval $[0, 24]$ is the depth of the water 3.9 m?

e For how long during the 24-hour period from midnight is the depth of the water more than 8.1 m?

5 Consider the function $f\colon [0, 2\pi] \to \mathbb{R}$ with rule $f(x) = 2\sin(3x) + 1$.

a Find the values of k such that the equation $f(x) = k$ has:

i six solutions for $x \in [0, 2\pi]$

ii three solutions for $x \in [0, 2\pi]$

iii no solutions for $x \in [0, 2\pi]$.

b Find a sequence of transformations which takes the graph of $y = f(x)$ to the graph of $y = \sin x$.

c Find the values of $h \in [0, 2\pi]$ such that the graph of $y = f(x + h)$ has:

i a maximum at the point $\left(\frac{\pi}{3}, 3\right)$

ii a minimum at the point $\left(\frac{\pi}{3}, -1\right)$.

6 **a** Find a sequence of transformations which takes the graph of $y = \cos x$ to the graph of $y = \sin x$.

b Find a sequence of transformations which takes the graph of $y = 2\cos x$ to the graph of $y = -\frac{1}{2}\sin(2x)$.

c **i** Find the rule for the image of the graph of $f(x) = \sin x$ under a dilation of factor $\frac{2}{\pi}$ from the y-axis, followed by a reflection in the line $y = 2$.

ii Find the range and period of the new function.

ISBN 978-1-316-63608-4

Review

7 The population, N, of a particular species of ant varies with the seasons. The population is modelled by the equation

$$N = 3000\sin\left(\frac{\pi(t-1)}{6}\right) + 4000$$

where t is the number of months after 1 January in a given year. The population, M, of a second species of ant also varies with time. Its population is modelled by the equation

$$M = 3000\sin\left(\frac{\pi(t-3.5)}{5}\right) + 5500$$

where t is again the number of months after 1 January in a given year.

Use your calculator to plot the graphs of both equations over a period of one year on the same axes, and hence:

a find the maximum and minimum populations of both species and the months in which those maximums and minimums occur

b find the month of the year during which the populations of both species are equal and find the population of each species at that time

c by formulating a third equation, find when the combined population of species N and M is at a maximum and find this maximum value

d by formulating a fourth equation, find when the difference between the two populations is at a maximum.

8 Passengers on a ferris wheel access their seats from a platform 5 m above the ground. As each seat is filled, the ferris wheel moves around so that the next seat can be filled. Once all seats are filled, the ride begins and lasts for 6 minutes. The height, h m, of Isobel's seat above the ground t seconds after the ride has begun is given by $h = 15\sin(10t - 45)^\circ + 16.5$.

a Use a calculator to sketch the graph of h against t for the first 2 minutes of the ride.

b How far above the ground is Isobel's seat at the commencement of the ride?

c After how many seconds does Isobel's seat pass the access platform?

d How many times will her seat pass the access platform in the first 2 minutes?

e How many times will her seat pass the access platform during the entire ride?

Due to a malfunction, the ferris wheel stops abruptly 1 minute 40 seconds into the ride.

f How far above the ground is Isobel stranded?

g If Isobel's brother Hamish had a seat 1.5 m above the ground at the commencement of the ride, how far above the ground is Hamish stranded?

ISBN 978-1-316-63608-4

Chapter 11

11 Trigonometric identities

Objectives

- To define the reciprocal circular functions **secant**, **cosecant** and **cotangent**.
- To evaluate simple trigonometric expressions using **trigonometric identities**.
- To prove simple trigonometric identities.
- To apply the **addition formulas** for circular functions.
- To apply the **double angle formulas** for circular functions.
- To simplify expressions of the form $a \cos x + b \sin x$.
- To sketch graphs of functions of the form $f(x) = a \cos x + b \sin x$.
- To solve equations of the form $a \cos x + b \sin x = c$.
- To apply the trigonometric identities for products of sines and cosines expressed as sums or differences, and vice versa.

In this chapter we build on our study of circular functions from Chapter 10.

There are many interesting and useful relationships between the circular functions. The most fundamental is the Pythagorean identity:

$$\sin^2 A + \cos^2 A = 1$$

Some of these identities were discovered a very long time ago. For example, the following two results were discovered by the Indian mathematician Bhāskara II in the twelfth century:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$
$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

They are of great importance in many areas of mathematics, including calculus.

ISBN 978-1-316-63608-4

11A Reciprocal functions and the Pythagorean identity

In this section we introduce the reciprocals of the basic circular functions. The graphs of these functions appear in Chapter 12, where reciprocal functions are studied in general. Here we use these functions in various forms of the Pythagorean identity.

▶ Reciprocal functions

The circular functions sine, cosine and tangent can be used to form three other functions, called the reciprocal circular functions.

Secant, cosecant and cotangent

- $\sec\theta = \dfrac{1}{\cos\theta}$ (for $\cos\theta \neq 0$)
- $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ (for $\sin\theta \neq 0$)
- $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ (for $\sin\theta \neq 0$)

Note: For $\cos\theta \neq 0$ and $\sin\theta \neq 0$, we have

$$\cot\theta = \frac{1}{\tan\theta} \quad \text{and} \quad \tan\theta = \frac{1}{\cot\theta}$$

Example 1

Find the exact value of each of the following:

a $\sec\left(\dfrac{2\pi}{3}\right)$ **b** $\cot\left(\dfrac{5\pi}{4}\right)$ **c** $\operatorname{cosec}\left(\dfrac{7\pi}{4}\right)$

Solution

a
$$\sec\left(\frac{2\pi}{3}\right) = \frac{1}{\cos\left(\frac{2\pi}{3}\right)} = \frac{1}{\cos\left(\pi - \frac{\pi}{3}\right)} = \frac{1}{-\cos\left(\frac{\pi}{3}\right)} = 1 \div \left(-\frac{1}{2}\right) = -2$$

b
$$\cot\left(\frac{5\pi}{4}\right) = \frac{\cos\left(\frac{5\pi}{4}\right)}{\sin\left(\frac{5\pi}{4}\right)} = \frac{\cos\left(\pi + \frac{\pi}{4}\right)}{\sin\left(\pi + \frac{\pi}{4}\right)} = \frac{-1}{\sqrt{2}} \div \left(\frac{-1}{\sqrt{2}}\right) = 1$$

c
$$\operatorname{cosec}\left(\frac{7\pi}{4}\right) = \frac{1}{\sin\left(2\pi - \frac{\pi}{4}\right)} = \frac{1}{-\sin\left(\frac{\pi}{4}\right)} = 1 \div \left(-\frac{1}{\sqrt{2}}\right) = -\sqrt{2}$$

Note: In this example, we are using symmetry properties and exact values of circular functions, which are revised in Section 10B.

ISBN 978-1-316-63608-4

Example 2

Find the values of x between 0 and 2π for which:

a $\sec x = -2$

b $\cot x = -1$

Solution

a
$$\sec x = -2$$
$$\frac{1}{\cos x} = -2$$
$$\cos x = \frac{-1}{2}$$
$$\therefore \quad x = \pi - \frac{\pi}{3} \quad \text{or} \quad x = \pi + \frac{\pi}{3}$$
$$\therefore \quad x = \frac{2\pi}{3} \quad \text{or} \quad x = \frac{4\pi}{3}$$

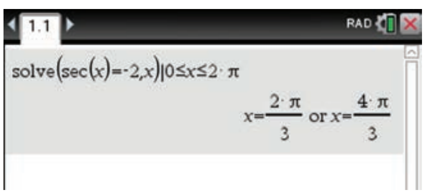
b
$$\cot x = -1$$
$$\tan x = -1$$
$$\therefore \quad x = \pi - \frac{\pi}{4} \quad \text{or} \quad x = 2\pi - \frac{\pi}{4}$$
$$\therefore \quad x = \frac{3\pi}{4} \quad \text{or} \quad x = \frac{7\pi}{4}$$

Using the TI-Nspire

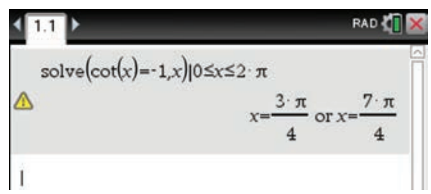
Check that your calculator is in radian mode. Use menu > **Algebra** > **Solve** as shown.

Note: Access sec and cot using trig. Access $\le$ using ctrl =.

a



b

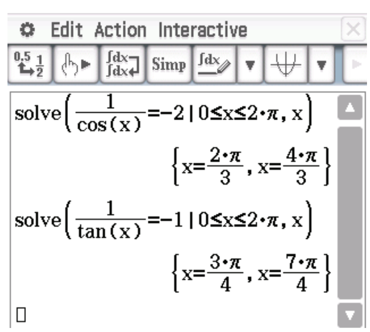


Using the Casio ClassPad

The ClassPad does not recognise sec x, cosec x and cot x. These functions must be entered as reciprocals of cos x, sin x and tan x respectively.

a
- Enter and highlight: $\dfrac{1}{\cos(x)} = -2 \mid 0 \le x \le 2\pi$
- Select **Interactive** > **Equation/Inequality** > **solve**, ensure the variable is set to x and tap OK.

b
- Enter and highlight: $\dfrac{1}{\tan(x)} = -1 \mid 0 \le x \le 2\pi$
- Select **Interactive** > **Equation/Inequality** > **solve**, ensure the variable is set to x and tap OK.



Note: The ‘for’ operator | is found in the Math3 keyboard and is used to specify a condition. In this case, the condition is the domain restriction.

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The Pythagorean identity

We introduced the Pythagorean identity in Section 10B.

Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

We can now derive two other forms of this identity:

- Dividing both sides by $\cos^2\theta$ gives

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\therefore \quad 1 + \tan^2\theta = \sec^2\theta$$

- Dividing both sides by $\sin^2\theta$ gives

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\therefore \quad \cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

Example 3

a If $\operatorname{cosec} x = \frac{7}{4}$, find $\cos x$.

b If $\sec x = -\frac{3}{2}$ and $\frac{\pi}{2} \le x \le \pi$, find $\sin x$.

Solution

a Since $\operatorname{cosec} x = \frac{7}{4}$, we have $\sin x = \frac{4}{7}$.

Now $\cos^2 x + \sin^2 x = 1$

$$\cos^2 x + \frac{16}{49} = 1$$

$$\cos^2 x = \frac{33}{49}$$

$$\therefore \quad \cos x = \pm\frac{\sqrt{33}}{7}$$

b Since $\sec x = -\frac{3}{2}$, we have $\cos x = -\frac{2}{3}$.

Now $\cos^2 x + \sin^2 x = 1$

$$\frac{4}{9} + \sin^2 x = 1$$

$$\therefore \quad \sin x = \pm\frac{\sqrt{5}}{3}$$

But $\sin x$ is positive for $P(x)$ in the 2nd quadrant, and so $\sin x = \frac{\sqrt{5}}{3}$.

Example 4

Prove the identity $\dfrac{1}{1-\cos\theta} + \dfrac{1}{1+\cos\theta} = 2\operatorname{cosec}^2\theta$.

Solution

$$\begin{aligned}
\text{LHS} &= \frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta} \\
&= \frac{1+\cos\theta+1-\cos\theta}{1-\cos^2\theta} \\
&= \frac{2}{1-\cos^2\theta} \\
&= \frac{2}{\sin^2\theta} \\
&= 2\operatorname{cosec}^2\theta \\
&= \text{RHS}
\end{aligned}$$

ISBN 978-1-316-63608-4

Section summary

- Reciprocal functions

$$\sec\theta = \frac{1}{\cos\theta} \quad (\text{for } \cos\theta \neq 0)$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \quad (\text{for } \sin\theta \neq 0)$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \quad (\text{for } \sin\theta \neq 0)$$

- Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

Exercise 11A

Example 1

1 Find the exact value of each of the following:

a $\cot\left(\frac{3\pi}{4}\right)$ **b** $\operatorname{cosec}\left(\frac{5\pi}{4}\right)$ **c** $\sec\left(\frac{5\pi}{6}\right)$ **d** $\operatorname{cosec}\left(\frac{\pi}{2}\right)$

e $\sec\left(\frac{4\pi}{3}\right)$ **f** $\operatorname{cosec}\left(\frac{13\pi}{6}\right)$ **g** $\cot\left(\frac{7\pi}{3}\right)$ **h** $\sec\left(\frac{5\pi}{3}\right)$

2 Without using a calculator, write down the exact value of each of the following:

a $\cot 135°$ **b** $\sec 150°$ **c** $\operatorname{cosec} 90°$ **d** $\cot 240°$ **e** $\operatorname{cosec} 225°$

f $\sec 330°$ **g** $\cot 315°$ **h** $\operatorname{cosec} 300°$ **i** $\cot 420°$

Example 2

3 Find the values of x between 0 and 2π for which:

a $\operatorname{cosec} x = 2$ **b** $\cot x = \sqrt{3}$ **c** $\sec x + \sqrt{2} = 0$ **d** $\operatorname{cosec} x = \sec x$

Example 3

4 If $\sec\theta = \frac{-17}{8}$ and $\frac{\pi}{2} < \theta < \pi$, find:

a $\cos\theta$ **b** $\sin\theta$ **c** $\tan\theta$

5 If $\tan\theta = \frac{-7}{24}$ and $\frac{3\pi}{2} < \theta < 2\pi$, find $\cos\theta$ and $\sin\theta$.

6 Find the value of $\sec\theta$ if $\tan\theta = 0.4$ and θ is not in the 1st quadrant.

7 If $\tan\theta = \frac{4}{3}$ and $\pi < \theta < \frac{3\pi}{2}$, evaluate $\frac{\sin\theta - 2\cos\theta}{\cot\theta - \sin\theta}$.

8 If $\cos\theta = \frac{2}{3}$ and θ is in the 4th quadrant, express $\frac{\tan\theta - 3\sin\theta}{\cos\theta - 2\cot\theta}$ in simplest surd form.

Example 4

9 Prove each of the following identities for suitable values of θ and φ:

a $(1 - \cos^2\theta)(1 + \cot^2\theta) = 1$ **b** $\cos^2\theta\tan^2\theta + \sin^2\theta\cot^2\theta = 1$

c $\frac{\tan\theta}{\tan\varphi} = \frac{\tan\theta + \cot\varphi}{\cot\theta + \tan\varphi}$ **d** $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2$

e $\frac{1 + \cot^2\theta}{\cot\theta\operatorname{cosec}\theta} = \sec\theta$ **f** $\sec\theta + \tan\theta = \frac{\cos\theta}{1 - \sin\theta}$

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11B Addition formulas and double angle formulas

▶ Addition formulas

Addition formulas for cosine

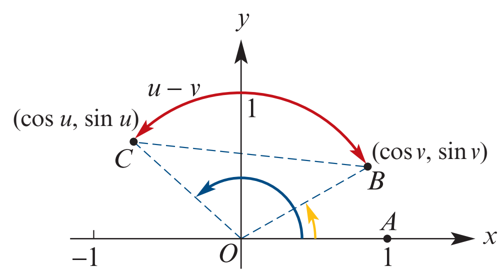
1. $\cos(u + v) = \cos u \cos v - \sin u \sin v$
2. $\cos(u - v) = \cos u \cos v + \sin u \sin v$

Proof Consider a unit circle as shown:

arc length $AB = v$ units

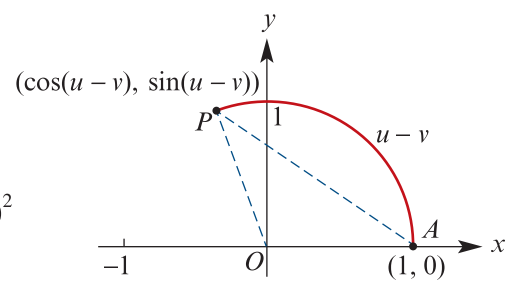
arc length $AC = u$ units

arc length $BC = u - v$ units



Rotate ΔOCB so that B is coincident with A. Then C is moved to

$$P(\cos(u - v), \sin(u - v))$$



Since the triangles CBO and PAO are congruent, we have $CB = PA$.

Using the coordinate distance formula:

$$\begin{aligned} CB^2 &= (\cos u - \cos v)^2 + (\sin u - \sin v)^2 \\ &= 2 - 2(\cos u \cos v + \sin u \sin v) \end{aligned}$$

$$\begin{aligned} PA^2 &= (\cos(u - v) - 1)^2 + (\sin(u - v) - 0)^2 \\ &= 2 - 2\cos(u - v) \end{aligned}$$

Since $CB = PA$, this gives

$$2 - 2\cos(u - v) = 2 - 2(\cos u \cos v + \sin u \sin v)$$

$$\therefore \quad \cos(u - v) = \cos u \cos v + \sin u \sin v$$

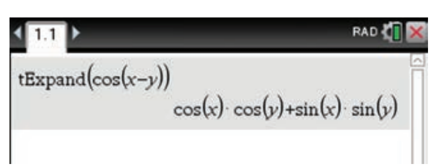
We can now obtain the first formula from the second by replacing v with $-v$:

$$\begin{aligned} \cos(u + v) &= \cos(u - (-v)) \\ &= \cos u \cos(-v) + \sin u \sin(-v) \\ &= \cos u \cos v - \sin u \sin v \end{aligned}$$

Note: Here we used $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.

Using the TI-Nspire

Access the **tExpand( )** command from menu > **Algebra** > **Trigonometry** > **Expand** and complete as shown.



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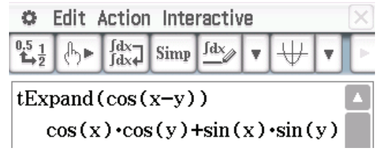
Using the Casio ClassPad

- In $\sqrt{\alpha}$ (Main), enter and highlight $\cos(x - y)$.
- Go to **Interactive** > **Transformation** > **tExpand** and tap OK.



Example 5

Evaluate $\cos 75°$.

Solution

$$\begin{aligned}\cos 75° &= \cos(45° + 30°)\\ &= \cos 45° \cos 30° - \sin 45° \sin 30°\\ &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Addition formulas for sine

1 $\sin(u + v) = \sin u \cos v + \cos u \sin v$

2 $\sin(u - v) = \sin u \cos v - \cos u \sin v$

Proof We use the symmetry properties $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ and $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$:

$$\begin{aligned}\sin(u+v) &= \cos\left(\frac{\pi}{2} - (u+v)\right)\\ &= \cos\left(\left(\frac{\pi}{2} - u\right) - v\right)\\ &= \cos\left(\frac{\pi}{2} - u\right)\cos v + \sin\left(\frac{\pi}{2} - u\right)\sin v\\ &= \sin u \cos v + \cos u \sin v\end{aligned}$$

We can now obtain the second formula from the first by replacing v with $-v$:

$$\begin{aligned}\sin(u-v) &= \sin u \cos(-v) + \cos u \sin(-v)\\ &= \sin u \cos v - \cos u \sin v\end{aligned}$$

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Example 6

Evaluate:

a $\sin 75^\circ$

b $\sin 15^\circ$

Solution

a
$$\begin{aligned}\sin 75^\circ &= \sin(30^\circ + 45^\circ)\\ &= \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ\\ &= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}\\ &= \frac{1+\sqrt{3}}{2\sqrt{2}}\\ &= \frac{1+\sqrt{3}}{2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{2}+\sqrt{6}}{4}\end{aligned}$$

b
$$\begin{aligned}\sin 15^\circ &= \sin(45^\circ - 30^\circ)\\ &= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ\\ &= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\cdot\frac{1}{2}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Addition formulas for tangent

1 $\tan(u+v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v}$

2 $\tan(u-v) = \dfrac{\tan u - \tan v}{1 + \tan u \tan v}$

Proof To obtain the first formula, we write

$$\tan(u+v) = \frac{\sin(u+v)}{\cos(u+v)} = \frac{\sin u \cos v + \cos u \sin v}{\cos u \cos v - \sin u \sin v}$$

Now divide the numerator and denominator by $\cos u \cos v$. The second formula can be obtained from the first by using $\tan(-\theta) = -\tan\theta$.

Example 7

If u and v are acute angles such that $\tan u = 4$ and $\tan v = \frac{3}{5}$, show that $u - v = \frac{\pi}{4}$.

Solution

$$\begin{aligned}\tan(u-v) &= \frac{\tan u - \tan v}{1 + \tan u \tan v}\\ &= \frac{4 - \frac{3}{5}}{1 + 4 \times \frac{3}{5}}\\ &= \frac{20-3}{5+4\times 3}\\ &= 1\end{aligned}$$

$$\therefore \quad u - v = \frac{\pi}{4}$$

Note: Since u and v are acute angles with $u > v$, the angle $u - v$ is in the interval $\left(0, \frac{\pi}{2}\right)$.

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Double angle formulas

Using the addition formulas, we can easily derive useful expressions for $\sin(2u)$, $\cos(2u)$ and $\tan(2u)$.

Double angle formulas for cosine

$$\begin{aligned}\cos(2u) &= \cos^2 u - \sin^2 u \\ &= 2\cos^2 u - 1 \qquad (\text{since } \sin^2 u = 1 - \cos^2 u) \\ &= 1 - 2\sin^2 u \qquad (\text{since } \cos^2 u = 1 - \sin^2 u)\end{aligned}$$

Proof $\cos(u + u) = \cos u \cos u - \sin u \sin u$

$$= \cos^2 u - \sin^2 u$$

Double angle formula for sine

$$\sin(2u) = 2\sin u \cos u$$

Proof $\sin(u + u) = \sin u \cos u + \cos u \sin u$

$$= 2\sin u \cos u$$

Double angle formula for tangent

$$\tan(2u) = \frac{2\tan u}{1 - \tan^2 u}$$

Proof $\tan(u + u) = \dfrac{\tan u + \tan u}{1 - \tan u \tan u}$

$$= \frac{2\tan u}{1 - \tan^2 u}$$

Example 8

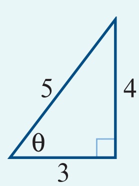
If $\tan\theta = \frac{4}{3}$ and $0 < \theta < \frac{\pi}{2}$, evaluate:

a $\sin(2\theta)$ **b** $\tan(2\theta)$

Solution

a $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$

$$\begin{aligned}\therefore \sin(2\theta) &= 2\sin\theta\cos\theta \\ &= 2 \times \frac{4}{5} \times \frac{3}{5} \\ &= \frac{24}{25}\end{aligned}$$



b
$$\begin{aligned}\tan(2\theta) &= \frac{2\tan\theta}{1 - \tan^2\theta} \\ &= \frac{2 \times \frac{4}{3}}{1 - \frac{16}{9}} \\ &= \frac{2 \times 4 \times 3}{9 - 16} \\ &= -\frac{24}{7}\end{aligned}$$

Cambridge Senior Maths AC
Specialist Mathematics Year 11
ISBN 978-1-316-63608-4

Example 9

Prove each of the following identities:

a $\dfrac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \tan(2\theta)$

b $\dfrac{\sin\theta}{\sin\varphi} + \dfrac{\cos\theta}{\cos\varphi} = \dfrac{2\sin(\theta+\varphi)}{\sin(2\varphi)}$

c $\dfrac{1}{\cos\theta + \sin\theta} + \dfrac{1}{\cos\theta - \sin\theta} = \tan(2\theta)\operatorname{cosec}\theta$

Solution

a
$$\begin{aligned}\text{LHS} &= \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}\\ &= \frac{\sin(2\theta)}{\cos(2\theta)}\\ &= \tan(2\theta)\\ &= \text{RHS}\end{aligned}$$

Note: Identity holds when $\cos(2\theta) \neq 0$.

b
$$\begin{aligned}\text{LHS} &= \frac{\sin\theta}{\sin\varphi} + \frac{\cos\theta}{\cos\varphi}\\ &= \frac{\sin\theta\cos\varphi + \cos\theta\sin\varphi}{\sin\varphi\cos\varphi}\\ &= \frac{\sin(\theta+\varphi)}{\frac{1}{2}\sin(2\varphi)}\\ &= \frac{2\sin(\theta+\varphi)}{\sin(2\varphi)}\\ &= \text{RHS}\end{aligned}$$

Note: Identity holds when $\sin(2\varphi) \neq 0$.

c
$$\begin{aligned}\text{LHS} &= \frac{1}{\cos\theta + \sin\theta} + \frac{1}{\cos\theta - \sin\theta}\\ &= \frac{\cos\theta - \sin\theta + \cos\theta + \sin\theta}{\cos^2\theta - \sin^2\theta}\\ &= \frac{2\cos\theta}{\cos(2\theta)}\\ &= \frac{2\cos\theta}{\cos(2\theta)} \cdot \frac{\sin\theta}{\sin\theta}\\ &= \frac{\sin(2\theta)}{\cos(2\theta)\sin\theta}\\ &= \frac{\tan(2\theta)}{\sin\theta}\\ &= \tan(2\theta)\operatorname{cosec}\theta\\ &= \text{RHS}\end{aligned}$$

Note: Identity holds when $\cos(2\theta) \neq 0$ and $\sin\theta \neq 0$.

Sometimes the easiest way to prove that two expressions are equal is to simplify each of them separately. This is demonstrated in the following example.

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Example 10

Prove that $(\sec A - \cos A)(\operatorname{cosec} A - \sin A) = \dfrac{1}{\tan A + \cot A}$.

Solution

$$\begin{aligned}\text{LHS} &= (\sec A - \cos A)(\operatorname{cosec} A - \sin A)\\ &= \left(\frac{1}{\cos A} - \cos A\right)\left(\frac{1}{\sin A} - \sin A\right)\\ &= \frac{1-\cos^2 A}{\cos A} \times \frac{1-\sin^2 A}{\sin A}\\ &= \frac{\sin^2 A\cos^2 A}{\cos A\sin A}\\ &= \cos A\sin A\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \frac{1}{\tan A + \cot A}\\ &= \frac{1}{\dfrac{\sin A}{\cos A} + \dfrac{\cos A}{\sin A}}\\ &= \frac{\cos A\sin A}{\sin^2 A + \cos^2 A}\\ &= \cos A\sin A\end{aligned}$$

We have shown that LHS = RHS.

Section summary

- **Addition formulas**

$$\cos(u+v) = \cos u\,\cos v - \sin u\,\sin v$$
$$\cos(u-v) = \cos u\,\cos v + \sin u\,\sin v$$
$$\sin(u+v) = \sin u\,\cos v + \cos u\,\sin v$$
$$\sin(u-v) = \sin u\,\cos v - \cos u\,\sin v$$
$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u\,\tan v}$$
$$\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u\,\tan v}$$

- **Double angle formulas**

$$\begin{aligned}\cos(2u) &= \cos^2 u - \sin^2 u\\ &= 2\cos^2 u - 1\\ &= 1 - 2\sin^2 u\end{aligned}$$
$$\sin(2u) = 2\sin u\,\cos u$$
$$\tan(2u) = \frac{2\tan u}{1-\tan^2 u}$$

Note: We investigate the triple angle formulas for sine and cosine in Chapter 14.

Exercise 11B

Skillsheet
Example 5

1 By using the appropriate addition formulas, find exact values for the following:

a $\cos 15°$ **b** $\cos 105°$

Example 6

2 By using the appropriate addition formulas, find exact values for the following:

a $\sin 165°$ **b** $\tan 75°$

3 Find the exact value of:

a $\cos\left(\dfrac{5\pi}{12}\right)$ **b** $\sin\left(\dfrac{11\pi}{12}\right)$ **c** $\tan\left(\dfrac{-\pi}{12}\right)$

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Example 7 **4** If $\sin u = \frac{12}{13}$ and $\sin v = \frac{3}{5}$, evaluate $\sin(u + v)$. (Note: There is more than one answer.)

5 Simplify the following:

a $\sin\left(\theta + \frac{\pi}{6}\right)$ **b** $\cos\left(\varphi - \frac{\pi}{4}\right)$ **c** $\tan\left(\theta + \frac{\pi}{3}\right)$ **d** $\sin\left(\theta - \frac{\pi}{4}\right)$

6 Simplify:

a $\cos(u - v)\sin v + \sin(u - v)\cos v$ **b** $\sin(u + v)\sin v + \cos(u + v)\cos v$

Example 8 **7** If $\sin\theta = \frac{-3}{5}$, with θ in the 3rd quadrant, and $\cos\varphi = \frac{-5}{13}$, with φ in the 2nd quadrant, evaluate each of the following without using a calculator:

a $\cos(2\varphi)$ **b** $\sin(2\theta)$ **c** $\tan(2\theta)$ **d** $\sec(2\varphi)$
e $\sin(\theta + \varphi)$ **f** $\cos(\theta - \varphi)$ **g** $\operatorname{cosec}(\theta + \varphi)$ **h** $\cot(2\theta)$

8 For acute angles u and v such that $\tan u = \frac{4}{3}$ and $\tan v = \frac{5}{12}$, evaluate:

a $\tan(u + v)$ **b** $\tan(2u)$ **c** $\cos(u - v)$ **d** $\sin(2u)$

9 If $\sin\alpha = \frac{3}{5}$ and $\sin\beta = \frac{24}{25}$, with $\frac{\pi}{2} < \beta < \alpha < \pi$, evaluate:

a $\cos(2\alpha)$ **b** $\sin(\alpha - \beta)$ **c** $\tan(\alpha + \beta)$ **d** $\sin(2\beta)$

10 If $\sin\theta = -\frac{\sqrt{3}}{2}$ and $\cos\theta = \frac{1}{2}$, evaluate:

a $\sin(2\theta)$ **b** $\cos(2\theta)$

11 Simplify each of the following expressions:

a $(\sin\theta - \cos\theta)^2$ **b** $\cos^4\theta - \sin^4\theta$

Example 9, 10 **12** Prove the following identities:

a $\sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right) = \sin\theta - \cos\theta$ **b** $\cos\left(\theta - \frac{\pi}{3}\right) + \cos\left(\theta + \frac{\pi}{3}\right) = \cos\theta$

c $\tan\left(\theta + \frac{\pi}{4}\right)\tan\left(\theta - \frac{\pi}{4}\right) = -1$ **d** $\cos\left(\theta + \frac{\pi}{6}\right) + \sin\left(\theta + \frac{\pi}{3}\right) = \sqrt{3}\cos\theta$

e $\tan\left(\theta + \frac{\pi}{4}\right) = \frac{1 + \tan\theta}{1 - \tan\theta}$ **f** $\frac{\sin(u + v)}{\cos u \cos v} = \tan v + \tan u$

g $\frac{\tan u + \tan v}{\tan u - \tan v} = \frac{\sin(u + v)}{\sin(u - v)}$ **h** $\cos(2\theta) + 2\sin^2\theta = 1$

i $\sin(4\theta) = 4\sin\theta\cos^3\theta - 4\cos\theta\sin^3\theta$

j $\frac{1 - \sin(2\theta)}{\sin\theta - \cos\theta} = \sin\theta - \cos\theta$

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11C Simplifying $a\cos x + b\sin x$

In this section, we see how to rewrite the rule of a function $f(x) = a\cos x + b\sin x$ in terms of a single circular function.

$$a\cos x + b\sin x = r\cos(x - \alpha) \quad \text{where } r = \sqrt{a^2 + b^2},\ \cos\alpha = \frac{a}{r} \text{ and } \sin\alpha = \frac{b}{r}$$

Proof Let $r = \sqrt{a^2 + b^2}$. Consider the point $P\left(\frac{a}{r}, \frac{b}{r}\right)$ and its distance from the origin O:

$$OP^2 = \left(\frac{a}{r}\right)^2 + \left(\frac{b}{r}\right)^2 = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} = 1$$

The point P is on the unit circle, and so $\frac{a}{r} = \cos\alpha$ and $\frac{b}{r} = \sin\alpha$, for some angle α.

We can now write

$$\begin{aligned} a\cos x + b\sin x &= r\left(\frac{a}{r}\cos x + \frac{b}{r}\sin x\right) \\ &= r(\cos\alpha\ \cos x + \sin\alpha\ \sin x) \\ &= r\cos(x - \alpha) \end{aligned}$$

Similarly, it may be shown that

$$a\cos x + b\sin x = r\sin(x + \beta) \quad \text{where } r = \sqrt{a^2 + b^2},\ \sin\beta = \frac{a}{r},\ \cos\beta = \frac{b}{r}$$

Example 11

Express $\cos x - \sqrt{3}\sin x$ in the form $r\cos(x - \alpha)$. Hence find the range of the function f with rule $f(x) = \cos x - \sqrt{3}\sin x$ and find the maximum and minimum values of f.

Solution

Here $a = 1$ and $b = -\sqrt{3}$. Therefore

$$r = \sqrt{1 + 3} = 2, \quad \cos\alpha = \frac{a}{r} = \frac{1}{2} \quad \text{and} \quad \sin\alpha = \frac{b}{r} = \frac{-\sqrt{3}}{2}$$

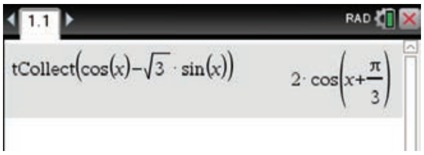
We see that $\alpha = -\frac{\pi}{3}$ and so

$$f(x) = \cos x - \sqrt{3}\sin x = 2\cos\left(x + \frac{\pi}{3}\right)$$

Thus the range of f is $[-2, 2]$, the maximum value is 2 and the minimum value is -2.

Using the TI-Nspire

Access the **tCollect( )** command from menu > **Algebra** > **Trigonometry** > **Collect** and complete as shown.



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Example 12

Solve $\cos x - \sqrt{3}\sin x = 1$ for $x \in [0, 2\pi]$.

Solution

From Example 11, we have

$$\cos x - \sqrt{3}\sin x = 2\cos\left(x + \frac{\pi}{3}\right)$$

$$\therefore \quad 2\cos\left(x + \frac{\pi}{3}\right) = 1$$

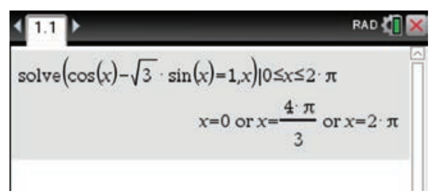
$$\cos\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$$

$$x + \frac{\pi}{3} = \frac{\pi}{3}, \frac{5\pi}{3} \text{ or } \frac{7\pi}{3}$$

$$x = 0, \frac{4\pi}{3} \text{ or } 2\pi$$

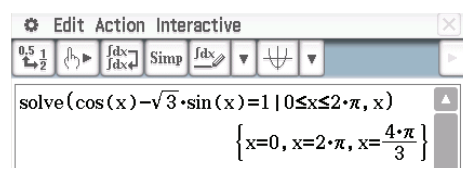
Using the TI-Nspire

Use **solve( )** from the **Algebra** menu as shown.



Using the Casio ClassPad

- In $\sqrt{\alpha}$ Main, enter and highlight the equation $\cos(x) - \sqrt{3}\sin(x) = 1 \mid 0 \le x \le 2\pi$
- Select **Interactive** > **Equation/Inequality** > **solve** and tap OK.



Example 13

Express $\sqrt{3}\sin(2x) - \cos(2x)$ in the form $r\sin(2x + \alpha)$.

Solution

A slightly different technique is used. Assume that

$$\sqrt{3}\sin(2x) - \cos(2x) = r\sin(2x + \alpha)$$

$$= r\left(\sin(2x)\cos\alpha + \cos(2x)\sin\alpha\right)$$

This is to hold for all x.

For $x = \frac{\pi}{4}$: $\quad \sqrt{3} = r\cos\alpha \qquad (1)$

For $x = 0$: $\quad -1 = r\sin\alpha \qquad (2)$

ISBN 978-1-316-63608-4

Squaring and adding (1) and (2) gives

$$r^2\cos^2\alpha + r^2\sin^2\alpha = 4$$
$$r^2 = 4$$
$$\therefore \quad r = \pm 2$$

We take the positive solution. Substituting in (1) and (2) gives

$$\frac{\sqrt{3}}{2} = \cos\alpha \quad \text{and} \quad -\frac{1}{2} = \sin\alpha$$

Thus $\alpha = -\frac{\pi}{6}$ and hence

$$\sqrt{3}\sin(2x) - \cos(2x) = 2\sin\left(2x - \frac{\pi}{6}\right)$$

Check: Expand the right-hand side of the equation using an addition formula.

Section summary

- $a\cos x + b\sin x = r\cos(x - \alpha)$ where $r = \sqrt{a^2 + b^2}$, $\cos\alpha = \frac{a}{r}$, $\sin\alpha = \frac{b}{r}$
- $a\cos x + b\sin x = r\sin(x + \beta)$ where $r = \sqrt{a^2 + b^2}$, $\sin\beta = \frac{a}{r}$, $\cos\beta = \frac{b}{r}$

Exercise 11C

Skillsheet
Example 11

1 Find the maximum and minimum values of the following:

a $4\cos x + 3\sin x$
b $\sqrt{3}\cos x + \sin x$
c $\cos x - \sin x$
d $\cos x + \sin x$
e $3\cos x + \sqrt{3}\sin x$
f $\sin x - \sqrt{3}\cos x$
g $\cos x - \sqrt{3}\sin x + 2$
h $5 + 3\sin x - 2\cos x$

Example 12

2 Solve each of the following for $x \in [0, 2\pi]$ or for $\theta \in [0, 360]$:

a $\sin x - \cos x = 1$
b $\sqrt{3}\sin x + \cos x = 1$
c $\sin x - \sqrt{3}\cos x = -1$
d $3\cos x - \sqrt{3}\sin x = 3$
e $4\sin\theta° + 3\cos\theta° = 5$
f $2\sqrt{2}\sin\theta° - 2\cos\theta° = 3$

3 Write $\sqrt{3}\cos(2x) - \sin(2x)$ in the form $r\cos(2x + \alpha)$.

Example 13

4 Write $\cos(3x) - \sin(3x)$ in the form $r\sin(3x - \alpha)$.

5 Sketch the graph of each of the following, showing one cycle:

a $f(x) = \sin x - \cos x$
b $f(x) = \sqrt{3}\sin x + \cos x$
c $f(x) = \sin x + \cos x$
d $f(x) = \sin x - \sqrt{3}\cos x$

ISBN 978-1-316-63608-4

11D Sums and products of sines and cosines

In Section 11B, we derived the addition formulas for sine and cosine. We use them in this section to obtain new identities which allow us to rewrite products of sines and cosines as sums or differences, and vice versa.

► Expressing products as sums or differences

Product-to-sum identities

$$2\cos A\cos B = \cos(A-B)+\cos(A+B)$$
$$2\sin A\sin B = \cos(A-B)-\cos(A+B)$$
$$2\sin A\cos B = \sin(A+B)+\sin(A-B)$$

Proof We use the addition formulas for sine and cosine:

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \qquad (1)$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B \qquad (2)$$
$$\sin(A+B) = \sin A\cos B + \cos A\sin B \qquad (3)$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B \qquad (4)$$

The first product-to-sum identity is obtained by adding (2) and (1), the second identity is obtained by subtracting (1) from (2), and the third by adding (3) and (4).

Example 14

Express each of the following products as sums or differences:

a $2\sin(3\theta)\cos(\theta)$

b $2\sin 50^\circ\cos 60^\circ$

c $2\cos\left(\theta+\frac{\pi}{4}\right)\cos\left(\theta-\frac{\pi}{4}\right)$

Solution

a Use the third product-to-sum identity:

$$\begin{aligned}2\sin(3\theta)\cos(\theta) &= \sin(3\theta+\theta)+\sin(3\theta-\theta)\\ &= \sin(4\theta)+\sin(2\theta)\end{aligned}$$

b Use the third product-to-sum identity:

$$\begin{aligned}2\sin 50^\circ\cos 60^\circ &= \sin 110^\circ + \sin(-10)^\circ\\ &= \sin 110^\circ - \sin 10^\circ\end{aligned}$$

c Use the first product-to-sum identity:

$$\begin{aligned}2\cos\left(\theta+\frac{\pi}{4}\right)\cos\left(\theta-\frac{\pi}{4}\right) &= \cos\left(\frac{\pi}{2}\right)+\cos(2\theta)\\ &= \cos(2\theta)\end{aligned}$$

ISBN 978-1-316-63608-4

► Expressing sums and differences as products

Sum-to-product identities

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

Proof Using the first product-to-sum identity, we have

$$\begin{aligned} 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) &= \cos\left(\frac{A+B}{2} - \frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2} + \frac{A-B}{2}\right) \\ &= \cos B + \cos A \\ &= \cos A + \cos B \end{aligned}$$

The other three sum-to-product identities can be obtained similarly.

Example 15

Express each of the following as products:

a $\sin 36° + \sin 10°$

b $\cos 36° + \cos 10°$

c $\sin 36° - \sin 10°$

d $\cos 36° - \cos 10°$

Solution

a $\sin 36° + \sin 10° = 2\sin 23° \cos 13°$

b $\cos 36° + \cos 10° = 2\cos 23° \cos 13°$

c $\sin 36° - \sin 10° = 2\cos 23° \sin 13°$

d $\cos 36° - \cos 10° = -2\sin 23° \sin 13°$

Example 16

Prove that $\dfrac{\cos(\theta) - \cos(3\theta)}{\sin(3\theta) - \sin(\theta)} = \tan(2\theta)$.

Solution

$$\begin{aligned} \text{LHS} &= \frac{\cos(\theta) - \cos(3\theta)}{\sin(3\theta) - \sin(\theta)} \\ &= \frac{-2\sin(2\theta)\sin(-\theta)}{2\sin(\theta)\cos(2\theta)} \\ &= \frac{2\sin(2\theta)\sin(\theta)}{2\sin(\theta)\cos(2\theta)} \\ &= \tan(2\theta) \\ &= \text{RHS} \end{aligned}$$

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Example 17

Solve the equation $\sin(3x) + \sin(11x) = 0$ for $x \in [0, \pi]$.

Solution

$$\sin(3x) + \sin(11x) = 0$$

$$\Leftrightarrow \quad 2\sin(7x)\cos(4x) = 0$$

$$\Leftrightarrow \quad \sin(7x) = 0 \quad \text{or} \quad \cos(4x) = 0$$

$$\Leftrightarrow \quad 7x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi, 7\pi \quad \text{or} \quad 4x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

$$\Leftrightarrow \quad x = 0, \frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}, \frac{4\pi}{7}, \frac{5\pi}{7}, \frac{6\pi}{7}, \pi, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8} \text{ or } \frac{7\pi}{8}$$

Section summary

- Product-to-sum identities

$$2\cos A\cos B = \cos(A - B) + \cos(A + B)$$

$$2\sin A\sin B = \cos(A - B) - \cos(A + B)$$

$$2\sin A\cos B = \sin(A + B) + \sin(A - B)$$

- Sum-to-product identities

$$\cos A + \cos B = 2\cos\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A + B}{2}\right)\sin\left(\frac{A - B}{2}\right)$$

$$\sin A + \sin B = 2\sin\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$$

$$\sin A - \sin B = 2\sin\left(\frac{A - B}{2}\right)\cos\left(\frac{A + B}{2}\right)$$

Exercise 11D

Example 14

1 Express each of the following products as sums or differences:

a $2\sin(3\pi t)\cos(2\pi t)$

b $\sin 20° \cos 30°$

c $2\cos\left(\frac{\pi x}{4}\right)\sin\left(\frac{3\pi x}{4}\right)$

d $2\sin\left(\frac{A + B + C}{2}\right)\cos\left(\frac{A - B - C}{2}\right)$

2 Express $2\sin(3\theta)\sin(2\theta)$ as a difference of cosines.

3 Use a product-to-sum identity to derive the expression for $2\sin\left(\frac{A - B}{2}\right)\cos\left(\frac{A + B}{2}\right)$ as a difference of sines.

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4 Show that $\sin 75^\circ \sin 15^\circ = \dfrac{1}{4}$.

Example 15 **5** Express each of the following as products:

a $\sin 56^\circ + \sin 22^\circ$
b $\cos 56^\circ + \cos 22^\circ$
c $\sin 56^\circ - \sin 22^\circ$
d $\cos 56^\circ - \cos 22^\circ$

6 Express each of the following as products:

a $\sin(6A) + \sin(2A)$
b $\cos(x) + \cos(2x)$
c $\sin(4x) - \sin(3x)$
d $\cos(3A) - \cos(A)$

Example 16 **7** Show that $\sin(A) + 2\sin(3A) + \sin(5A) = 4\cos^2(A)\sin(3A)$.

8 For any three angles α, β and γ, show that

$$\sin(\alpha + \beta)\sin(\alpha - \beta) + \sin(\beta + \gamma)\sin(\beta - \gamma) + \sin(\gamma + \alpha)\sin(\gamma - \alpha) = 0$$

9 Show that $\cos 70^\circ + \sin 40^\circ = \cos 10^\circ$.

10 Show that $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0$.

Example 17 **11** Solve each of the following equations for $x \in [-\pi, \pi]$:

a $\cos(5x) + \cos(x) = 0$
b $\cos(5x) - \cos(x) = 0$
c $\sin(5x) + \sin(x) = 0$
d $\sin(5x) - \sin(x) = 0$

12 Solve each of the following equations for $\theta \in [0, \pi]$:

a $\cos(2\theta) - \sin(\theta) = 0$
b $\sin(5\theta) - \sin(3\theta) + \sin(\theta) = 0$
c $\sin(7\theta) - \sin(\theta) = \sin(3\theta)$
d $\cos(3\theta) - \cos(5\theta) + \cos(7\theta) = 0$

13 Prove that $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan\left(\dfrac{A + B}{2}\right)$.

14 Prove the identity:

$$4\sin(A + B)\sin(B + C)\sin(C + A) = \sin(2A) + \sin(2B) + \sin(2C) - \sin(2A + 2B + 2C)$$

15 Prove that $\dfrac{\cos(2A) - \cos(2B)}{\sin(2A - 2B)} = -\dfrac{\sin(A + B)}{\cos(A - B)}$.

16 Prove each of the following identities:

a $\dfrac{\sin(A) + \sin(3A) + \sin(5A)}{\cos(A) + \cos(3A) + \cos(5A)} = \tan(3A)$

b $\cos^2(A) + \cos^2(B) - 1 = \cos(A + B)\cos(A - B)$

c $\cos^2(A - B) - \cos^2(A + B) = \sin(2A)\sin(2B)$

d $\cos^2(A - B) - \sin^2(A + B) = \cos(2A)\cos(2B)$

17 Find the sum $\sin(x) + \sin(3x) + \sin(5x) + \cdots + \sin(99x)$.
Hint: First multiply this sum by $2\sin(x)$.

ISBN 978-1-316-63608-4

Cambridge University Press

Review

AS

Nrich

Chapter summary

- Reciprocal circular functions

$$\sec\theta = \frac{1}{\cos\theta} \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$= \frac{1}{\tan\theta} \qquad (\text{if } \cos\theta \neq 0)$$

- Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

- Addition formulas

$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$

$$\cos(u-v) = \cos u \cos v + \sin u \sin v$$

$$\sin(u+v) = \sin u \cos v + \cos u \sin v$$

$$\sin(u-v) = \sin u \cos v - \cos u \sin v$$

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

$$\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

- Double angle formulas

$$\cos(2u) = \cos^2 u - \sin^2 u$$

$$= 2\cos^2 u - 1$$

$$= 1 - 2\sin^2 u$$

$$\sin(2u) = 2\sin u \cos u$$

$$\tan(2u) = \frac{2\tan u}{1 - \tan^2 u}$$

- Linear combinations

$$a\cos x + b\sin x = r\cos(x - \alpha) \qquad \text{where } r = \sqrt{a^2 + b^2}, \quad \cos\alpha = \frac{a}{r}, \quad \sin\alpha = \frac{b}{r}$$

$$a\cos x + b\sin x = r\sin(x + \beta) \qquad \text{where } r = \sqrt{a^2 + b^2}, \quad \sin\beta = \frac{a}{r}, \quad \cos\beta = \frac{b}{r}$$

- Product-to-sum identities

$$2\cos A \cos B = \cos(A - B) + \cos(A + B)$$

$$2\sin A \sin B = \cos(A - B) - \cos(A + B)$$

$$2\sin A \cos B = \sin(A + B) + \sin(A - B)$$

- Sum-to-product identities

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

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Short-answer questions

1 Find the maximum and minimum values of each of the following:

a $3 + 2\sin\theta$ **b** $1 - 3\cos\theta$ **c** $4\sin\left(\frac{3\theta}{2}\right)$ **d** $2\sin^2\left(\frac{\theta}{2}\right)$ **e** $\frac{1}{2 + \cos\theta}$

2 Find the values of $\theta \in [0, 2\pi]$ for which:

a $\tan^2\theta = \frac{1}{3}$ **b** $\tan(2\theta) = -1$ **c** $\sin(3\theta) = -1$ **d** $\sec(2\theta) = \sqrt{2}$

3 Prove each of the following identities:

a $\sec\theta + \operatorname{cosec}\theta\cot\theta = \sec\theta\operatorname{cosec}^2\theta$ **b** $\sec\theta - \sin\theta = \frac{\tan^2\theta + \cos^2\theta}{\sec\theta + \sin\theta}$

4 If $\sin A = \frac{5}{13}$ and $\sin B = \frac{8}{17}$, where A and B are acute, find:

a $\cos(A + B)$ **b** $\sin(A - B)$ **c** $\tan(A + B)$

5 Find:

a $\cos 80° \cos 20° + \sin 80° \sin 20°$ **b** $\frac{\tan 15° + \tan 30°}{1 - \tan 15° \tan 30°}$

6 If $A + B = \frac{\pi}{2}$, find the value of:

a $\sin A\cos B + \cos A\sin B$ **b** $\cos A\cos B - \sin A\sin B$

7 Prove each of the following:

a $\sin^2 A\cos^2 B - \cos^2 A\sin^2 B = \sin^2 A - \sin^2 B$

b $\frac{\sin\theta}{1 + \cos\theta} + \frac{1 + \cos\theta}{\sin\theta} = \frac{2}{\sin\theta}$ **c** $\frac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta} = \tan\theta$

8 Given that $\sin A = \frac{\sqrt{5}}{3}$ and that A is obtuse, find the value of:

a $\cos(2A)$ **b** $\sin(2A)$ **c** $\sin(4A)$

9 Prove:

a $\frac{1 - \tan^2 A}{1 + \tan^2 A} = \cos(2A)$ **b** $\sqrt{2r^2(1 - \cos\theta)} = 2r\sin\left(\frac{\theta}{2}\right)$ for $r > 0$ and θ acute

10 Find $\tan 15°$ in simplest surd form.

11 Solve each of the following equations for $x \in [0, 2\pi]$:

a $\sin x + \cos x = 1$ **b** $\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right) = -\frac{1}{4}$

c $3\tan(2x) = 2\tan x$ **d** $\sin^2 x = \cos^2 x + 1$

e $\sin(3x)\cos x - \cos(3x)\sin x = \frac{\sqrt{3}}{2}$ **f** $2\cos\left(2x - \frac{\pi}{3}\right) = -\sqrt{3}$

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Review

12 Sketch the graph of:

a $y = 2\cos^2 x$ **b** $y = 1 - 2\sin\left(\frac{\pi}{2} - \frac{x}{2}\right)$ **c** $f(x) = \tan(2x)$

13 If $\tan A = 2$ and $\tan(\theta + A) = 4$, find the exact value of $\tan\theta$.

14 **a** Express $2\cos\theta + 9\sin\theta$ in the form $r\cos(\theta - \alpha)$, where $r > 0$ and $0 < \alpha < \frac{\pi}{2}$.

b **i** Give the maximum value of $2\cos\theta + 9\sin\theta$.

ii Give the cosine of θ for which this maximum occurs.

iii Find the smallest positive solution of the equation $2\cos\theta + 9\sin\theta = 1$.

15 Solve each of the following equations for $\theta \in [0, \pi]$:

a $\sin(4\theta) + \sin(2\theta) = 0$ **b** $\sin(2\theta) - \sin(\theta) = 0$

16 Prove that $\dfrac{\cos A - \cos B}{\sin A + \sin B} = \tan\left(\dfrac{B - A}{2}\right)$.

Multiple-choice questions

1 $\operatorname{cosec} x - \sin x$ is equal to

A $\cos x \cot x$ **B** $\operatorname{cosec} x \tan x$ **C** $1 - \sin^2 x$

D $\sin x \operatorname{cosec} x$ **E** $\dfrac{1 - \sin x}{\sin x}$

2 If $\cos x = \dfrac{-1}{3}$, then the possible values of $\sin x$ are

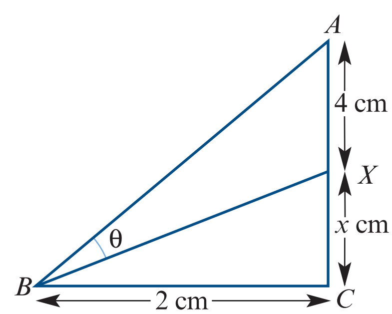
A $\dfrac{-2\sqrt{2}}{3}, \dfrac{2\sqrt{2}}{3}$ **B** $\dfrac{-2}{3}, \dfrac{2}{3}$ **C** $\dfrac{-8}{9}, \dfrac{8}{9}$

D $\dfrac{-\sqrt{2}}{3}, \dfrac{\sqrt{2}}{3}$ **E** $\dfrac{1}{2}, \dfrac{-1}{2}$

3 If $\cos\theta = \dfrac{a}{b}$ and $0 < \theta < \dfrac{\pi}{2}$, then $\tan\theta$ is equal to

A $\dfrac{\sqrt{a^2 + b^2}}{b}$ **B** $\dfrac{\sqrt{b^2 - a^2}}{a}$ **C** $\dfrac{a}{\sqrt{b^2 - a^2}}$ **D** $\dfrac{a}{\sqrt{b^2 + a^2}}$ **E** $\dfrac{a}{b\sqrt{b^2 + a^2}}$

4 In the diagram, the magnitude of $\angle ABX$ is θ, $AX = 4$ cm, $XC = x$ cm and $BC = 2$ cm. Therefore $\tan\theta$ is equal to



A $\dfrac{8}{(x + 2)^2}$ **B** $\dfrac{4}{x}$ **C** $8 - x$

D $8 + x$ **E** $\dfrac{8}{\sqrt{x^2 + 4}}$

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5 If $\frac{\pi}{2} < A < \pi$ and $\pi < B < \frac{3\pi}{2}$ with $\cos A = t$ and $\sin B = t$, then $\sin(B + A)$ equals

A 0 **B** 1 **C** $2t^2 - 1$ **D** $1 - 2t^2$ **E** -1

6 $\dfrac{\sin(2A)}{\cos(2A) - 1}$ is equal to

A $\cot(2A) - 1$ **B** $\sin(2A) + \sec(2A)$ **C** $\dfrac{\sin A}{\cos A - 1}$

D $\sin(2A) - \tan(2A)$ **E** $-\cot A$

7 $(1 + \cot x)^2 + (1 - \cot x)^2$ is equal to

A $2 + \cot(x) + 2\cot(2x)$ **B** 2 **C** $-4\cot x$

D $2 + \cot(2x)$ **E** $2\operatorname{cosec}^2 x$

8 If $\sin(2A) = m$ and $\cos A = n$, then $\tan A$ is equal to

A $\dfrac{m}{2n^2}$ **B** $\dfrac{n}{m}$ **C** $\dfrac{2n}{m^2}$ **D** $\dfrac{2n}{m}$ **E** $\dfrac{2n^2}{m}$

9 Expressing $-\cos x + \sin x$ in the form $r\sin(x + \alpha)$, where $r > 0$, gives

A $\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$ **B** $-\sin\left(x + \frac{\pi}{4}\right)$ **C** $\sqrt{2}\sin\left(x + \frac{5\pi}{4}\right)$

D $\sqrt{2}\sin\left(x + \frac{7\pi}{4}\right)$ **E** $\sqrt{2}\sin\left(x + \frac{3\pi}{4}\right)$

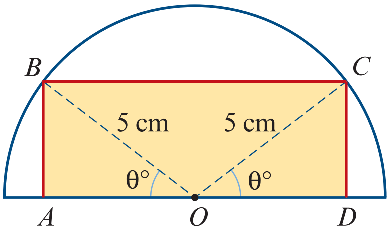
10 The product $\sin 25^\circ \cos 75^\circ$ can be rewritten as

A $\sin 100^\circ - \sin 50^\circ$ **B** $2(\sin 100^\circ + \sin 50^\circ)$ **C** $2(\sin 100^\circ - \sin 50^\circ)$

D $\frac{1}{2}(\sin 100^\circ + \sin 50^\circ)$ **E** $\frac{1}{2}(\sin 100^\circ - \sin 50^\circ)$

Extended-response questions

1 The diagram shows a rectangle $ABCD$ inside a semicircle, centre O and radius 5 cm, with $\angle BOA = \angle COD = \theta^\circ$.



a Show that the perimeter, P cm, of the rectangle is given by $P = 20\cos\theta + 10\sin\theta$.

b Express P in the form $r\cos(\theta - \alpha)$ and hence find the value of θ for which $P = 16$.

c Find the value of k for which the area of the rectangle is $k\sin(2\theta)$ cm^2.

d Find the value of θ for which the area is a maximum.

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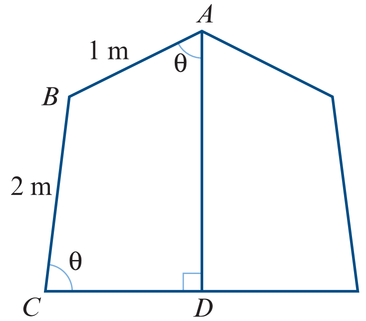
2 The diagram shows a vertical section through a tent in which $AB = 1$ m, $BC = 2$ m and $\angle BAD = \angle BCD = \theta$. The line CD is horizontal, and the diagram is symmetrical about the vertical AD.



a Obtain an expression for AD in terms of θ.

b Express AD in the form $r\cos(\theta - \alpha)$, where r is positive.

c State the maximum length of AD and the corresponding value of θ.

d Given that $AD = 2.15$ m, find the value of θ for which $\theta > \alpha$.

3 **a** Prove the identity

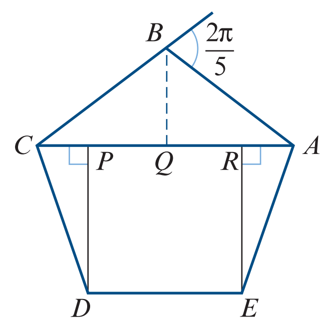
$$\cos(2\theta) = \frac{1 - \tan^2\theta}{1 + \tan^2\theta}$$

b **i** Use the result of **a** to show that $1 + x^2 = \sqrt{2}x^2 - \sqrt{2}$, where $x = \tan(67\tfrac{1}{2})°$.

ii Hence find the values of integers a and b such that $\tan(67\tfrac{1}{2})° = a + b\sqrt{2}$.

c Find the value of $\tan(7\tfrac{1}{2})°$.

4 $ABCDE$ is a regular pentagon with side length one unit. The exterior angles of a regular pentagon each have magnitude $\dfrac{2\pi}{5}$.



a **i** Show that the magnitude of $\angle BCA$ is $\dfrac{\pi}{5}$.

ii Find the length of CA.

b **i** Show the magnitude of $\angle DCP$ is $\dfrac{2\pi}{5}$.

ii Use the fact that $AC = 2CQ = 2CP + PR$ to show that $2\cos\left(\dfrac{\pi}{5}\right) = 2\cos\left(\dfrac{2\pi}{5}\right) + 1$.

iii Use the identity $\cos(2\theta) = 2\cos^2\theta - 1$ to form a quadratic equation in terms of $\cos\left(\dfrac{\pi}{5}\right)$.

iv Find the exact value of $\cos\left(\dfrac{\pi}{5}\right)$.

5 **a** Prove each of the identities:

i $\cos\theta = \dfrac{1 - \tan^2(\frac{1}{2}\theta)}{1 + \tan^2(\frac{1}{2}\theta)}$ **ii** $\sin\theta = \dfrac{2\tan(\frac{1}{2}\theta)}{1 + \tan^2(\frac{1}{2}\theta)}$

b Use the results of **a** to find the value of $\tan(\tfrac{1}{2}\theta)$, given that $8\cos\theta - \sin\theta = 4$.

ISBN 978-1-316-63608-4

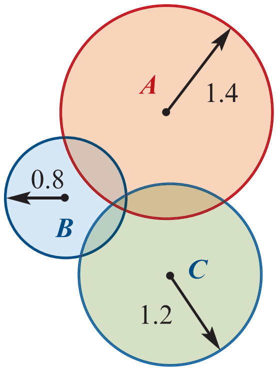
12 Graphing techniques

Objectives

- To sketch graphs of **reciprocal functions**, including those of polynomial functions and circular functions.
- To apply transformations to the graphs of the **reciprocal circular functions**.
- To give **locus definitions** of lines, circles, parabolas, ellipses and hyperbolas, and to find the **Cartesian equations** of these curves.
- To use **parametric equations** to describe curves in the plane.
- To understand **polar coordinates** and their relationship to **Cartesian coordinates**.
- To sketch graphs in polar form.

The extensive use of mobile phones has led to an increased awareness of potential threats to the privacy of their users. For example, a little basic mathematics can be employed to track the movements of someone in possession of a mobile phone.

Suppose that there are three transmission towers within range of your mobile phone. By measuring the time taken for signals to travel between your phone and each transmission tower, it is possible to estimate the distance from your phone to each tower.



In the diagram, there are transmission towers at points A, B and C. If it is estimated that a person is no more than 1.4 km from A, no more than 0.8 km from B and no more than 1.2 km from C, then the person can be located in the intersection of the three circles.

In this chapter, we will look at different ways of describing circles and various other interesting figures.

ISBN 978-1-316-63608-4

Cambridge University Press

12A Reciprocal functions

▶ Reciprocals of polynomials

You have learned in previous years that the **reciprocal** of a non-zero number a is $\frac{1}{a}$. Likewise, we have the following definition.

If $y = f(x)$ is a polynomial function, then its **reciprocal function** is defined by the rule

$$y = \frac{1}{f(x)}$$

For example, the reciprocal of the function $y = x^3$ is $y = \frac{1}{x^3}$.

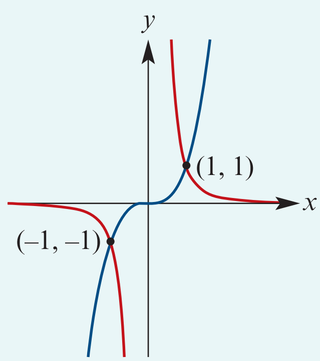
In this section, we will find relationships between the graph of a function and the graph of its reciprocal. Let's consider some specific examples, from which we will draw general conclusions.

Example 1

Sketch the graphs of $y = x^3$ and $y = \frac{1}{x^3}$ on the same set of axes.

Solution

We first sketch the graph of $y = x^3$. This is shown in blue.



Horizontal asymptotes

If $x \to \pm\infty$, then $\frac{1}{x^3} \to 0$. Therefore the line $y = 0$ is a horizontal asymptote of the reciprocal function.

Vertical asymptotes

Notice that $x^3 = 0$ when $x = 0$.

If x is a small positive number, then $\frac{1}{x^3}$ is a large positive number.

If x is a small negative number, then $\frac{1}{x^3}$ is a large negative number.

Therefore the line $x = 0$ is a vertical asymptote of the reciprocal function.

Observations from the example

This example highlights behaviour typical of reciprocal functions:

- If $y = f(x)$ is a non-zero polynomial function, then the graph of $y = \frac{1}{f(x)}$ will have vertical asymptotes where $f(x) = 0$.
- The graphs of a function and its reciprocal are always on the same side of the x-axis.
- If the graphs of a function and its reciprocal intersect, then it must be where $f(x) = \pm 1$.

ISBN 978-1-316-63608-4

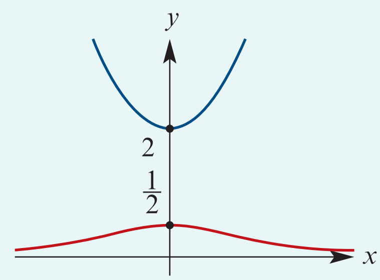
The following example is perhaps easier, because the reciprocal graph has no vertical asymptotes. This time we are interested in turning points.

Example 2

Consider the function $f(x) = x^2 + 2$. Sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ on the same set of axes.

Solution

We first sketch $y = x^2 + 2$. This is shown in blue.



Horizontal asymptotes If $x \to \pm\infty$, then $\frac{1}{f(x)} \to 0$.

Therefore the line $y = 0$ is a horizontal asymptote of the reciprocal function.

Vertical asymptotes There are no vertical asymptotes, as there is no solution to the equation $f(x) = 0$.

Turning points Notice that the graph of $y = x^2 + 2$ has a minimum at $(0, 2)$. The reciprocal function therefore has a maximum at $(0, \frac{1}{2})$.

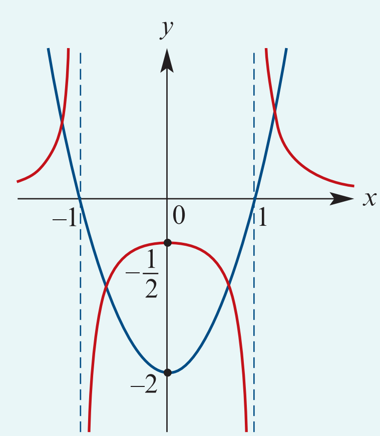
- If the graph of $y = f(x)$ has a local minimum at $x = a$, then the graph of $y = \frac{1}{f(x)}$ will have a local maximum at $x = a$.
- If the graph of $y = f(x)$ has a local maximum at $x = a$, then the graph of $y = \frac{1}{f(x)}$ will have a local minimum at $x = a$.

Example 3

Consider the function $f(x) = 2(x - 1)(x + 1)$. Sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ on the same set of axes.

Solution

We first sketch $y = 2(x - 1)(x + 1)$. This is shown in blue.



Horizontal asymptotes If $x \to \pm\infty$, then $\frac{1}{f(x)} \to 0$.

Therefore the line $y = 0$ is a horizontal asymptote of the reciprocal function.

Vertical asymptotes We have $f(x) = 0$ when $x = -1$ or $x = 1$. Therefore the lines $x = -1$ and $x = 1$ are vertical asymptotes of the reciprocal function.

Turning points The graph of $y = f(x)$ has a minimum at $(0, -2)$. Therefore the reciprocal has a local maximum at $(0, -\frac{1}{2})$.

ISBN 978-1-316-63608-4

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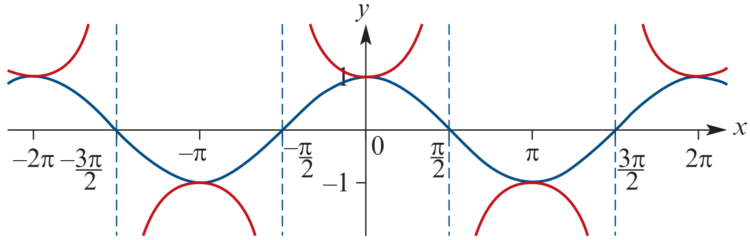
Reciprocals of circular functions

We now briefly consider reciprocal circular functions, which were introduced in Chapter 11. We will investigate these functions in more detail in the next section.

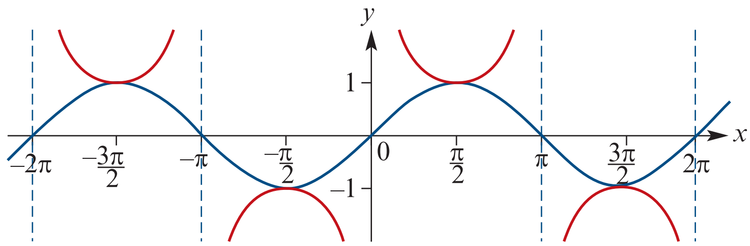
Secant

$$\sec x = \frac{1}{\cos x}$$



Cosecant

$$\operatorname{cosec} x = \frac{1}{\sin x}$$



The first graph shows $y = \cos x$ and $y = \sec x$; the second shows $y = \sin x$ and $y = \operatorname{cosec} x$.

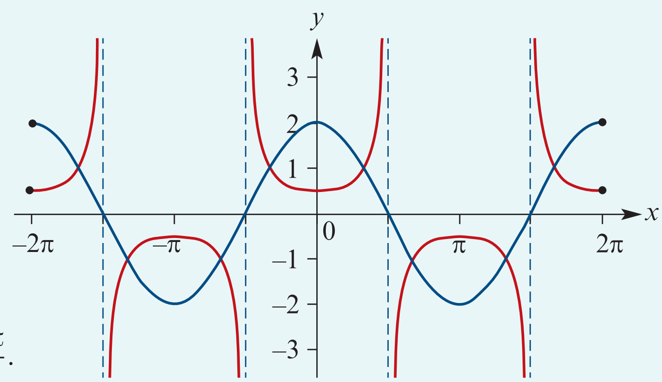
Note: The x-axis intercepts become vertical asymptotes.
Local maximums become local minimums, and vice versa.

Example 4

Let $f(x) = 2\cos x$ for $-2\pi \le x \le 2\pi$. Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes.

Solution

We first sketch $y = 2\cos x$ for $x \in [-2\pi, 2\pi]$. This is shown in blue.



Vertical asymptotes

Vertical asymptotes of the reciprocal function will occur when $f(x) = 0$.

These are given by $x = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}$.

Turning points

The points $(0, 2)$ and $(\pm 2\pi, 2)$ are local maximums of $y = f(x)$. Therefore the points $(0, \frac{1}{2})$ and $(\pm 2\pi, \frac{1}{2})$ are local minimums of the reciprocal.

The points $(\pm\pi, -2)$ are local minimums of $y = f(x)$. Therefore the points $(\pm\pi, -\frac{1}{2})$ are local maximums of the reciprocal.

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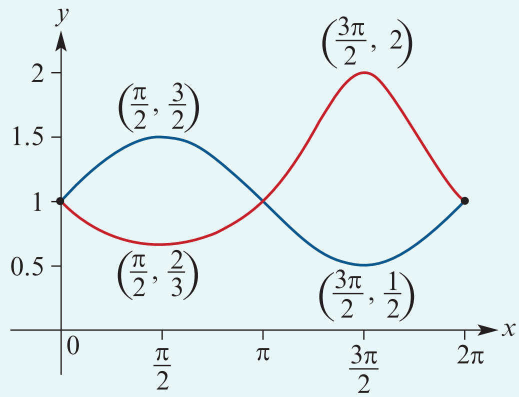
The graph of the next function has no x-axis intercepts, and so its reciprocal has no vertical asymptotes.

Example 5

Let $f(x) = 0.5 \sin x + 1$ for $0 \le x \le 2\pi$. Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes.

Solution

We first sketch $y = 0.5 \sin x + 1$ for $x \in [0, 2\pi]$. This is shown in blue.



Turning points

The point $(\frac{\pi}{2}, \frac{3}{2})$ is a local maximum of $y = f(x)$. Therefore the point $(\frac{\pi}{2}, \frac{2}{3})$ is a local minimum of the reciprocal.

The point $(\frac{3\pi}{2}, \frac{1}{2})$ is a local minimum of $y = f(x)$. Therefore the point $(\frac{3\pi}{2}, 2)$ is a local maximum of the reciprocal.

Section summary

Given the graph of a continuous function $y = f(x)$, we can sketch the graph of $y = \dfrac{1}{f(x)}$ with the help of the following observations:

Function $y = f(x)$	Reciprocal function $y = \dfrac{1}{f(x)}$
x-axis intercept at $x = a$	vertical asymptote $x = a$
local maximum at $x = a$	local minimum at $x = a$
local minimum at $x = a$	local maximum at $x = a$
above the x-axis	above the x-axis
below the x-axis	below the x-axis
increasing over an interval	decreasing over the interval
decreasing over an interval	increasing over the interval
values approach ∞	values approach 0 from above
values approach $-\infty$	values approach 0 from below
values approach 0 from above	values approach ∞
values approach 0 from below	values approach $-\infty$

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Exercise 12A

Skillsheet

1 For each of the following functions, sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes:

Example 1, 2
Example 3

a $f(x) = x + 3$ **b** $f(x) = x^2$ **c** $f(x) = x^2 + 4$
d $f(x) = (x - 1)(x + 1)$ **e** $f(x) = 4 - x^2$ **f** $f(x) = (x - 1)^2 - 1$
g $f(x) = x^2 - 2x - 3$ **h** $f(x) = -x^2 - 2x + 3$ **i** $f(x) = x^3 + 1$

Example 4, 5

2 For each of the following functions, sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes. Label asymptotes, turning points and endpoints.

a $f(x) = \sin x$ for $0 \le x \le 2\pi$ **b** $f(x) = \cos x$ for $0 \le x \le 2\pi$
c $f(x) = -2\cos x$ for $-\pi \le x \le \pi$ **d** $f(x) = \cos x + 1$ for $0 \le x \le 4\pi$
e $f(x) = -\sin x - 1$ for $-2\pi \le x \le 2\pi$ **f** $f(x) = \cos x - 2$ for $0 \le x \le 2\pi$
g $f(x) = -\sin x + 2$ for $0 \le x \le 2\pi$ **h** $f(x) = -2\cos x + 3$ for $-\pi \le x \le \pi$

3 Consider the quadratic function $f(x) = x^2 + 2x + 2$.

a By completing the square, find the turning point of the graph of $y = f(x)$.

b Hence sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes.

4 Consider the quadratic function $f(x) = 5x(1 - x)$.

a Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes.

b Locate all points of intersection of the two graphs by solving $f(x) = 1$ and $f(x) = -1$.

5 Sketch the graphs of $y = 2\sin^2 x$ and $y = \dfrac{1}{2\sin^2 x}$ on the same set of axes, over the interval $0 \le x \le 2\pi$.

6 Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ where $f(x) = 2^x - 1$. Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes.

7 Let $k \in \mathbb{R}$ and consider the function $f(x) = x^2 + 2kx + 1$.

a By completing the square, show that the graph of $y = f(x)$ has a minimum turning point at $(-k, 1 - k^2)$.

b For what values of k does the graph of $y = f(x)$ have:

i no x-axis intercept **ii** one x-axis intercept **iii** two x-axis intercepts?

c Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ when the graph of $y = f(x)$ has:

i no x-axis intercept **ii** one x-axis intercept **iii** two x-axis intercepts.

Hint: It helps to ignore the y-axis.

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12B Graphing the reciprocal circular functions

We now consider the graphs of the **reciprocal circular functions** in greater depth. In this section we will apply basic transformations to the graphs of these functions.

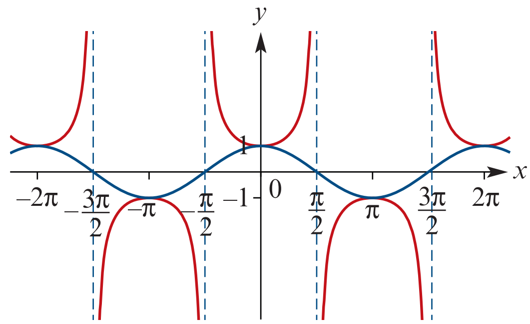
▶ The secant function

The secant function is defined by

$$\sec x = \frac{1}{\cos x}$$

provided $\cos x \neq 0$.

The graphs of $y = \cos x$ and $y = \sec x$ are shown here on the same axes.



The significant features of the two graphs are listed in the following table.

Function $y = \cos x$	Reciprocal function $y = \sec x$
x-axis intercepts at $x = \frac{(2n+1)\pi}{2}$, $n \in \mathbb{Z}$	vertical asymptotes at $x = \frac{(2n+1)\pi}{2}$, $n \in \mathbb{Z}$
domain $= \mathbb{R}$	domain $= \mathbb{R} \setminus \left\{ \frac{(2n+1)\pi}{2} : n \in \mathbb{Z} \right\}$
local maximums at $(2n\pi, 1)$, $n \in \mathbb{Z}$	local minimums at $(2n\pi, 1)$, $n \in \mathbb{Z}$
local minimums at $((2n+1)\pi, -1)$, $n \in \mathbb{Z}$	local maximums at $((2n+1)\pi, -1)$, $n \in \mathbb{Z}$
range $= [-1, 1]$	range $= (-\infty, -1] \cup [1, \infty)$

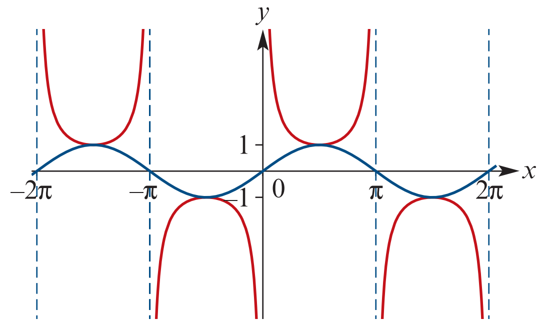
▶ The cosecant function

The cosecant function is defined by

$$\operatorname{cosec} x = \frac{1}{\sin x}$$

provided $\sin x \neq 0$.

The graphs of $y = \sin x$ and $y = \operatorname{cosec} x$ are shown here on the same axes.



Function $y = \sin x$	Reciprocal function $y = \operatorname{cosec} x$
x-axis intercepts at $x = n\pi$, $n \in \mathbb{Z}$	vertical asymptotes at $x = n\pi$, $n \in \mathbb{Z}$
domain $= \mathbb{R}$	domain $= \mathbb{R} \setminus \{ n\pi : n \in \mathbb{Z} \}$
local maximums at $\left(2n\pi + \frac{\pi}{2}, 1\right)$, $n \in \mathbb{Z}$	local minimums at $\left(2n\pi + \frac{\pi}{2}, 1\right)$, $n \in \mathbb{Z}$
local minimums at $\left(2n\pi - \frac{\pi}{2}, -1\right)$, $n \in \mathbb{Z}$	local maximums at $\left(2n\pi - \frac{\pi}{2}, -1\right)$, $n \in \mathbb{Z}$
range $= [-1, 1]$	range $= (-\infty, -1] \cup [1, \infty)$

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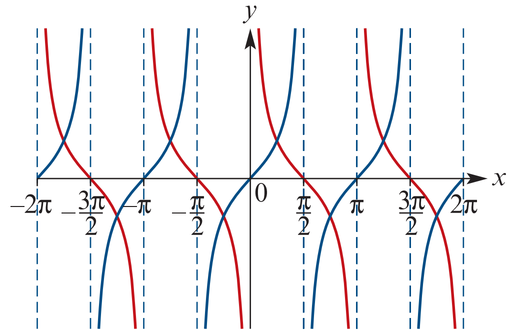
▶ The cotangent function

The cotangent function is defined by

$$\cot x = \frac{\cos x}{\sin x}$$

provided $\sin x \neq 0$.

This diagram shows the graph of $y = \tan x$ in blue and the graph of $y = \cot x$ in red.



Function $y = \tan x$	Function $y = \cot x$
x-axis intercepts at $x = n\pi,\ n \in \mathbb{Z}$	vertical asymptotes at $x = n\pi,\ n \in \mathbb{Z}$
vertical asymptotes at $x = \dfrac{(2n+1)\pi}{2},\ n \in \mathbb{Z}$	x-axis intercepts at $x = \dfrac{(2n+1)\pi}{2},\ n \in \mathbb{Z}$
domain $= \mathbb{R} \setminus \left\{ \dfrac{(2n+1)\pi}{2} : n \in \mathbb{Z} \right\}$	domain $= \mathbb{R} \setminus \{ n\pi : n \in \mathbb{Z} \}$
range $= \mathbb{R}$	range $= \mathbb{R}$

Note the similarity between the graphs of $y = \cot x$ and $y = \tan x$. Using the complementary relationship between sine and cosine, we have

$$\cot x = \frac{\cos x}{\sin x} = \frac{\sin\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right)} = \tan\left(\frac{\pi}{2} - x\right) = \tan\left(-\left(x - \frac{\pi}{2}\right)\right) = -\tan\left(x - \frac{\pi}{2}\right)$$

Therefore the graph of $y = \cot x$ can be obtained from the graph of $y = \tan x$ by a reflection in the x-axis followed by a translation of $\frac{\pi}{2}$ units in the positive direction of the x-axis.

▶ Transformations of the reciprocal circular functions

We now look at the effect of the basic transformations (dilations, reflections and translations) on the reciprocal circular functions.

Example 6

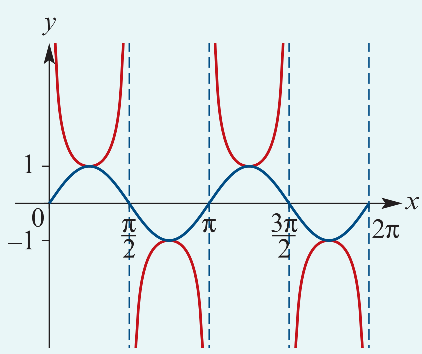
Sketch the graph of each of the following over the interval $[0, 2\pi]$:

a $y = \operatorname{cosec}(2x)$ **b** $y = 2\sec x$ **c** $y = -\cot x$

Solution

a The graph of $y = \operatorname{cosec}(2x)$ can be obtained from the graph of $y = \operatorname{cosec} x$ by a dilation of factor $\frac{1}{2}$ from the y-axis.

It is helpful to draw the graph of $y = \sin(2x)$ on the same axes.



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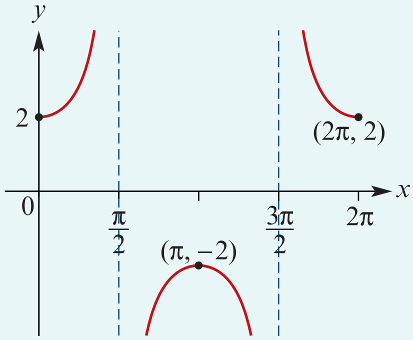
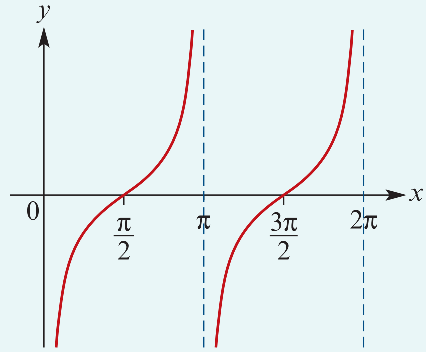
b The graph of $y = 2\sec x$ can be obtained from the graph of $y = \sec x$ by a dilation of factor 2 from the x-axis.



c The graph of $y = -\cot x$ can be obtained from the graph of $y = \cot x$ by a reflection in the x-axis.



Example 7

Sketch the graph of each of the following over the interval $[0, 2\pi]$:

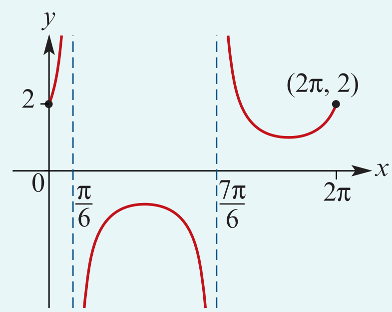
a $y = \sec\left(x + \frac{\pi}{3}\right)$ **b** $y = \operatorname{cosec}(x) - 2$ **c** $y = \cot\left(x - \frac{\pi}{4}\right)$

Solution

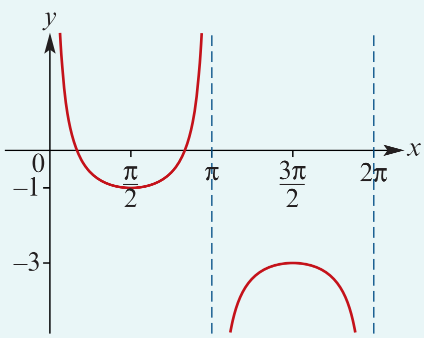
a The graph of $y = \sec\left(x + \frac{\pi}{3}\right)$ can be obtained from the graph of $y = \sec x$ by a translation of $\frac{\pi}{3}$ units in the negative direction of the x-axis.

The y-axis intercept is $\sec\left(\frac{\pi}{3}\right) = 2$.

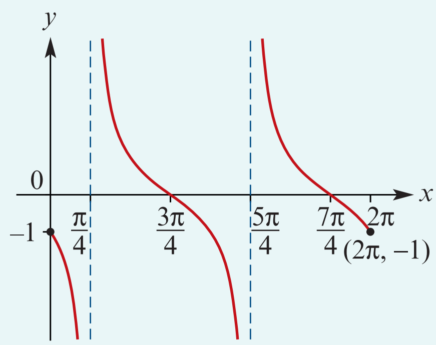
The asymptotes are $x = \frac{\pi}{6}$ and $x = \frac{7\pi}{6}$.



b The graph of $y = \operatorname{cosec}(x) - 2$ can be obtained from the graph of $y = \operatorname{cosec} x$ by a translation of 2 units in the negative direction of the y-axis.



c The graph of $y = \cot\left(x - \frac{\pi}{4}\right)$ can be obtained from the graph of $y = \cot x$ by a translation of $\frac{\pi}{4}$ units in the positive direction of the x-axis.



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Example 8

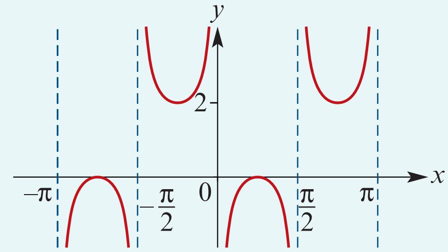
Describe a sequence of transformations that will take the graph of $y = \sec x$ to the graph of $y = -\sec\left(2x - \frac{\pi}{2}\right) + 1$. Sketch the transformed graph over the interval $[-\pi, \pi]$.

Solution

It helps to write the equation of the transformed graph as $y = -\sec\left(2\left(x - \frac{\pi}{4}\right)\right) + 1$.

An appropriate sequence is:

1. reflection in the x-axis
2. dilation of factor $\frac{1}{2}$ from the y-axis
3. translation of $\frac{\pi}{4}$ units to the right and 1 unit up.



Section summary

Reciprocal circular functions

- $\sec x = \dfrac{1}{\cos x}$ (provided $\cos x \neq 0$)
- $\operatorname{cosec} x = \dfrac{1}{\sin x}$ (provided $\sin x \neq 0$)
- $\cot x = \dfrac{\cos x}{\sin x}$ (provided $\sin x \neq 0$)

Exercise 12B

Example 6

1 Sketch the graph of each of the following over the interval $[0, 2\pi]$:

a $y = \sec(2x)$ **b** $y = \cot(2x)$ **c** $y = 3\sec x$
d $y = 2\operatorname{cosec} x$ **e** $y = -\operatorname{cosec} x$ **f** $y = -2\sec x$

Example 7

2 Sketch the graph of each of the following over the interval $[0, 2\pi]$:

a $y = \sec\left(x - \frac{\pi}{2}\right)$ **b** $y = \cot\left(x + \frac{\pi}{4}\right)$ **c** $y = -\operatorname{cosec}\left(x + \frac{\pi}{2}\right)$
d $y = 1 + \sec x$ **e** $y = 2 - \operatorname{cosec} x$ **f** $y = 1 + \cot\left(x + \frac{\pi}{4}\right)$

Example 8

3 Describe a sequence of transformations that will take the graph of $y = \sec x$ to the graph of $y = -2\sec\left(x - \frac{\pi}{2}\right)$. Sketch the transformed graph over the interval $[-\pi, \pi]$.

4 Describe a sequence of transformations that will take the graph of $y = \operatorname{cosec} x$ to the graph of $y = \operatorname{cosec}(-2x) + 1$. Sketch the transformed graph over the interval $[0, 2\pi]$.

5 Describe a sequence of transformations that will take the graph of $y = \cot x$ to the graph of $y = -\cot\left(2x - \frac{\pi}{2}\right) - 1$. Sketch the transformed graph over the interval $[0, 2\pi]$.

6 On the one set of axes, sketch the graphs of $y = \sec x$ and $y = \operatorname{cosec} x$ over the interval $[0, 2\pi]$. Find and label the points of intersection.

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12C Locus of points

Until now, all the curves we have studied have been described by an algebraic relationship between the x- and y-coordinates, such as $y = x^2 + 1$. In this section, we are interested in sets of points described by a geometric condition. A set described in this way is often called a **locus**. Many of these descriptions will give curves that are already familiar.

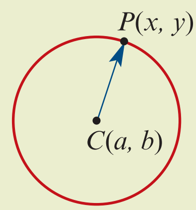
▶ Circles

Circles have a very simple geometric description.

Locus definition of a circle

A **circle** is the locus of a point $P(x, y)$ that moves so that its distance from a fixed point $C(a, b)$ is constant.

Note: The constant distance is called the **radius** and the fixed point $C(a, b)$ is called the **centre** of the circle.



This definition can be used to find the equation of a circle.

Recall that the distance between points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Let r be the radius of the circle. Then

$$\begin{aligned} CP &= r \\ \sqrt{(x-a)^2 + (y-b)^2} &= r \\ (x-a)^2 + (y-b)^2 &= r^2 \end{aligned}$$

The circle with radius r and centre $C(a, b)$ has equation

$$(x-a)^2 + (y-b)^2 = r^2$$

Example 9

a Find the locus of points $P(x, y)$ whose distance from $C(2, -1)$ is 3.

b Find the centre and radius of the circle with equation $x^2 + 2x + y^2 - 4y = 1$.

Solution

a We know that the point $P(x, y)$ satisfies

$$\begin{aligned} CP &= 3 \\ \sqrt{(x-2)^2 + (y+1)^2} &= 3 \\ (x-2)^2 + (y+1)^2 &= 3^2 \end{aligned}$$

This is a circle with centre $(2, -1)$ and radius 3.

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b We must complete the square in both variables. This gives

$$x^2 + 2x + y^2 - 4y = 1$$
$$(x^2 + 2x + 1) - 1 + (y^2 - 4y + 4) - 4 = 1$$
$$(x + 1)^2 + (y - 2)^2 = 6$$

Therefore the centre of the circle is $(-1, 2)$ and its radius is $\sqrt{6}$.

Straight lines

You have learned in previous years that a straight line is the set of points (x, y) satisfying

$$ax + by = c$$

for some constants a, b, c with $a \neq 0$ or $b \neq 0$.

Lines can also be described geometrically as follows.

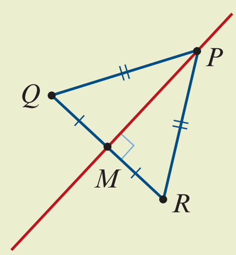
Locus definition of a straight line

Suppose that points Q and R are fixed.

A **straight line** is the locus of a point P that moves so that its distance from Q is the same as its distance from R. That is,

$$QP = RP$$

We can say that point P is **equidistant** from points Q and R.



Note: This straight line is the **perpendicular bisector** of line segment QR. To see this, we note that the midpoint M of QR is on the line. If P is any other point on the line, then

$$QP = RP, \quad QM = RM \quad \text{and} \quad MP = MP$$

and so $\triangle QMP$ is congruent to $\triangle RMP$. Therefore $\angle QMP = \angle RMP = 90°$.

Example 10

a Find the locus of points $P(x, y)$ that are equidistant from the points $Q(1, 1)$ and $R(3, 5)$.

b Show that this is the perpendicular bisector of line segment QR.

Solution

a We know that the point $P(x, y)$ satisfies

$$QP = RP$$
$$\sqrt{(x-1)^2 + (y-1)^2} = \sqrt{(x-3)^2 + (y-5)^2}$$
$$(x-1)^2 + (y-1)^2 = (x-3)^2 + (y-5)^2$$
$$x + 2y = 8$$
$$y = -\tfrac{1}{2}x + 4$$

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b This line has gradient $-\frac{1}{2}$. The line through $Q(1, 1)$ and $R(3, 5)$ has gradient $\dfrac{5-1}{3-1} = 2$.

Because the product of the two gradients is -1, the two lines are perpendicular.

We also need to check that the line $y = -\frac{1}{2}x + 4$ passes through the midpoint of QR, which is $(2, 3)$. When $x = 2$, $y = -\frac{1}{2} \times 2 + 4 = 3$. Thus $(2, 3)$ is on the line.

Section summary

- A **locus** is the set of points described by a geometric condition.
- A **circle** is the locus of a point P that moves so that its distance from a fixed point C is constant.
- A **straight line** is the locus of a point P that moves so that it is equidistant from two fixed points Q and R.

Exercise 12C

Skillsheet

Example 9

1 Find the locus of points $P(x, y)$ whose distance from $Q(1, -2)$ is 4.

2 Find the locus of points $P(x, y)$ whose distance from $Q(-4, 3)$ is 5.

Example 10

3 **a** Find the locus of points $P(x, y)$ that are equidistant from $Q(-1, -1)$ and $R(1, 1)$.
b Show that this is the perpendicular bisector of line segment QR.

4 **a** Find the locus of points $P(x, y)$ that are equidistant from $Q(0, 2)$ and $R(1, 0)$.
b Show that this is the perpendicular bisector of line segment QR.

5 Point P is equidistant from points $Q(0, 1)$ and $R(2, 3)$. Moreover, its distance from point $S(3, 3)$ is 3. Find the possible coordinates of P.

6 Point P is equidistant from points $Q(0, 1)$ and $R(2, 0)$. Moreover, it is also equidistant from points $S(-1, 0)$ and $T(0, 2)$. Find the coordinates of P.

7 A valuable item is buried in a forest. It is 10 metres from a tree stump located at coordinates $T(0, 0)$ and 2 metres from a rock at coordinates $R(6, 10)$. Find the possible coordinates of the buried item.

8 Consider the three points $R(4, 5)$, $S(6, 1)$ and $T(1, -4)$.
a Find the locus of points $P(x, y)$ that are equidistant from the points R and S.
b Find the locus of points $P(x, y)$ that are equidistant from the points S and T.
c Hence find the point that is equidistant from the points R, S and T.
d Hence find the equation of the circle through the points R, S and T.

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9 Given two fixed points $A(0, 1)$ and $B(2, 5)$, find the locus of P if the gradient of AB equals that of BP.

10 A triangle OAP has vertices $O(0, 0)$, $A(4, 0)$ and $P(x, y)$, where $y > 0$. The triangle has area 12 square units. Find the locus of P.

11 **a** Determine the locus of a point $P(x, y)$ that moves so that its distance from the origin is equal to the sum of its x- and y-coordinates.

b Determine the locus of a point $P(x, y)$ that moves so that the *square* of its distance from the origin is equal to the sum of its x- and y-coordinates.

12 $A(0, 0)$ and $B(3, 0)$ are two vertices of a triangle ABP. The third vertex P is such that $AP : BP = 2$. Find the locus of P.

13 Find the locus of the point P that moves so that its distance from the line $y = 3$ is always 2 units.

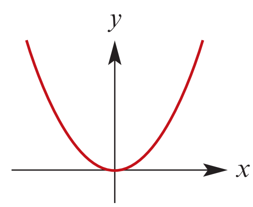
14 A steel pipe is too heavy to drag, but can be lifted at one end and rotated about its opposite end. How many moves are required to rotate the pipe into the parallel position indicated by the dotted line? The distance between the parallel lines is less than the length of the pipe.

12D Parabolas

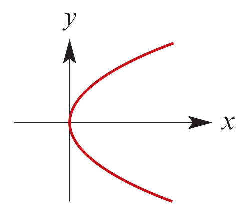
The parabola has been studied since antiquity and is admired for its range of applications, one of which we will explore at the end of this section.

The standard form of a parabola is $y = ax^2$.

Rotating the figure by 90° gives a parabola with equation $x = ay^2$.



$y = ax^2$



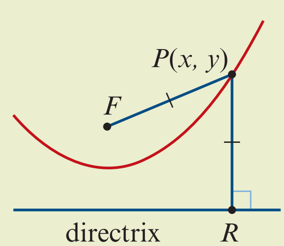
$x = ay^2$

The parabola can also be defined geometrically.

Locus definition of a parabola

A **parabola** is the locus of a point P that moves so that its distance from a fixed point F is equal to its perpendicular distance from a fixed line.

Note: The fixed point is called the **focus** and the fixed line is called the **directrix**.



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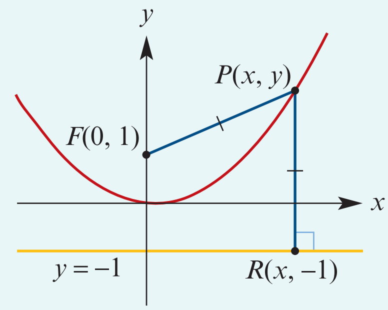
Example 11

Verify that the set of all points $P(x, y)$ that are equidistant from the point $F(0, 1)$ and the line $y = -1$ is a parabola.

Solution

We know that the point $P(x, y)$ satisfies



$$FP = RP$$
$$\sqrt{x^2 + (y-1)^2} = \sqrt{(y-(-1))^2}$$
$$x^2 + (y-1)^2 = (y+1)^2$$
$$x^2 + y^2 - 2y + 1 = y^2 + 2y + 1$$
$$x^2 - 2y = 2y$$
$$x^2 = 4y$$
$$y = \frac{x^2}{4}$$

Therefore the set of points is the parabola with equation $y = \dfrac{x^2}{4}$.

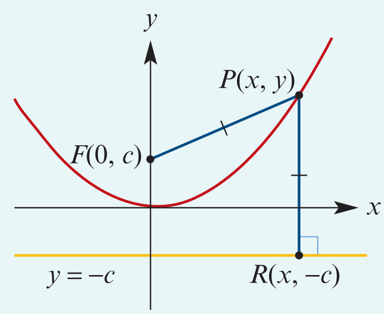
Example 12

a Find the equation of the parabola with focus $F(0, c)$ and directrix $y = -c$.

b Hence find the focus of the parabola with equation $y = 2x^2$.

Solution

a A point $P(x, y)$ on the parabola satisfies



$$FP = RP$$
$$\sqrt{x^2 + (y-c)^2} = \sqrt{(y-(-c))^2}$$
$$x^2 + (y-c)^2 = (y+c)^2$$
$$x^2 + y^2 - 2cy + c^2 = y^2 + 2cy + c^2$$
$$x^2 - 2cy = 2cy$$
$$x^2 = 4cy$$

The parabola has equation $4cy = x^2$.

b Since $\dfrac{y}{2} = x^2$, we solve $\dfrac{1}{2} = 4c$, giving $c = \dfrac{1}{8}$.

Hence the focus is $F\left(0, \dfrac{1}{8}\right)$.

In the previous example, we proved the following result:

The parabola with focus $F(0, c)$ and directrix $y = -c$ has equation $4cy = x^2$.

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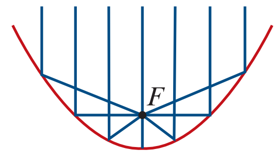
A remarkable application

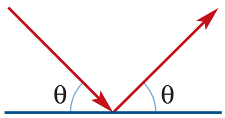
Parabolas have a remarkable property that makes them extremely useful. Light travelling parallel to the axis of symmetry of a reflective parabola is always reflected to its focus.



Parabolas can therefore be used to make reflective telescopes. Low intensity signals from outer space will reflect off the dish and converge at a receiver located at the focus.

To see how this works, we require a simple law of physics:

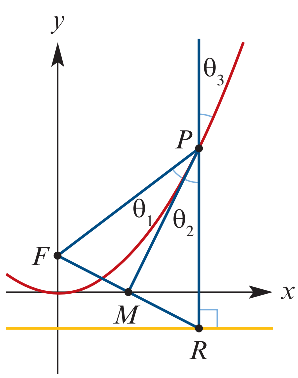
- When light is reflected off a surface, the angle between the ray and the tangent to the surface is preserved after reflection.



Reflective property of the parabola

Any ray of light parallel to the axis of symmetry of the parabola that reflects off the parabola at point P will pass through the focus at F.

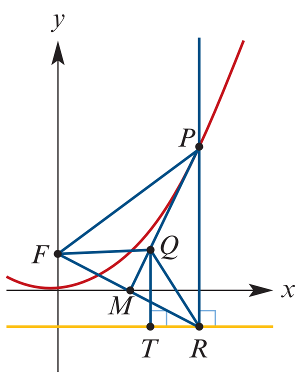
Proof Since point P is on the parabola, the distance to the focus F is the same as the distance to the directrix. Therefore $FP = RP$, and so $\triangle FPR$ is isosceles.



Let M be the midpoint of FR. Then $\triangle FMP$ is congruent to $\triangle RMP$ (by SSS). Therefore MP is the perpendicular bisector of FR and

$$\theta_1 = \theta_2 \quad (\text{as } \triangle FMP \equiv \triangle RMP)$$
$$= \theta_3 \quad (\text{vertically opposite angles})$$

However, we also need to ensure that line MP is tangent to the parabola. To see this, we will show that point P is the only point common to the parabola and line MP.



Take any other point Q on line MP. Suppose that point T is the point on the directrix closest to Q. Then

$$FQ = RQ > TQ$$

and so point Q is not on the parabola.

Section summary

- A **parabola** is the locus of a point P that moves so that its distance from a fixed point F is equal to its perpendicular distance from a fixed line.
- The fixed point is called the **focus** and the fixed line is called the **directrix**.
- The parabola with equation $4cy = x^2$ has focus $F(0, c)$ and directrix $y = -c$.

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Exercise 12D

Example 11 **1** Find the equation of the locus of points $P(x, y)$ whose distance to the point $F(0, 3)$ is equal to the perpendicular distance to the line with equation $y = -3$.

2 Find the equation of the locus of points $P(x, y)$ whose distance to the point $F(0, -4)$ is equal to the perpendicular distance to the line with equation $y = 2$.

3 Find the equation of the locus of points $P(x, y)$ whose distance to the point $F(2, 0)$ is equal to the perpendicular distance to the line with equation $x = -4$.

Example 12 **4** **a** Find the equation of the parabola with focus $F(c, 0)$ and directrix $x = -c$.
b Hence find the focus of the parabola with equation $x = 3y^2$.

5 **a** Find the equation of the locus of points $P(x, y)$ whose distance to the point $F(a, b)$ is equal to the perpendicular distance to the line with equation $y = c$.
b Hence find the equation of the parabola with focus $(1, 2)$ and directrix $y = 3$.

6 A parabola goes through the point $P(7, 9)$ and its focus is $F(1, 1)$. The axis of symmetry of the parabola is $x = 1$. Find the equation of its directrix.
Hint: The directrix will be a horizontal line, $y = c$. Expect to find two answers.

7 A parabola goes through the point $(1, 1)$, its axis of symmetry is the line $x = 2$ and its directrix is the line $y = 3$. Find the coordinates of its focus.
Hint: The focus must lie on the axis of symmetry.

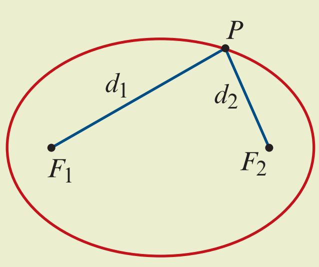
12E Ellipses

A ball casts a shadow that looks like a squashed circle. This figure – called an ellipse – is of considerable geometric significance. For instance, the planets in our solar system have elliptic orbits.

Locus definition of an ellipse

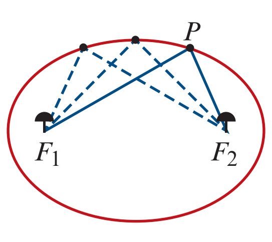
An **ellipse** is the locus of a point P that moves so that the sum of its distances from two fixed points F_1 and F_2 is a constant. That is,

$$F_1P + F_2P = k$$



Note: Points F_1 and F_2 are called the **foci** of the ellipse.

Drawing an ellipse An ellipse can be drawn by pushing two pins into paper. These will be the foci. A string of length k is tied to each of the two pins and the tip of a pen is used to pull the string taut and form a triangle. The pen will trace an ellipse if it is moved around the pins while keeping the string taut.



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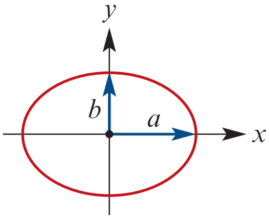
► Cartesian equations of ellipses

The standard form of the Cartesian equation of an ellipse centred at the origin is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



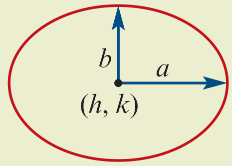
This ellipse has x-axis intercepts $\pm a$ and y-axis intercepts $\pm b$.

Applying the translation defined by $(x, y) \to (x + h, y + k)$, we can see the following result:

The graph of

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

is an ellipse centred at the point (h, k).



Example 13

For each of the following equations, sketch the graph of the corresponding ellipse. Give the coordinates of the centre and the axis intercepts.

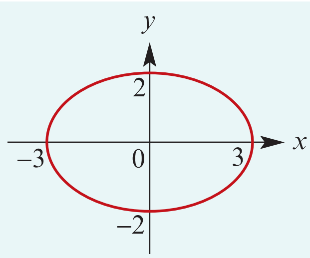
a $\frac{x^2}{9} + \frac{y^2}{4} = 1$ **b** $4x^2 + 9y^2 = 1$ **c** $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$

Solution

a The equation can be written as

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

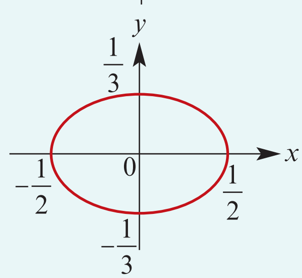
This is an ellipse with centre $(0, 0)$ and axis intercepts at $x = \pm 3$ and $y = \pm 2$.



b The equation can be written as

$$\frac{x^2}{(\frac{1}{2})^2} + \frac{y^2}{(\frac{1}{3})^2} = 1$$

This is an ellipse with centre $(0, 0)$ and axis intercepts at $x = \pm\frac{1}{2}$ and $y = \pm\frac{1}{3}$.



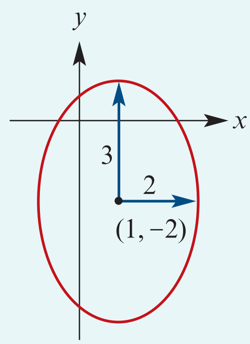
c This is an ellipse with centre $(1, -2)$.

To find the x-axis intercepts, let $y = 0$. Then solving for x gives

$$x = \frac{3 \pm 2\sqrt{5}}{3}$$

Likewise, to find the y-axis intercepts, let $x = 0$. This gives

$$y = \frac{-4 \pm 3\sqrt{3}}{2}$$



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▶ Using the locus definition

Example 14

Consider points $A(-2, 0)$ and $B(2, 0)$. Find the equation of the locus of points P satisfying $AP + BP = 8$.

Solution

Let (x, y) be the coordinates of point P. If $AP + BP = 8$, then

$$\sqrt{(x+2)^2 + y^2} + \sqrt{(x-2)^2 + y^2} = 8$$

and so
$$\sqrt{(x+2)^2 + y^2} = 8 - \sqrt{(x-2)^2 + y^2}$$

Square both sides, then expand and simplify:

$$(x+2)^2 + y^2 = 64 - 16\sqrt{(x-2)^2 + y^2} + (x-2)^2 + y^2$$

$$x^2 + 4x + 4 + y^2 = 64 - 16\sqrt{(x-2)^2 + y^2} + x^2 - 4x + 4 + y^2$$

$$x - 8 = -2\sqrt{(x-2)^2 + y^2}$$

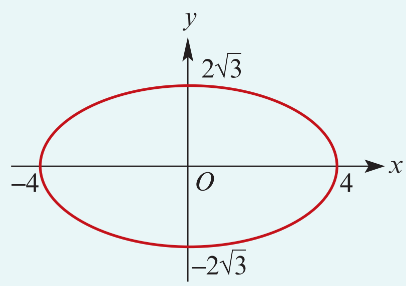
Square both sides again:

$$x^2 - 16x + 64 = 4(x^2 - 4x + 4 + y^2)$$

Simplifying yields

$$3x^2 + 4y^2 = 48$$

i.e. $$\frac{x^2}{16} + \frac{y^2}{12} = 1$$



This is an ellipse with centre the origin and axis intercepts at $x = \pm 4$ and $y = \pm 2\sqrt{3}$.

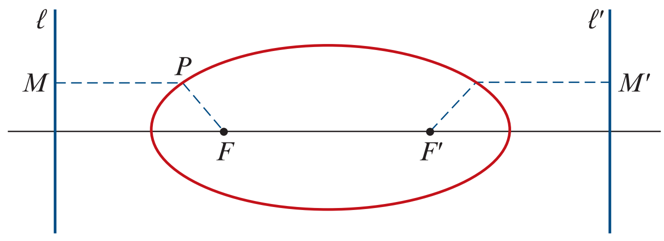
Every point P on the ellipse satisfies $AP + BP = 8$.

Note: You might like to consider the general version of this example with $A(-c, 0)$, $B(c, 0)$ and $AP + BP = 2a$, where $a > c > 0$.

It can also be shown that an ellipse is the locus of points $P(x, y)$ satisfying

$$FP = eMP$$

where F is a fixed point, $0 < e < 1$ and MP is the perpendicular distance from P to a fixed line ℓ. From the symmetry of the ellipse, it is clear that there is a second point F' and a second line ℓ' such that $F'P = eM'P$ defines the same locus, where $M'P$ is the perpendicular distance from P to ℓ'.



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Example 15

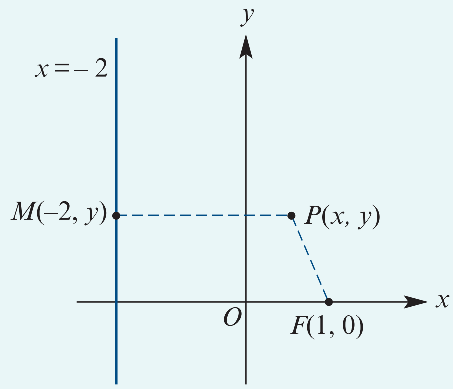
Find the equation of the locus of points $P(x, y)$ if the distance from P to the point $F(1, 0)$ is half the distance MP, the perpendicular distance from P to the line with equation $x = -2$. That is, $FP = \frac{1}{2}MP$.

Solution

Let (x, y) be the coordinates of point P.

If $FP = \frac{1}{2}MP$, then

$$\sqrt{(x-1)^2 + y^2} = \frac{1}{2}\sqrt{(x+2)^2}$$



Square both sides:

$$(x-1)^2 + y^2 = \frac{1}{4}(x+2)^2$$

$$4(x^2 - 2x + 1) + 4y^2 = x^2 + 4x + 4$$

$$3x^2 - 12x + 4y^2 = 0$$

Complete the square:

$$3(x^2 - 4x + 4) + 4y^2 = 12$$

$$3(x-2)^2 + 4y^2 = 12 \quad \text{or equivalently} \quad \frac{(x-2)^2}{4} + \frac{y^2}{3} = 1$$

This is an ellipse with centre $(2, 0)$.

Section summary

- An **ellipse** is the locus of a point P that moves so that the sum of its distances d_1 and d_2 from two fixed points F_1 and F_2 (called the **foci**) is equal to a fixed positive constant.
- The graph of

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

is an ellipse centred at the point (h, k).

Exercise 12E

Skillsheet

Example 13

1 Sketch the graph of each ellipse, labelling the axis intercepts:

a $\frac{x^2}{9} + \frac{y^2}{64} = 1$ **b** $\frac{x^2}{100} + \frac{y^2}{25} = 1$ **c** $\frac{y^2}{9} + \frac{x^2}{64} = 1$ **d** $25x^2 + 9y^2 = 225$

2 Sketch the graph of each ellipse, labelling the centre and the axis intercepts:

a $\frac{(x-3)^2}{9} + \frac{(y-4)^2}{16} = 1$ **b** $\frac{(x+3)^2}{9} + \frac{(y+4)^2}{25} = 1$

c $\frac{(y-3)^2}{16} + \frac{(x-2)^2}{4} = 1$ **d** $25(x-5)^2 + 9y^2 = 225$

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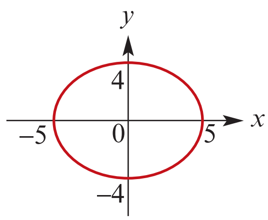
3 Find the Cartesian equations of the following ellipses:

a



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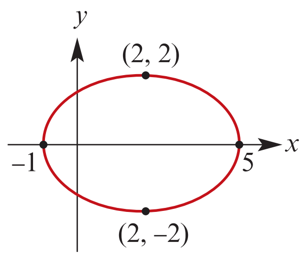
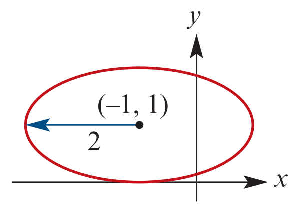


c



Example 14 **4** Find the locus of the point P as it moves such that the sum of its distances from two fixed points $A(1, 0)$ and $B(-1, 0)$ is 4 units.

5 Find the locus of the point P as it moves such that the sum of its distances from two fixed points $A(0, 2)$ and $B(0, -2)$ is 6 units.

Example 15 **6** Find the equation of the locus of points $P(x, y)$ such that the distance from P to the point $F(2, 0)$ is half the distance MP, the perpendicular distance from P to the line with equation $x = -4$. That is, $FP = \frac{1}{2}MP$.

7 A circle has equation $x^2 + y^2 = 1$. It is then dilated by a factor of 3 from the x-axis and by a factor of 5 from the y-axis. Find the equation of the image and sketch its graph.

12F Hyperbolas

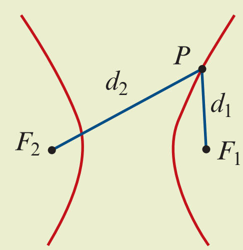
Hyperbolas are defined analogously to ellipses, but using the difference instead of the sum.

Locus definition of a hyperbola

A **hyperbola** is the locus of a point P that moves so that the difference between its distances from two fixed points F_1 and F_2 is a constant. That is,

$$\left|F_2P - F_1P\right| = k$$

Note: Points F_1 and F_2 are called the **foci** of the hyperbola.



The standard form of the Cartesian equation of a hyperbola centred at the origin is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Applying the translation defined by $(x, y) \to (x + h, y + k)$, we can see the following result:

The graph of

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

is a hyperbola centred at the point (h, k).

Note: Interchanging x and y in this equation produces another hyperbola (rotated by 90°).

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Asymptotes of the hyperbola

We now investigate the behaviour of the hyperbola as $x \to \pm\infty$. We first show that the hyperbola with equation

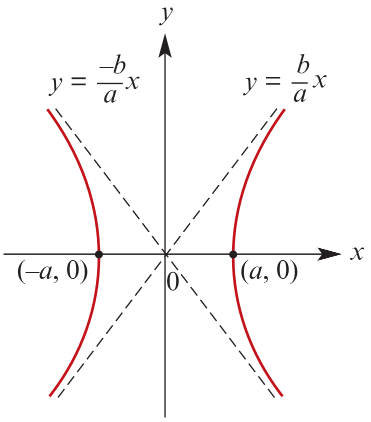
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has asymptotes

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x$$

To see why this should be the case, we rearrange the equation of the hyperbola as follows:

$$\begin{aligned} \frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1 \\ \frac{y^2}{b^2} &= \frac{x^2}{a^2} - 1 \\ y^2 &= \frac{b^2x^2}{a^2} - b^2 \\ &= \frac{b^2x^2}{a^2}\left(1 - \frac{a^2}{x^2}\right) \end{aligned}$$



If $x \to \pm\infty$, then $\frac{a^2}{x^2} \to 0$. Therefore $y^2 \to \frac{b^2x^2}{a^2}$ as $x \to \pm\infty$. That is,

$$y \to \pm\frac{bx}{a} \quad \text{as} \quad x \to \pm\infty$$

Applying the translation defined by $(x, y) \to (x + h, y + k)$, we obtain the following result:

The hyperbola with equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

has asymptotes given by

$$y - k = \pm\frac{b}{a}(x - h)$$

Example 16

For each of the following equations, sketch the graph of the corresponding hyperbola. Give the coordinates of the centre, the axis intercepts and the equations of the asymptotes.

a $\frac{x^2}{9} - \frac{y^2}{4} = 1$

b $\frac{y^2}{9} - \frac{x^2}{4} = 1$

c $(x-1)^2 - (y+2)^2 = 1$

d $\frac{(y-1)^2}{4} - \frac{(x+2)^2}{9} = 1$

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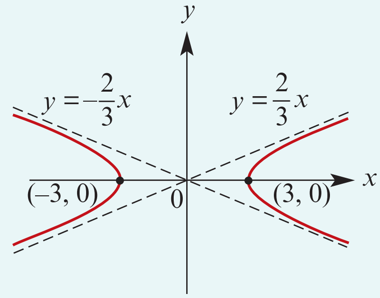
Solution

a Since $\frac{x^2}{9} - \frac{y^2}{4} = 1$, we have

$$y^2 = \frac{4x^2}{9}\left(1 - \frac{9}{x^2}\right)$$

Thus the equations of the asymptotes are $y = \pm\frac{2}{3}x$.

If $y = 0$, then $x^2 = 9$ and so $x = \pm 3$. The x-axis intercepts are $(3, 0)$ and $(-3, 0)$. The centre is $(0, 0)$.

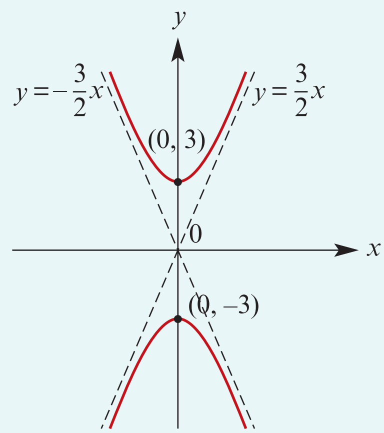


b Since $\frac{y^2}{9} - \frac{x^2}{4} = 1$, we have

$$y^2 = \frac{9x^2}{4}\left(1 + \frac{4}{x^2}\right)$$

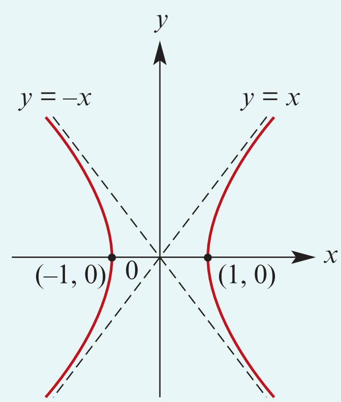
Thus the equations of the asymptotes are $y = \pm\frac{3}{2}x$.

The y-axis intercepts are $(0, 3)$ and $(0, -3)$.
The centre is $(0, 0)$.



c First sketch the graph of $x^2 - y^2 = 1$. The asymptotes are $y = x$ and $y = -x$. The centre is $(0, 0)$ and the axis intercepts are $(1, 0)$ and $(-1, 0)$.

Note: This hyperbola is called a **rectangular hyperbola**, as its asymptotes are perpendicular.



Now to sketch the graph of

$$(x - 1)^2 - (y + 2)^2 = 1$$

we apply the translation $(x, y) \to (x + 1, y - 2)$.

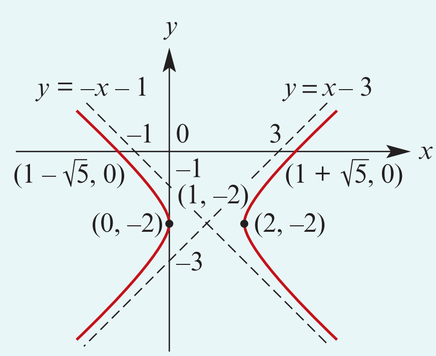
The new centre is $(1, -2)$ and the asymptotes have equations $y + 2 = \pm(x - 1)$. That is, $y = x - 3$ and $y = -x - 1$.

Axis intercepts

If $x = 0$, then $y = -2$.

If $y = 0$, then $(x - 1)^2 = 5$ and so $x = 1 \pm \sqrt{5}$.

Therefore the axis intercepts are $(0, -2)$ and $(1 \pm \sqrt{5}, 0)$.



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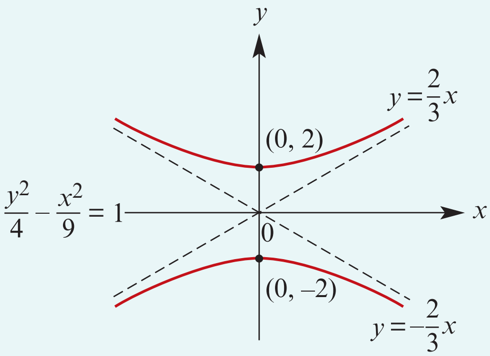
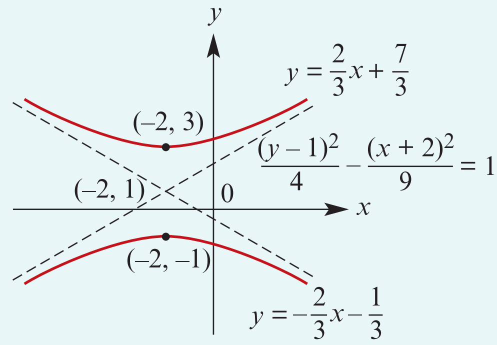
d The graph of $\frac{(y-1)^2}{4} - \frac{(x+2)^2}{9} = 1$ is obtained from the hyperbola $\frac{y^2}{4} - \frac{x^2}{9} = 1$ through the translation $(x, y) \to (x-2, y+1)$. Its centre will be $(-2, 1)$.





Using the locus definition

Example 17

Consider the points $A(-2, 0)$ and $B(2, 0)$. Find the equation of the locus of points P satisfying $AP - BP = 3$.

Solution

Let (x, y) be the coordinates of point P.

If $AP - BP = 3$, then

$$\sqrt{(x+2)^2 + y^2} - \sqrt{(x-2)^2 + y^2} = 3$$

and so

$$\sqrt{(x+2)^2 + y^2} = 3 + \sqrt{(x-2)^2 + y^2}$$

Square both sides, then expand and simplify:

$$(x+2)^2 + y^2 = 9 + 6\sqrt{(x-2)^2 + y^2} + (x-2)^2 + y^2$$

$$x^2 + 4x + 4 + y^2 = 9 + 6\sqrt{(x-2)^2 + y^2} + x^2 - 4x + 4 + y^2$$

$$8x - 9 = 6\sqrt{(x-2)^2 + y^2}$$

Note that this only holds if $x \geq \frac{9}{8}$. Squaring both sides again gives

$$64x^2 - 144x + 81 = 36(x^2 - 4x + 4 + y^2)$$

$$28x^2 - 36y^2 = 63$$

$$\frac{4x^2}{9} - \frac{4y^2}{7} = 1 \qquad \text{for } x \geq \frac{3}{2}$$

This is the right branch of a hyperbola with centre the origin and x-axis intercept $\frac{3}{2}$.

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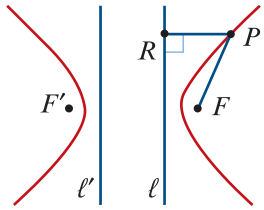
It can also be shown that a hyperbola is the locus of points $P(x, y)$ satisfying

$$FP = eRP$$

where F is a fixed point, $e > 1$ and RP is the perpendicular distance from P to a fixed line ℓ.



From the symmetry of the hyperbola, it is clear that there is a second point F' and a second line ℓ' such that $F'P = eR'P$ defines the same locus, where $R'P$ is the perpendicular distance from P to ℓ'.

Example 18

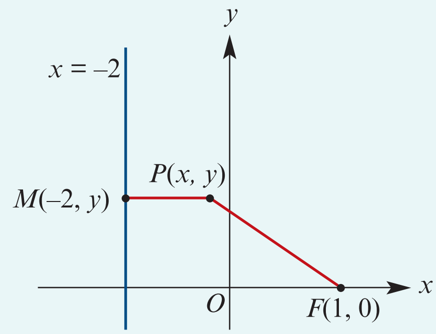
Find the equation of the locus of points $P(x, y)$ that satisfy the property that the distance from P to the point $F(1, 0)$ is twice the distance MP, the perpendicular distance from P to the line with equation $x = -2$. That is, $FP = 2MP$.

Solution

Let (x, y) be the coordinates of point P.

If $FP = 2MP$, then

$$\sqrt{(x-1)^2 + y^2} = 2\sqrt{(x+2)^2}$$



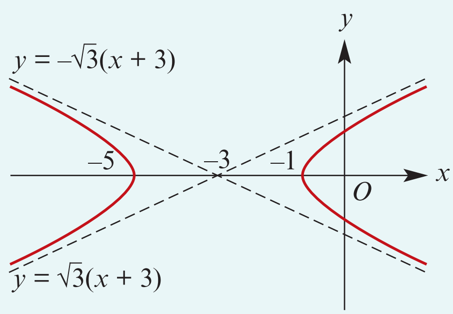
Squaring both sides gives

$$(x-1)^2 + y^2 = 4(x+2)^2$$
$$x^2 - 2x + 1 + y^2 = 4(x^2 + 4x + 4)$$
$$3x^2 + 18x - y^2 + 15 = 0$$

By completing the square, we obtain

$$3(x^2 + 6x + 9) - 27 - y^2 + 15 = 0$$
$$3(x+3)^2 - y^2 = 12$$
$$\frac{(x+3)^2}{4} - \frac{y^2}{12} = 1$$

This is a hyperbola with centre $(-3, 0)$.



Section summary

- A **hyperbola** is the locus of a point P that moves so that the difference between its distances from two fixed points F_1 and F_2 (called the **foci**) is a constant. That is, $|F_2P - F_1P| = k$.
- The graph of

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

is a hyperbola centred at the point (h, k). The asymptotes are $y - k = \pm\frac{b}{a}(x - h)$.

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Exercise 12F

Skillsheet

Example 16

1 Sketch the graph of each of the following hyperbolas. Label axis intercepts and give the equations of the asymptotes.

a $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$

b $x^2 - \dfrac{y^2}{4} = 1$

c $\dfrac{y^2}{25} - \dfrac{x^2}{100} = 1$

d $25x^2 - 9y^2 = 225$

2 Sketch the graph of each of the following hyperbolas. State the centre and label axis intercepts and asymptotes.

a $(x-1)^2 - (y+2)^2 = 1$

b $\dfrac{(x+1)^2}{4} - \dfrac{(y-2)^2}{16} = 1$

c $\dfrac{(y-3)^2}{9} - (x-2)^2 = 1$

d $25(x-4)^2 - 9y^2 = 225$

e $x^2 - 4y^2 - 4x - 8y - 16 = 0$

f $9x^2 - 25y^2 - 90x + 150y = 225$

Example 17

3 Consider the points $A(4, 0)$ and $B(-4, 0)$. Find the equation of the locus of points P satisfying $AP - BP = 6$.

4 Find the equation of the locus of points $P(x, y)$ satisfying $AP - BP = 4$, given coordinates $A(-3, 0)$ and $B(3, 0)$.

Example 18

5 Find the equation of the locus of points $P(x, y)$ that satisfy the property that the distance to P from the point $F(5, 0)$ is twice the distance MP, the perpendicular distance to P from the line with equation $x = -1$. That is, $FP = 2MP$.

6 Find the equation of the locus of points $P(x, y)$ that satisfy the property that the distance to P from the point $F(0, -1)$ is twice the distance MP, the perpendicular distance to P from the line with equation $y = -4$. That is, $FP = 2MP$.

12G Parametric equations

A **parametric curve** in the plane is a pair of functions

$$x = f(t) \quad \text{and} \quad y = g(t)$$

The variable t is called the **parameter**, and for each choice of t we get a point in the plane $(f(t), g(t))$. The set of all such points will be a curve in the plane.

It is sometimes useful to think of t as being *time*, so that the equations $x = f(t)$ and $y = g(t)$ give the position of an object at time t. Points on the curve can be plotted by substituting various values of t into the two equations.

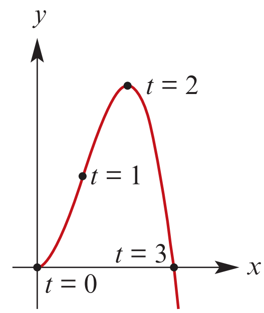
ISBN 978-1-316-63608-4

For instance, we can plot points on the curve defined by the parametric equations

$$x = t \quad \text{and} \quad y = 3t^2 - t^3$$

by letting $t = 0, 1, 2, 3$.

t	0	1	2	3
x	0	1	2	3
y	0	2	4	0



In this instance, it is possible to eliminate the parameter t to obtain a Cartesian equation in x and y alone. Substituting $t = x$ into the second equation gives $y = 3x^2 - x^3$.

► Lines

Example 19

a Find the Cartesian equation for the curve defined by the parametric equations

$$x = t + 2 \quad \text{and} \quad y = 2t - 3$$

b Find parametric equations for the line through the points $A(2, 3)$ and $B(4, 7)$.

Solution

a Substitute $t = x - 2$ into the second equation to give

$$\begin{aligned} y &= 2(x - 2) - 3 \\ &= 2x - 7 \end{aligned}$$

Thus every point lies on the straight line with equation $y = 2x - 7$.

b The gradient of the straight line through points $A(2, 3)$ and $B(4, 7)$ is

$$m = \frac{7 - 3}{4 - 2} = 2$$

Therefore the line has equation

$$\begin{aligned} y - 3 &= 2(x - 2) \\ y &= 2x - 1 \end{aligned}$$

We can simply let $x = t$ and so $y = 2t - 1$.

Note: There are infinitely many pairs of parametric equations that describe the same curve. In part **b**, we could also let $x = 2t$ and $y = 4t - 1$. These parametric equations describe exactly the same set of points. As t increases, the point moves along the same line twice as fast.

► Parabolas

Example 20

Find the Cartesian equation of the parabola defined by the parametric equations

$$x = t - 1 \quad \text{and} \quad y = t^2 + 1$$

Solution

Substitute $t = x + 1$ into the second equation to give $y = (x + 1)^2 + 1$.

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Circles

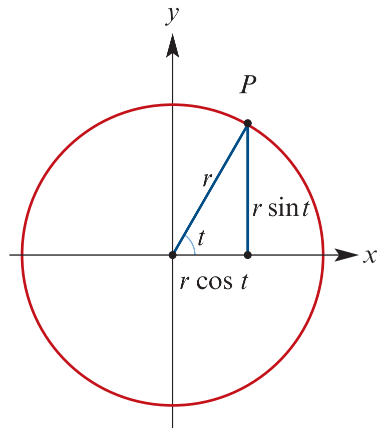
We have seen that the circle with radius r and centre at the origin can be written in Cartesian form as

$$x^2 + y^2 = r^2$$

We now introduce the parameter t and let

$$x = r\cos t \quad \text{and} \quad y = r\sin t$$

As t increases from 0 to 2π, the point $P(x, y)$ travels from $(r, 0)$ anticlockwise around the circle and returns to its original position.



To demonstrate that this parameterises the circle, we evaluate

$$\begin{aligned} x^2 + y^2 &= r^2\cos^2 t + r^2\sin^2 t \\ &= r^2(\cos^2 t + \sin^2 t) \\ &= r^2 \end{aligned}$$

where we have used the Pythagorean identity $\cos^2 t + \sin^2 t = 1$.

Example 21

a Find the Cartesian equation of the circle defined by the parametric equations

$$x = \cos t + 1 \quad \text{and} \quad y = \sin t - 2$$

b Find parametric equations for the circle with Cartesian equation

$$(x + 1)^2 + (y + 3)^2 = 4$$

Solution

a We rearrange each equation to isolate $\cos t$ and $\sin t$ respectively. This gives

$$x - 1 = \cos t \quad \text{and} \quad y + 2 = \sin t$$

Using the Pythagorean identity:

$$(x - 1)^2 + (y + 2)^2 = \cos^2 t + \sin^2 t = 1$$

So every point on the graph lies on the circle with equation $(x - 1)^2 + (y + 2)^2 = 1$.

b We let

$$\cos t = \frac{x + 1}{2} \quad \text{and} \quad \sin t = \frac{y + 3}{2}$$

giving

$$x = 2\cos t - 1 \quad \text{and} \quad y = 2\sin t - 3$$

We can easily check that these equations parameterise the given circle.

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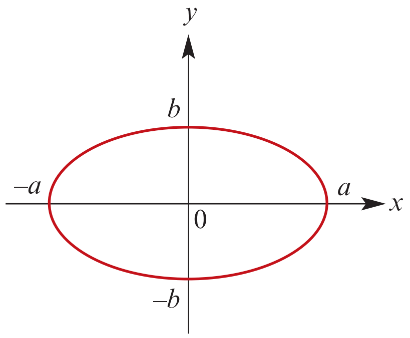
▶ Ellipses

An ellipse can be thought of as a squashed circle.

This is made apparent from the parametric equations for an ellipse:

$$x = a\cos t \quad \text{and} \quad y = b\sin t$$



As with the circle, we see the sine and cosine functions, but these are now scaled by different constants, giving different dilations from the x- and y-axes.

We can turn this pair of parametric equations into one Cartesian equation as follows:

$$\frac{x}{a} = \cos t \quad \text{and} \quad \frac{y}{b} = \sin t$$

giving

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 t + \sin^2 t = 1$$

which is the standard form of an ellipse centred at the origin with axis intercepts at $x = \pm a$ and $y = \pm b$.

Example 22

a Find the Cartesian equation of the ellipse defined by the parametric equations

$$x = 3\cos t + 1 \quad \text{and} \quad y = 2\sin t - 1$$

b Find parametric equations for the ellipse with Cartesian equation

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{16} = 1$$

Solution

a We rearrange each equation to isolate $\cos t$ and $\sin t$ respectively. This gives

$$\frac{x-1}{3} = \cos t \quad \text{and} \quad \frac{y+1}{2} = \sin t$$

Using the Pythagorean identity:

$$\left(\frac{x-1}{3}\right)^2 + \left(\frac{y+1}{2}\right)^2 = \cos^2 t + \sin^2 t = 1$$

So every point on the graph lies on the ellipse with equation $\dfrac{(x-1)^2}{3^2} + \dfrac{(y+1)^2}{2^2} = 1$.

b We let

$$\cos t = \frac{x-1}{2} \quad \text{and} \quad \sin t = \frac{y+2}{4}$$

giving

$$x = 2\cos t + 1 \quad \text{and} \quad y = 4\sin t - 2$$

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Hyperbolas

We can parameterise a hyperbola using the equations

$$x = a \sec t \quad \text{and} \quad y = b \tan t$$

From these two equations, we can find the more familiar Cartesian equation:

$$\frac{x}{a} = \sec t \quad \text{and} \quad \frac{y}{b} = \tan t$$

giving

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \sec^2 t - \tan^2 t = 1$$

which is the standard form of a hyperbola centred at the origin.

Example 23

a Find the Cartesian equation of the hyperbola defined by the parametric equations

$$x = 3 \sec t - 1 \quad \text{and} \quad y = 2 \tan t + 2$$

b Find parametric equations for the hyperbola with Cartesian equation

$$\frac{(x+2)^2}{4} - \frac{(y-3)^2}{16} = 1$$

Solution

a We rearrange each equation to isolate $\sec t$ and $\tan t$ respectively. This gives

$$\frac{x+1}{3} = \sec t \quad \text{and} \quad \frac{y-2}{2} = \tan t$$

and therefore

$$\left(\frac{x+1}{3}\right)^2 - \left(\frac{y-2}{2}\right)^2 = \sec^2 t - \tan^2 t = 1$$

So each point on the graph lies on the hyperbola with equation $\dfrac{(x+1)^2}{3^2} - \dfrac{(y-2)^2}{2^2} = 1.$

b We let

$$\sec t = \frac{x+2}{2} \quad \text{and} \quad \tan t = \frac{y-3}{4}$$

giving

$$x = 2 \sec t - 2 \quad \text{and} \quad y = 4 \tan t + 3$$

Parametric equations with restricted domains

Example 24

Eliminate the parameter to determine the graph of the parameterised curve

$$x = t - 1, \quad y = t^2 - 2t + 1 \qquad \text{for } 0 \le t \le 2$$

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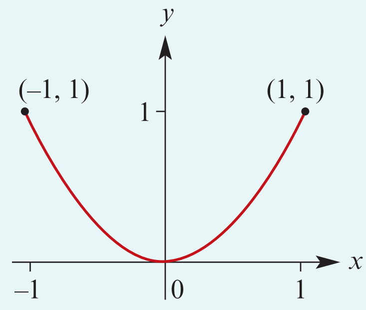
Solution

Substitute $t = x + 1$ from the first equation into the second equation, giving

$$y = (x+1)^2 - 2(x+1) + 1$$
$$= x^2 + 2x + 1 - 2x - 2 + 1$$
$$= x^2$$

Since $0 \le t \le 2$, it follows that $-1 \le x \le 1$.

Therefore, as t increases from 0 to 2, the point travels along the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$.



Intersections of curves defined parametrically

It is often difficult to find the intersection of two curves defined parametrically. This is because, although the curves may intersect, they might do so for different values of the parameter t.

In many instances, it is easiest to find the points of intersection using the Cartesian equations for the two curves.

Example 25

Find the points of intersection of the circle and line defined by the parametric equations:

circle $\quad x = 5\cos t \quad$ and $\quad y = 5\sin t$

line $\quad x = t - 3 \quad$ and $\quad y = 2t - 8$

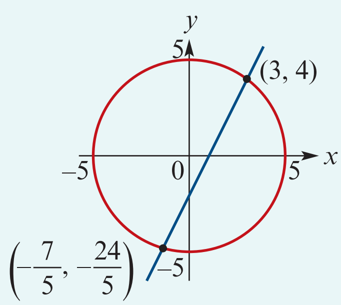
Solution

The Cartesian equation of the circle is $x^2 + y^2 = 25$.

The Cartesian equation of the line is $y = 2x - 2$.

Substituting the second equation into the first gives

$$x^2 + (2x-2)^2 = 25$$
$$x^2 + 4x^2 - 8x + 4 = 25$$
$$5x^2 - 8x - 21 = 0$$
$$(x-3)(5x+7) = 0$$



This gives solutions $x = 3$ and $x = -\dfrac{7}{5}$.

Substituting these into the equation $y = 2x - 2$ gives $y = 4$ and $y = -\dfrac{24}{5}$ respectively.

The points of intersection are $(3, 4)$ and $\left(-\dfrac{7}{5}, -\dfrac{24}{5}\right)$.

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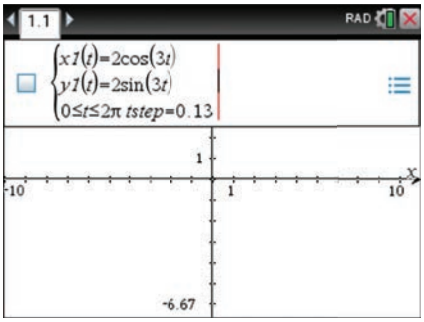
▶ Using a CAS calculator with parametric equations

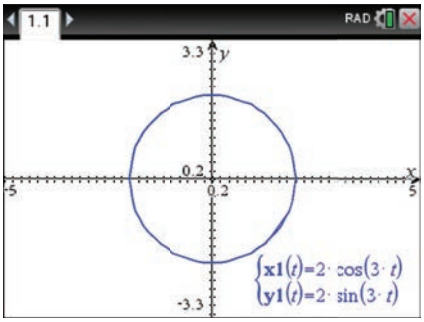
Example 26

Plot the graph of the parametric curve given by $x = 2\cos(3t)$ and $y = 2\sin(3t)$.

Using the TI-Nspire

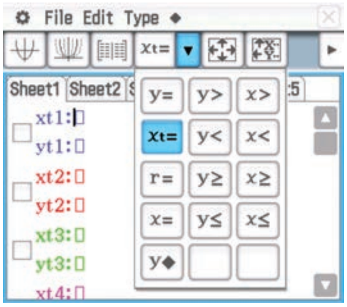
- Open a **Graphs** application (on > **New Document** > **Add Graphs**).
- Use menu > **Graph Entry/Edit** > **Parametric** to show the entry line for parametric equations.
- Enter $x1(t) = 2\cos(3t)$ and $y1(t) = 2\sin(3t)$ as shown.



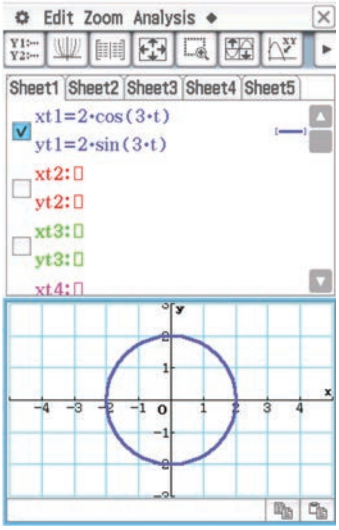


Using the Casio ClassPad

- Open the **Graph & Table** application.
- Clear all equations and graphs.
- Tap on y= in the toolbar and select xt=.



- Enter the equations in $xt1$ and $yt1$ as shown.
- Tick the box and tap .



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Section summary

- A **parametric curve** in the plane is a pair of functions

$$x = f(t) \quad \text{and} \quad y = g(t)$$

where t is called the **parameter** of the curve. For example:

	Cartesian equation	Parametric equations
Circle	$x^2 + y^2 = r^2$	$x = r\cos t$ and $y = r\sin t$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$x = a\cos t$ and $y = b\sin t$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$x = a\sec t$ and $y = b\tan t$

- We can sometimes find the Cartesian equation of a parametric curve by eliminating t and solving for y in terms of x.

Exercise 12G

Skillsheet

1 Consider the parametric equations

$$x = t - 1 \quad \text{and} \quad y = t^2 - 1$$

Example 19

a Find the Cartesian equation of the curve described by these equations.

b Sketch the curve and label the points on the curve corresponding to $t = 0, 1, 2$.

Example 20

2 For each of the following pairs of parametric equations, find the Cartesian equation and sketch the curve:

a $x = t + 1$ and $y = 2t + 1$

b $x = t - 1$ and $y = 2t^2 + 1$

c $x = t^2$ and $y = t^6$

d $x = t + 2$ and $y = \frac{1}{t+1}$

Example 21

3 **a** Find the Cartesian equation of the circle defined by the parametric equations

$$x = 2\cos t \quad \text{and} \quad y = 2\sin t$$

Example 22

b Find the Cartesian equation of the ellipse defined by the parametric equations

$$x = 3\cos t - 1 \quad \text{and} \quad y = 2\sin t + 2$$

c Find parametric equations for the circle with Cartesian equation

$$(x + 3)^2 + (y - 2)^2 = 9$$

d Find parametric equations for the ellipse with Cartesian equation

$$\frac{(x+2)^2}{9} + \frac{(y-1)^2}{4} = 1$$

4 Find parametric equations for the line through the points $A(-1, -2)$ and $B(1, 4)$.

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Example 24

5 **a** Eliminate the parameter t to determine the equation of the parameterised curve

$$x = t - 1 \quad \text{and} \quad y = -2t^2 + 4t - 2 \qquad \text{for } 0 \le t \le 2$$

b Sketch the graph of this curve over an appropriate domain.

Example 25

6 Find the points of intersection of the circle and line defined by the parametric equations:

circle $\quad x = \cos t \quad \text{and} \quad y = \sin t$

line $\quad x = 3t + 6 \quad \text{and} \quad y = 4t + 8$

7 A curve is parameterised by the equations

$$x = \sin t \quad \text{and} \quad y = 2\sin^2 t + 1 \qquad \text{for } 0 \le t \le 2\pi$$

a Find the curve's Cartesian equation.
b What is the domain of the curve?
c What is the range of the curve?
d Sketch the graph of the curve.

8 A curve is parameterised by the equations

$$x = 2^t \quad \text{and} \quad y = 2^{2t} + 1 \qquad \text{for } t \in \mathbb{R}$$

a Find the curve's Cartesian equation.
b What is the domain of the curve?
c What is the range of the curve?
d Sketch the graph of the curve.

9 Eliminate the parameter to determine the graph of the parameterised curve

$$x = \cos t \quad \text{and} \quad y = 1 - 2\sin^2 t \qquad \text{for } 0 \le t \le 2\pi$$

10 Consider the parametric equations

$$x = 2^t + 2^{-t} \quad \text{and} \quad y = 2^t - 2^{-t}$$

a Show that the Cartesian equation of the curve is $\dfrac{x^2}{4} - \dfrac{y^2}{4} = 1$ for $x \ge 2$.
b Sketch the graph of the curve.

11 Consider the circle with Cartesian equation $x^2 + (y - 1)^2 = 1$.

a Sketch the graph of the circle.
b Show that the parametric equations $x = \cos t$ and $y = \sin t + 1$ define the same circle.
c A different parameterisation of the circle can be found without the use of the cosine and sine functions. Suppose that t is any real number and let $P(x, y)$ be the point of intersection of the line $y = 2 - tx$ with the circle. Solve for x and y in terms of t, assuming that $x \ne 0$.
d Verify that the equations found in part **c** parameterise the same circle.

12 The curve with parametric equations $x = \dfrac{t}{2\pi}\cos t$ and $y = \dfrac{t}{2\pi}\sin t$ is called an **Archimedean spiral**.

a With the help of your calculator, sketch the curve over the interval $0 \le t \le 6\pi$.
b Label the points on the curve corresponding $t = 0, 1, 2, 3, 4, 5, 6$.

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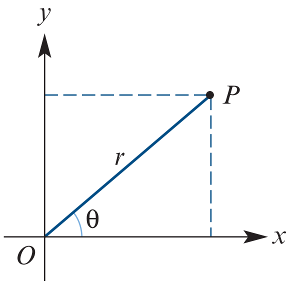
12H Polar coordinates

Until now, we have described each point in the plane by a pair of numbers (x, y). These are called Cartesian coordinates, and take their name from the French intellectual René Descartes (1596–1650) who introduced them. However, they are not the only way to describe points in the plane. In fact, for many situations it is more convenient to use **polar coordinates**.

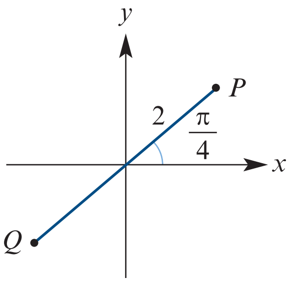
Using polar coordinates, every point P in the plane is described by a pair of numbers (r, θ).

- The number r is the distance from the origin O to P.
- The number θ measures the angle between the positive direction of the x-axis and the ray OP, as shown.



For example, the diagram on the right shows the point P with polar coordinates $\left(2, \frac{\pi}{4}\right)$.

We can even make sense of polar coordinates such as $Q\left(-2, \frac{\pi}{4}\right)$: go to the direction $\frac{\pi}{4}$ and then move a distance of 2 in the opposite direction.



Converting between the two coordinate systems requires little more than basic trigonometry.

- If a point P has polar coordinates (r, θ), then its Cartesian coordinates (x, y) satisfy

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

- If a point P has Cartesian coordinates (x, y), then its polar coordinates (r, θ) satisfy

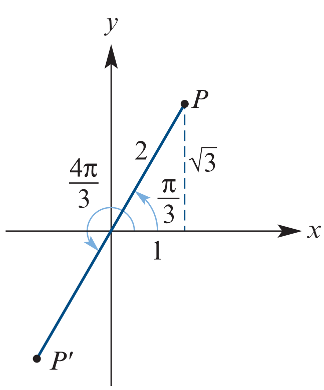
$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x} \qquad (\text{if } x \neq 0)$$

Non-uniqueness of polar coordinates

Polar coordinates differ from Cartesian coordinates in that each point in the plane has more than one representation in polar coordinates.

For example, the following polar coordinates all represent the same point:

$$\left(2, \frac{\pi}{3}\right), \quad \left(-2, \frac{4\pi}{3}\right) \quad \text{and} \quad \left(2, \frac{7\pi}{3}\right)$$



The point $P(r, \theta)$ can be described in infinitely many ways:

$$(r, \theta + 2n\pi) \quad \text{and} \quad (-r, \theta + (2n+1)\pi) \qquad \text{for all } n \in \mathbb{Z}$$

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Example 27

Convert polar coordinates $\left(2, \frac{5\pi}{6}\right)$ into Cartesian coordinates.

Solution

$$x = r\cos\theta = 2\cos\left(\frac{5\pi}{6}\right) = -\sqrt{3}$$

$$y = r\sin\theta = 2\sin\left(\frac{5\pi}{6}\right) = 1$$

The Cartesian coordinates are $(-\sqrt{3}, 1)$.

Example 28

For each pair of Cartesian coordinates, find two representations using polar coordinates, one with $r > 0$ and the other with $r < 0$.

a $(3, 3)$ **b** $(1, -\sqrt{3})$ **c** $(-5, 0)$ **d** $(0, 3)$

Solution

a $r = \sqrt{3^2 + 3^2} = 3\sqrt{2}$

$\theta = \tan^{-1}(1) = \frac{\pi}{4}$

The point has polar coordinates $\left(3\sqrt{2}, \frac{\pi}{4}\right)$.

We could also let $r = -3\sqrt{2}$ and add π to the angle, giving $\left(-3\sqrt{2}, \frac{5\pi}{4}\right)$.

b $r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$

$\theta = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

The point has polar coordinates $\left(2, -\frac{\pi}{3}\right)$.

We could also let $r = -2$ and add π to the angle, giving $\left(-2, \frac{2\pi}{3}\right)$.

c $r = 5$ and $\theta = \pi$

The point has polar coordinates $(5, \pi)$.

We could also let $r = -5$ and subtract π from the angle, giving $(-5, 0)$.

d $r = 3$ and $\theta = \frac{\pi}{2}$

The point has polar coordinates $\left(3, \frac{\pi}{2}\right)$.

We could also let $r = -3$ and subtract π from the angle, giving $\left(-3, -\frac{\pi}{2}\right)$.

Section summary

- Each point P in the plane can be represented using polar coordinates (r, θ), where:
  - r is the distance from the origin O to P
  - θ is the angle between the positive direction of the x-axis and the ray OP.
- If a point P has polar coordinates (r, θ), then its Cartesian coordinates (x, y) satisfy

$$x = r\cos\theta \quad \text{and} \quad y = r\cos\theta$$

- If a point P has Cartesian coordinates (x, y), then all its polar coordinates (r, θ) satisfy

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x} \qquad (\text{if } x \neq 0)$$

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- Each point in the plane has more than one representation in polar coordinates. For example, the coordinates $\left(2, \frac{\pi}{4}\right)$, $\left(2, \frac{9\pi}{4}\right)$ and $\left(-2, \frac{5\pi}{4}\right)$ all represent the same point.

Exercise 12H

Example 27

1 Plot the points with the following polar coordinates and then find their Cartesian coordinates:

a $A\left(1, \frac{\pi}{2}\right)$ **b** $B\left(2, \frac{3\pi}{4}\right)$ **c** $C\left(3, \frac{-\pi}{2}\right)$ **d** $D\left(-2, \frac{\pi}{4}\right)$ **e** $E(-1, \pi)$

f $F\left(0, \frac{\pi}{4}\right)$ **g** $G\left(4, -\frac{5\pi}{6}\right)$ **h** $H\left(-2, \frac{2\pi}{3}\right)$ **i** $I\left(-2, -\frac{\pi}{4}\right)$

Example 28

2 For each of the following pairs of Cartesian coordinates, find two representations using polar coordinates, one with $r > 0$ and the other with $r < 0$:

a $(1, -1)$ **b** $(1, \sqrt{3})$ **c** $(2, -2)$

d $(-\sqrt{2}, -\sqrt{2})$ **e** $(3, 0)$ **f** $(0, -2)$

3 Two points have polar coordinates $P\left(2, \frac{\pi}{6}\right)$ and $Q\left(3, \frac{\pi}{2}\right)$ respectively. Find the exact length of line segment PQ.

4 Two points have polar coordinates $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$. Find a formula for the length of PQ.

12I Graphing using polar coordinates

Polar coordinates are useful for describing and sketching curves in the plane, especially in situations that involve symmetry with respect to the origin. Suppose that f is a function. The graph of f in polar coordinates is simply the set of all points (r, θ) such that $r = f(\theta)$.

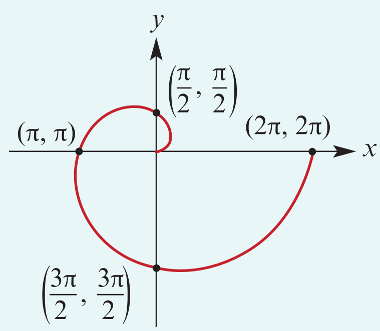
Example 29

Sketch the spiral with polar equation $r = \theta$, for $0 \le \theta \le 2\pi$.

Solution

The distance r from the origin exactly matches the angle θ. So as the angle increases, so too does the distance from the origin.

Note that the coordinates on the graph are in polar form.



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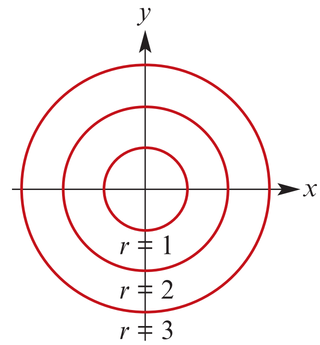
Cambridge University Press

Circles

If a circle is centred at the origin, then its polar equation could not be simpler.



A circle of radius a centred at the origin has polar equation

$$r = a$$

That is, the distance r from the origin is constant, having no dependence on the angle θ. This illustrates rather forcefully the utility of polar coordinates: they simplify situations that involve symmetry with respect to the origin.

For circles not centred at the origin, the polar equations are less obvious.

Example 30

A curve has polar equation $r = 2\sin\theta$. Show that its Cartesian equation is $x^2 + (y-1)^2 = 1$.

Solution

The trick here is to first multiply both sides of the polar equation by r to get

$$r^2 = 2r\sin\theta$$

Since $r^2 = x^2 + y^2$ and $r\sin\theta = y$, this equation becomes

$$\begin{aligned} x^2 + y^2 &= 2y \\ x^2 + y^2 - 2y &= 0 \\ x^2 + (y^2 - 2y + 1) - 1 &= 0 \qquad \text{(completing the square)} \\ x^2 + (y-1)^2 &= 1 \end{aligned}$$

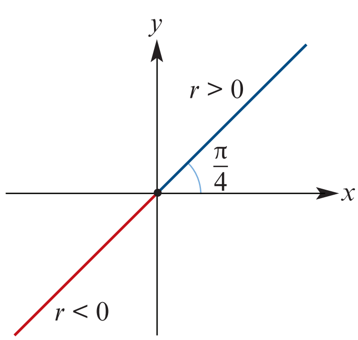
This is a circle with centre (0, 1) and radius 1.

Lines

For a straight line through the origin, the angle θ is fixed and the distance r varies. Because we have allowed negative values of r, the straight line goes in both directions.

The straight line shown has equation

$$\theta = \frac{\pi}{4}$$



For a straight line that does not go through the origin, the equation is more complicated. A line in Cartesian form $ax + by = c$ can be converted into polar form by substituting $x = r\cos\theta$ and $y = r\sin\theta$.

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Example 31

a Express $x + y = 1$ in polar form.

b Express $r = \dfrac{2}{3\cos\theta - 4\sin\theta}$ in Cartesian form.

Solution

a Since $x = r\cos\theta$ and $y = r\sin\theta$, the equation $x + y = 1$ becomes

$$r\cos\theta + r\sin\theta = 1$$
$$r(\cos\theta + \sin\theta) = 1$$

Therefore the straight line has polar equation

$$r = \frac{1}{\cos\theta + \sin\theta}$$

b Since $\dfrac{x}{r} = \cos\theta$ and $\dfrac{y}{r} = \sin\theta$, the equation becomes

$$r = \frac{2}{\dfrac{3x}{r} - \dfrac{4y}{r}}$$
$$r = \frac{2r}{3x - 4y}$$
$$1 = \frac{2}{3x - 4y}$$

Therefore the Cartesian equation is

$$3x - 4y = 2$$

Section summary

- For a function f, the graph of f in polar coordinates is the set of all points (r, θ) such that $r = f(\theta)$.
- Polar coordinates are useful for describing graphs that are symmetric about the origin. For example:
  - The circle centred at the origin with radius a has equation $r = a$.
  - The line through the origin at angle α to the positive x-axis has equation $\theta = \alpha$.
- To convert between the polar form and the Cartesian form of an equation, substitute $x = r\cos\theta$ and $y = r\sin\theta$.

Exercise 12I

Skillsheet

Example 29

1 Sketch the spiral with polar equation $r = \dfrac{\theta}{2\pi}$, for $0 \le \theta \le 4\pi$.

2 Express each of the following Cartesian equations in polar form:

a $x = 4$ **b** $xy = 1$ **c** $y = x^2$

d $x^2 + y^2 = 9$ **e** $x^2 - y^2 = 1$ **f** $2x - 3y = 5$

3 Express each of the following polar equations in Cartesian form:

a $r = \dfrac{2}{\cos\theta}$ **b** $r = 2$ **c** $\theta = \dfrac{\pi}{4}$ **d** $r = \dfrac{4}{3\cos\theta - 2\sin\theta}$

Example 30

4 By finding the Cartesian equation, show that each of the following polar equations describes a circle:

a $r = 6\cos\theta$ **b** $r = 4\sin\theta$ **c** $r = -6\cos\theta$ **d** $r = -8\sin\theta$

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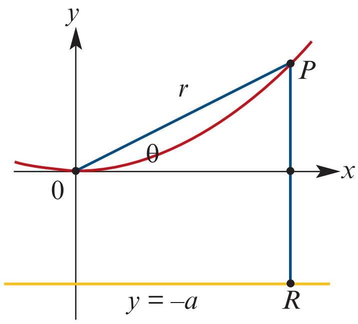
5 Show that the graph of $r = 2a\cos\theta$ is a circle of radius a centred at $(a, 0)$.

Example 31

6 **a** Show that the graph of $r = \dfrac{a}{\cos\theta}$ is a vertical line.

b Find the polar form of the horizontal line $y = a$.

7 A set of points $P(x, y)$ is such that the distance from P to the origin O is equal to the perpendicular distance from P to the line $y = -a$, where $a > 0$. This set of points is a parabola. Suppose that point P has polar coordinates (r, θ).



a Show that the distance from P to the line is $a + r\sin\theta$.

b Conclude that the equation for the parabola can be written as $r = \dfrac{a}{1 - \sin\theta}$.

12J Further graphing using polar coordinates

Various geometrically significant and beautiful figures are best described using polar coordinates.

▶ Cardioids

The name **cardioid** comes from the Greek word for heart. A cardioid is the curve traced by a point on the perimeter of a circle that is rolling around a fixed circle of the same radius.

Example 32

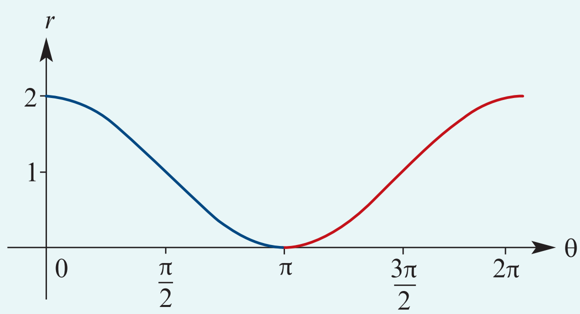
Graph the cardioid with equation $r = 1 + \cos\theta$, for $\theta \in [0, 2\pi]$.

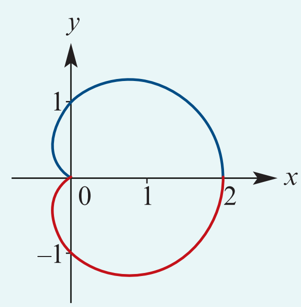
Solution

To help sketch this curve, we first graph the function $r = 1 + \cos\theta$ using Cartesian coordinates, as shown on the left. This allows us to see how r changes as θ increases.

- As the angle θ increases from 0 to π, the distance r decreases from 2 to 0.
- As the angle θ increases from π to 2π, the distance r increases from 0 to 2.

This gives the graph of the cardioid shown on the right.





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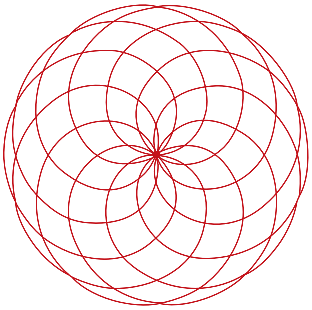
Roses



This impressive curve is fittingly called a **rose**. It belongs to the family of curves with polar equations of the form

$$r = \cos(n\theta)$$

For the example shown, $n = \frac{5}{8}$.

Example 33

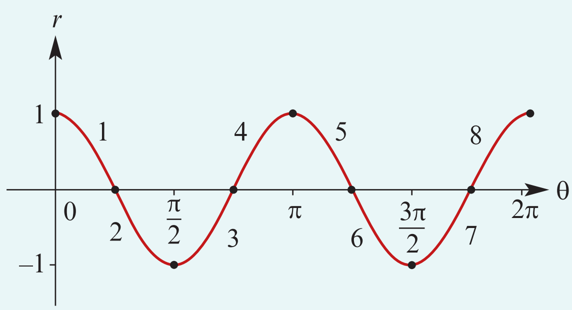
A curve has polar equation $r = \cos(2\theta)$.

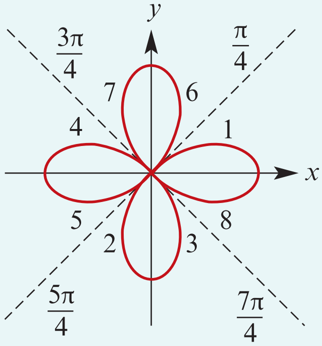
a Sketch the graph of the curve.

b Show that its Cartesian equation is $(x^2 + y^2)^3 = (x^2 - y^2)^2$.

Solution

a To help sketch this curve, we first graph the function $r = \cos(2\theta)$ using Cartesian coordinates, as shown on the left. This allows us to see how r changes as θ increases. Using numbers, we have labelled how each section of this graph corresponds to a section of the rose shown on the right.





b Using the double angle formula $\cos(2\theta) = \cos^2\theta - \sin^2\theta$, we have

$$\begin{aligned} r &= \cos(2\theta) \\ r &= \cos^2\theta - \sin^2\theta \\ r &= \left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2 \\ r^3 &= x^2 - y^2 \\ \left(\sqrt{x^2 + y^2}\right)^3 &= x^2 - y^2 \\ (x^2 + y^2)^3 &= (x^2 - y^2)^2 \end{aligned}$$

Note: This example further illustrates how polar coordinates can give more pleasing equations than their Cartesian counterparts.

The curve in this example is a **four-leaf rose**. More generally, the equations $r = \cos(n\theta)$ and $r = \sin(n\theta)$ give $2n$-leaf roses if n is even, and give n-leaf roses if n is odd.

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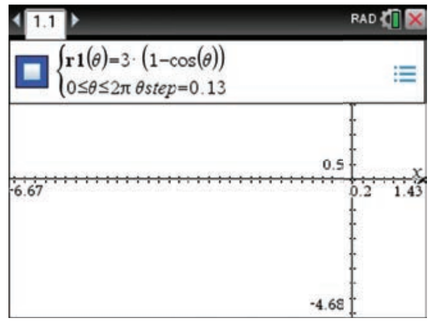
▶ Using a CAS calculator with polar coordinates

Example 34

Plot the graph of $r = 3(1 - \cos\theta)$.

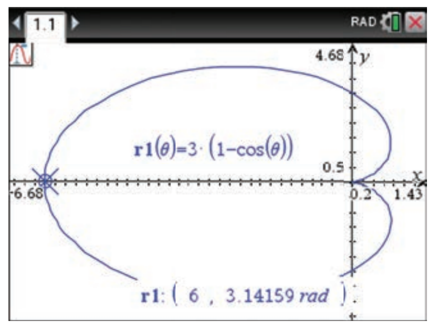
Using the TI-Nspire

- Open a **Graphs** application ([⌂ on] > **New Document** > **Add Graphs**) and set to polar using [menu] > **Graph Entry/Edit** > **Polar**.
- Enter $r1(\theta) = 3(1 - \cos(\theta))$ as shown. The variable θ is entered using [π▸] or the Symbols palette ([ctrl] [📖]).



Note: The domain and the step size can be adjusted in this window.

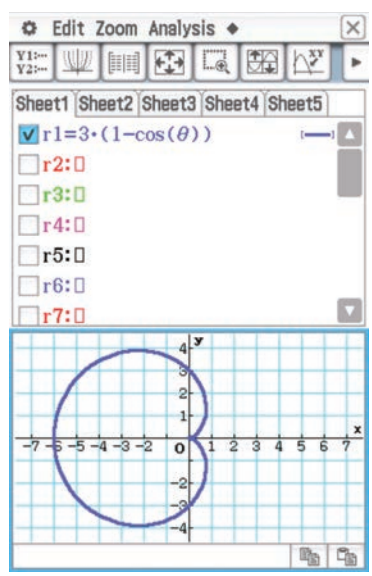
- Set the scale using [menu] > **Window/Zoom** > **Zoom – Fit**.
- You can see the polar coordinates (r, θ) of points on the graph using [menu] > **Trace** > **Graph Trace**.
- To go to the point where $\theta = \pi$, simply type π and then press [enter]. The cursor will move to the point $(r, \theta) = (6, \pi)$ as shown.



Using the Casio ClassPad

- Open the **Graph & Table** application [Graph& Table].
- Clear all equations and graphs.
- Tap on [y=] in the toolbar and select [r=].

- Enter $3(1 - \cos(\theta))$ in $r1$.
- Tick the box and tap [graph].
- Select **Zoom** > **Initialize** to adjust the window.



Note: The variable θ is found in the [Trig] keyboard.

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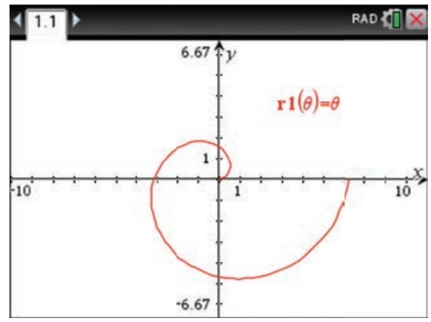
Example 35

Plot the graph of $r = \theta$ for $0 \le \theta \le 6\pi$.

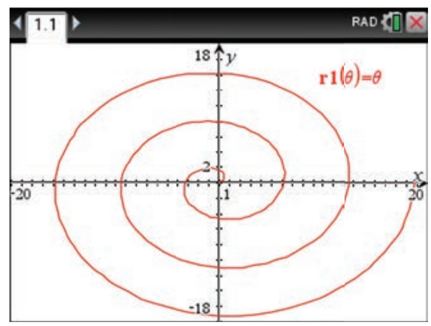
Using the TI-Nspire

- Open a **Graphs** application (on > **New Document** > **Add Graphs**) and set to polar using menu > **Graph Entry/Edit** > **Polar**.
- Enter $r1(\theta) = \theta$.
- The graph is shown.



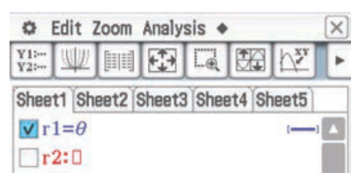
- If the domain is extended, the graph continues to spiral out. This can be observed by extending the domain to $0 \le \theta \le 6\pi$.

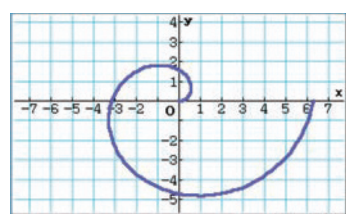
Note: Change the domain in the graph entry line; it is not the window setting.



Using the Casio ClassPad

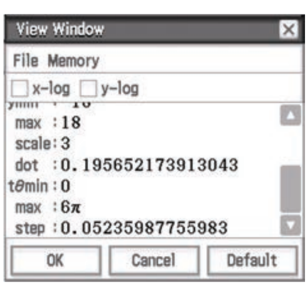
- Open the **Graph & Table** application.
- Clear all equations and graphs.
- Tap y= in the toolbar and select r=.
- Enter θ in $r1$. Tick the box and tap .

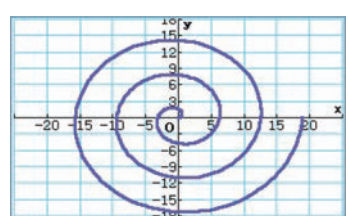




To extend the domain of the graph:

- Tap in the toolbar.
- Scroll down to the bottom of the list of settings.
- Set $t\theta$ max to 6π as shown below.





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Section summary

- To sketch the curve $r = f(\theta)$ in polar coordinates, it helps to first sketch the graph in Cartesian coordinates.

Exercise 12J

1 Consider the polar equation

$$r = \frac{e}{1 + e\sin\theta}$$

When $0 < e < 1$, it can be shown that this equation describes an ellipse.

a With the help of your calculator, sketch the graphs of this polar equation when $e = 0.7, 0.8, 0.9$ on the same set of axes.

b The number e is called the **eccentricity** of the ellipse. What happens to the shape of the ellipse as the value of e is increased?

Example 32

2 The curve with polar equation $r = 1 - \sin\theta$ is a cardioid.

a Sketch the graph of the cardioid.

b Show that its Cartesian equation is $(x^2 + y^2 + y)^2 = x^2 + y^2$.

Example 33

3 Sketch the graphs of the roses with the following polar equations:

a $r = \cos(3\theta)$

b $r = \sin(3\theta)$

4 The polar equation $r = \sin(2\theta)$ defines a four-leaf rose.

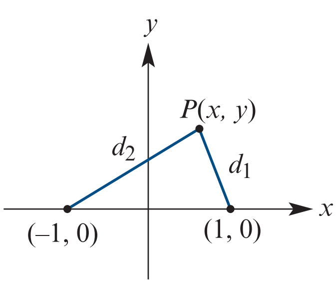
a Sketch the graph of the rose.

b Using a suitable double angle formula, show that its Cartesian equation is $(x^2 + y^2)^3 = 4x^2y^2$.

5 **a** With the help of your calculator, sketch the graph of the **lemniscate**, which has polar equation $r = \sqrt{2\cos(2\theta)}$.

b Using an appropriate double angle formula, find its equation in Cartesian coordinates.

c Show that the lemniscate can also be described as the set of all points $P(x, y)$ in the diagram that satisfy $d_1 d_2 = 1$.



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Chapter summary

AS
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Reciprocal functions

- If $y = f(x)$ is a function, then the **reciprocal function** is defined by the rule $y = \dfrac{1}{f(x)}$.
- To sketch the graph of $y = \dfrac{1}{f(x)}$, we first sketch the graph of $y = f(x)$.
- The x-axis intercepts of $y = f(x)$ will become vertical asymptotes of $y = \dfrac{1}{f(x)}$.
- Local maximums of $y = f(x)$ will become local minimums of $y = \dfrac{1}{f(x)}$, and vice versa.

Parabolas, ellipses and hyperbolas

- A **locus** is the set of points described by a geometric condition.
- A **circle** is the locus of a point P that moves so that its distance from a fixed point C is constant.
- A **straight line** is the locus of a point P that moves so that it is equidistant from two fixed points Q and R.
- A **parabola** is the locus of a point P that moves so that its distance from a fixed point F is equal to its perpendicular distance from a fixed line.
- An **ellipse** is the locus of a point P that moves so that the sum of its distances d_1 and d_2 from two fixed points F_1 and F_2 is a constant.
- The graph of
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$
is an ellipse centred at the point (h, k).
- A **hyperbola** is the locus of a point P that moves so that the difference between its distances from two fixed points F_1 and F_2 is a constant.
- The graph of
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$
is a hyperbola centred at the point (h, k). The asymptotes are $y - k = \pm\dfrac{b}{a}(x - h)$.

Parametric curves

- A **parametric curve** in the plane is a pair of functions
$$x = f(t) \quad \text{and} \quad y = g(t)$$
where t is called the **parameter** of the curve.
- It can be helpful to think of the parameter t as describing time. Parametric curves are then useful for describing the motion of an object.
- We can sometimes find the Cartesian equation of a parametric curve by eliminating t and solving for y in terms of x.

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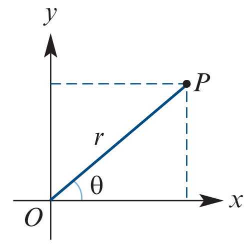
Polar coordinates



- Each point P in the plane can be represented using polar coordinates (r, θ), where:
  - r is the distance from the origin O to P
  - θ is the angle between the positive direction of the x-axis and the ray OP.
- If a point P has polar coordinates (r, θ), then its Cartesian coordinates (x, y) satisfy

$$x = r\cos\theta \quad \text{and} \quad y = r\cos\theta \qquad (1)$$

- If a point P has Cartesian coordinates (x, y), then all its polar coordinates (r, θ) satisfy

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x} \quad (\text{if } x \neq 0) \qquad (2)$$

- If f is a function, then the graph of f in polar coordinates is the set of all points (r, θ) such that $r = f(\theta)$.
- To convert between the polar form and the Cartesian form of an equation, use formulas (1) and (2) above.

Short-answer questions

1 For each of the following, sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same axes:

a $f(x) = \frac{1}{2}(x^2 - 4)$ **b** $f(x) = (x+1)^2 + 1$

c $f(x) = \cos(x) + 1,\ x \in [0, 2\pi]$ **d** $f(x) = \sin(x) + 2,\ x \in [0, 2\pi]$

2 Sketch the graph of each of the following over the interval $[0, 2\pi]$:

a $y = \cot(4x)$ **b** $y = 2\operatorname{cosec}(2x)$ **c** $y = -3\sec(2x)$

d $y = \operatorname{cosec}\left(x - \frac{\pi}{3}\right)$ **e** $y = \sec(x) - 3$ **f** $y = \cot\left(x + \frac{\pi}{3}\right)$

3 Describe a sequence of transformations that will take the graph of $y = \operatorname{cosec} x$ to the graph of $y = -3\operatorname{cosec}(2x) + 1$. Sketch the transformed graph over the interval $[-\pi, \pi]$.

4 The point $P(x, y)$ moves so that it is equidistant from $Q(2, -1)$ and $R(1, 2)$. Find the locus of the point P.

5 Find the locus of the point $P(x, y)$ such that $AP = 6$, given point $A(3, 2)$.

6 A circle has equation $x^2 + 4x + y^2 - 8y = 0$. Find the coordinates of the centre and the radius of the circle.

7 Sketch the graph of each ellipse and find the coordinates of its axis intercepts:

a $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ **b** $\dfrac{(x-2)^2}{4} + \dfrac{(y+1)^2}{9} = 1$

8 An ellipse has equation $x^2 + 4x + 2y^2 = 0$. Find the coordinates of the centre and the axis intercepts of the ellipse.

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Review

9 Sketch the graph of each hyperbola and write down the equations of its asymptotes:

a $x^2 - \frac{y^2}{4} = 1$ **b** $\frac{(y-1)^2}{16} - \frac{(x+2)^2}{4} = 1$

10 A point $P(x, y)$ moves so that its distance from the point $K(-2, 5)$ is twice its distance from the line $x = 1$. Find its locus.

11 For each of the following pairs of parametric equations, find the corresponding Cartesian equation:

a $x = 2t - 1$ and $y = 6 - 4t$ **b** $x = 2\cos t$ and $y = 2\sin t$

c $x = 3\cos t + 1$ and $y = 5\sin t - 1$ **d** $x = \cos t$ and $y = 3\sin^2 t - 2$

12 A curve has parametric equations $x = t - 1$ and $y = 2t^2 - 1$ for $0 \le t \le 2$.

a Find the curve's Cartesian equation. **b** What is the domain of the curve?

c What is the range of the curve? **d** Sketch the graph of the curve.

13 Convert the polar coordinates $\left(2, \frac{3\pi}{4}\right)$ into Cartesian coordinates.

14 The point P has Cartesian coordinates $(2, -2\sqrt{3})$. Find two representations of P using polar coordinates, one with $r > 0$ and the other with $r < 0$.

15 Find an equation for the straight line $2x + 3y = 5$ using polar coordinates.

16 Show that the graph of $r = 6\sin\theta$ is a circle of radius 3 centred at $(0, 3)$.
Hint: First multiply both sides of the equation by r.

Multiple-choice questions

1 The graph of $y = \frac{1}{x^2 + 8x + k}$ will have two vertical asymptotes provided

A $k = 16$ **B** $k < 16$ **C** $k > 16$

D $k < -4$ or $k > 4$ **E** $-4 < k < 4$

2 An equation for the graph shown is

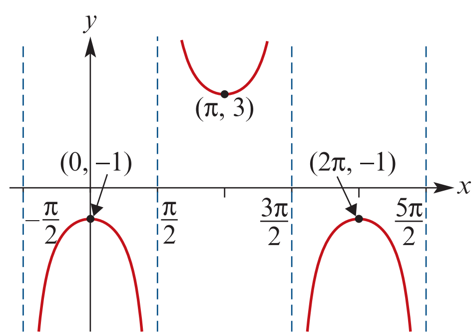
A $y = -2\operatorname{cosec}(x) + 1$

B $y = -2\sec(x) + 1$

C $y = 2\operatorname{cosec}(x) + 1$

D $y = 2\sec(x) + 1$

E $y = 2\operatorname{cosec}(x) - 1$



3 The locus of points $P(x, y)$ which satisfy the property that $AP = BP$, given points $A(2, -5)$ and $B(-4, 1)$, is described by the equation

A $y = x - 1$ **B** $y = x - 6$ **C** $y = -x - 3$ **D** $y = x + 1$ **E** $y = 3 - x$

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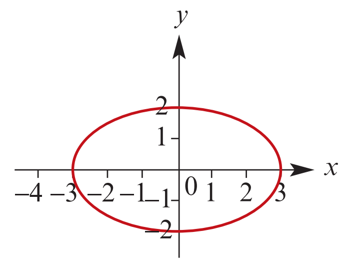
Review

4 A parabola has focus $(0, 2)$ and directrix $y = -2$. Which of the following is not true about the parabola?

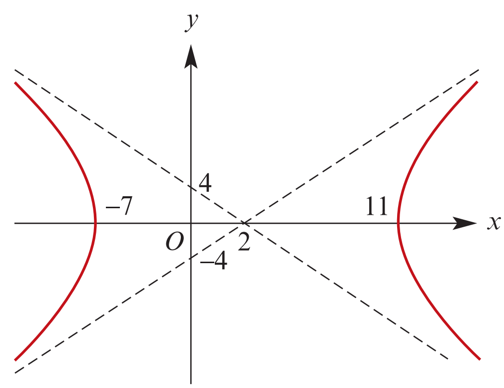
A Its axis of symmetry is the line $x = 0$.
B It passes through the origin.
C It contains no point below the x-axis.
D The point $(2, 1)$ lies on the parabola.
E The point $(4, 2)$ lies on the parabola.

5 The equation of the graph shown is



A $\dfrac{x^2}{3} + \dfrac{y^2}{2} = 1$
B $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$
C $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$
D $\dfrac{x^2}{3} - \dfrac{y^2}{2} = 1$
E $3x^2 + 2y^2 = 1$

6 The equation of the graph shown is



A $\dfrac{(x+2)^2}{27} - \dfrac{y^2}{108} = 1$
B $\dfrac{(x-2)^2}{9} - \dfrac{y^2}{34} = 1$
C $\dfrac{(x+2)^2}{81} - \dfrac{y^2}{324} = 1$
D $\dfrac{(x-2)^2}{81} - \dfrac{y^2}{324} = 1$
E $\dfrac{(x+2)^2}{9} - \dfrac{y^2}{36} = 1$

7 The asymptotes of the hyperbola with equation $\dfrac{(y-2)^2}{9} - \dfrac{(x+3)^2}{4} = 1$ intersect at the point

A $(3, 2)$ **B** $(3, -2)$ **C** $(-3, 2)$ **D** $(2, -3)$ **E** $(-2, 3)$

8 An ellipse is parameterised by the equations $x = 4\cos t + 1$ and $y = 2\sin t - 1$. The coordinates of its x-axis intercepts are

A $(1 - 3\sqrt{2}, 0),\ (1 + 3\sqrt{2}, 0)$
B $(-3, 0),\ (5, 0)$
C $(1 - 2\sqrt{3}, 0),\ (1 + 2\sqrt{3}, 0)$
D $(0, -3),\ (0, 5)$
E $(0, 1 - 2\sqrt{3}),\ (0, 1 + 2\sqrt{3})$

9 Which of the following pairs of polar coordinates represent the same point?

A $\left(2, \dfrac{\pi}{4}\right)$ and $\left(2, \dfrac{3\pi}{4}\right)$
B $\left(3, \dfrac{\pi}{2}\right)$ and $\left(-3, \dfrac{\pi}{2}\right)$
C $\left(2, \dfrac{\pi}{3}\right)$ and $\left(-2, \dfrac{2\pi}{3}\right)$
D $\left(3, \dfrac{\pi}{4}\right)$ and $\left(3, \dfrac{5\pi}{4}\right)$
E $\left(1, \dfrac{\pi}{6}\right)$ and $\left(-1, \dfrac{7\pi}{6}\right)$

10 A curve has polar equation $r = 1 + \cos\theta$. Its equation in Cartesian coordinates is

A $xy = x^2 + y^2$
B $(x^2 + y^2 - x)^2 = x^2 + y^2$
C $x = x^2 + y^2$
D $(x^2 + y^2 - y)^2 = x^2 + y^2$
E $y = x^2 + y^2$

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Extended-response questions

1 Consider points $A(0, 3)$ and $B(6, 0)$. Find the locus of the point $P(x, y)$ given that:

a $AP = BP$

b $AP = 2BP$

2 Find the equation of the locus of points $P(x, y)$ which satisfy the property that the distance to P from the point $F(0, 4)$ is equal to:

a MP, the perpendicular distance from the line with equation $y = -2$

b half the distance MP, the perpendicular distance from the line $y = -2$

c twice the distance MP, the perpendicular distance from the line $y = -2$.

3 A ball is thrown into the air. The position of the ball at time $t \geq 0$ is given by the parametric equations $x = 10t$ and $y = 20t - 5t^2$.

a Find the Cartesian equation of the ball's flight.

b Sketch the graph of the ball's path.

c What is the maximum height reached by the ball?

A second ball is thrown into the air. Its position at time $t \geq 0$ is given by the parametric equations $x = 60 - 10t$ and $y = 20t - 5t^2$.

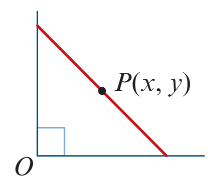
d Find the Cartesian equation of the second ball's flight.

e Sketch the graph of the second ball's path on the same set of axes.

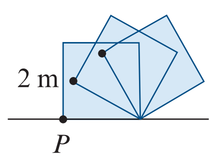
f Find the points of intersection of the two paths.

g Do the balls collide?

4 A ladder of length 6 metres stands against a vertical wall. The ladder then slides along on the floor until it lies flat. Show that the midpoint $P(x, y)$ of the ladder moves along a circular path.



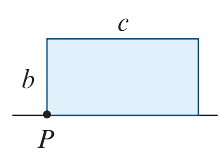
5 A square box of side length 2 metres is too heavy to lift, but can be rolled along the flat ground, using each edge as a pivot. The box is rolled one full revolution.



a Sketch the full path of the point P.

b Find the total distance travelled by the point P.

c A second rectangular box has width b metres and length c metres. Sketch the path taken by the point P when the box is rolled one full revolution, and find the total distance travelled by this point.



d For the second box, find the area between the path taken by P and the ground.

 ISBN 978-1-316-63608-4 Cambridge University Press

Chapter 13

13 Complex numbers

Objectives

- To understand the **imaginary number** i and the set of **complex numbers** $\mathbb{C}$.
- To find the **real part** and the **imaginary part** of a complex number.
- To perform **addition**, **subtraction**, **multiplication** and **division** of complex numbers.
- To find the **conjugate** of a complex number.
- To represent complex numbers graphically on an **Argand diagram**.
- To work with complex numbers in **polar form**, and to understand the geometric interpretation of multiplication and division of complex numbers in this form.
- To factorise quadratic polynomials over $\mathbb{C}$.
- To solve quadratic equations over $\mathbb{C}$.

In this chapter we introduce a new set of numbers, called *complex numbers*. These numbers first arose in the search for solutions to polynomial equations.

In the sixteenth century, mathematicians including Girolamo Cardano began to consider square roots of negative numbers. Although these numbers were regarded as ‘impossible’, they arose in calculations to find real solutions of cubic equations.

For example, the cubic equation $x^3 - 15x - 4 = 0$ has three real solutions. Cardano’s formula gives the solution

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

which you can show equals 4.

Today complex numbers are widely used in physics and engineering, such as in the study of aerodynamics.

ISBN 978-1-316-63608-4

13A Starting to build the complex numbers

Mathematicians in the eighteenth century introduced the imaginary number i with the property that

$$i^2 = -1$$

The equation $x^2 = -1$ has two solutions, namely i and $-i$.

By declaring that $i = \sqrt{-1}$, we can find square roots of all negative numbers.

For example:

$$\begin{aligned}\sqrt{-4} &= \sqrt{4 \times (-1)}\\ &= \sqrt{4} \times \sqrt{-1}\\ &= 2i\end{aligned}$$

Note: The identity $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ holds for positive real numbers a and b, but does not hold when both a and b are negative. In particular, $\sqrt{-1} \times \sqrt{-1} \neq \sqrt{(-1) \times (-1)}$.

Now consider the equation $x^2 + 2x + 3 = 0$. Using the quadratic formula gives

$$\begin{aligned}x &= \frac{-2 \pm \sqrt{4 - 12}}{2}\\ &= \frac{-2 \pm \sqrt{-8}}{2}\\ &= -1 \pm \sqrt{-2}\end{aligned}$$

This equation has no real solutions. However, using complex numbers we obtain solutions

$$x = -1 \pm \sqrt{2}i$$

▶ The set of complex numbers

A **complex number** is an expression of the form $a + bi$, where a and b are real numbers.

The set of all complex numbers is denoted by $\mathbb{C}$. That is,

$$\mathbb{C} = \{\, a + bi : a, b \in \mathbb{R} \,\}$$

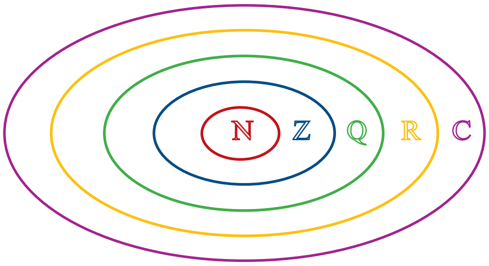
The letter often used to denote a complex number is z.

Therefore if $z \in \mathbb{C}$, then $z = a + bi$ for some $a, b \in \mathbb{R}$.

- If $a = 0$, then $z = bi$ is said to be an **imaginary number**.
- If $b = 0$, then $z = a$ is a **real number**.

The real numbers and the imaginary numbers are subsets of $\mathbb{C}$.

We can now extend the diagram from Chapter 2 to include the complex numbers.



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Real and imaginary parts

For a complex number $z = a + bi$, we define

$$\text{Re}(z) = a \quad \text{and} \quad \text{Im}(z) = b$$

where $\text{Re}(z)$ is called the **real part** of z and $\text{Im}(z)$ is called the **imaginary part** of z.

For example, for the complex number $z = 2 + 5i$, we have $\text{Re}(z) = 2$ and $\text{Im}(z) = 5$.

Note: Both $\text{Re}(z)$ and $\text{Im}(z)$ are real numbers. That is, $\text{Re}\colon \mathbb{C} \to \mathbb{R}$ and $\text{Im}\colon \mathbb{C} \to \mathbb{R}$.

Equality of complex numbers

Two complex numbers are defined to be **equal** if both their real parts and their imaginary parts are equal:

$$a + bi = c + di \quad \text{if and only if} \quad a = c \text{ and } b = d$$

Example 1

If $4 - 3i = 2a + bi$, find the real values of a and b.

Solution

$$2a = 4 \quad \text{and} \quad b = -3$$
$$\therefore \quad a = 2 \quad \text{and} \quad b = -3$$

Example 2

Find the real values of a and b such that $(2a + 3b) + (a - 2b)i = -1 + 3i$.

Solution

$$2a + 3b = -1 \qquad (1)$$
$$a - 2b = 3 \qquad (2)$$

Multiply (2) by 2:

$$2a - 4b = 6 \qquad (3)$$

Subtract (3) from (1):

$$7b = -7$$

Therefore $b = -1$ and $a = 1$.

Operations on complex numbers

Addition and subtraction

Addition of complex numbers

If $z_1 = a + bi$ and $z_2 = c + di$, then

$$z_1 + z_2 = (a + c) + (b + d)i$$

The **zero** of the complex numbers can be written as $0 = 0 + 0i$.

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If $z = a + bi$, then we define $-z = -a - bi$.

Subtraction of complex numbers

If $z_1 = a + bi$ and $z_2 = c + di$, then

$$z_1 - z_2 = z_1 + (-z_2) = (a - c) + (b - d)i$$

It is easy to check that the following familiar properties of the real numbers extend to the complex numbers:

- $z_1 + z_2 = z_2 + z_1$
- $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
- $z + 0 = z$
- $z + (-z) = 0$

Example 3

If $z_1 = 2 - 3i$ and $z_2 = -4 + 5i$, find:

a $z_1 + z_2$ **b** $z_1 - z_2$

Solution

a $z_1 + z_2 = (2 - 3i) + (-4 + 5i)$
$= (2 + (-4)) + (-3 + 5)i$
$= -2 + 2i$

b $z_1 - z_2 = (2 - 3i) - (-4 + 5i)$
$= (2 - (-4)) + (-3 - 5)i$
$= 6 - 8i$

Multiplication by a real constant

If $z = a + bi$ and $k \in \mathbb{R}$, then

$$kz = k(a + bi) = ka + kbi$$

For example, if $z = 3 - 6i$, then $3z = 9 - 18i$.

Powers of *i*

Successive multiplication by i gives the following:

- $i^0 = 1$
- $i^1 = i$
- $i^2 = -1$
- $i^3 = -i$
- $i^4 = (-1)^2 = 1$
- $i^5 = i$
- $i^6 = -1$
- $i^7 = -i$

In general, for $n = 0, 1, 2, 3, \dots$

- $i^{4n} = 1$
- $i^{4n+1} = i$
- $i^{4n+2} = -1$
- $i^{4n+3} = -i$

Example 4

Simplify:

a i^{13} **b** $3i^4 \times (-2i)^3$

Solution

a $i^{13} = i^{4\times3+1}$
$= i$

b $3i^4 \times (-2i)^3 = 3 \times (-2)^3 \times i^4 \times i^3$
$= -24i^7$
$= 24i$

ISBN 978-1-316-63608-4

Section summary

- The imaginary number i satisfies $i^2 = -1$.
- If a is a positive real number, then $\sqrt{-a} = i\sqrt{a}$.
- The set of **complex numbers** is $\mathbb{C} = \{ a + bi : a, b \in \mathbb{R} \}$.
- For a complex number $z = a + bi$:
  - the **real part** of z is $\text{Re}(z) = a$
  - the **imaginary part** of z is $\text{Im}(z) = b$.
- Equality of complex numbers:

  $$a + bi = c + di \quad \text{if and only if} \quad a = c \text{ and } b = d$$
- If $z_1 = a + bi$ and $z_2 = c + di$, then

  $$z_1 + z_2 = (a + c) + (b + d)i \quad \text{and} \quad z_1 - z_2 = (a - c) + (b - d)i$$
- When simplifying powers of i, remember that $i^4 = 1$.

Exercise 13A

1 State the values of Re(z) and Im(z) for each of the following:

a $z = 2 + 3i$ **b** $z = 4 + 5i$ **c** $z = \frac{1}{2} - \frac{3}{2}i$

d $z = -4$ **e** $z = 3i$ **f** $z = \sqrt{2} - 2\sqrt{2}i$

Example 1, 2

2 Find the real values of a and b in each of the following:

a $2a - 3bi = 4 + 6i$ **b** $a + b - 2abi = 5 - 12i$

c $2a + bi = 10$ **d** $3a + (a - b)i = 2 + i$

Example 3

3 Simplify:

a $(2 - 3i) + (4 - 5i)$ **b** $(4 + i) + (2 - 2i)$

c $(-3 - i) - (3 + i)$ **d** $(2 - \sqrt{2}i) + (5 - \sqrt{8}i)$

e $(1 - i) - (2i + 3)$ **f** $(2 + i) - (-2 - i)$

g $4(2 - 3i) - (2 - 8i)$ **h** $-(5 - 4i) + (1 + 2i)$

i $5(i + 4) + 3(2i - 7)$ **j** $\frac{1}{2}(4 - 3i) - \frac{3}{2}(2 - i)$

Example 4

4 Simplify:

a $\sqrt{-16}$ **b** $2\sqrt{-9}$ **c** $\sqrt{-2}$

d i^3 **e** i^{14} **f** i^{20}

g $-2i \times i^3$ **h** $4i^4 \times 3i^2$ **i** $\sqrt{8}i^5 \times \sqrt{-2}$

5 Simplify:

a $i(2 - i)$ **b** $i^2(3 - 4i)$

c $\sqrt{2}i(i - \sqrt{2})$ **d** $-\sqrt{3}(\sqrt{-3} + \sqrt{2})$

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13B Multiplication and division of complex numbers

In the previous section, we defined addition and subtraction of complex numbers. We begin this section by defining multiplication.

▶ Multiplication of complex numbers

Let $z_1 = a + bi$ and $z_2 = c + di$ (where $a, b, c, d \in \mathbb{R}$). Then

$$\begin{aligned} z_1 \times z_2 &= (a + bi)(c + di) \\ &= ac + bci + adi + bdi^2 \\ &= (ac - bd) + (ad + bc)i \qquad \text{(since } i^2 = -1\text{)} \end{aligned}$$

We carried out this calculation with an assumption that we are in a system where all the usual rules of algebra apply. However, it should be understood that the following is a *definition* of multiplication for $\mathbb{C}$.

Multiplication of complex numbers

Let $z_1 = a + bi$ and $z_2 = c + di$. Then

$$z_1 \times z_2 = (ac - bd) + (ad + bc)i$$

The multiplicative identity for $\mathbb{C}$ is $1 = 1 + 0i$.

It is easy to check that the following familiar properties of the real numbers extend to the complex numbers:

- $z_1 z_2 = z_2 z_1$
- $z \times 1 = z$
- $(z_1 z_2) z_3 = z_1 (z_2 z_3)$
- $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Example 5

If $z_1 = 3 - 2i$ and $z_2 = 1 + i$, find $z_1 z_2$.

Solution

$$\begin{aligned} z_1 z_2 &= (3 - 2i)(1 + i) \\ &= 3 - 2i + 3i - 2i^2 \\ &= 5 + i \end{aligned}$$

Explanation

Expand the brackets in the usual way.

Remember that $i^2 = -1$.

▶ The conjugate of a complex number

Let $z = a + bi$. The **conjugate** of z is denoted by $\bar{z}$ and is given by

$$\bar{z} = a - bi$$

For example, the conjugate of $-4 + 3i$ is $-4 - 3i$, and vice versa.

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For a complex number $z = a + bi$, we have

$$\begin{aligned} z\bar{z} &= (a+bi)(a-bi) \\ &= a^2 + abi - abi - b^2 i^2 \\ &= a^2 + b^2 \qquad \text{where } a^2 + b^2 \text{ is a real number} \end{aligned}$$

The **modulus** of the complex number $z = a + bi$ is denoted by $|z|$ and is given by

$$|z| = \sqrt{a^2 + b^2}$$

The calculation above shows that

$$z\bar{z} = |z|^2$$

Note: In the case that z is a real number, this definition of $|z|$ agrees with the definition of the modulus of a real number given in Chapter 2.

Example 6

If $z_1 = 2 - 3i$ and $z_2 = -1 + 2i$, find:

a $\overline{z_1 + z_2}$ and $\overline{z_1} + \overline{z_2}$

b $\overline{z_1 \cdot z_2}$ and $\overline{z_1} \cdot \overline{z_2}$

Solution

We have $\overline{z_1} = 2 + 3i$ and $\overline{z_2} = -1 - 2i$.

a
$$\begin{aligned} z_1 + z_2 &= (2-3i) + (-1+2i) \\ &= 1 - i \end{aligned}$$

$$\overline{z_1 + z_2} = 1 + i$$

$$\begin{aligned} \overline{z_1} + \overline{z_2} &= (2+3i) + (-1-2i) \\ &= 1 + i \end{aligned}$$

b
$$\begin{aligned} z_1 \cdot z_2 &= (2-3i)(-1+2i) \\ &= 4 + 7i \end{aligned}$$

$$\overline{z_1 \cdot z_2} = 4 - 7i$$

$$\begin{aligned} \overline{z_1} \cdot \overline{z_2} &= (2+3i)(-1-2i) \\ &= 4 - 7i \end{aligned}$$

- The conjugate of a sum is equal to the sum of the conjugates:

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

- The conjugate of a product is equal to the product of the conjugates:

$$\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$$

Division of complex numbers

Multiplicative inverse

We begin with some familiar algebra that will motivate the definition:

$$\frac{1}{a+bi} = \frac{1}{a+bi} \times \frac{a-bi}{a-bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2+b^2}$$

We can see that

$$(a+bi) \times \frac{a-bi}{a^2+b^2} = 1$$

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Although we have carried out this arithmetic, we have not yet defined what $\dfrac{1}{a + bi}$ means.

Multiplicative inverse of a complex number

If $z = a + bi$ with $z \neq 0$, then

$$z^{-1} = \frac{a - bi}{a^2 + b^2} = \frac{\bar{z}}{|z|^2}$$

Note: We can check that $(z_1 z_2)^{-1} = z_1^{-1} z_2^{-1}$.

Division

The formal definition of division in the complex numbers is via the multiplicative inverse:

Division of complex numbers

$$\frac{z_1}{z_2} = z_1 z_2^{-1} = \frac{z_1 \overline{z_2}}{|z_2|^2} \qquad (\text{for } z_2 \neq 0)$$

Here is the procedure that is used in practice:

Assume that $z_1 = a + bi$ and $z_2 = c + di$ (where $a, b, c, d \in \mathbb{R}$). Then

$$\frac{z_1}{z_2} = \frac{a + bi}{c + di}$$

Multiply the numerator and denominator by the conjugate of z_2:

$$\frac{z_1}{z_2} = \frac{a + bi}{c + di} \times \frac{c - di}{c - di}$$

$$= \frac{(a + bi)(c - di)}{c^2 + d^2}$$

Complete the division by simplifying. This process is demonstrated in the next example.

Example 7

If $z_1 = 2 - i$ and $z_2 = 3 + 2i$, find $\dfrac{z_1}{z_2}$.

Solution

$$\frac{z_1}{z_2} = \frac{2 - i}{3 + 2i} \times \frac{3 - 2i}{3 - 2i}$$

$$= \frac{6 - 3i - 4i + 2i^2}{3^2 + 2^2}$$

$$= \frac{4 - 7i}{13}$$

$$= \frac{1}{13}(4 - 7i)$$

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Example 8

Solve for z in the equation $(2 + 3i)z = -1 - 2i$.

Solution

$$(2+3i)z = -1-2i$$

$$\therefore \quad z = \frac{-1-2i}{2+3i}$$

$$= \frac{-1-2i}{2+3i} \times \frac{2-3i}{2-3i}$$

$$= \frac{-8-i}{13}$$

$$= -\frac{1}{13}(8+i)$$

There is an obvious similarity between the process for expressing a complex number with a real denominator and the process for rationalising the denominator of a surd expression.

Example 9

If $z = 2 - 5i$, find z^{-1} and express with a real denominator.

Solution

$$z^{-1} = \frac{1}{z}$$

$$= \frac{1}{2-5i}$$

$$= \frac{1}{2-5i} \times \frac{2+5i}{2+5i}$$

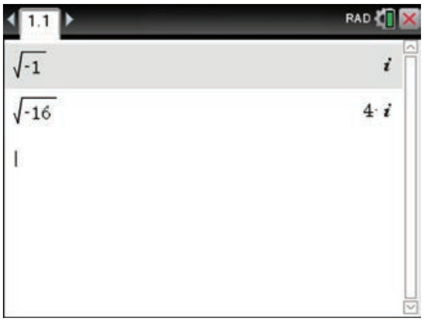
$$= \frac{2+5i}{29}$$

$$= \frac{1}{29}(2+5i)$$

Using the TI-Nspire

Set to complex mode using ⌂ on > **Settings** > **Document Settings**. Select **Rectangular** from the **Real or Complex** field.

Note: The square root of a negative number can be found only in complex mode. But most computations with complex numbers can also be performed in real mode.



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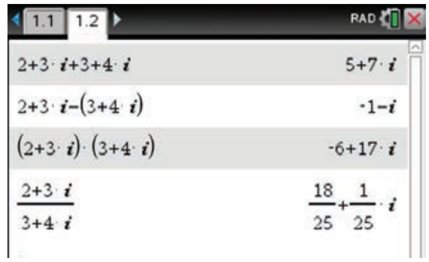
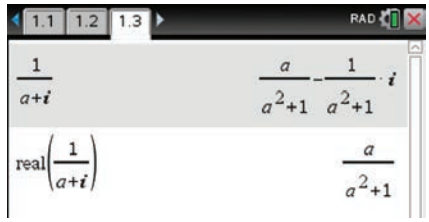
- The results of the arithmetic operations +, −, × and ÷ are illustrated using the two complex numbers $2 + 3i$ and $3 + 4i$.



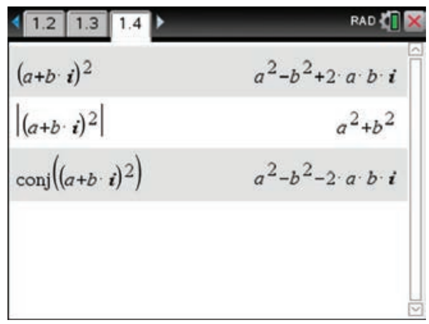
Note: Do not use the text i for the imaginary constant. The symbol i is found using [π▸] or the Symbols palette ([ctrl] [📖]).

- To find the real part of a complex number, use [menu] > **Number** > **Complex Number Tools** > **Real Part** as shown.



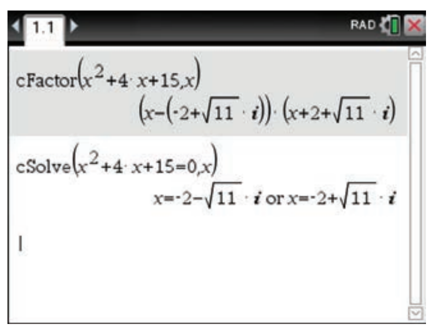
Hint: Type real(.

- To find the modulus of a complex number, use [menu] > **Number** > **Complex Number Tools** > **Magnitude** as shown.
  Alternatively, use |□| from the 2D-template palette [⊞] or type abs(.
- To find the conjugate of a complex number, use [menu] > **Number** > **Complex Number Tools** > **Complex Conjugate** as shown.



Hint: Type conj(.

There are also commands for factorising polynomials over the complex numbers and for solving polynomial equations over the complex numbers. These are available from [menu] > **Algebra** > **Complex**.

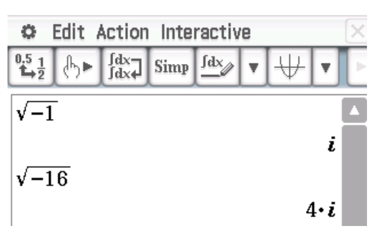


Note: You must use this menu even if the calculator is in complex mode.
When using **cFactor**, you must include the variable as shown.

Using the Casio ClassPad

In Main $\sqrt{\alpha}$, tap **Real** in the status bar at the bottom of the screen to change to **Cplx** mode.

- Enter $\sqrt{-1}$ and tap [EXE] to obtain the answer i.
- Enter $\sqrt{-16}$ to obtain the answer $4i$.



Note: The symbol i is found in both the [Math2] and the [Math3] keyboards.

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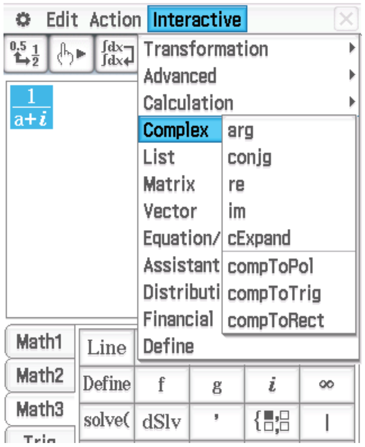
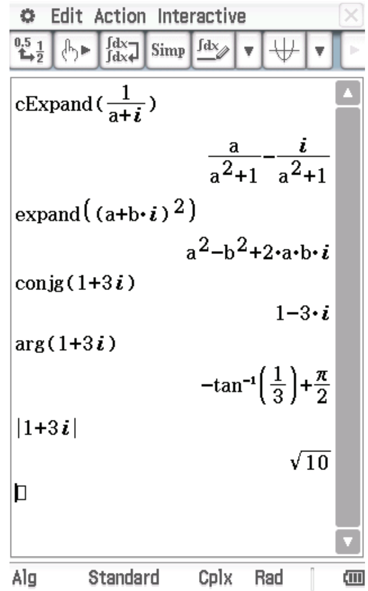
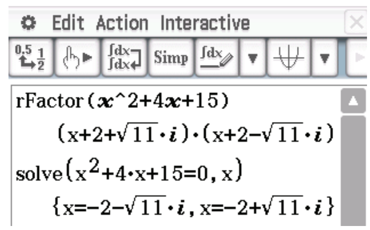
With the calculator set to complex mode, various operations on complex numbers can be carried out using options from **Interactive** > **Complex**.





Polynomials can be factorised over the complex numbers (**Interactive** > **Transformation** > **factor** > **rFactor**).

Equations can be solved over the complex numbers (**Interactive** > **Equation/Inequality** > **solve**).



Section summary

- **Multiplication** To find a product $(a + bi)(c + di)$, expand the brackets in the usual way, remembering that $i^2 = -1$.
- **Conjugate** If $z = a + bi$, then $\bar{z} = a - bi$.
- **Division** To perform a division, start with

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \times \frac{c-di}{c-di}$$
$$= \frac{(a+bi)(c-di)}{c^2+d^2}$$

and then simplify.
- **Multiplicative inverse** To find z^{-1}, calculate $\frac{1}{z}$.

Exercise 13B

Example 5

1 Expand and simplify:

a $(4 + i)^2$ **b** $(2 - 2i)^2$ **c** $(3 + 2i)(2 + 4i)$
d $(-1 - i)^2$ **e** $(\sqrt{2} - \sqrt{3}i)(\sqrt{2} + \sqrt{3}i)$ **f** $(5 - 2i)(-2 + 3i)$

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2 Write down the conjugate of each of the following complex numbers:

a $2 - 5i$ **b** $-1 + 3i$ **c** $\sqrt{5} - 2i$ **d** $-5i$

Example 6

3 If $z_1 = 2 - i$ and $z_2 = -3 + 2i$, find:

a $\overline{z_1}$ **b** $\overline{z_2}$ **c** $z_1 \cdot z_2$ **d** $\overline{z_1 \cdot z_2}$

e $\overline{z_1} \cdot \overline{z_2}$ **f** $z_1 + z_2$ **g** $\overline{z_1 + z_2}$ **h** $\overline{z_1} + \overline{z_2}$

4 If $z = 2 - 4i$, express each of the following in the form $x + yi$:

a $\overline{z}$ **b** $z\overline{z}$ **c** $z + \overline{z}$ **d** $z(z + \overline{z})$

Example 9

e $z - \overline{z}$ **f** $i(z - \overline{z})$ **g** z^{-1} **h** $\dfrac{z}{i}$

5 Find the real values of a and b such that $(a + bi)(2 + 5i) = 3 - i$.

Example 7

6 Express in the form $x + yi$:

a $\dfrac{2 - i}{4 + i}$ **b** $\dfrac{3 + 2i}{2 - 3i}$ **c** $\dfrac{4 + 3i}{1 + i}$ **d** $\dfrac{2 - 2i}{4i}$ **e** $\dfrac{1}{2 - 3i}$ **f** $\dfrac{i}{2 + 6i}$

7 Find the real values of a and b if $(3 - i)(a + bi) = 6 - 7i$.

Example 8

8 Solve each of the following for z:

a $(2 - i)z = 42i$ **b** $(1 + 3i)z = -2 - i$ **c** $(3i + 5)z = 1 + i$

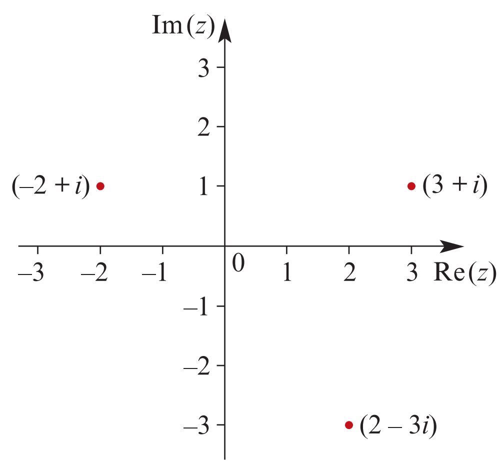
d $2(4 - 7i)z = 5 + 2i$ **e** $z(1 + i) = 4$

13C Argand diagrams

An **Argand diagram** is a geometric representation of the set of complex numbers. A complex number has two dimensions: the real part and the imaginary part. Therefore a plane is required to represent $\mathbb{C}$.

An Argand diagram is drawn with two perpendicular axes. The horizontal axis represents $\text{Re}(z)$, for $z \in \mathbb{C}$, and the vertical axis represents $\text{Im}(z)$, for $z \in \mathbb{C}$.

Each point on an Argand diagram represents a complex number. The complex number $a + bi$ is situated at the point (a, b) on the equivalent Cartesian axes, as shown by the examples in this figure.



A complex number written as $a + bi$ is said to be in **Cartesian form**.

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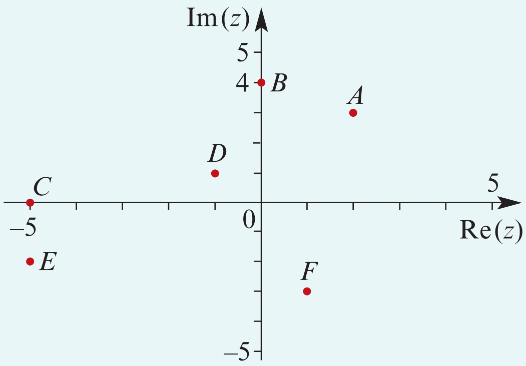
Example 10

Write down the complex number represented by each of the points shown on this Argand diagram.



Solution

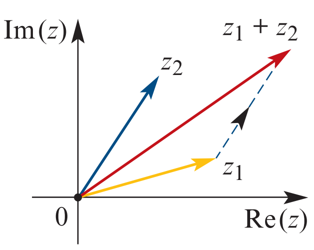
A $2 + 3i$ **B** $4i$ **C** -5

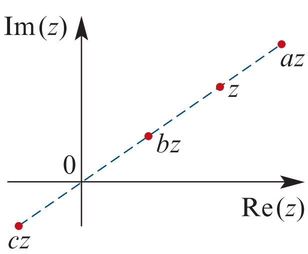
D $-1 + i$ **E** $-5 - 2i$ **F** $1 - 3i$

▶ Geometric representation of the basic operations on complex numbers

In an Argand diagram, the sum of two complex numbers z_1 and z_2 can be found geometrically by placing the 'tail' of z_2 on the 'tip' of z_1, as shown in the diagram on the left. We will see in Chapter 17 that this is analogous to vector addition.

When a complex number is multiplied by a real constant, it maintains the same 'direction', but its distance from the origin is scaled. This is shown in the diagram on the right.





$a > 1$

$0 < b < 1$

$c < 0$

The difference $z_1 - z_2$ is represented by the sum $z_1 + (-z_2)$.

Example 11

a Represent the following complex numbers as points on an Argand diagram:

i 2 **ii** $-3i$ **iii** $2 - i$ **iv** $-(2 + 3i)$ **v** $-1 + 2i$

b Let $z_1 = 2 + i$ and $z_2 = -1 + 3i$.
Represent the complex numbers z_1, z_2, $z_1 + z_2$ and $z_1 - z_2$ on an Argand diagram and show the geometric interpretation of the sum and difference.

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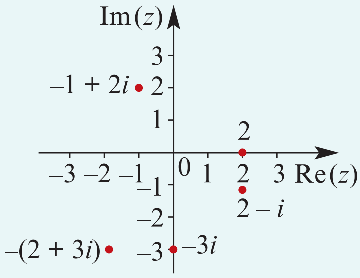
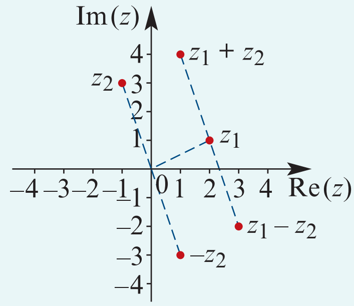
Solution

a



b



$z_1 + z_2 = (2 + i) + (-1 + 3i) = 1 + 4i$

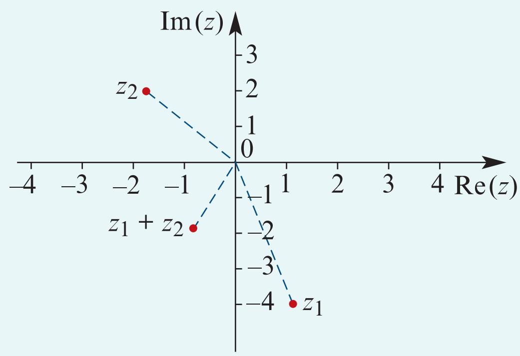
$z_1 - z_2 = (2 + i) - (-1 + 3i) = 3 - 2i$

Example 12

Let $z_1 = 1 - 4i$ and $z_2 = -2 + 2i$. Find $z_1 + z_2$ algebraically and illustrate $z_1 + z_2$ on an Argand diagram.

Solution

$$z_1 + z_2 = (1 - 4i) + (-2 + 2i)$$
$$= -1 - 2i$$

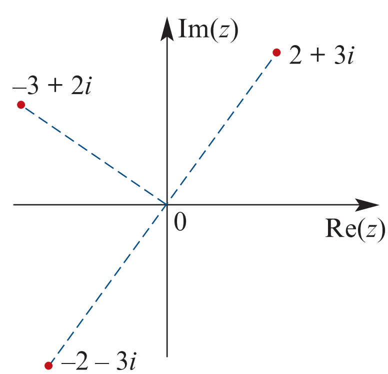


Rotation about the origin

When the complex number $2 + 3i$ is multiplied by -1, the result is $-2 - 3i$. This is achieved through a rotation of 180° about the origin.

When the complex number $2 + 3i$ is multiplied by i, we obtain

$$i(2 + 3i) = 2i + 3i^2$$
$$= 2i - 3$$
$$= -3 + 2i$$



The result is achieved through a rotation of 90° anticlockwise about the origin.

If $-3 + 2i$ is multiplied by i, the result is $-2 - 3i$. This is again achieved through a rotation of 90° anticlockwise about the origin.

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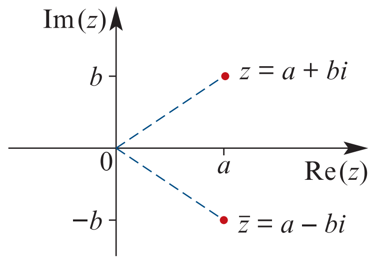
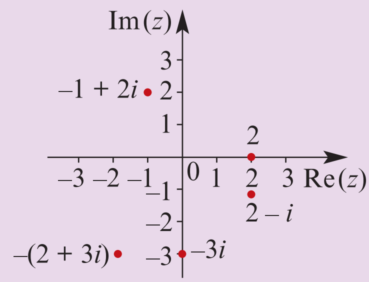
Reflection in the horizontal axis

The conjugate of a complex number $z = a + bi$ is $\bar{z} = a - bi$. Therefore $\bar{z}$ is the reflection of z in the horizontal axis of an Argand diagram.



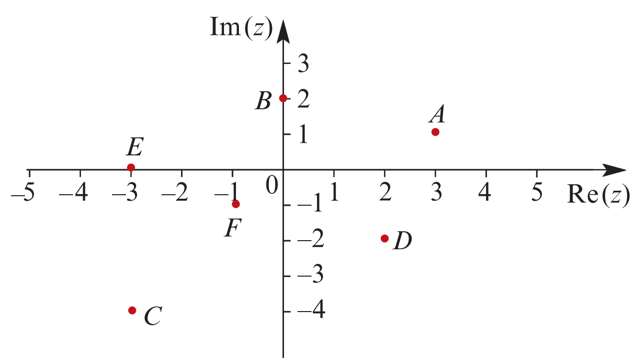
Section summary

- An **Argand diagram** is a geometric representation of the set of complex numbers.
- The horizontal axis represents Re(z) and the vertical axis represents Im(z), for $z \in \mathbb{C}$.
- The operations of addition, subtraction and multiplication by a real constant all have geometric interpretations on an Argand diagram.
- Multiplication of a complex number by i corresponds to a rotation of 90° anticlockwise about the origin.
- Complex conjugate corresponds to reflection in the horizontal axis.



Exercise 13C

Example 10 **1** Write down the complex numbers represented on this Argand diagram.



Example 11 **2** Represent each of the following complex numbers as points on an Argand diagram:

a $3 - 4i$ **b** $-4 + i$ **c** $4 + i$ **d** -3 **e** $-2i$ **f** $-5 - 2i$

Example 12 **3** If $z_1 = 6 - 5i$ and $z_2 = -3 + 4i$, represent each of the following on an Argand diagram:

a $z_1 + z_2$ **b** $z_1 - z_2$

4 If $z = 1 + 3i$, represent each of the following on an Argand diagram:

a z **b** $\bar{z}$ **c** z^2 **d** $-z$ **e** $\frac{1}{z}$

5 If $z = 2 - 5i$, represent each of the following on an Argand diagram:

a z **b** zi **c** zi^2 **d** zi^3 **e** zi^4

ISBN 978-1-316-63608-4

Cambridge University Press

13D Solving equations over the complex numbers

Quadratic equations with a negative discriminant have no real solutions. The introduction of complex numbers enables us to solve such quadratic equations.

► Sum of two squares

Since $i^2 = -1$, we can rewrite a sum of two squares as a difference of two squares:

$$\begin{aligned} z^2 + a^2 &= z^2 - (ai)^2 \\ &= (z + ai)(z - ai) \end{aligned}$$

This allows us to solve equations of the form $z^2 + a^2 = 0$.

Example 13

Solve the equations:

a $z^2 + 16 = 0$

b $2z^2 + 6 = 0$

Solution

a

$$\begin{aligned} z^2 + 16 &= 0 \\ z^2 - 16i^2 &= 0 \\ (z + 4i)(z - 4i) &= 0 \\ \therefore \quad z &= \pm 4i \end{aligned}$$

b

$$\begin{aligned} 2z^2 + 6 &= 0 \\ z^2 + 3 &= 0 \\ z^2 - 3i^2 &= 0 \\ (z + \sqrt{3}i)(z - \sqrt{3}i) &= 0 \\ \therefore \quad z &= \pm\sqrt{3}i \end{aligned}$$

► Solution of quadratic equations

To solve quadratic equations which have a negative discriminant, we can use the quadratic formula in the usual way.

Example 14

Solve the equation $3z^2 + 5z + 3 = 0$.

Solution

Using the quadratic formula:

$$\begin{aligned} z &= \frac{-5 \pm \sqrt{25 - 36}}{6} \\ &= \frac{-5 \pm \sqrt{-11}}{6} \\ &= \frac{1}{6}(-5 \pm \sqrt{11}i) \end{aligned}$$

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Factorisation of quadratics

To factorise a quadratic over the complex numbers, we can complete the square in the usual way. We may then need to rewrite a sum of two squares as a difference of two squares:

$$\begin{aligned}(z+a)^2+b^2 &= (z+a)^2-(bi)^2\\ &= (z+a+bi)(z+a-bi)\end{aligned}$$

Example 15

Factorise:

a z^2+z+3 **b** $2z^2-z+1$

Solution

a By completing the square, we have

$$\begin{aligned}z^2+z+3 &= \left(z^2+z+\frac{1}{4}\right)+3-\frac{1}{4}\\ &= \left(z+\frac{1}{2}\right)^2+\frac{11}{4}\\ &= \left(z+\frac{1}{2}\right)^2-\frac{11}{4}i^2\\ &= \left(z+\frac{1}{2}+\frac{\sqrt{11}}{2}i\right)\left(z+\frac{1}{2}-\frac{\sqrt{11}}{2}i\right)\end{aligned}$$

b By completing the square, we have

$$\begin{aligned}2z^2-z+1 &= 2\left(z^2-\frac{1}{2}z+\frac{1}{2}\right)\\ &= 2\left(\left(z^2-\frac{1}{2}z+\frac{1}{16}\right)+\frac{1}{2}-\frac{1}{16}\right)\\ &= 2\left(\left(z-\frac{1}{4}\right)^2+\frac{7}{16}\right)\\ &= 2\left(\left(z-\frac{1}{4}\right)^2-\frac{7}{16}i^2\right)\\ &= 2\left(z-\frac{1}{4}+\frac{\sqrt{7}}{4}i\right)\left(z-\frac{1}{4}-\frac{\sqrt{7}}{4}i\right)\end{aligned}$$

Note: From this example, we can see that given any quadratic $p(z)=az^2+bz+c$, where the coefficients a, b and c are real numbers, we can rewrite $p(z)$ as a product of two linear factors over the complex numbers:

$$az^2+bz+c=a(z-\alpha)(z-\beta)$$

for some complex numbers α and β. Moreover, if $\alpha, \beta \in \mathbb{C}\setminus\mathbb{R}$, then α and β will be conjugates of each other.

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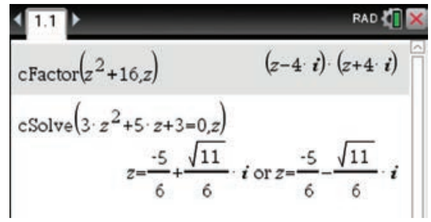
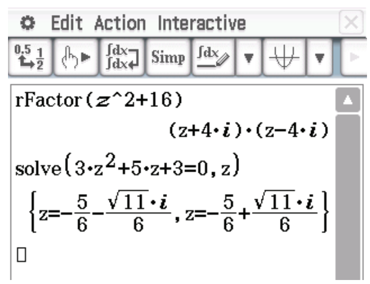
Using the TI-Nspire



- To factorise polynomials over the complex numbers, use (menu) > **Algebra** > **Complex** > **Factor** as shown.
- To solve polynomial equations over the complex numbers, use (menu) > **Algebra** > **Complex** > **Solve** as shown.

Using the Casio ClassPad



To factorise:

- Ensure the mode is set to **Cplx**.
- Enter and highlight the expression $z^2 + 16$.
- Select **Interactive** > **Transformation** > **factor** > **rFactor**.

To solve:

- Enter and highlight $3z^2 + 5z + 3$.
- Select **Interactive** > **Equation/Inequality** > **solve**.
- Ensure the variable selected is z.

Section summary

- Quadratic equations can be solved over the complex numbers using the same techniques as for the real numbers.
- Two properties of complex numbers that are useful when solving equations:
  - $z^2 + a^2 = z^2 - (ai)^2 = (z + ai)(z - ai)$
  - $\sqrt{-a} = i\sqrt{a}$, where a is a positive real number.

Exercise 13D

Skillsheet

Example 13, 14

1 Solve each of the following equations over $\mathbb{C}$:

a $z^2 + 4 = 0$ **b** $2z^2 + 18 = 0$ **c** $3z^2 = -15$
d $(z - 2)^2 + 16 = 0$ **e** $(z + 1)^2 = -49$ **f** $z^2 - 2z + 3 = 0$
g $z^2 + 3z + 3 = 0$ **h** $2z^2 + 5z + 4 = 0$ **i** $3z^2 = z - 2$
j $2z = z^2 + 5$ **k** $2z^2 - 6z = -10$ **l** $z^2 - 6z = -14$

Example 15

2 Factorise each of the following quadratics over $\mathbb{C}$:

a $z^2 + 9$ **b** $z^2 + 3$ **c** $3z^2 + 12$
d $z^2 + 2z + 5$ **e** $z^2 - 3z + 6$ **f** $2z^2 + 2z + 1$

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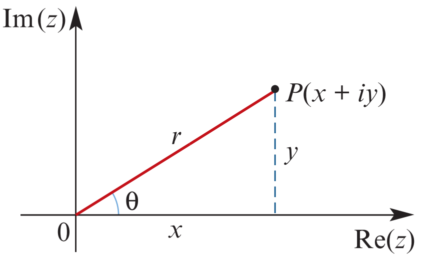
13E Polar form of a complex number

Polar coordinates for points in the plane were introduced in Chapter 12. Similarly, each complex number may be described by an angle and a distance from the origin. In this section, we will see that this is a very useful way to describe complex numbers.

The diagram shows the point P corresponding to the complex number $z = x + yi$. We see that $x = r\cos\theta$ and $y = r\sin\theta$, and so we can write

$$\begin{aligned} z &= x + yi \\ &= r\cos\theta + (r\sin\theta)\,i \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$



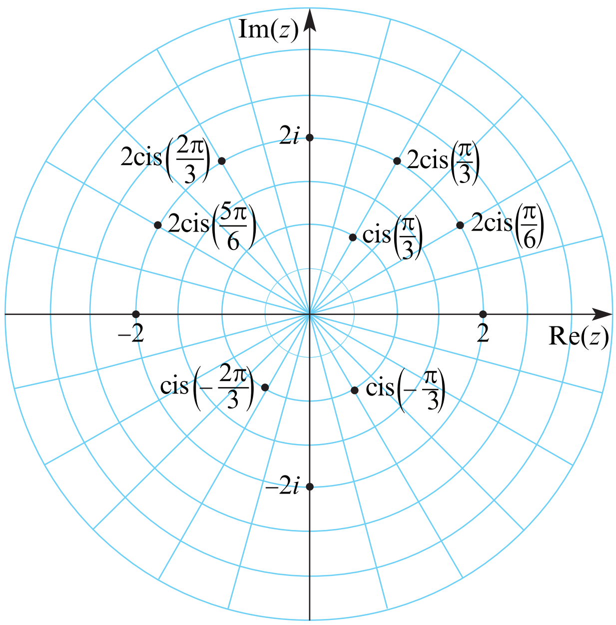
This is called the **polar form** of the complex number. The polar form is abbreviated to

$$z = r\operatorname{cis}\theta$$

- The distance $r = \sqrt{x^2 + y^2}$ is called the **modulus** of z and is denoted by $|z|$.
- The angle θ, measured anticlockwise from the horizontal axis, is called the **argument** of z.

Polar form for complex numbers is also called **modulus–argument form**.

This Argand diagram uses a polar grid with rays at intervals of $\frac{\pi}{12} = 15^\circ$.



Non-uniqueness of polar form

Each complex number has more than one representation in polar form.

Since $\cos\theta = \cos(\theta + 2n\pi)$ and $\sin\theta = \sin(\theta + 2n\pi)$, for all $n \in \mathbb{Z}$, we can write

$$z = r\operatorname{cis}\theta = r\operatorname{cis}(\theta + 2n\pi) \quad \text{for all } n \in \mathbb{Z}$$

The convention is to use the angle θ such that $-\pi < \theta \le \pi$. This value of θ is called the **principal value** of the argument of z and is denoted by $\operatorname{Arg} z$. That is,

$$-\pi < \operatorname{Arg} z \le \pi$$

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Example 16

Express each of the following complex numbers in polar form:

a $z = 1 + \sqrt{3}i$

b $z = 2 - 2i$

Solution

a We have $x = 1$ and $y = \sqrt{3}$, giving

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{1 + 3} \\ &= 2 \end{aligned}$$

The point $z = 1 + \sqrt{3}i$ is in the 1st quadrant, and so $0 < \theta < \frac{\pi}{2}$.

We know that

$$\cos\theta = \frac{x}{r} = \frac{1}{2}$$

and $$\sin\theta = \frac{y}{r} = \frac{\sqrt{3}}{2}$$

Hence $\theta = \frac{\pi}{3}$ and therefore

$$\begin{aligned} z &= 1 + \sqrt{3}i \\ &= 2\operatorname{cis}\left(\frac{\pi}{3}\right) \end{aligned}$$

b We have $x = 2$ and $y = -2$, giving

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{4 + 4} \\ &= 2\sqrt{2} \end{aligned}$$

The point $z = 2 - 2i$ is in the 4th quadrant, and so $-\frac{\pi}{2} < \theta < 0$.

We know that

$$\cos\theta = \frac{x}{r} = \frac{1}{\sqrt{2}}$$

and $$\sin\theta = \frac{y}{r} = \frac{-1}{\sqrt{2}}$$

Hence $\theta = -\frac{\pi}{4}$ and therefore

$$\begin{aligned} z &= 2 - 2i \\ &= 2\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right) \end{aligned}$$

Example 17

Express $z = 2\operatorname{cis}\left(\frac{-2\pi}{3}\right)$ in Cartesian form.

Solution

$$\begin{aligned} x &= r\cos\theta \\ &= 2\cos\left(\frac{-2\pi}{3}\right) \\ &= 2 \times \left(-\frac{1}{2}\right) \\ &= -1 \end{aligned}$$

$$\begin{aligned} y &= r\sin\theta \\ &= 2\sin\left(\frac{-2\pi}{3}\right) \\ &= 2 \times \left(\frac{-\sqrt{3}}{2}\right) \\ &= -\sqrt{3} \end{aligned}$$

Hence

$$\begin{aligned} z &= 2\operatorname{cis}\left(\frac{-2\pi}{3}\right) \\ &= -1 - \sqrt{3}i \end{aligned}$$

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Multiplication and division in polar form

We can give a simple geometric interpretation of multiplication and division of complex numbers in polar form.

If $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$, then

$$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \qquad \text{(multiply the moduli and add the angles)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2) \qquad \text{(divide the moduli and subtract the angles)}$$

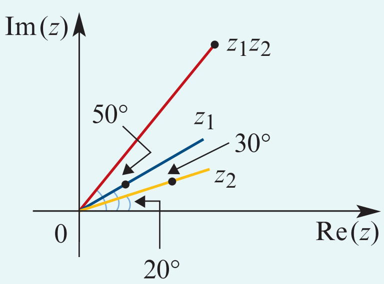
These two results can be proved using the addition formulas for sine and cosine established in Chapter 11.

Example 18

Let $z_1 = 2 \operatorname{cis} 30^\circ$ and $z_2 = 4 \operatorname{cis} 20^\circ$. Find the product $z_1 z_2$ and represent it on an Argand diagram.

Solution

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \\ &= 2 \times 4 \operatorname{cis}(30^\circ + 20^\circ) \\ &= 8 \operatorname{cis} 50^\circ \end{aligned}$$



Example 19

Let $z_1 = 3 \operatorname{cis}\left(\frac{\pi}{2}\right)$ and $z_2 = 2 \operatorname{cis}\left(\frac{5\pi}{6}\right)$. Find the product $z_1 z_2$.

Solution

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \\ &= 6 \operatorname{cis}\left(\frac{\pi}{2} + \frac{5\pi}{6}\right) \\ &= 6 \operatorname{cis}\left(\frac{4\pi}{3}\right) \end{aligned}$$

$$\therefore \quad z_1 z_2 = 6 \operatorname{cis}\left(\frac{-2\pi}{3}\right) \qquad \text{since } -\pi < \operatorname{Arg} z \le \pi$$

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Example 20

If $z_1 = -\sqrt{3} + i$ and $z_2 = 2\sqrt{3} + 2i$, find the quotient $\frac{z_1}{z_2}$ and express in Cartesian form.

Solution

First express z_1 and z_1 in polar form:

- $|z_1| = \sqrt{3+1} = 2$

  Let $\theta_1 = \text{Arg}\, z_1$. Then $\cos\theta_1 = \frac{-\sqrt{3}}{2}$ and $\sin\theta_1 = \frac{1}{2}$, giving $\theta_1 = \frac{5\pi}{6}$.

- $|z_2| = \sqrt{12+4} = 4$

  Let $\theta_2 = \text{Arg}\, z_2$. Then $\cos\theta_2 = \frac{\sqrt{3}}{2}$ and $\sin\theta_2 = \frac{1}{2}$, giving $\theta_2 = \frac{\pi}{6}$.

Hence $z_1 = 2\,\text{cis}\left(\frac{5\pi}{6}\right)$ and $z_2 = 4\,\text{cis}\left(\frac{\pi}{6}\right)$. Therefore

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{r_1}{r_2}\,\text{cis}(\theta_1 - \theta_2)\\ &= \frac{2}{4}\,\text{cis}\left(\frac{5\pi}{6} - \frac{\pi}{6}\right)\\ &= \frac{1}{2}\,\text{cis}\left(\frac{2\pi}{3}\right)\end{aligned}$$

In Cartesian form:

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{1}{2}\cos\left(\frac{2\pi}{3}\right) + \frac{1}{2}\sin\left(\frac{2\pi}{3}\right)i\\ &= \frac{1}{2}\left(\frac{-1}{2}\right) + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)i\\ &= -\frac{1}{4}(1 - \sqrt{3}i)\end{aligned}$$

Section summary

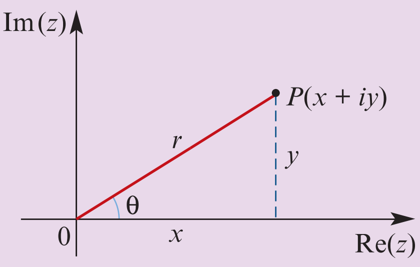
- **Polar form**

  A complex number in Cartesian form

  $$z = x + yi$$

  can be written in polar form as

  $$\begin{aligned} z &= r(\cos\theta + i\sin\theta)\\ &= r\,\text{cis}\,\theta\end{aligned}$$



  - The distance $r = \sqrt{x^2 + y^2}$ is called the **modulus** of z and is denoted by $|z|$.
  - The angle θ, measured anticlockwise from the horizontal axis, is called the **argument** of z.

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- The polar form of a complex number is not unique. The argument θ of z such that $-\pi < \theta \le \pi$ is called the **principal value** of the argument of z and is denoted by $\operatorname{Arg} z$.
- Multiplication and division in polar form

  If $z_1 = r_1 \operatorname{cis}\theta_1$ and $z_2 = r_2 \operatorname{cis}\theta_2$, then

$$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \quad \text{and} \quad \frac{z_1}{z_2} = \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)$$

Exercise 13E

Skillsheet

Example 16

1 Express each of the following in polar form $r \operatorname{cis}\theta$ with $-\pi < \theta \le \pi$:

a $1 + \sqrt{3}i$ **b** $1 - i$ **c** $-2\sqrt{3} + 2i$

d $-4 - 4i$ **e** $12 - 12\sqrt{3}i$ **f** $-\frac{1}{2} + \frac{1}{2}i$

Example 17

2 Express each of the following in the form $x + yi$:

a $3\operatorname{cis}\left(\frac{\pi}{2}\right)$ **b** $\sqrt{2}\operatorname{cis}\left(\frac{\pi}{3}\right)$ **c** $2\operatorname{cis}\left(\frac{\pi}{6}\right)$

d $5\operatorname{cis}\left(\frac{3\pi}{4}\right)$ **e** $12\operatorname{cis}\left(\frac{5\pi}{6}\right)$ **f** $3\sqrt{2}\operatorname{cis}\left(\frac{-\pi}{4}\right)$

g $5\operatorname{cis}\left(\frac{4\pi}{3}\right)$ **h** $5\operatorname{cis}\left(\frac{-2\pi}{3}\right)$

Example 18, 19

3 Simplify the following and express the answers in Cartesian form:

a $2\operatorname{cis}\left(\frac{\pi}{6}\right) \cdot 3\operatorname{cis}\left(\frac{\pi}{12}\right)$ **b** $4\operatorname{cis}\left(\frac{\pi}{12}\right) \cdot 3\operatorname{cis}\left(\frac{\pi}{4}\right)$

c $\operatorname{cis}\left(\frac{\pi}{4}\right) \cdot 5\operatorname{cis}\left(\frac{5\pi}{12}\right)$ **d** $12\operatorname{cis}\left(\frac{-\pi}{3}\right) \cdot 3\operatorname{cis}\left(\frac{2\pi}{3}\right)$

e $12\operatorname{cis}\left(\frac{5\pi}{6}\right) \cdot 3\operatorname{cis}\left(\frac{\pi}{2}\right)$ **f** $(\sqrt{2}\operatorname{cis}\pi) \cdot \sqrt{3}\operatorname{cis}\left(\frac{-3\pi}{4}\right)$

Example 20

g $\dfrac{10\operatorname{cis}\left(\frac{\pi}{4}\right)}{5\operatorname{cis}\left(\frac{\pi}{12}\right)}$ **h** $\dfrac{12\operatorname{cis}\left(\frac{-\pi}{3}\right)}{3\operatorname{cis}\left(\frac{2\pi}{3}\right)}$ **i** $\dfrac{12\sqrt{8}\operatorname{cis}\left(\frac{3\pi}{4}\right)}{3\sqrt{2}\operatorname{cis}\left(\frac{\pi}{12}\right)}$ **j** $\dfrac{20\operatorname{cis}\left(\frac{-\pi}{6}\right)}{8\operatorname{cis}\left(\frac{5\pi}{6}\right)}$

ISBN 978-1-316-63608-4

Chapter summary

AS

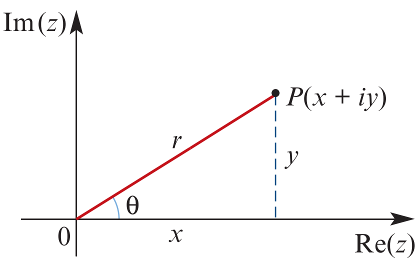
Nrich

- The imaginary number i has the property $i^2 = -1$.
- The set of **complex numbers** is $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$.
- For a complex number $z = a + bi$:
  - the **real part** of z is $\text{Re}(z) = a$
  - the **imaginary part** of z is $\text{Im}(z) = b$.
- Complex numbers z_1 and z_2 are equal if and only if $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.
- An **Argand diagram** is a geometric representation of $\mathbb{C}$.
- The **modulus** of z, denoted by $|z|$, is the distance from the origin to the point representing z in an Argand diagram. Thus $|a + bi| = \sqrt{a^2 + b^2}$.
- The **argument** of z is an angle measured anticlockwise about the origin from the positive direction of the x-axis to the line joining the origin to z.
- The **principal value** of the argument, denoted by $\text{Arg}\, z$, is the angle in the interval $(-\pi, \pi]$.
- The complex number $z = x + yi$ can be expressed in **polar form** as

$$z = r(\cos\theta + i\sin\theta)$$
$$= r\,\text{cis}\,\theta$$

where $r = |z| = \sqrt{x^2 + y^2}$, $\cos\theta = \dfrac{x}{r}$, $\sin\theta = \dfrac{y}{r}$.

This is also called modulus–argument form.



- The **complex conjugate** of $z = a + bi$ is given by $\bar{z} = a - bi$. Note that $z\bar{z} = |z|^2$.
- Division of complex numbers:

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \times \frac{\overline{z_2}}{\overline{z_2}} = \frac{z_1\overline{z_2}}{|z_2|^2}$$

- Multiplication and division in polar form:
  Let $z_1 = r_1\,\text{cis}\,\theta_1$ and $z_2 = r_2\,\text{cis}\,\theta_2$. Then

$$z_1 z_2 = r_1 r_2\,\text{cis}(\theta_1 + \theta_2) \quad \text{and} \quad \frac{z_1}{z_2} = \frac{r_1}{r_2}\,\text{cis}(\theta_1 - \theta_2)$$

Short-answer questions

1 For $z_1 = m + ni$ and $z_2 = p + qi$, express each of the following in the form $a + bi$:

a $2z_1 + 3z_2$ **b** $\overline{z_2}$ **c** $z_1\overline{z_2}$

d $\dfrac{z_1}{z_2}$ **e** $z_1 + \overline{z_1}$ **f** $(z_1 + z_2)(z_1 - z_2)$

g $\dfrac{1}{z_1}$ **h** $\dfrac{z_2}{z_1}$ **i** $\dfrac{3z_1}{z_2}$

ISBN 978-1-316-63608-4

Review

2 Let $z = 1 - \sqrt{3}i$. For each of the following, express in the form $a + bi$ and mark on an Argand diagram:

a z **b** z^2 **c** z^3 **d** $\frac{1}{z}$ **e** $\bar{z}$ **f** $\frac{1}{\bar{z}}$

3 Write each of the following in polar form:

a $1 + i$ **b** $1 - \sqrt{3}i$ **c** $2\sqrt{3} + i$

d $3\sqrt{2} + 3\sqrt{2}i$ **e** $-3\sqrt{2} - 3\sqrt{2}i$ **f** $\sqrt{3} - i$

4 Write each of the following in Cartesian form:

a $-2\operatorname{cis}\left(\frac{\pi}{3}\right)$ **b** $3\operatorname{cis}\left(\frac{\pi}{4}\right)$ **c** $3\operatorname{cis}\left(\frac{3\pi}{4}\right)$

d $-3\operatorname{cis}\left(\frac{-3\pi}{4}\right)$ **e** $3\operatorname{cis}\left(\frac{-5\pi}{6}\right)$ **f** $\sqrt{2}\operatorname{cis}\left(\frac{-\pi}{4}\right)$

5 Let $z = \operatorname{cis}\left(\frac{\pi}{3}\right)$. On an Argand diagram, carefully plot:

a z^2 **b** $\bar{z}$ **c** $\frac{1}{z}$ **d** $\operatorname{cis}\left(\frac{2\pi}{3}\right)$

6 Let $z = \operatorname{cis}\left(\frac{\pi}{4}\right)$. On an Argand diagram, carefully plot:

a iz **b** $\bar{z}$ **c** $\frac{1}{z}$ **d** $-iz$

Multiple-choice questions

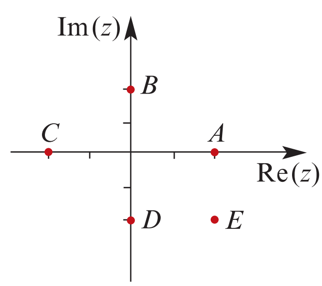
1 If $u = 1 + i$, then $\frac{1}{2 - u}$ is equal to

A $-\frac{1}{2} - \frac{1}{2}i$ **B** $\frac{1}{5} + \frac{2}{5}i$ **C** $\frac{1}{2} + \frac{1}{2}i$ **D** $-\frac{1}{2} + \frac{1}{5}i$ **E** $1 + 5i$

2 The point C on the Argand diagram represents the complex number z. Which point represents the complex number $i \times z$?

A A **B** B **C** C

D D **E** E



3 If $|z| = 5$, then $\left|\frac{1}{z}\right| =$

A $\frac{1}{\sqrt{5}}$ **B** $-\frac{1}{\sqrt{5}}$ **C** $\frac{1}{5}$ **D** $-\frac{1}{5}$ **E** $\sqrt{5}$

4 If $(x + yi)^2 = -32i$ for real values of x and y, then

A $x = 4, y = 4$ **B** $x = -4, y = 4$

C $x = 4, y = -4$ **D** $x = 4, y = -4$ or $x = -4, y = 4$

E $x = 4, y = 4$ or $x = -4, y = -4$

ISBN 978-1-316-63608-4

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Review

5 The linear factors of $z^2 + 6z + 10$ over $\mathbb{C}$ are

A $(z + 3 + i)^2$ **B** $(z + 3 - i)^2$ **C** $(z + 3 + i)(z - 3 + i)$
D $(z + 3 - i)(z + 3 + i)$ **E** $(z + 3 + i)(z - 3 - i)$

6 Let $z = \dfrac{1}{1 - i}$. If $r = |z|$ and $\theta = \operatorname{Arg} z$, then

A $r = 2$ and $\theta = \dfrac{\pi}{4}$ **B** $r = \dfrac{1}{2}$ and $\theta = \dfrac{\pi}{4}$ **C** $r = \sqrt{2}$ and $\theta = -\dfrac{\pi}{4}$
D $r = \dfrac{1}{\sqrt{2}}$ and $\theta = -\dfrac{\pi}{4}$ **E** $r = \dfrac{1}{\sqrt{2}}$ and $\theta = \dfrac{\pi}{4}$

7 The solution of the equation $\dfrac{z - 2i}{z - (3 - 2i)} = 2$, where $z \in \mathbb{C}$, is

A $z = 6 + 2i$ **B** $z = 6 - 2i$ **C** $z = -6 - 6i$
D $z = 6 - 6i$ **E** $z = -6 + 2i$

8 Let $z = a + bi$, where $a, b \in \mathbb{R}$. If $z^2(1 + i) = 2 - 2i$, then the Cartesian form of one value of z could be

A $\sqrt{2i}$ **B** $-\sqrt{2i}$ **C** $-1 - i$ **D** $-1 + i$ **E** $\sqrt{-2}$

9 The value of the discriminant for the quadratic expression $(2 + 2i)z^2 + 8iz - 4(1 - i)$ is

A -32 **B** 0 **C** 64 **D** 32 **E** -64

10 If $\operatorname{Arg}(ai + 1) = \dfrac{\pi}{6}$, then the real number a is

A $\sqrt{3}$ **B** $-\sqrt{3}$ **C** 1 **D** $\dfrac{1}{\sqrt{3}}$ **E** $-\dfrac{1}{\sqrt{3}}$

Extended-response questions

1 **a** Find the exact solutions in $\mathbb{C}$ for the equation $z^2 - 2\sqrt{3}z + 4 = 0$.

b **i** Plot the two solutions from part **a** on an Argand diagram.

ii Find the equation of the circle, with centre the origin, which passes through these two points.

iii Find the value of $a \in \mathbb{Z}$ such that the circle passes through $(0, \pm a)$.

2 Let z be a complex number with $|z| = 6$. Let A be the point representing z and let B be the point representing $(1 + i)z$.

a Find:

i $|(1 + i)z|$

ii $|(1 + i)z - z|$

b Prove that OAB is a right-angled isosceles triangle.

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Review

3 Let $z = \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$.

a On an Argand diagram, the points O, A, Z, P and Q represent the complex numbers 0, 1, z, $1 + z$ and $1 - z$ respectively. Show these points on a diagram.

b Prove that the magnitude of $\angle POQ$ is $\dfrac{\pi}{2}$. Find the ratio $\dfrac{OP}{OQ}$.

4 Let z_1 and z_2 be two complex numbers. Prove the following:

a $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + z_1\overline{z_2} + \overline{z_1}z_2$

b $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - (z_1\overline{z_2} + \overline{z_1}z_2)$

c $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

State a geometric theorem from the result of **c**.

5 Let z_1 and z_2 be two complex numbers.

a Prove the following:

i $\overline{\overline{z_1}z_2} = z_1\overline{z_2}$

ii $z_1\overline{z_2} + \overline{z_1}z_2$ is a real number

iii $z_1\overline{z_2} - \overline{z_1}z_2$ is an imaginary number

iv $(z_1\overline{z_2} + \overline{z_1}z_2)^2 - (z_1\overline{z_2} - \overline{z_1}z_2)^2 = 4|z_1z_2|^2$

b Use the results from part **a** and Question 4 to prove that $|z_1 + z_2| \le |z_1| + |z_2|$.
Hint: Show that $(|z_1| + |z_2|)^2 - |z_1 + z_2|^2 \ge 0$.

c Hence prove that $|z_1 - z_2| \ge |z_1| - |z_2|$.

6 Assume that $|z| = 1$ and that the argument of z is θ, where $0 < \theta < \pi$. Find the modulus and argument of:

a $z + 1$ **b** $z - 1$ **c** $\dfrac{z-1}{z+1}$

7 The quadratic expression $ax^2 + bx + c$ has real coefficients.

a Find the discriminant of $ax^2 + bx + c$.

b Find the condition in terms of a, b and c for which the equation $ax^2 + bx + c = 0$ has no real solutions.

c If this condition is fulfilled, let z_1 and z_2 be the complex solutions of the equation and let P_1 and P_2 the corresponding points on an Argand diagram.

i Find $z_1 + z_2$ and $|z_1|$ in terms of a, b and c.

ii Find $\cos(\angle P_1OP_2)$ in terms of a, b and c.

8 Let z_1 and z_2 be the solutions of the quadratic equation $z^2 + z + 1 = 0$.

a Find z_1 and z_2.

b Prove that $z_1 = z_2^2$ and $z_2 = z_1^2$.

c Find the modulus and the principal value of the argument of z_1 and z_2.

d Let P_1 and P_2 be the points on an Argand diagram corresponding to z_1 and z_2. Find the area of triangle P_1OP_2.

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14 Revision of Chapters 10–13

14A Short-answer questions

1 Solve each of the following equations:

a $\sin\left(\frac{x}{3}\right) = \frac{1}{2}$ for $x \in [0, 12\pi]$

b $\sqrt{2}\cos\left(2x + \frac{\pi}{6}\right) + 1 = 0$ for $x \in [-\pi, \pi]$

2 Solve the equation $\cos x = \frac{\sqrt{3}}{2}$, giving the general solution.

3 Given that $\sin A = \frac{3}{5}$, where A is acute, and that $\cos B = -\frac{1}{2}$, where B is obtuse, find the exact values of:

a $\sec A$ **b** $\cot A$ **c** $\cot B$ **d** $\operatorname{cosec} B$

4 Given that $\cos A = \frac{1}{3}$, find the possible values of $\cos\left(\frac{A}{2}\right)$.

5 Prove the identity $\frac{1}{1 + \sin A} + \frac{1}{1 - \sin A} = 2\sec^2 A$.

6 **a** Write the product $\cos(3x)\sin(x)$ as a sum or a difference.

b Find the general solution of the equation $\cos(3\theta) + \cos(\theta) = 0$.

7 Prove each of the following:

a $\frac{\sin(3x) + \sin(x)}{\cos(3x) + \cos(x)} = \tan(2x)$

b $\frac{\sin(x) + \sin(2x)}{1 + \cos(x) + \cos(2x)} = \tan(x)$

8 If $w = 3 + 2i$ and $z = 3 - 2i$, express each of the following in the form $a + bi$, where a and b are real numbers:

a $w + z$ **b** $w - z$ **c** wz **d** $w^2 + z^2$

e $(w + z)^2$ **f** $(w - z)^2$ **g** $w^2 - z^2$ **h** $(w - z)(w + z)$

ISBN 978-1-316-63608-4

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9 If $w = 1 - 2i$ and $z = 2 - 3i$, express each of the following in the form $a + bi$, where a and b are real numbers:

a $w + z$ **b** $w - z$ **c** wz **d** $\frac{w}{z}$ **e** iw **f** $\frac{i}{w}$

g $\frac{w}{i}$ **h** $\frac{z}{w}$ **i** $\frac{w}{w + z}$ **j** $(1 + i)w$ **k** $\frac{w}{1 + i}$ **l** w^2

10 Write each polynomial as a product of linear factors:

a $z^2 + 49$ **b** $z^2 - 2z + 10$ **c** $9z^2 - 6z + 5$ **d** $4z^2 + 12z + 13$

11 **a** Find the two square roots of $3 + 4i$.

b Use the quadratic formula to solve $(2 - i)z^2 + (4 + 3i)z + (-1 + 3i) = 0$ for z.

12 For each of the following functions, sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ on the same set of axes:

a $f(x) = x^2 + 3x + 2$ **b** $f(x) = (x - 1)^2 + 1$

c $f(x) = \sin(x) + 1,\ x \in [0, 2\pi]$ **d** $f(x) = \cos(x) + 2,\ x \in [0, 2\pi]$

13 Sketch the graphs of the following ellipses, and find the coordinates of their axis intercepts:

a $\frac{x^2}{4^2} + \frac{y^2}{5^2} = 1$ **b** $\frac{(x + 1)^2}{2^2} + \frac{(y - 2)^2}{3^2} = 1$

14 Sketch the graphs of the following hyperbolas, and write down the equations of their asymptotes:

a $x^2 - \frac{y^2}{3^2} = 1$ **b** $\frac{(y + 1)^2}{4^2} - \frac{(x - 2)^2}{2^2} = 1$

15 A point $P(x, y)$ moves so that it is equidistant from points $A(2, 2)$ and $B(3, 4)$. Find the locus of the point P.

16 A point $P(x, y)$ moves so that its distance from the point $K(0, 1)$ is half its distance from the line $x = -3$. Find its locus.

17 Each point $P(x, y)$ on a curve has the following property: the distance to $P(x, y)$ from the point $F(0, 1)$ is the same as the shortest distance to $P(x, y)$ from the line $y = -3$. Find the equation of the curve.

18 Find the Cartesian equation corresponding to each pair of parametric equations:

a $x = 2t + 1$ and $y = 2 - 3t$ **b** $x = \cos(2t)$ and $y = \sin(2t)$

c $x = 2\cos t + 2$ and $y = 3\sin t + 3$ **d** $x = 2\tan t$ and $y = 3\sec t$

19 A curve has parametric equations

$$x = t - 1 \quad \text{and} \quad y = 1 - 2t^2 \qquad \text{for } 0 \le t \le 2$$

a Find the curve's Cartesian equation. **b** What is the domain of the curve?

c What is the range of the curve? **d** Sketch the graph of the curve.

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20 Convert the polar coordinates $\left(2, \frac{7\pi}{6}\right)$ into Cartesian coordinates.

21 A point P has Cartesian coordinates $(2, -2)$. Find two representations of P using polar coordinates, one with $r > 0$ and the other with $r < 0$.

22 Convert the following polar equations into Cartesian equations:

a $r = 5$ **b** $\theta = \frac{\pi}{3}$ **c** $r = \frac{3}{\sin\theta}$

d $r = \frac{2}{3\sin\theta + 4\cos\theta}$ **e** $r^2 = \frac{1}{\sin(2\theta)}$

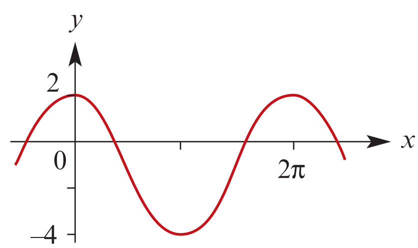
23 **a** Sketch the circle with equation $x^2 + (y-2)^2 = 2^2$.

b Show that this circle has polar equation $r = 4\sin\theta$.

14B Multiple-choice questions

1 The graph shown has amplitude

A 2 **B** 3 **C** 4
D 6 **E** 2π

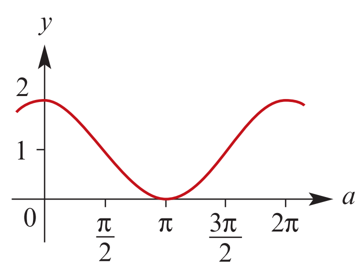


2 If $\sin A = \frac{5}{13}$ and $\sin B = \frac{8}{17}$, where A and B are acute, then $\sin(A - B)$ is given by

A $\frac{140}{221}$ **B** $-\frac{21}{221}$ **C** $\frac{34\,209}{23\,560}$ **D** $-\frac{107}{140}$ **E** $\frac{107}{140}$

3 The graph shown is best described by

A $y = \sin(a)$ **B** $y = 2\cos(a)$
C $y = \sin(a) + 1$ **D** $y = \cos(2a)$
E $y = \cos(a) + 1$



4 If $\sin A = \frac{5}{13}$ and $\sin B = \frac{8}{17}$, where A and B acute, then $\tan(A + B)$ is given by

A $\frac{140}{221}$ **B** $-\frac{21}{221}$ **C** $\frac{34\,209}{23\,560}$ **D** $-\frac{171}{140}$ **E** $\frac{171}{140}$

5 If $2\sin\left(x - \frac{\pi}{6}\right) = \sqrt{3}$ and $0 \le x \le 2\pi$, then x is equal to

A $\frac{\pi}{3}$ or $\frac{2\pi}{3}$ **B** $\frac{\pi}{6}$ or $\frac{5\pi}{6}$ **C** $\frac{\pi}{6}$ or $\frac{\pi}{2}$ **D** $\frac{\pi}{2}$ or $\frac{5\pi}{6}$ **E** $\frac{\pi}{3}$ or π

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6 If $\cos\theta = c$ and θ is acute, then $\cot\theta$ can be expressed in terms of c as

A $c\sqrt{1-c^2}$ **B** $\sqrt{1-c^2}$ **C** $\dfrac{1}{\sqrt{1-c^2}}$ **D** $\dfrac{c}{\sqrt{1-c^2}}$ **E** $2c\sqrt{1-c^2}$

7 A trigonometric graph has the following properties:

- period is 120°
- amplitude is 3
- range is $[-4, 2]$
- $y = -1$ when $a = 0$

This graph would be described by the equation

A $y = 3\sin(a)° + 1$ **B** $y = 3\cos(3a)° - 1$ **C** $y = -3\sin(3a)° - 1$
D $y = 3\sin(3a)° + 1$ **E** $y = \cos(3a)° - 2$

8 Compared with the graph of $y = \sin\theta$, the graph of $y = \sin\left(\frac{1}{2}\theta\right)$ has

A the same amplitude, but double the period
B the same amplitude, but half the period
C double the amplitude, but the same period
D half the amplitude, but the same period
E the same amplitude, but shifted $\frac{1}{2}$ unit to the left.

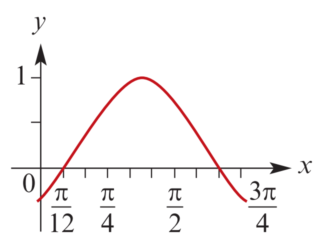
9 If $A + B = \dfrac{\pi}{2}$, then the value of $\cos A\cos B - \sin A\sin B$ is

A -2 **B** 1 **C** -1 **D** 0 **E** 2

10 Given that $\sin A = \dfrac{\sqrt{5}}{3}$ and that A is obtuse, the value of $\sin(2A)$ is

A $\dfrac{16\sqrt{5}}{243}$ **B** $-\dfrac{1}{9}$ **C** $-\dfrac{8\sqrt{5}}{27}$ **D** $\dfrac{5}{9}$ **E** $-\dfrac{4\sqrt{5}}{9}$

11 A possible equation for the graph shown is



A $y = \sin\left(2\left(x - \dfrac{\pi}{12}\right)\right)$ **B** $y = \cos\left(2\left(x - \dfrac{\pi}{12}\right)\right)$
C $y = \sin\left(2\left(x + \dfrac{\pi}{12}\right)\right)$ **D** $y = \cos\left(2\left(x + \dfrac{\pi}{12}\right)\right)$
E $y = -\sin\left(2\left(x - \dfrac{\pi}{12}\right)\right)$

12 Which of the following statements are true for $f(x) = -2\tan(3x)°$?

I The period is 60°. II The amplitude is 2. III The period is 30°.
IV The graph is a reflection of the graph of $h(x) = 2\tan(3x)°$ in the x-axis.
V The graph is a reflection of the graph of $g(x) = \tan(x)°$ in the y-axis.

A I and IV only **B** I, II and IV only **C** I, IV and V only
D II and IV only **E** II and IV only

13 If $\cos\theta = c$ and θ is acute, then $\sin(2\theta)$ can be expressed in terms of c as

A $c\sqrt{1-c^2}$ **B** $\sqrt{1-c^2}$ **C** $\dfrac{1}{\sqrt{1-c^2}}$ **D** $\dfrac{c}{\sqrt{1-c^2}}$ **E** $2c\sqrt{1-c^2}$

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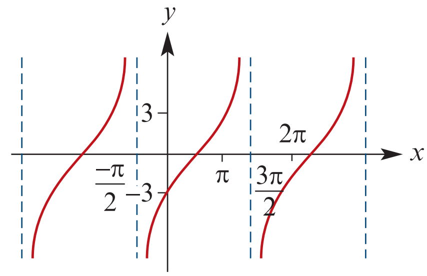
14 The angles between 0° and 360° which satisfy the equation $4\cos x - 3\sin x = 1$, given correct to two decimal places, are

A 53.13° and 126.87° **B** 48.41° and 205.33° **C** 41.59° and 244.67°
D 131.59° and 334.67° **E** 154.67° and 311.59°

15 The expression $8\sin\theta\cos^3\theta - 8\sin^3\theta\cos\theta$ is equal to

A $8\sin\theta\cos\theta$ **B** $\sin(8\theta)$ **C** $2\sin(4\theta)$
D $4\cos(2\theta)$ **E** $2\sin(2\theta)\cos(2\theta)$

16 A possible equation for the graph shown is



A $y = \tan\left(\frac{1}{2}x - \frac{\pi}{4}\right) + 3$

B $y = \tan\left(\frac{1}{2}x + \frac{\pi}{4}\right) - 3$

C $y = 3\tan\left(\frac{1}{2}x - \frac{\pi}{4}\right)$

D $y = 3\tan\left(\frac{1}{2}x + \frac{\pi}{4}\right)$

E $y = \tan(3x)$

17 If v, w and z are complex numbers such that $v = 4\operatorname{cis}(-0.3\pi)$, $w = 5\operatorname{cis}(0.6\pi)$ and $z = v\overline{w}$, then $\operatorname{Arg} z$ is equal to

A 0.9π **B** -0.9π **C** 0.3π **D** -0.3π **E** 1.8π

18 The complex number $2\operatorname{cis}\left(\frac{2\pi}{3}\right)$ is written in Cartesian form as

A $\sqrt{3} - i$ **B** $-\sqrt{3} + i$ **C** $1 - \sqrt{3}i$ **D** $-1 + \sqrt{3}i$ **E** $\frac{1}{3} - \frac{\sqrt{3}}{2}i$

19 If $z = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$, then $\operatorname{Arg} z$ is equal to

A $\frac{4\pi}{3}$ **B** $\frac{7\pi}{6}$ **C** $-\frac{\pi}{6}$ **D** $-\frac{2\pi}{3}$ **E** $-\frac{5\pi}{6}$

20 The imaginary part of the complex number $-2 - 3i$ is

A -3 **B** $-3i$ **C** 3 **D** -2 **E** $3i$

21 If $u = 3\operatorname{cis}\left(\frac{\pi}{2}\right)$ and $v = 5\operatorname{cis}\left(\frac{2\pi}{3}\right)$, then uv is equal to

A $15\operatorname{cis}\left(\frac{\pi}{3}\right)$ **B** $15\operatorname{cis}\left(\frac{\pi^2}{3}\right)$ **C** $15\operatorname{cis}\left(\frac{-5\pi}{6}\right)$ **D** $8\operatorname{cis}\left(\frac{\pi^2}{3}\right)$ **E** $8\operatorname{cis}\left(\frac{7\pi}{6}\right)$

22 The modulus of $12 - 5i$ is

A 169 **B** 7 **C** 13 **D** $\sqrt{119}$ **E** $\sqrt{7}$

23 Let $z = x + yi$, where x and y are real numbers which are not both zero. Which one of the following expressions does not necessarily represent a real number?

A z^2 **B** $z\overline{z}$ **C** $z^{-1}z$ **D** $\operatorname{Im}(z)$ **E** $z + \overline{z}$

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24 If $z = -14 - 7i$, then the complex conjugate of z is equal to

A $7 - 14i$ **B** $14 + 7i$ **C** $-14 + 7i$ **D** $14 - 7i$ **E** $-7 + 14i$

25 The expression $3z^2 + 9$ is factorised over $\mathbb{C}$. Which one of the following is a factor?

A $3z$ **B** $z + 3$ **C** $z + 3i$ **D** $z - 3i$ **E** $z + \sqrt{3}i$

26 $(1 + 2i)^2$ is equal to

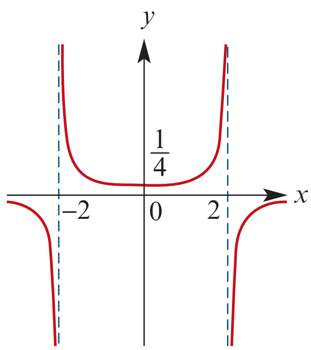
A -3 **B** $-3 + 2i$ **C** $-3 + 4i$ **D** $-1 + 4i$ **E** $5 + 4i$

27 Which of the following equations has the graph shown?

A $y = \dfrac{1}{4 - x^2}$ **B** $y = \dfrac{1}{x^2 - 4}$

C $y = \dfrac{1}{2 - x^2}$ **D** $y = \dfrac{1}{x^2 - 2}$

E $y = \dfrac{1}{(x - 2)^2}$



28 If a and b are positive real numbers, then the graph of the reciprocal of $y = a\sin(x) + b$, where $0 \le x \le 2\pi$, will have two vertical asymptotes provided

A $b > a$ **B** $a > b$ **C** $b > -a$ **D** $a > -b$ **E** $a > 0$

29 The graph of $f(x) = \sec(2x)$, for $-\pi \le x \le \pi$, has its local minimum points at

A $x = 0$ **B** $x = -\pi, \pi$ **C** $x = -\pi, 0, \pi$

D $x = -\dfrac{\pi}{2}, \dfrac{\pi}{2}$ **E** $x = -\pi, -\dfrac{\pi}{2}, 0, \dfrac{\pi}{2}, \pi$

30 Given point $A(1, -2)$, a set of points $P(x, y)$ satisfy $AP = 3$. This set of points is a

A line **B** circle **C** parabola **D** ellipse **E** hyperbola

31 A line has equation $y = x + 1$. For some pair of points A and B, each point $P(x, y)$ on the line satisfies $AP = BP$. The coordinates of A and B could be

A $A(0, 0)$ and $B(0, 1)$ **B** $A(0, 0)$ and $B(-1, 1)$ **C** $A(-1, 0)$ and $B(0, 1)$

D $A(0, 1)$ and $B(1, 0)$ **E** $A(0, 1)$ and $B(-1, 0)$

32 A parabola has focus $F(0, 2)$ and directrix $y = -4$. Which of the following is true?

A The parabola has axis of symmetry $y = 0$.

B The parabola goes through the origin.

C The parabola goes through the point $(0, -1)$.

D The parabola goes through the point $(1, 2)$.

E The parabola has equation $y = 2x^2 - 4$.

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33 Let a and b be positive real numbers. The graphs of $x^2 - y^2 = 1$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ will have four points of intersection provided

A $b > a$ **B** $a > 1$ **C** $a < 1$ **D** $b > 1$ **E** $b < 1$

34 A hyperbola has asymptotes $y = 2x + 1$ and $y = -2x + 1$ and has no x-axis intercepts. The equation of the hyperbola could be

A $x^2 - \frac{(y-1)^2}{4} = 1$ **B** $\frac{x^2}{4} - (y-1)^2 = 1$ **C** $\frac{(x-1)^2}{4} - y^2 = 1$

D $\frac{(y-1)^2}{4} - x^2 = 1$ **E** $\frac{y^2}{4} - (x-1)^2 = 1$

35 A curve is parameterised by the equations $x = 1 + t$ and $y = \frac{1-t}{1+t}$. The Cartesian equation of the curve is

A $y = \frac{2}{x} - 1$ **B** $y = \frac{1}{x} - 1$ **C** $y = \frac{1}{x} - 2$ **D** $y = \frac{1}{x} + 2$ **E** $y = \frac{2}{x} + 1$

36 The ellipse with equation $\frac{(x-1)^2}{4} + \frac{(y+1)^2}{9} = 1$ can be parameterised by the pair of equations

A $x = 4\cos(t) - 1$ and $y = 9\sin(t) + 1$ **B** $x = 4\cos(t) + 1$ and $y = 9\sin(t) - 1$

C $x = 4\cos(t) - 1$ and $y = 9\sin(t) - 1$ **D** $x = 2\cos(t) - 1$ and $y = 3\sin(t) - 1$

E $x = 2\cos(t) + 1$ and $y = 3\sin(t) - 1$

37 A curve is parameterised by the equations $x = 2t - 3$ and $y = t^2 - 3t$. Which of the following points does the curve pass through?

A $(5, 1)$ **B** $(5, 2)$ **C** $(5, 3)$ **D** $(5, 4)$ **E** $(5, 5)$

38 The Cartesian equation $y = x^2$ written in polar form is

A $r = \sec\theta \tan\theta$ **B** $r = \cos\theta \cot\theta$ **C** $r = \sec\theta \cot\theta$

D $r = \cos\theta \tan\theta$ **E** $r = \sin\theta \tan\theta$

14C Extended-response questions

1 A particle oscillates along a straight line. Its displacement, x m, from a point O at time t s is given by $x = 5 + 3\sin\left(\frac{\pi}{6}t\right)$.

a Find its displacement at time:

i $t = 0$ **ii** $t = 3$

b Sketch the graph of x against t for $t \in [0, 24]$, labelling clearly all turning points.

c **i** State the maximum distance of the particle from O.

ii State the minimum distance of the particle from O.

d Correct to two decimal places, at what times $t \in [0, 24]$ is the particle:

i 5 m from O **ii** 6 m from O?

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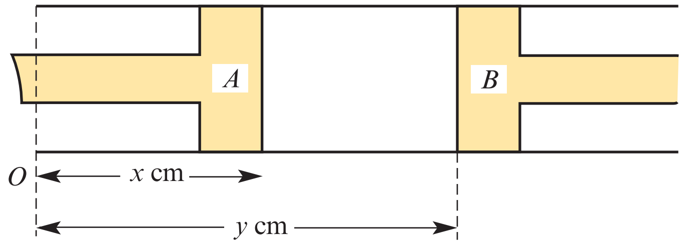
2 Two pistons A and B move backwards and forwards in a cylinder as shown.



The distance, x cm, of the right-hand end of piston A from the point O at time t seconds is modelled by the rule

$$x = 4\sin(3t) + 4$$

The distance, y cm, of the left-hand end of piston B from the point O at time t seconds is modelled by the rule

$$y = 2\sin\left(2t - \frac{\pi}{6}\right) + 10$$

The two pistons are set in motion at time $t = 0$.

a State the value of x and the value of y when $t = 0$.

b **i** State the amplitude of the motion of piston A.

ii State the amplitude of the motion of piston B.

c **i** State the maximum and minimum values of x.

ii State the maximum and minimum values of y.

d **i** State the period of the motion of piston A.

ii State the period of the motion of piston B.

e Find the time(s) in the first cycle of A that its distance from O is a maximum.

f Find the next four values of t for which x takes its maximum value.

g Find the values of $t \in [0, 4\pi]$ for which y attains its minimum value.

h On one set of axes, sketch the graphs of $x = 4\sin(3t) + 4$ and $y = 2\sin\left(2t - \frac{\pi}{6}\right) + 10$ over the interval $[0, \pi]$.

i State the time when the pistons first touch each other.

j How many seconds are there between the first and second times the pistons touch?

3 The pistons A and B (from Question 2) are adjusted so that the distance, x cm, of the right-hand end of piston A from the point O at time t seconds is modelled by the rule

$$x = a\sin(nt) + b$$

and the distance, y cm, of the left-hand end of piston B from the point O at time t seconds is modelled by the rule

$$y = c\sin(mt) + d$$

The pistons meet every second at a point 8 cm from O. The right-hand end of piston A cannot go to the left of the point O.

a Find one possible set of values of a, b, n and c, d, m and explain your solution.

b Using the set of values found in part **a**, sketch the graphs of x against t and y against t on the one set of axes.

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4 Suppose that k is a real number and consider the function $f\colon [0, 2\pi] \to \mathbb{R}$ where $f(x) = 2\sin(x) + k$.

a Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ when:

i $k = 1$ **ii** $k = 3$

b For what value of k does the graph of $y = \dfrac{1}{f(x)}$ have only one vertical asymptote?

c For this value of k, sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$.

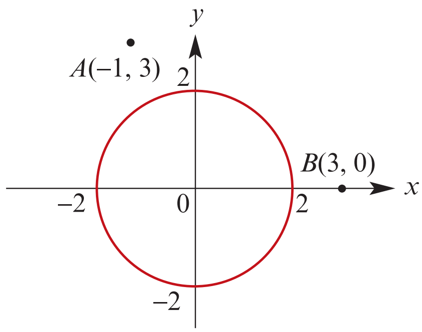
5 Two towns A and B are located on a rectangular grid with coordinates $A(1, 2)$ and $B(2, -2)$, where the units are kilometres. A straight section of road is to be constructed so that each point $P(x, y)$ on the road is equidistant from the two towns.

a Find the equation of the road.

b Show that the road is the perpendicular bisector of the line segment AB.

c Hence find the shortest distance from town A to the road.

6 The circle with equation $x^2 + y^2 = 4$ is shown. We will say that point A is **visible** to point B if the line AB does not intersect the circle.



a Consider points $A(-1, 3)$ and $B(3, 0)$. Show that the equations

$$x = 4t - 1 \quad \text{and} \quad y = 3 - 3t$$

parameterise the line AB.

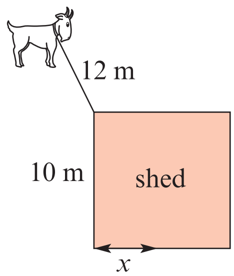
b Show that A is not visible to B by showing that there are two values of t for which the line AB intersects the circle.

c Find parametric equations for the line that goes through points $C(-1, 4)$ and $B(3, 0)$.

d Show that C is visible to B by showing that there is no value of t for which the line CB intersects the circle.

e Find the range of values k for which the point $D(-1, k)$ is visible to B.

7 A shed has a square base of side length 10 metres. A goat is tied to a corner of the shed by a rope of length 12 metres. As the goat pulls tightly on the rope and walks around the shed in both directions, a path is traced by the goat.



a Sketch the shed and the path described above.

b Find the size of the area over which the goat can walk.

c The goat is now tied to a point on the shed x metres from the corner, where $0 \le x \le 5$. Find a formula for the area A over which the goat can walk, in terms of x.

Hint: Consider the two cases $0 \le x \le 2$ and $2 < x \le 5$.

d Sketch the graph of A against x for $0 \le x \le 5$.

e Where should the goat be tied if the area is to be:

i a maximum **ii** a minimum?

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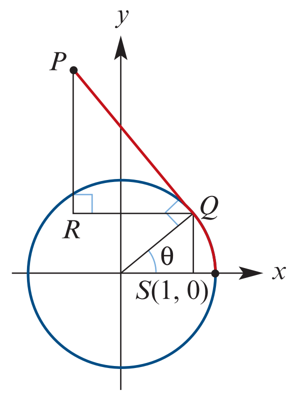
8 A rope of length π is affixed to point $S(1, 0)$ on one side of a circle of radius 1 centred at the origin. The rope can be pulled tight and wrapped around the circle in both directions. The end of the rope traces out a curve.



a Explain why the rope can reach to the opposite side of the circle.

b Sketch the unit circle and the curve described above.

We now find parametric equations to describe the part of the curve obtained when some of the rope is wrapped anticlockwise around the unit circle.

c Referring to the diagram, find the following in terms of θ:

i Arc length SQ **ii** Length PQ **iii** Angle RPQ **iv** Length RQ **v** Length RP

d Hence, by finding the coordinates of point P, give parametric equations for the curve in terms of θ.

Triple angle formulas

9 **a** Using the addition formulas and the double angle formulas, prove each of the following identities:

i $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$ **ii** $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$

Now let $\alpha = \cos\left(\frac{\pi}{9}\right)$.

b Use an identity from part **a** to show that $x = \alpha$ is a solution of the equation $f(x) = 0$, where $f(x) = x^3 - \frac{3}{4}x - \frac{1}{8}$.

c Use your calculator to plot the graph of $y = f(x)$. Hence determine that $x = \alpha$ is the unique positive real solution of the cubic equation $f(x) = 0$.

For a cubic equation of the form $z^3 + pz + q = 0$, we can use Cardano's formula to find a solution over $\mathbb{C}$:

$$z = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

d Use Cardano's formula to find a solution of the equation $f(z) = 0$ over $\mathbb{C}$. Express your answer in the form $z = a(\sqrt[3]{1 + bi} + \sqrt[3]{1 - bi})$, for $a, b \in \mathbb{R}^+$.

Note: It can be shown that an expression of the form obtained in part **d** must be a positive real number. Thus this expression is equal to $\alpha = \cos\left(\frac{\pi}{9}\right)$.

10 **a** Use the rule for multiplying complex numbers in polar form to show that:

i $(\text{cis}\,\theta)^2 = \text{cis}(2\theta)$ **ii** $(\text{cis}\,\theta)^3 = \text{cis}(3\theta)$

b From part **a ii**, we now have the equation

$$(\cos\theta + i\sin\theta)^3 = \cos(3\theta) + i\sin(3\theta)$$

Expand and simplify the left-hand side of this equation. Then, by equating real and imaginary parts, derive the two triple angle formulas from Question 9.

 ISBN 978-1-316-63608-4 Cambridge University Press

Chapter 15

15 Matrices

Objectives

- To identify when two matrices are **equal**.
- To **add** and **subtract** matrices of the same size.
- To multiply a matrix by a real number.
- To identify when the multiplication of two given matrices is possible.
- To perform **multiplication** of two suitable matrices.
- To find the **inverse** of a 2×2 matrix.
- To find the **determinant** of a 2×2 matrix.
- To solve **simultaneous linear equations** in two unknowns by using an inverse matrix.

A **matrix** is a rectangular array of numbers. An example of a matrix is

$$\begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & -1 \\ 1 & 2 & -2 \end{bmatrix}$$

Matrix algebra was first studied in England in the middle of the nineteenth century. Matrices are now used in many areas of science: for example, in physics, medical research, encryption and internet search engines.

In this chapter we will show how addition and multiplication of matrices can be defined and how matrices can be used to solve simultaneous linear equations. In Chapter 16 we will see how they can be used to study transformations of the plane.

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15A Matrix notation

A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** of the matrix.

The following are examples of matrices:

$$\begin{bmatrix} -1 & 2 \\ -3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} \sqrt{2} & \pi & 3 \\ 0 & 0 & 1 \\ \sqrt{2} & 0 & \pi \end{bmatrix} \qquad \begin{bmatrix} 5 \end{bmatrix}$$

▶ The size of a matrix

Matrices vary in size. The **size** of the matrix is described by specifying the number of **rows** (horizontal lines) and **columns** (vertical lines) that occur in the matrix.

The sizes of the above matrices are, in order:

$$3 \times 2, \quad 1 \times 4, \quad 3 \times 3, \quad 1 \times 1$$

The first number represents the number of rows, and the second the number of columns.

An $m \times n$ matrix has m rows and n columns.

Example 1

Write down the sizes of the following matrices:

a $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ **c** $\begin{bmatrix} 2 & 2 & 3 \end{bmatrix}$

Solution

a 2×3 **b** 4×1 **c** 1×3

▶ Storing information in matrices

The use of matrices to store information is demonstrated by the following example.

Four exporters A, B, C and D sell refrigerators (r), CD players (c), washing machines (w) and televisions (t). The sales in a particular month can be represented by a 4×4 array of numbers. This array of numbers is called a matrix.

	r	*c*	*w*	*t*	
A	120	95	370	250	row 1
B	430	380	950	900	row 2
C	60	50	150	100	row 3
D	200	100	470	50	row 4
	column 1	column 2	column 3	column 4	

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From this matrix it can be seen that:

- Exporter A sold 120 refrigerators, 95 CD players, 370 washing machines, 250 televisions.
- Exporter B sold 430 refrigerators, 380 CD players, 950 washing machines, 900 televisions.

The entries for the sales of refrigerators are in column 1.

The entries for the sales of exporter A are in row 1.

Example 2

A minibus has four rows of seats, with three seats in each row. If 0 indicates that a seat is vacant and 1 indicates that a seat is occupied, write down a matrix to represent:

a the 1st and 3rd rows are occupied, but the 2nd and 4th rows are vacant

b only the seat at the front-left corner of the minibus is occupied.

Solution

a $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

b $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Example 3

There are four clubs in a local football league:

- Club A has 2 senior teams and 3 junior teams.
- Club B has 2 senior teams and 4 junior teams.
- Club C has 1 senior team and 2 junior teams.
- Club D has 3 senior teams and 3 junior teams.

Represent this information in a matrix.

Solution

$$\begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 2 \\ 3 & 3 \end{bmatrix}$$

Explanation

The rows represent clubs A, B, C, D and the columns represent the number of senior and junior teams.

► Entries and equality

We will use uppercase letters $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$ to denote matrices.

If $\mathbf{A}$ is a matrix, then a_{ij} will be used to denote the entry that occurs in row i and column j of $\mathbf{A}$. Thus a 3×4 matrix may be written as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

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Two matrices **A** and **B** are **equal**, and we can write **A** = **B**, when:

- they have the same number of rows and the same number of columns, and
- they have the same entry at corresponding positions.

For example:

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1+1 & 1 & -1 \\ 1-1 & 1 & \frac{6}{2} \end{bmatrix}$$

Example 4

If matrices **A** and **B** are equal, find the values of x and y.

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ x & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ -3 & y \end{bmatrix}$$

Solution

$x = -3$ and $y = 4$

Although a matrix is made from a set of numbers, it is important to think of a matrix as a single entity, somewhat like a 'super number'.

Section summary

- A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** of the matrix.
- The **size** of a matrix is described by specifying the number of rows and the number of columns. An $m \times n$ matrix has m rows and n columns.
- Two matrices **A** and **B** are equal when:
  - they have the same number of rows and the same number of columns, and
  - they have the same entry at corresponding positions.

Exercise 15A

Example 1

1 Write down the sizes of the following matrices:

a $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} a & b & c & d \end{bmatrix}$ **d** $\begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}$

Example 2

2 There are 25 seats arranged in five rows and five columns. Using 0 to indicate that a seat is vacant and 1 to indicate that a seat is occupied, write down a matrix to represent the situation when:

a only the seats on the two diagonals are occupied

b all seats are occupied.

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3 Seating arrangements are again represented by matrices, as in Question 2. Describe the seating arrangement represented by each of the following matrices:

a the entry a_{ij} is 1 if $i = j$, but 0 if $i \neq j$

b the entry a_{ij} is 1 if $i > j$, but 0 if $i \leq j$

c the entry a_{ij} is 1 if $i = j + 1$, but 0 otherwise.

Example 3

4 At a certain school there are 200 girls and 110 boys in Year 7. The numbers of girls and boys in the other year levels are 180 and 117 in Year 8, 135 and 98 in Year 9, 110 and 89 in Year 10, 56 and 53 in Year 11, and 28 and 33 in Year 12. Summarise this information in a matrix.

Example 4

5 From the following, select those pairs of matrices which could be equal, and write down the values of x and y which would make them equal:

a $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ x \end{bmatrix}$, $\begin{bmatrix} 0 & x \end{bmatrix}$, $\begin{bmatrix} 0 & 4 \end{bmatrix}$

b $\begin{bmatrix} 4 & 7 \\ 1 & -2 \end{bmatrix}$, $\begin{bmatrix} 1 & -2 \\ 4 & x \end{bmatrix}$, $\begin{bmatrix} x & 7 \\ 1 & -2 \end{bmatrix}$, $\begin{bmatrix} 4 & x & 1 & -2 \end{bmatrix}$

c $\begin{bmatrix} 2 & x & 4 \\ -1 & 10 & 3 \end{bmatrix}$, $\begin{bmatrix} y & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix}$

6 Find the values of the pronumerals so that matrices **A** and **B** are equal:

a $\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} x & 1 & -1 \\ 0 & 1 & y \end{bmatrix}$

b $\mathbf{A} = \begin{bmatrix} x \\ 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 \\ y \end{bmatrix}$

c $\mathbf{A} = \begin{bmatrix} -3 & x \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} y & 4 \end{bmatrix}$

d $\mathbf{A} = \begin{bmatrix} 1 & y \\ 4 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -2 \\ 4 & x \end{bmatrix}$

7 The statistics for five members of a basketball team are recorded as follows:

Player A points 21, rebounds 5, assists 5
Player B points 8, rebounds 2, assists 3
Player C points 4, rebounds 1, assists 1
Player D points 14, rebounds 8, assists 60
Player E points 0, rebounds 1, assists 2

Express this information in a 5×3 matrix.

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15B Addition, subtraction and multiplication by a real number

Addition of matrices

If **A** and **B** are two matrices of the same size, then the sum **A** + **B** is the matrix obtained by adding together the corresponding entries of the two matrices.

For example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 4 & 3 \end{bmatrix}$$

and

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \\ a_{31} + b_{31} & a_{32} + b_{32} \end{bmatrix}$$

Multiplication of a matrix by a real number

If **A** is any matrix and k is a real number, then the product k**A** is the matrix obtained by multiplying each entry of **A** by k.

For example:

$$3\begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 0 & 3 \end{bmatrix}$$

Note: If a matrix is added to itself, then the result is twice the matrix, i.e. **A** + **A** = 2**A**. Similarly, for any natural number n, the sum of n matrices each equal to **A** is n**A**.

If **B** is any matrix, then **−B** denotes the product (−1)**B**.

Subtraction of matrices

If **A** and **B** are matrices of the same size, then **A** − **B** is defined to be the sum

$$\mathbf{A} + (-\mathbf{B}) = \mathbf{A} + (-1)\mathbf{B}$$

For two matrices **A** and **B** of the same size, the difference **A** − **B** can be found by subtracting corresponding entries.

Example 5

Find:

a $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -4 & 1 \end{bmatrix}$

b $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$

Solution

a $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 6 & -1 \end{bmatrix}$

b $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

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Zero matrix

The $m \times n$ matrix with all entries equal to zero is called the **zero matrix**, and will be denoted by **O**.

For any $m \times n$ matrix **A** and the $m \times n$ zero matrix **O**, we have

$$\mathbf{A} + \mathbf{O} = \mathbf{A} \qquad \text{and} \qquad \mathbf{A} + (-\mathbf{A}) = \mathbf{O}$$

Example 6

Let $\mathbf{X} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix}$.

Find $\mathbf{X} + \mathbf{Y}$, $2\mathbf{X}$, $4\mathbf{Y} + \mathbf{X}$, $\mathbf{X} - \mathbf{Y}$, $-3\mathbf{A}$ and $3\mathbf{A} + \mathbf{B}$.

Solution

$$\mathbf{X} + \mathbf{Y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

$$2\mathbf{X} = 2\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$4\mathbf{Y} + \mathbf{X} = 4\begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 \\ 24 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ 28 \end{bmatrix}$$

$$\mathbf{X} - \mathbf{Y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$-3\mathbf{A} = -3\begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 3 & -6 \end{bmatrix}$$

$$-3\mathbf{A} + \mathbf{B} = \begin{bmatrix} -6 & 0 \\ 3 & -6 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$$

Example 7

If $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -4 \\ -2 & 8 \end{bmatrix}$, find the matrix **X** such that $2\mathbf{A} + \mathbf{X} = \mathbf{B}$.

Solution

If $2\mathbf{A} + \mathbf{X} = \mathbf{B}$, then $\mathbf{X} = \mathbf{B} - 2\mathbf{A}$. Therefore

$$\begin{aligned} \mathbf{X} &= \begin{bmatrix} 0 & -4 \\ -2 & 8 \end{bmatrix} - 2\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 - 2 \times 3 & -4 - 2 \times 2 \\ -2 - 2 \times (-1) & 8 - 2 \times 1 \end{bmatrix} \\ &= \begin{bmatrix} -6 & -8 \\ 0 & 6 \end{bmatrix} \end{aligned}$$

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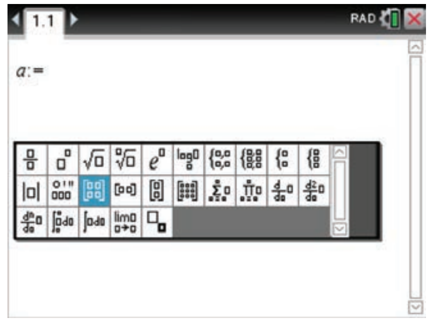
Using the TI-Nspire

The matrix template

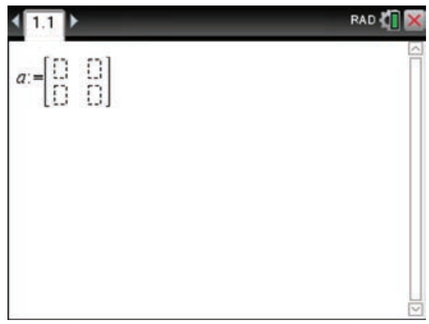
Matrices can be assigned (or stored) as variables for further computations.

Assign matrix $\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$ as follows:

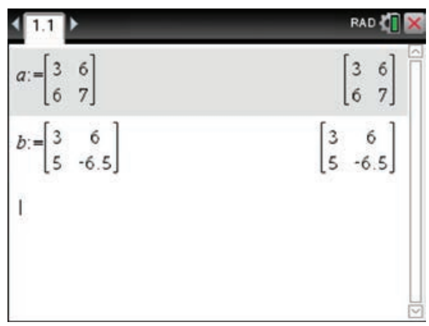
- In a **Calculator** page, type $a :=$ and then enter the matrix. (Access the assignment symbol := using (ctrl) (templates key).)



- The simplest way to enter a 2 × 2 matrix is using the 2 × 2 matrix template as shown. (Access the templates using either (templates key) or (ctrl) (menu) > **Math Templates**.)
- Notice that there is also a template for entering $m \times n$ matrices.
- Use the touchpad arrows (or (tab)) to move between the entries of the 2 × 2 matrix template.

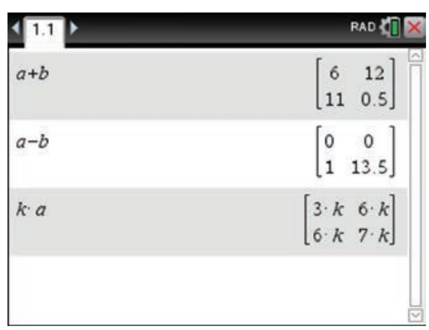


Assign the matrix $\mathbf{B} = \begin{bmatrix} 3 & 6 \\ 5 & -6.5 \end{bmatrix}$ similarly.



Operations on matrices

Once **A** and **B** are defined as above, the matrices **A** + **B**, **A** − **B** and k**A** can easily be determined.



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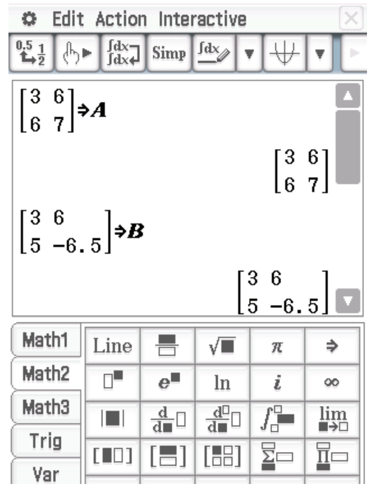
Using the Casio ClassPad

Entering a matrix

- In Main $\sqrt{\alpha}$, select the Math2 keyboard.
- To enter a 2×2 matrix, tap [■■].
- Type the values into the matrix template.

Note: Tap at each new position to enter the value, or use the black cursor key on the hard keyboard to navigate to each new position.



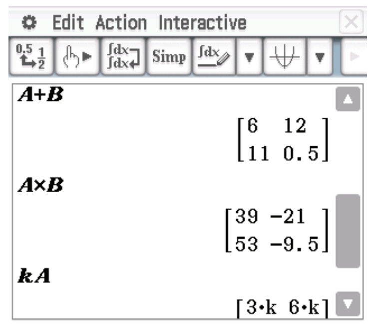
Assigning a matrix

- Move the cursor to the right-hand side of the matrix. Then tap the variable assignment key ⇒ followed by Var CAPS A.
- Tap EXE to confirm your choice.
- Enter the second matrix and assign it the variable name B as shown.

Note: Until it is reassigned, the variable A will represent the matrix as defined above.

Operations on matrices

- Calculate $\mathbf{A} + \mathbf{B}$, $\mathbf{AB}$ and $k\mathbf{A}$ as shown. (Use the Var keyboard to enter the variable names.)



Section summary

- If $\mathbf{A}$ and $\mathbf{B}$ are matrices of the same size, then:
  - the matrix $\mathbf{A} + \mathbf{B}$ is obtained by adding the corresponding entries of $\mathbf{A}$ and $\mathbf{B}$
  - the matrix $\mathbf{A} - \mathbf{B}$ is obtained by subtracting the corresponding entries of $\mathbf{A}$ and $\mathbf{B}$.
- If $\mathbf{A}$ is any matrix and k is a real number, then the matrix $k\mathbf{A}$ is obtained by multiplying each entry of $\mathbf{A}$ by k.

Exercise 15B

Example 6

1 Let $\mathbf{X} = \begin{bmatrix}1\\-2\end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix}3\\0\end{bmatrix}$, $\mathbf{A} = \begin{bmatrix}1 & -1\\2 & 3\end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix}4 & 0\\-1 & 2\end{bmatrix}$.

Find $\mathbf{X} + \mathbf{Y}$, $2\mathbf{X}$, $4\mathbf{Y} + \mathbf{X}$, $\mathbf{X} - \mathbf{Y}$, $-3\mathbf{A}$ and $-3\mathbf{A} + \mathbf{B}$.

2 Let $\mathbf{A} = \begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}$. Find $2\mathbf{A}$, $-3\mathbf{A}$ and $-6\mathbf{A}$.

3 For $m \times n$ matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, is it always true that:

a $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

b $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$?

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4 Let $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -2 & -2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -3 \\ 4 & 1 \end{bmatrix}$. Calculate:

a $2\mathbf{A}$ **b** $3\mathbf{B}$ **c** $2\mathbf{A} + 3\mathbf{B}$ **d** $3\mathbf{B} - 2\mathbf{A}$

5 Let $\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$ and $\mathbf{R} = \begin{bmatrix} 0 & 4 \\ 1 & 1 \end{bmatrix}$. Calculate:

a $\mathbf{P} + \mathbf{Q}$ **b** $\mathbf{P} + 3\mathbf{Q}$ **c** $2\mathbf{P} - \mathbf{Q} + \mathbf{R}$

Example 7 **6** If $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -10 \\ -2 & 17 \end{bmatrix}$, find matrices $\mathbf{X}$ and $\mathbf{Y}$ such that $2\mathbf{A} - 3\mathbf{X} = \mathbf{B}$ and $3\mathbf{A} + 2\mathbf{Y} = 2\mathbf{B}$.

7 Matrices $\mathbf{X}$ and $\mathbf{Y}$ show the production of four models of cars a, b, c, d at two factories P, Q in successive weeks. Find $\mathbf{X} + \mathbf{Y}$ and describe what this sum represents.

$$\text{Week 1: } \mathbf{X} = \begin{matrix} & a & b & c & d \\ P & 150 & 90 & 100 & 50 \\ Q & 100 & 0 & 75 & 0 \end{matrix} \qquad \text{Week 2: } \mathbf{Y} = \begin{matrix} & a & b & c & d \\ P & 160 & 90 & 120 & 40 \\ Q & 100 & 0 & 50 & 0 \end{matrix}$$

15C Multiplication of matrices

Multiplication of a matrix by a real number has been discussed in the previous section. The definition for multiplication of matrices is less straightforward. The procedure for multiplying two 2×2 matrices is shown first.

Let $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix}$.

Then
$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \times 5 + 3 \times 6 & 1 \times 1 + 3 \times 3 \\ 4 \times 5 + 2 \times 6 & 4 \times 1 + 2 \times 3 \end{bmatrix} \\ &= \begin{bmatrix} 23 & 10 \\ 32 & 10 \end{bmatrix} \end{aligned}$$

and
$$\begin{aligned} \mathbf{BA} &= \begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 5 \times 1 + 1 \times 4 & 5 \times 3 + 1 \times 2 \\ 6 \times 1 + 3 \times 4 & 6 \times 3 + 3 \times 2 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 17 \\ 18 & 24 \end{bmatrix} \end{aligned}$$

Note that $\mathbf{AB} \neq \mathbf{BA}$.

ISBN 978-1-316-63608-4

If **A** is an $m \times n$ matrix and **B** is an $n \times r$ matrix, then the product **AB** is the $m \times r$ matrix whose entries are determined as follows:

> To find the entry in row i and column j of **AB**, single out row i in matrix **A** and column j in matrix **B**. Multiply the corresponding entries from the row and column and then add up the resulting products.

Note: The product **AB** is defined only if the number of columns of **A** is the same as the number of rows of **B**.

Example 8

For $\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, find **AB**.

Solution

A is a 2×2 matrix and **B** is a 2×1 matrix. Therefore the product **AB** is defined and will be a 2×1 matrix.

$$\mathbf{AB} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \times 5 + 4 \times 3 \\ 3 \times 5 + 6 \times 3 \end{bmatrix} = \begin{bmatrix} 22 \\ 33 \end{bmatrix}$$

Example 9

Matrix **X** shows the number of cars of models a and b bought by four dealers A, B, C, D. Matrix **Y** shows the cost in dollars of cars a and b. Find **XY** and explain what it represents.

$$\mathbf{X} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} \end{array} \qquad \mathbf{Y} = \begin{bmatrix} 26\,000 \\ 32\,000 \end{bmatrix}\begin{array}{c} a \\ b \end{array}$$

Solution

X is a 4×2 matrix and **Y** is a 2×1 matrix. Therefore **XY** is a 4×1 matrix.

$$\mathbf{XY} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} \end{array}\begin{bmatrix} 26\,000 \\ 32\,000 \end{bmatrix}\begin{array}{c} a \\ b \end{array}$$

$$= \begin{bmatrix} 3 \times 26\,000 + 1 \times 32\,000 \\ 2 \times 26\,000 + 2 \times 32\,000 \\ 1 \times 26\,000 + 4 \times 32\,000 \\ 1 \times 26\,000 + 1 \times 32\,000 \end{bmatrix} = \begin{bmatrix} 110\,000 \\ 116\,000 \\ 154\,000 \\ 58\,000 \end{bmatrix}$$

The matrix **XY** shows that dealer A spent \$110 000, dealer B spent \$116 000, dealer C spent \$154 000 and dealer D spent \$58 000.

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Example 10

For $\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}$, find $\mathbf{AB}$.

Solution

$\mathbf{A}$ is a 2×3 matrix and $\mathbf{B}$ is a 3×2 matrix. Therefore $\mathbf{AB}$ is a 2×2 matrix.

$$\mathbf{AB} = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 4 + 3 \times 1 + 4 \times 0 & 2 \times 0 + 3 \times 2 + 4 \times 3 \\ 5 \times 4 + 6 \times 1 + 7 \times 0 & 5 \times 0 + 6 \times 2 + 7 \times 3 \end{bmatrix} = \begin{bmatrix} 11 & 18 \\ 26 & 33 \end{bmatrix}$$

Section summary

- If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times r$ matrix, then the product $\mathbf{AB}$ is the $m \times r$ matrix whose entries are determined as follows:

  To find the entry in row i and column j of $\mathbf{AB}$, single out row i in matrix $\mathbf{A}$ and column j in matrix $\mathbf{B}$. Multiply the corresponding entries from the row and column and then add up the resulting products.
- The product $\mathbf{AB}$ is defined only if the number of columns of $\mathbf{A}$ is the same as the number of rows of $\mathbf{B}$.

Exercise 15C

Skillsheet

Example 8, 10

1 Let $\mathbf{X} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Find the products $\mathbf{AX}$, $\mathbf{BX}$, $\mathbf{AY}$, $\mathbf{IX}$, $\mathbf{AC}$, $\mathbf{CA}$, $(\mathbf{AC})\mathbf{X}$, $\mathbf{C}(\mathbf{BX})$, $\mathbf{AI}$, $\mathbf{IB}$, $\mathbf{AB}$, $\mathbf{BA}$, $\mathbf{A}^2$, $\mathbf{B}^2$, $\mathbf{A}(\mathbf{CA})$ and $\mathbf{A}^2\mathbf{C}$.

2 Which of the following products of matrices from Question 1 are defined?

$\mathbf{AY}$, $\mathbf{YA}$, $\mathbf{XY}$, $\mathbf{X}^2$, $\mathbf{CI}$, $\mathbf{XI}$

3 If $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ -3 & 2 \end{bmatrix}$, find $\mathbf{AB}$.

4 Let $\mathbf{A}$ and $\mathbf{B}$ be 2×2 matrices and let $\mathbf{O}$ be the 2×2 zero matrix. Is the following argument correct?

'If $\mathbf{AB} = \mathbf{O}$ and $\mathbf{A} \neq \mathbf{O}$, then $\mathbf{B} = \mathbf{O}$.'

5 Find a matrix $\mathbf{A}$ such that $\mathbf{A} \neq \mathbf{O}$ but $\mathbf{A}^2 = \mathbf{O}$.

6 If $\mathbf{L} = \begin{bmatrix} 2 & -1 \end{bmatrix}$ and $\mathbf{X} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, find $\mathbf{LX}$ and $\mathbf{XL}$.

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7 Assume that both **A** and **B** are $m \times n$ matrices. Are **AB** and **BA** defined and, if so, how many rows and columns do they have?

8 Suppose that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

a Show that $ad - bc = 1$.

b What is the product matrix if the order of multiplication on the left-hand side is reversed?

9 Using the result of Question 8, write down a pair of matrices **A** and **B** such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, where $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

10 Choose any three 2×2 matrices **A**, **B** and **C**. Find $\mathbf{A}(\mathbf{B} + \mathbf{C})$, $\mathbf{AB} + \mathbf{AC}$ and $(\mathbf{B} + \mathbf{C})\mathbf{A}$.

11 Find matrices **A** and **B** such that $(\mathbf{A} + \mathbf{B})^2 \neq \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$.

Example 9 **12** It takes John 5 minutes to drink a milk shake which costs \$2.50, and 12 minutes to eat a banana split which costs \$3.00.

a Find the product $\begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and interpret the result in milk-bar economics.

b Two friends join John. Find $\begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ and interpret the result.

13 Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. Find $\mathbf{A}^2$ and use your answer to find $\mathbf{A}^4$ and $\mathbf{A}^8$.

14 Let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find $\mathbf{A}^2$, $\mathbf{A}^3$ and $\mathbf{A}^4$. Write down a formula for $\mathbf{A}^n$.

15D Identities, inverses and determinants for 2×2 matrices

▶ Identities

A matrix with the same number of rows and columns is called a **square matrix**. For square matrices of a given size (e.g. 2×2), a multiplicative identity **I** exists.

For 2×2 matrices, the **identity matrix** is $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

For example, if $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, then $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$, and this result holds for any square matrix multiplied by the appropriate multiplicative identity.

For 3×3 matrices, the identity matrix is $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

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Specialist Mathematics Year 11
ISBN 978-1-316-63608-4

▶ Inverses

Given a 2×2 matrix $\mathbf{A}$, is there a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$?

For example, consider $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ and let $\mathbf{B} = \begin{bmatrix} x & y \\ u & v \end{bmatrix}$.

Then $\mathbf{AB} = \mathbf{I}$ implies

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e. $$\begin{bmatrix} 2x + 3u & 2y + 3v \\ x + 4u & y + 4v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ $\quad 2x + 3u = 1 \quad$ and $\quad 2y + 3v = 0$

$\quad x + 4u = 0 \qquad y + 4v = 1$

These simultaneous equations can be solved to find x, y, u, v and hence $\mathbf{B}$.

$$\mathbf{B} = \begin{bmatrix} 0.8 & -0.6 \\ -0.2 & 0.4 \end{bmatrix}$$

In general:

If $\mathbf{A}$ is a square matrix and if a matrix $\mathbf{B}$ can be found such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}$$

then $\mathbf{A}$ is said to be **invertible** and $\mathbf{B}$ is called the **inverse** of $\mathbf{A}$.

We leave it as an exercise to show that the inverse of an invertible matrix is unique.

We will denote the inverse of $\mathbf{A}$ by $\mathbf{A}^{-1}$.

For an invertible matrix $\mathbf{A}$, we have

$$\mathbf{AA}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$$

The inverse of a general 2×2 matrix

Now consider $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let $\mathbf{B} = \begin{bmatrix} x & y \\ u & v \end{bmatrix}$.

Then $\mathbf{AB} = \mathbf{I}$ implies

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e. $$\begin{bmatrix} ax + bu & ay + bv \\ cx + du & cy + dv \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ $\quad ax + bu = 1 \quad$ and $\quad ay + bv = 0$

$\quad cx + du = 0 \qquad cy + dv = 1$

These form two pairs of simultaneous equations, the first for x, u and the second for y, v.

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The first pair of equations gives

$$(ad - bc)x = d \qquad \text{(eliminating } u\text{)}$$

$$(bc - ad)u = c \qquad \text{(eliminating } x\text{)}$$

These two equations can be solved for x and u provided $ad - bc \neq 0$:

$$x = \frac{d}{ad - bc} \qquad \text{and} \qquad u = \frac{c}{bc - ad} = \frac{-c}{ad - bc}$$

In a similar way, we obtain

$$y = \frac{-b}{ad - bc} \qquad \text{and} \qquad v = \frac{-a}{bc - ad} = \frac{a}{ad - bc}$$

We have established the following result.

Inverse of a 2 × 2 matrix

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the inverse of $\mathbf{A}$ is given by

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \text{(provided } ad - bc \neq 0\text{)}$$

The determinant

The quantity $ad - bc$ that appears in the formula for $\mathbf{A}^{-1}$ has a name: the **determinant** of $\mathbf{A}$. This is denoted $\det(\mathbf{A})$.

Determinant of a 2 × 2 matrix

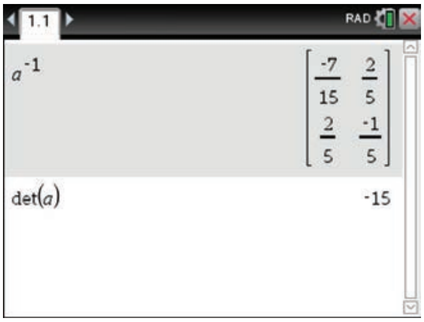
If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(\mathbf{A}) = ad - bc$.

A 2×2 matrix $\mathbf{A}$ has an inverse only if $\det(\mathbf{A}) \neq 0$.

Using the TI-Nspire

- The inverse of a matrix is obtained by raising the matrix to the power of -1.
- The determinant command (menu > **Matrix and Vector** > **Determinant**) is used as shown.

Hint: You can also type in det(a).



(Here a is the matrix $\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$ defined on page 430.)

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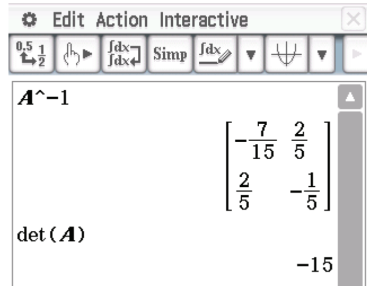
Using the Casio ClassPad

- To find the inverse matrix, type A^−1 and tap EXE.

Note: If the matrix has no inverse, then the calculator will give the message **Undefined**.

- To find the determinant, enter and highlight A. Select **Interactive** > **Matrix** > **Calculation** > **det**.



Example 11

For the matrix $\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$, find:

a $\det(\mathbf{A})$

b $\mathbf{A}^{-1}$

Solution

a $\det(\mathbf{A}) = 5 \times 1 - 2 \times 3 = -1$

b $\mathbf{A}^{-1} = \dfrac{1}{-1}\begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}$

Example 12

For the matrix $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix}$, find:

a $\det(\mathbf{A})$

b $\mathbf{A}^{-1}$

c $\mathbf{X}$, if $\mathbf{AX} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

d $\mathbf{Y}$, if $\mathbf{YA} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

Solution

a $\det(\mathbf{A}) = 3 \times 6 - 2 = 16$

b $\mathbf{A}^{-1} = \dfrac{1}{16}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}$

c $\mathbf{AX} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

Multiply both sides (on the left) by $\mathbf{A}^{-1}$.

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$$

$$\therefore \quad \mathbf{IX} = \mathbf{X} = \frac{1}{16}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 16 & 32 \\ 16 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

d $\mathbf{YA} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

Multiply both sides (on the right) by $\mathbf{A}^{-1}$.

$$\mathbf{YAA}^{-1} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}\mathbf{A}^{-1}$$

$$\therefore \quad \mathbf{YI} = \mathbf{Y} = \frac{1}{16}\begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 24 & 8 \\ 40 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{5}{2} & -\frac{1}{2} \end{bmatrix}$$

ISBN 978-1-316-63608-4

Section summary

- For a 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:
  - the inverse of $\mathbf{A}$ is given by
  $$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad (\text{if } ad - bc \neq 0)$$
  - the determinant of $\mathbf{A}$ is given by
  $$\det(\mathbf{A}) = ad - bc$$
- A 2×2 matrix $\mathbf{A}$ has an inverse only if $\det(\mathbf{A}) \neq 0$.

Exercise 15D

Skillsheet

Example 11

1 For the matrices $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 & -2 \\ 3 & 2 \end{bmatrix}$, find:

a $\det(\mathbf{A})$ **b** $\mathbf{A}^{-1}$ **c** $\det(\mathbf{B})$ **d** $\mathbf{B}^{-1}$

2 Find the inverse of each of the following invertible matrices (where k is any non-zero real number):

a $\begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ **d** $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

3 If the matrix $\mathbf{A}$ is invertible, show that the inverse is unique.

4 Let $\mathbf{A}$ and $\mathbf{B}$ be the invertible matrices $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$.

a Find $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$.

b Find $\mathbf{AB}$ and hence find, if possible, $(\mathbf{AB})^{-1}$.

c From $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$, find the products $\mathbf{A}^{-1}\mathbf{B}^{-1}$ and $\mathbf{B}^{-1}\mathbf{A}^{-1}$. What do you notice?

Example 12

5 Let $\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$.

a Find $\mathbf{A}^{-1}$. **b** If $\mathbf{AX} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$, find $\mathbf{X}$. **c** If $\mathbf{YA} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$, find $\mathbf{Y}$.

6 Let $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix}$.

a Find $\mathbf{X}$ such that $\mathbf{AX} + \mathbf{B} = \mathbf{C}$. **b** Find $\mathbf{Y}$ such that $\mathbf{YA} + \mathbf{B} = \mathbf{C}$.

7 Assume that $\mathbf{A}$ is a 2×2 matrix such that $a_{12} = a_{21} = 0$, $a_{11} \neq 0$ and $a_{22} \neq 0$. Show that $\mathbf{A}$ is invertible and find $\mathbf{A}^{-1}$.

8 Let $\mathbf{A}$ be an invertible 2×2 matrix, let $\mathbf{B}$ be a 2×2 matrix and assume that $\mathbf{AB} = \mathbf{O}$. Show that $\mathbf{B} = \mathbf{O}$.

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9 Find all 2×2 matrices such that $\mathbf{A}^{-1} = \mathbf{A}$.

10 For what values of a does the matrix $\mathbf{A} = \begin{bmatrix} a & 1 \\ 2 & a \end{bmatrix}$ not have an inverse?

11 Let n be a natural number and let

$$\mathbf{A} = \begin{bmatrix} \frac{1}{n} & \frac{1}{n+1} \\ \frac{1}{n+1} & \frac{1}{n+2} \end{bmatrix}$$

Show that all the entries of the inverse matrix $\mathbf{A}^{-1}$ are integers.

15E Solution of simultaneous equations using matrices

Inverse matrices can be used to solve some systems of simultaneous linear equations.

Simultaneous equations with a unique solution

For example, consider the pair of simultaneous equations

$$3x - 2y = 5$$
$$5x - 3y = 9$$

This can be written as a matrix equation:

$$\begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

Let $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix}$. The determinant of $\mathbf{A}$ is $3(-3) - (-2)5 = 1$.

Since the determinant is non-zero, the inverse matrix exists:

$$\mathbf{A}^{-1} = \begin{bmatrix} -3 & 2 \\ -5 & 3 \end{bmatrix}$$

Now multiply both sides of the original matrix equation on the left by $\mathbf{A}^{-1}$:

$$\begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$\mathbf{A}^{-1}\mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$\mathbf{I} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} 5 \\ 9 \end{bmatrix} \qquad \text{since } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

$$\therefore \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

This is the solution to the simultaneous equations. Check by substituting $x = 3$ and $y = 2$ into the two equations.

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Simultaneous equations without a unique solution

If a pair of simultaneous linear equations in two variables corresponds to two parallel lines, then a non-invertible matrix results.

For example, the following pair of simultaneous equations has no solution:

$$x + 2y = 3$$
$$-2x - 4y = 6$$

The associated matrix equation is

$$\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

The determinant of the matrix $\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$ is $1(-4) - 2(-2) = 0$, so the matrix has no inverse.

Example 13

Let $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{K} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Solve the system $\mathbf{AX} = \mathbf{K}$, where $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Solution

If $\mathbf{AX} = \mathbf{K}$, then

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$$
$$= \frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Example 14

Solve the following simultaneous equations:

$$3x - 2y = 6$$
$$7x + 4y = 7$$

Solution

The matrix equation is

$$\begin{bmatrix} 3 & -2 \\ 7 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

Let $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 7 & 4 \end{bmatrix}$. Then $\mathbf{A}^{-1} = \dfrac{1}{26}\begin{bmatrix} 4 & 2 \\ -7 & 3 \end{bmatrix}$.

Therefore

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{26}\begin{bmatrix} 4 & 2 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \end{bmatrix} = \frac{1}{26}\begin{bmatrix} 38 \\ -21 \end{bmatrix}$$

ISBN 978-1-316-63608-4

Exercise 15E

Skillsheet

1 Let $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$ and $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$. Solve the system $\mathbf{AX} = \mathbf{K}$, where:

Example 13

a $\mathbf{K} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

b $\mathbf{K} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

Example 14

2 Use matrices to solve each of the following pairs of simultaneous equations:

a $-2x + 4y = 6$
$3x + y = 1$

b $-x + 2y = -1$
$-x + 4y = 2$

3 Use matrices to find the point of intersection of the lines given by the equations $2x - 3y = 7$ and $3x + y = 5$.

4 Two children spend their pocket money buying some books and some CDs. One child spends \$120 and buys four books and four CDs. The other child spends \$114 and buys three CDs and five books. Set up a system of simultaneous equations and use matrices to find the cost of a single book and a single CD.

5 Consider the system

$$2x - 3y = 3$$
$$4x - 6y = 6$$

a Write this system in matrix form, as $\mathbf{AX} = \mathbf{K}$.

b Is $\mathbf{A}$ an invertible matrix?

c Can any solutions be found for this system of equations?

d How many pairs does the solution set contain?

6 Suppose that $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{X}$ are 2×2 matrices and that both $\mathbf{A}$ and $\mathbf{B}$ are invertible. Solve the following for $\mathbf{X}$:

a $\mathbf{AX} = \mathbf{C}$

b $\mathbf{ABX} = \mathbf{C}$

c $\mathbf{AXB} = \mathbf{C}$

d $\mathbf{A}(\mathbf{X} + \mathbf{B}) = \mathbf{C}$

e $\mathbf{AX} + \mathbf{B} = \mathbf{C}$

f $\mathbf{XA} + \mathbf{B} = \mathbf{A}$

ISBN 978-1-316-63608-4

Chapter summary

AS

Nrich

- A **matrix** is a rectangular array of numbers.
- Two matrices **A** and **B** are equal when:
  - they have the same number of rows and the same number of columns, and
  - they have the same entry at corresponding positions.
- The **size** of a matrix is described by specifying the number of rows and the number of columns. An $m \times n$ matrix has m rows and n columns.
- Addition is defined for two matrices only when they have the same size. The sum is found by adding corresponding entries.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

  Subtraction is performed in a similar way.
- If **A** is any matrix and k is a real number, then the matrix $k\mathbf{A}$ is obtained by multiplying each entry of **A** by k.

$$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

- If **A** is an $m \times n$ matrix and **B** is an $n \times r$ matrix, then the product **AB** is the $m \times r$ matrix whose entries are determined as follows:

  To find the entry in row i and column j of **AB**, single out row i in matrix **A** and column j in matrix **B**. Multiply the corresponding entries from the row and column and then add up the resulting products.

  Note that the product **AB** is defined only if the number of columns of **A** is the same as the number of rows of **B**.
- If **A** is a square matrix and if a matrix **B** can be found such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, then **A** is said to be **invertible** and **B** is called the **inverse** of **A**.
- For a 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:
  - the inverse of **A** is given by

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad (\text{if } ad - bc \neq 0)$$

  - the determinant of **A** is given by

$$\det(\mathbf{A}) = ad - bc$$

- A 2×2 matrix **A** has an inverse if and only if $\det(\mathbf{A}) \neq 0$.
- Simultaneous equations can be solved using inverse matrices. For example, the system of equations

$$ax + by = c$$
$$dx + ey = f$$

  can be written as $\begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$ and solved using $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}^{-1}\begin{bmatrix} c \\ f \end{bmatrix}$.

ISBN 978-1-316-63608-4

Short-answer questions

1 If $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, find:

a $(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B})$ **b** $\mathbf{A}^2 - \mathbf{B}^2$

2 Find all possible matrices $\mathbf{A}$ which satisfy the equation $\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}\mathbf{A} = \begin{bmatrix} 8 \\ 16 \end{bmatrix}$.

3 Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & -1 & 2 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 2 & 4 \end{bmatrix}$ and $\mathbf{E} = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}$.

a State whether or not each of the following products exists: **AB**, **AC**, **CD**, **BE**.

b Find **DA** and $\mathbf{A}^{-1}$.

4 If $\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \\ -5 & 1 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -4 \\ 1 & -6 \\ 3 & -8 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find **AB** and $\mathbf{C}^{-1}$.

5 Find the 2×2 matrix $\mathbf{A}$ such that $\mathbf{A}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 12 & 14 \end{bmatrix}$.

6 If $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$, find $\mathbf{A}^2$ and hence find $\mathbf{A}^{-1}$.

7 If the matrix $\begin{bmatrix} 1 & 2 \\ 4 & x \end{bmatrix}$ does not have an inverse, find the value of x.

8 **a** If $\mathbf{M} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, find:

i $\mathbf{MM} = \mathbf{M}^2$ **ii** $\mathbf{MMM} = \mathbf{M}^3$ **iii** $\mathbf{M}^{-1}$

b Find x and y, given that $\mathbf{M}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

Multiple-choice questions

1 The matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ -2 & 3 \\ 3 & 0 \end{bmatrix}$ has size

A 8 **B** 4×2 **C** 2×4 **D** 1×4 **E** 3×4

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2 If $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & -3 & 4 \\ -1 & -3 & -1 \end{bmatrix}$, then $\mathbf{A} + \mathbf{B} =$

A $\begin{bmatrix} 3 & -3 \\ -2 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 3 & 4 \\ -2 & 2 \end{bmatrix}$ **C** $\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$ **D** $\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$ **E** undefined

3 If $\mathbf{C} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 0 & -2 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 1 & -3 & 1 \\ 2 & 3 & -1 \end{bmatrix}$, then $\mathbf{D} - \mathbf{C} =$

A $\begin{bmatrix} 1 & 0 & 0 \\ -1 & -3 & -1 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & -6 & 4 \\ -2 & 0 & -4 \end{bmatrix}$ **C** $\begin{bmatrix} -1 & 0 & 0 \\ 1 & 3 & 1 \end{bmatrix}$

D $\begin{bmatrix} 1 & -6 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ **E** undefined

4 If $\mathbf{M} = \begin{bmatrix} -4 & 0 \\ -2 & -6 \end{bmatrix}$, then $-\mathbf{M} =$

A $\begin{bmatrix} -4 & 0 \\ -2 & -6 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & -4 \\ -6 & -2 \end{bmatrix}$ **C** $\begin{bmatrix} 4 & 0 \\ -2 & -6 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 4 \\ 6 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} 4 & 0 \\ 2 & 6 \end{bmatrix}$

5 If $\mathbf{M} = \begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} 0 & 4 \\ 3 & 0 \end{bmatrix}$, then $2\mathbf{M} - 2\mathbf{N} =$

A $\begin{bmatrix} 0 & 0 \\ -9 & 2 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & -2 \\ -6 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & -4 \\ -12 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 4 \\ 12 & -2 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & 2 \\ 6 & -1 \end{bmatrix}$

6 If both $\mathbf{A}$ and $\mathbf{B}$ are $m \times n$ matrices, where $m \neq n$, then $\mathbf{A} + \mathbf{B}$ is

A an $m \times n$ matrix **B** an $m \times m$ matrix **C** an $n \times n$ matrix

D a $2m \times 2n$ matrix **E** not defined

7 If $\mathbf{P}$ is an $m \times n$ matrix and $\mathbf{Q}$ is an $n \times p$ matrix, where $m \neq p$, then $\mathbf{QP}$ is

A an $n \times n$ matrix **B** an $m \times p$ matrix **C** an $n \times p$ matrix

D an $m \times n$ matrix **E** not defined

8 The determinant of the matrix $\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$ is

A 4 **B** 0 **C** −4 **D** 1 **E** 2

9 The inverse of the matrix $\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$ is

A −1 **B** $\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$

10 If $\mathbf{M} = \begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}$, then $\mathbf{NM} =$

A $\begin{bmatrix} 0 & -4 \\ -9 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} -4 & -2 \\ 2 & -8 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 4 \\ 9 & 1 \end{bmatrix}$ **D** $\begin{bmatrix} -6 & 2 \\ -3 & -5 \end{bmatrix}$ **E** $\begin{bmatrix} 6 & -2 \\ -3 & -5 \end{bmatrix}$

ISBN 978-1-316-63608-4

Review

Extended-response questions

1 **a** Consider the system of equations

$$2x - 3y = 3$$
$$4x + y = 5$$

i Write this system in matrix form, as $\mathbf{AX} = \mathbf{K}$.

ii Find $\det(\mathbf{A})$ and $\mathbf{A}^{-1}$.

iii Solve the system of equations.

iv Interpret your solution geometrically.

b Consider the system of equations

$$2x + y = 3$$
$$4x + 2y = 8$$

i Write this system in matrix form, as $\mathbf{AX} = \mathbf{K}$.

ii Find $\det(\mathbf{A})$ and explain why $\mathbf{A}^{-1}$ does not exist.

c Interpret your findings in part **b** geometrically.

2 The final grades for Physics and Chemistry are made up of three components: tests, practical work and exams. Each semester, a mark out of 100 is awarded for each component. Wendy scored the following marks in the three components for Physics:

Semester 1 tests 79, practical work 78, exam 80
Semester 2 tests 80, practical work 78, exam 82

a Represent this information in a 2×3 matrix.

To calculate the final grade for each semester, the three components are weighted: tests are worth 20%, practical work is worth 30% and the exam is worth 50%.

b Represent this information in a 3×1 matrix.

c Calculate Wendy's final grade for Physics in each semester.

Wendy also scored the following marks in the three components for Chemistry:

Semester 1 tests 86, practical work 82, exam 84
Semester 2 tests 81, practical work 80, exam 70

d Calculate Wendy's final grade for Chemistry in each semester.

Students who gain a total score of 320 or more for Physics and Chemistry over the two semesters are awarded a Certificate of Merit in Science.

e Will Wendy be awarded a Certificate of Merit in Science?

She asks her teacher to re-mark her Semester 2 Chemistry exam, hoping that she will gain the necessary marks to be awarded a Certificate of Merit.

f How many extra marks on the exam does she need?

ISBN 978-1-316-63608-4

3 A company runs computing classes and employs full-time and part-time teaching staff, as well as technical staff, catering staff and cleaners. The number of staff employed depends on demand from term to term.
In one year the company employed the following teaching staff:

Term 1 full-time 10, part-time 2
Term 2 full-time 8, part-time 4
Term 3 full-time 8, part-time 8
Term 4 full-time 6, part-time 10

a Represent this information in a 4×2 matrix.

Full-time teachers are paid \$70 per hour and part-time teachers are paid \$60 per hour.

b Represent this information in a 2×1 matrix.

c Calculate the cost per hour to the company for teaching staff for each term.

In the same year the company also employed the following support staff:

Term 1 technical 2, catering 2, cleaning 1
Term 2 technical 2, catering 2, cleaning 1
Term 3 technical 3, catering 4, cleaning 2
Term 4 technical 3, catering 4, cleaning 2

d Represent this information in a 4×3 matrix.

Technical staff are paid \$60 per hour, catering staff are paid \$55 per hour and cleaners are paid \$40 per hour.

e Represent this information in a 3×1 matrix.

f Calculate the cost per hour to the company for support staff for each term.

g Calculate the total cost per hour to the company for teaching and support staff for each term.

4 Suppose that $\mathbf{A}$ and $\mathbf{B}$ are 2×2 matrices.

a Prove that $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

b Hence prove that if both $\mathbf{A}$ and $\mathbf{B}$ are invertible, then $\mathbf{AB}$ is invertible.

ISBN 978-1-316-63608-4

Cambridge University Press

Chapter 16

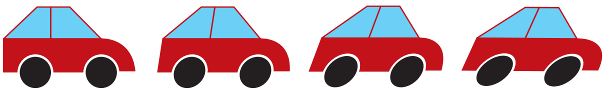
16 Transformations of the plane

Objectives

- To define **linear transformations**.
- To represent a linear transformation as a 2×2 matrix.
- To study the effect of important transformations, including **dilations**, **reflections**, **rotations**, **shears** and **translations**.
- To investigate **compositions** and **inverses** of transformations.
- To investigate the connection between the **determinant** of a transformation matrix and area.
- To investigate the effect of transformations on regions of the plane, including points, shapes and graphs.

Modern animations are largely created with the use of computers. Many basic visual effects can be understood in terms of simple transformations of the plane.

For example, suppose that an animator wants to give the car below a sense of movement. This can be achieved by gradually tilting the car so that it leans forwards. We will see later how this can easily be done using a transformation called a **shear**.



Aside from computer graphics, linear transformations play an important role in many diverse fields such as mathematics, physics, engineering and economics.

ISBN 978-1-316-63608-4

16A Linear transformations

Each point in the plane can be denoted by an ordered pair (x, y). The set of all ordered pairs is often called the **Cartesian plane**: $\mathbb{R}^2 = \{ (x, y) : x, y \in \mathbb{R} \}$.

A **transformation** of the plane maps each point (x, y) in the plane to a new point (x', y'). We say that (x', y') is the **image** of (x, y).

We will mainly be concerned with **linear transformations**, which have rules of the form

$$(x, y) \to (ax + by, cx + dy)$$

Example 1

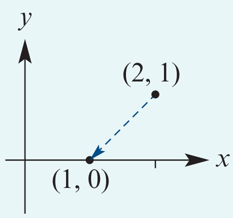
Find the image of the point (2, 1) under the transformation with rule

$$(x, y) \to (3x - 5y, 2x - 4y)$$

Solution

We let $x = 2$ and $y = 1$, giving

$$(2, 1) \to (3 \times 2 - 5 \times 1, 2 \times 2 - 4 \times 1) = (1, 0)$$



Matrices and linear transformations

Each ordered pair can also be written as a 2×1 matrix, which we will call a **column vector**:

$$(x, y) = \begin{bmatrix} x \\ y \end{bmatrix}$$

This is a very useful observation, since we can now easily perform the linear transformation $(x, y) \to (ax + by, cx + dy)$ by using matrix multiplication:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Example 2

a Find the matrix of the linear transformation with rule $(x, y) \to (x - 2y, 3x + y)$.

b Use the matrix to find the image of the point (2, 3) under the transformation.

Solution

a $\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$

b $\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \times 2 - 2 \times 3 \\ 3 \times 2 + 1 \times 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 9 \end{bmatrix}$

Therefore the image of (2, 3) is (−4, 9).

Explanation

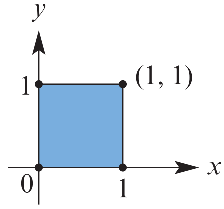
The rows of the matrix are given by the coefficients of x and y.

We write the point (2, 3) as a column vector and multiply by the transformation matrix.

ISBN 978-1-316-63608-4

▶ Transforming the unit square

The **unit square** is the square with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$. The effect of a linear transformation can often be demonstrated by studying its effect on the unit square.



Example 3

A linear transformation is represented by the matrix $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$.

a Find the image of the unit square under this transformation.

b Sketch the unit square and its image.

Solution

a We could find the images of the four vertices of the square one at a time:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

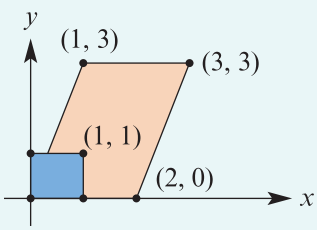
However, this can be done in a single step by multiplying the transformation matrix by a rectangular matrix whose columns are the coordinates of each vertex of the square:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

The columns of the result give the images of the vertices:

$(0, 0), \quad (2, 0), \quad (1, 3), \quad (3, 3)$

b The unit square is shown in blue and its image in red.



▶ Mapping the standard unit vectors

Let's express the points $(1, 0)$ and $(0, 1)$ as column vectors:

$$(1, 0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad (0, 1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

These are called the **standard unit vectors** in $\mathbb{R}^2$.

We now consider the images of these points under the transformation with matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

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We have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} = \text{first column of the matrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} = \text{second column of the matrix}$$

To find the matrix of a linear transformation:

- The first column is the image of $(1, 0)$, written as a column vector.
- The second column is the image of $(0, 1)$, written as a column vector.

This observation allows us to write down the matrix of a linear transformation given just two pieces of information.

Example 4

A linear transformation maps the points $(1, 0)$ and $(0, 1)$ to the points $(1, 1)$ and $(-2, 3)$ respectively.

a Find the matrix of the transformation.

b Find the image of the point $(-3, 4)$.

Solution

a $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

b $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} -11 \\ 9 \end{bmatrix}$

Therefore $(-3, 4) \to (-11, 9)$.

Explanation

The image of $(1, 0)$ is $(1, 1)$, and the image of $(0, 1)$ is $(-2, 3)$. We write these images as the columns of a matrix.

Write the point $(-3, 4)$ as a column vector and multiply by the transformation matrix.

Section summary

- A **transformation** maps each point (x, y) in the plane to a new point (x', y').
- A **linear transformation** is defined by a rule of the form $(x, y) \to (ax + by, cx + dy)$.
- Linear transformations can be represented using matrix multiplication:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

- The **unit square** has vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$. The effect of a linear transformation can be seen by looking at the image of the unit square.
- In the matrix of a linear transformation:
  - the first column is the image of $(1, 0)$, written as a column vector
  - the second column is the image of $(0, 1)$, written as a column vector.

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Exercise 16A

Skillsheet
Example 1

1 Find the image of the point $(2, -4)$ under the transformation with rule:

a $(x, y) \to (x + y, x - y)$ **b** $(x, y) \to (2x + 3y, 3x - 4y)$

c $(x, y) \to (3x - 5y, x)$ **d** $(x, y) \to (y, -x)$

Example 2

2 Find the image of the point $(2, 3)$ under the linear transformation with matrix:

a $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ **d** $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

3 Find the matrix of the linear transformation defined by the rule:

a $(x, y) \to (2x + 3y, 4x + 5y)$ **b** $(x, y) \to (11x - 3y, 3x - 8y)$

c $(x, y) \to (2x, x - 3y)$ **d** $(x, y) \to (y, -x)$

Example 3

4 Find and sketch the image of the unit square under the linear transformation represented by the matrix:

a $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ **d** $\begin{bmatrix} -1 & 3 \\ 2 & -1 \end{bmatrix}$

5 Find the image of the triangle with vertices $(1, 1)$, $(1, 2)$ and $(2, 1)$ under the linear transformation represented by the matrix $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$.

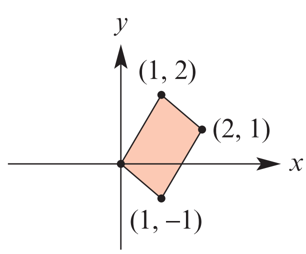
Example 4

6 Find the matrix of the linear transformation that maps the points $(1, 0)$ and $(0, 1)$ to the points $(3, 4)$ and $(5, 6)$ respectively. Hence find the image of the point $(-2, 4)$.

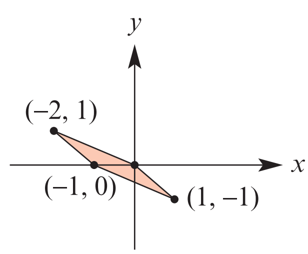
7 Find the matrix of the linear transformation that maps the points $(1, 0)$ and $(0, 1)$ to the points $(-3, 2)$ and $(1, -1)$ respectively. Hence find the image of the point $(2, 3)$.

8 Find a matrix that transforms the unit square to each of the following parallelograms.
Note: There are two possible answers for each part.

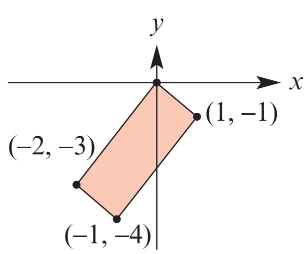
a



b



c



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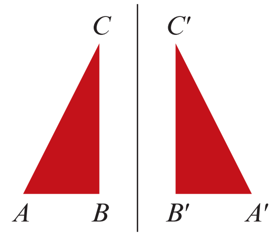
16B Geometric transformations

We now look at various important transformations that are geometric in nature.

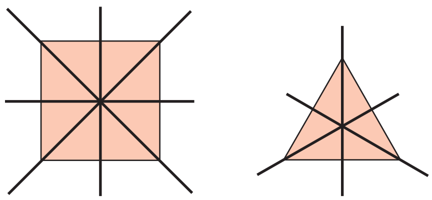
▶ Reflections

A **reflection** in a line ℓ maps each point in the plane to its mirror image on the other side of the line. The point A and its image A' are the same distance from ℓ and the line AA' is perpendicular to ℓ.



These transformations are important for studying figures with **reflective symmetry**, that is, figures that look the same when reflected in a **line of symmetry**.

A square has four lines of symmetry, while an equilateral triangle has just three.

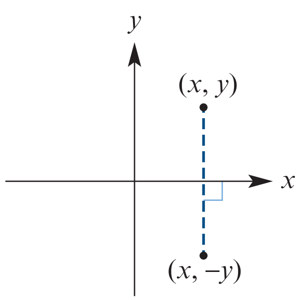


Note: A reflection is an example of a transformation that does not change lengths. Such a transformation is called an **isometry**.

Reflection in the x-axis

A reflection in the x-axis is defined by

$$(x, y) \to (x, -y)$$



So if (x', y') is the image of the point (x, y), then

$$x' = x \quad \text{and} \quad y' = -y$$

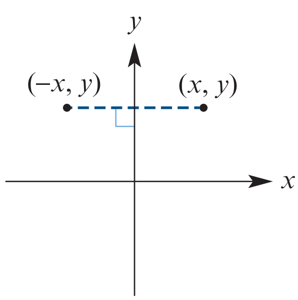
This transformation can also be represented using matrix multiplication:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Reflection in the y-axis

A reflection in the y-axis is defined by

$$(x, y) \to (-x, y)$$



So if (x', y') is the image of the point (x, y), then

$$x' = -x \quad \text{and} \quad y' = y$$

Once again, this transformation can be represented using matrix multiplication:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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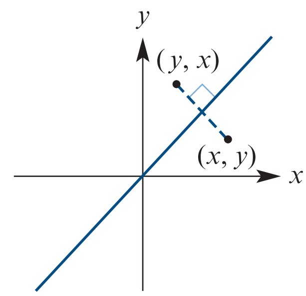
Reflection in the line $y = x$

If the point (x, y) is reflected in the line $y = x$, then it is mapped to the point (y, x). So if (x', y') is the image of (x, y), then

$$x' = y \quad \text{and} \quad y' = x$$

Expressing this using matrix multiplication gives

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



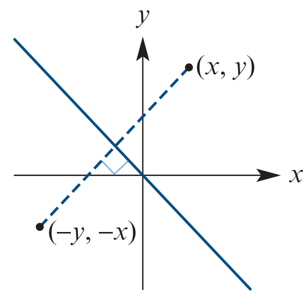
Reflection in the line $y = -x$

If the point (x, y) is reflected in the line $y = -x$, it is mapped to $(-y, -x)$. So if (x', y') is the image of (x, y), then

$$x' = -y \quad \text{and} \quad y' = -x$$

Expressing this using matrix multiplication gives

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Transformation	Rule	Matrix
Reflection in the x-axis	$x' = 1x + 0y$ $y' = 0x - 1y$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection in the y-axis	$x' = -1x + 0y$ $y' = 0x + 1y$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection in the line $y = x$	$x' = 0x + 1y$ $y' = 1x + 0y$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Reflection in the line $y = -x$	$x' = 0x - 1y$ $y' = -1x + 0y$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

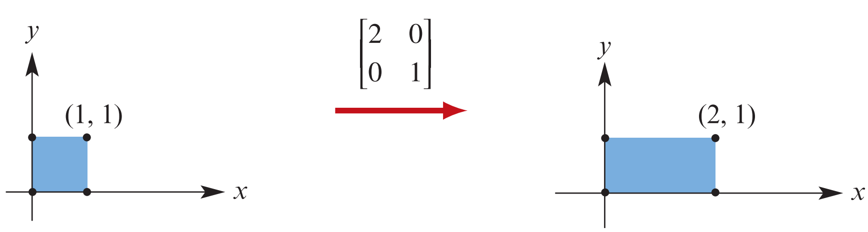
▶ Dilations

Dilation from the y-axis

A dilation from the y-axis is a transformation of the form

$$(x, y) \to (cx, y)$$

where $c > 0$. The x-coordinate is scaled by a factor of c, but the y-coordinate is unchanged.



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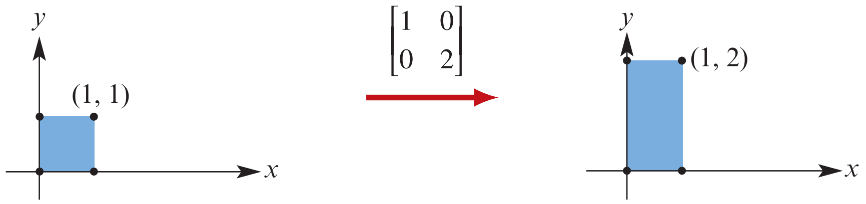
Dilation from the x-axis

Likewise, a dilation from the x-axis is a transformation of the form

$$(x, y) \to (x, cy)$$

where $c > 0$. The y-coordinate is scaled by a factor of c, but the x-coordinate is unchanged.



Dilation from the x- and y-axes

We can also simultaneously scale along the x- and y-axes using the transformation

$$(x, y) \to (cx, dy)$$

with scale factors $c > 0$ and $d > 0$.

Transformation	Rule	Matrix
Dilation from the y-axis	$x' = cx + 0y$ $y' = 0x + 1y$	$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$
Dilation from the x-axis	$x' = 1x + 0y$ $y' = 0x + cy$	$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$
Dilation from the x- and y-axes	$x' = cx + 0y$ $y' = 0x + dy$	$\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$

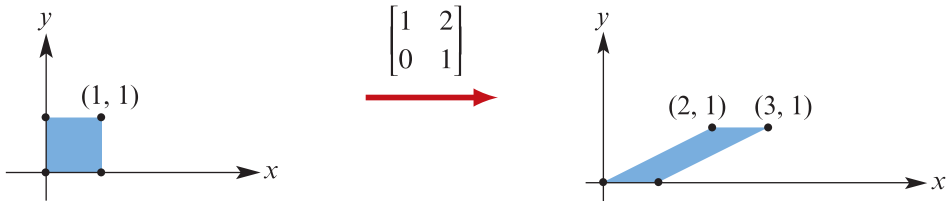
▶ Shears

Shear parallel to the x-axis

A shear parallel to the x-axis is a transformation of the form

$$(x, y) \to (x + cy, y)$$

Notice that each point is moved in the x-direction by an amount proportional to the distance from the x-axis. This means that the unit square is tilted in the x-direction.



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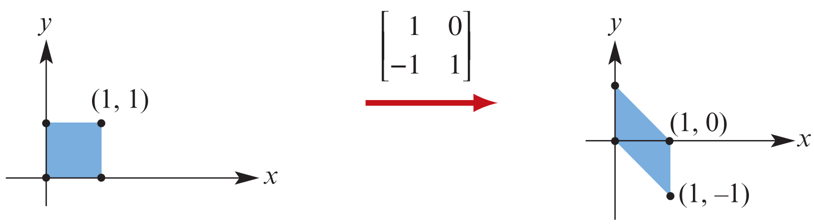
Shear parallel to the y-axis

A shear parallel to the y-axis is a transformation of the form

$$(x, y) \to (x, cx + y)$$

Here, each point is moved in the y-direction by an amount proportional to the distance from the y-axis. Now the unit square is tilted in the y-direction.

Note that if $c < 0$, then we obtain a shear in the negative direction.



Transformation	Rule	Matrix
Shear parallel to the x-axis	$x' = 1x + cy$ $y' = 0x + 1y$	$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$
Shear parallel to the y-axis	$x' = 1x + 0y$ $y' = cx + 1y$	$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$

► Projections

The transformation defined by

$$(x, y) \to (x, 0)$$

will project the point (x, y) onto the x-axis.

Likewise, the transformation defined by

$$(x, y) \to (0, y)$$

will project the point (x, y) onto the y-axis.

Transformation	Rule	Matrix
Projection onto the x-axis	$x' = 1x + 0y$ $y' = 0x + 0y$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Projection onto the y-axis	$x' = 0x + 0y$ $y' = 0x + 1y$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Projections are an important class of transformations. For example, the image on a television screen is the projection of a three-dimensional scene onto a two-dimensional surface.

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Example 5

Find the image of the point (3, 4) under each of the following transformations:

a reflection in the y-axis

b dilation of factor 2 from the y-axis

c shear of factor 4 parallel to the x-axis

d projection onto the y-axis

Solution

a $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$(3, 4) \to (-3, 4)$

b $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

$(3, 4) \to (6, 4)$

c $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 19 \\ 4 \end{bmatrix}$

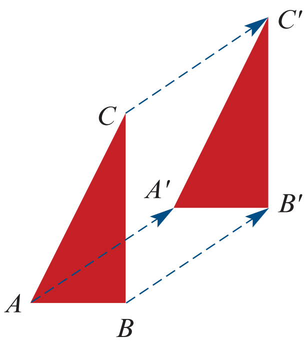
$(3, 4) \to (19, 4)$

d $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

$(3, 4) \to (0, 4)$

▶ Translations

A **translation** moves a figure so that every point in the figure moves in the same direction and over the same distance.



A translation of a units in the x-direction and b units in the y-direction is defined by the rule

$$(x, y) \to (x + a, y + b)$$

This can be expressed using vector addition:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

Note: Translations cannot be represented using matrix multiplication. To see this, note that matrix multiplication will always map the point (0, 0) to itself. Therefore, there is no matrix that will translate the point (0, 0) to (a, b), unless $a = b = 0$.

Example 6

Find the rule for a translation of 2 units in the x-direction and −1 units in the y-direction, and sketch the image of the unit square under this translation.

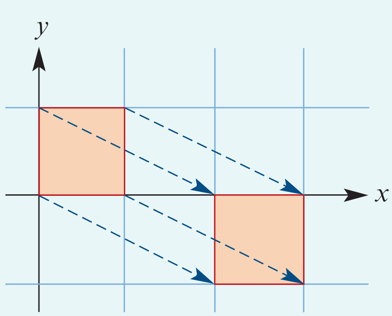
Solution

Using vector addition, this translation can be defined by the rule

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} x + 2 \\ y - 1 \end{bmatrix}$$

or equivalently

$$x' = x + 2 \quad \text{and} \quad y' = y - 1$$



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Section summary

- Important geometric transformation matrices are summarised in the table below.

Transformation	Matrix	Transformation	Matrix
Reflection in the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Reflection in the y-axis	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection in the line $y = x$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	Reflection in the line $y = -x$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
Dilation from the y-axis	$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$	Dilation from the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$
Shear parallel to the x-axis	$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$	Shear parallel to the y-axis	$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$
Projection onto the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	Projection onto the y-axis	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

- A translation of a units in the x-direction and b units in the y-direction is defined by the rule $(x, y) \to (x + a, y + b)$. This can be expressed using vector addition:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

Exercise 16B

Skillsheet

Example 5

1 For each of the transformations described below:

i find the matrix of the transformation

ii sketch the image of the unit square under this transformation.

a dilation of factor 2 from the x-axis

b dilation of factor 3 from the y-axis

c shear of factor 3 parallel to the x-axis

d shear of factor -1 parallel to the y-axis

e reflection in the x-axis

f reflection in the line $y = -x$

Example 6

2 For each of the translations described below:

i find the rule for the translation using column vectors

ii sketch the image of the unit square under this translation.

a translation of 2 units in the x-direction

b translation of -3 units in the y-direction

c translation of -2 units in the x-direction and -4 units in the y-direction

d translation by the vector $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

e translation by the vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

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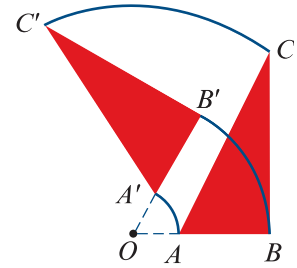
16C Rotations and general reflections

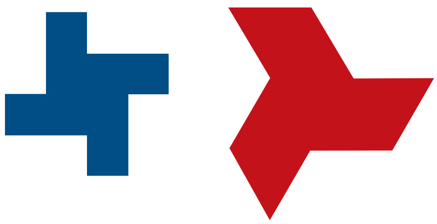
▶ Rotations



A **rotation** turns an object about a point, but keeps its distance to the point fixed. A rotation does not change lengths, and so is another example of an isometry.

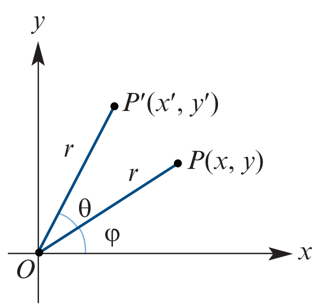
Rotations are important for studying figures with **rotational symmetry**, that is, figures that look the same when rotated through a certain angle.

These two figures have rotational symmetry, but no reflective symmetry.



Finding the rotation matrix

Consider the transformation that rotates each point in the plane about the origin by θ degrees anticlockwise. We will show that this is a linear transformation and find its matrix.



Let O be the origin and let $P(x, y)$ be a point in the plane.

Then we can write

$$x = r\cos\varphi \quad \text{and} \quad y = r\sin\varphi$$

where r is the distance OP and φ is the angle between OP and the positive direction of the x-axis.

Now let $P'(x', y')$ be the image of $P(x, y)$ under a rotation about O by angle θ anticlockwise.

As $OP' = r$, we can use the addition formulas to show that

$$\begin{aligned} x' &= r\cos(\varphi + \theta) \\ &= r\cos\varphi\,\cos\theta - r\sin\varphi\,\sin\theta \\ &= x\cos\theta - y\sin\theta \end{aligned}$$

and

$$\begin{aligned} y' &= r\sin(\varphi + \theta) \\ &= r\sin\varphi\,\cos\theta + r\cos\varphi\,\sin\theta \\ &= y\cos\theta + x\sin\theta \end{aligned}$$

Writing this using matrix multiplication gives

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

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Example 7

Find the matrix that represents a rotation of the plane about the origin by:

a 90° anticlockwise

b 45° clockwise.

Solution

a $\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Explanation

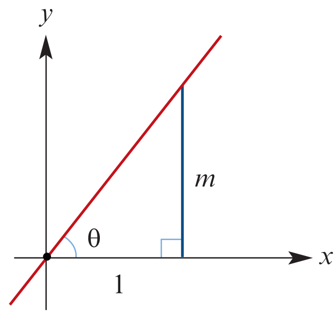
An anticlockwise rotation means that we let $\theta = 90^\circ$ in the formula for the rotation matrix.

b $\begin{bmatrix} \cos(-45^\circ) & -\sin(-45^\circ) \\ \sin(-45^\circ) & \cos(-45^\circ) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

A clockwise rotation means that we let $\theta = -45^\circ$ in the formula for the rotation matrix.

▶ Reflection in the line $y = mx$

Reflection in a line that passes through the origin is also a linear transformation. We will find the matrix that will reflect the point (x, y) in the line $y = mx$.

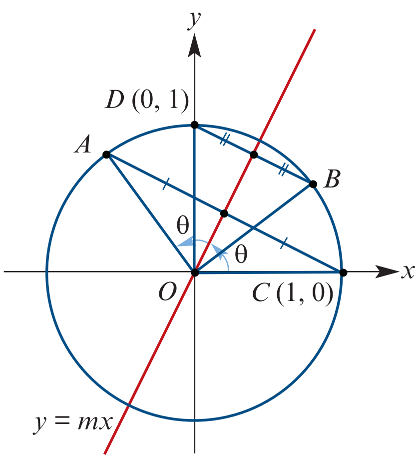


Let's suppose that the angle between the positive direction of the x-axis and the line $y = mx$ is θ. Then $\tan\theta = m$ and so

$$y = mx = x\tan\theta$$

Finding the reflection matrix

We will use the fact that the first column of the required matrix will be the image A of $C(1, 0)$, written as a column vector, and the second column will be the image B of $D(0, 1)$, written as a column vector.



Since $\angle AOC = 2\theta$, we have

$$(1, 0) \to (\cos(2\theta), \sin(2\theta))$$

Moreover, since $\angle BOC = 2\theta - 90^\circ$, we have

$$\begin{aligned}(0, 1) &\to (\cos(2\theta - 90^\circ), \sin(2\theta - 90^\circ)) \\ &= (\sin(2\theta), -\cos(2\theta))\end{aligned}$$

Writing these images as column vectors gives the reflection matrix:

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

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Example 8

a Find the matrix that will reflect the point (x, y) in the line through the origin at an angle of 30° to the positive direction of the x-axis.

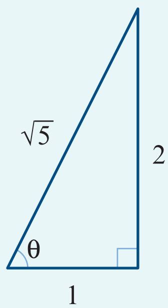
b Find the matrix that will reflect the point (x, y) in the line $y = 2x$.

Solution

a We simply let $\theta = 30°$, and so the required reflection matrix is

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} = \begin{bmatrix} \cos 60° & \sin 60° \\ \sin 60° & -\cos 60° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{bmatrix}$$

b Since $\tan\theta = 2 = \frac{2}{1}$, we draw a right-angled triangle with opposite and adjacent lengths 2 and 1 respectively.



Pythagoras' theorem gives the hypotenuse as $\sqrt{5}$. Therefore

$$\cos\theta = \frac{1}{\sqrt{5}} \quad \text{and} \quad \sin\theta = \frac{2}{\sqrt{5}}$$

We then use the double angle formulas to show that

$$\cos(2\theta) = 2\cos^2\theta - 1 = 2\left(\frac{1}{\sqrt{5}}\right)^2 - 1 = \frac{2}{5} - 1 = -\frac{3}{5}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta = 2 \times \frac{2}{\sqrt{5}} \times \frac{1}{\sqrt{5}} = \frac{4}{5}$$

Therefore the required reflection matrix is

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

Section summary

Rotation matrix

The matrix that will rotate the plane about the origin by angle θ anticlockwise is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Reflection matrix

The matrix that will reflect the plane in the line $y = mx = x\tan\theta$ is

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

Exercise 16C

Example 7

1 Find the matrix for each of the following rotations about the origin:

a 270° anticlockwise

b 30° anticlockwise

c 60° clockwise

d 135° clockwise

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2 Find the image of the point $(2, 3)$ under a rotation about the origin by:

a $90°$ anticlockwise **b** $45°$ clockwise.

Example 8a

3 Find the matrix that will reflect the point (x, y) in the line through the origin that makes an angle with the positive direction of the x-axis of:

a $45°$ **b** $60°$ **c** $-30°$ **d** $15°$

Example 8b

4 Find the matrix that will reflect the point (x, y) in the line:

a $y = 3x$ **b** $y = 5x$ **c** $y = \frac{2x}{3}$ **d** $y = -3x$

5 **a** Find a formula (in terms of m) for the matrix that will reflect the point (x, y) in the line $y = mx$.

b Hence find the image of $(1, 1)$ when reflected in the line $y = 6x$.

6 The unit square is rotated about the origin by $45°$ anticlockwise.

a Find the matrix of this transformation.

b Draw the unit square and its image on the same set of axes.

c Find the area of the overlapping region.

7 The point $A(1, 0)$ is rotated to point B through angle $120°$ anticlockwise. The point $A(1, 0)$ is also rotated to point C through angle $240°$ anticlockwise.

a Find the coordinates of B and C.

b What sort of triangle is ABC?

c Triangle ABC has three lines of symmetry. Find the equations of all three lines.

16D Composition of transformations

▶ Composition as matrix multiplication

The matrix representation of linear transformations makes it easy to find the effect of one transformation followed by another. Suppose that we would like to:

1. reflect the point (x, y) in the y-axis
2. then dilate the result by a factor of 2 from the x-axis.

To reflect the point (x, y), we would first evaluate

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

To dilate the result by a factor of 2 from the x-axis, we would then multiply

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Therefore the matrix that will achieve both transformations is the product

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

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If matrices **A** and **B** correspond to two different linear transformations, then:

- **AB** is the matrix of transformation **B** followed by **A**
- **BA** is the matrix of transformation **A** followed by **B**.

Example 9

Find the matrix that corresponds to:

a a reflection in the x-axis and then a rotation about the origin by 90° anticlockwise

b a rotation about the origin by 90° anticlockwise and then a reflection in the x-axis.

Solution

a $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

b $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

Explanation

A reflection in the x-axis has matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

An anticlockwise rotation by 90° has matrix

$$\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

We then multiply these two matrices together in the correct order.

Note: In this example, we get a different matrix when the same two transformations take place in reverse order. This should not be a surprise, as matrix multiplication is not commutative in general.

► Compositions involving translations

Example 10

a Find the rule for the transformation that will reflect (x, y) in the x-axis and then translate the result by the vector $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

b Find the rule for the transformation if the translation takes place before the reflection.

Solution

a
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$
$$= \begin{bmatrix} x \\ -y \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$
$$= \begin{bmatrix} x-3 \\ -y+4 \end{bmatrix}$$

Therefore the transformation is $(x, y) \to (x-3, -y+4)$.

b
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix}\right)$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x-3 \\ y+4 \end{bmatrix}$$
$$= \begin{bmatrix} x-3 \\ -y-4 \end{bmatrix}$$

Therefore the transformation is $(x, y) \to (x-3, -y-4)$.

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Section summary

If matrices **A** and **B** correspond to two different linear transformations, then:

- **AB** is the matrix of transformation **B** followed by **A**
- **BA** is the matrix of transformation **A** followed by **B**.

The order is important, as matrix multiplication is not commutative in general.

Exercise 16D

Skillsheet

Example 9

1 Find the matrix that represents a reflection in the y-axis followed by a dilation of factor 3 from the x-axis.

2 Find the matrix that represents a rotation about the origin by 90° anticlockwise followed by a reflection in the x-axis.

3 **a** Find the matrix that represents a reflection in the x-axis followed by a reflection in the y-axis.

b Show that this matrix corresponds to a rotation about the origin by 180°.

4 Consider these two transformations:

- T_1: A reflection in the x-axis.
- T_2: A dilation of factor 2 from the y-axis.

a Find the matrix of T_1 followed by T_2. **b** Find the matrix of T_2 followed by T_1.

c Does the order of transformation matter in this instance?

5 Consider these two transformations:

- T_1: A rotation about the origin by 90° clockwise.
- T_2: A reflection in the line $y = x$.

a Find the matrix of T_1 followed by T_2. **b** Find the matrix of T_2 followed by T_1.

c Does the order of transformation matter in this instance?

Example 10

6 Consider these two transformations:

- T_1: A reflection in the y-axis.
- T_2: A translation of −3 units in the x-direction and 5 units in the y-direction.

a Find the rule for T_1 followed by T_2. **b** Find the rule for T_2 followed by T_1.

c Does the order of transformation matter in this instance?

7 Express each of the following transformation matrices as the product of a dilation matrix and a reflection matrix:

a $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$ **c** $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ **d** $\begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}$

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8 **a** Find the matrix for the transformation that is a reflection in the x-axis followed by a reflection in the line $y = x$.

b Show that these two reflections can be achieved with one rotation.

9 Suppose that matrix $\mathbf{A}$ gives a rotation about the origin by θ degrees anticlockwise and that matrix $\mathbf{B}$ gives a reflection in the line $y = x$. If $\mathbf{AB} = \mathbf{BA}$, find the angle θ.

10 Suppose that matrix $\mathbf{A}$ rotates the plane about the origin by angle θ anticlockwise.

a Through what angle will the matrix $\mathbf{A}^2$ rotate the plane?

b Evaluate $\mathbf{A}^2$.

c Hence find formulas for $\cos(2\theta)$ and $\sin(2\theta)$.

11 A transformation T consists of a reflection in the line $y = x$ followed by a translation by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

a Find the rule for the transformation T.

b Show that the transformation T can also be obtained by a translation and then a reflection in the line $y = x$. Find the translation vector.

12 **a** Find the rotation matrix for an angle of 60° anticlockwise.

b Find the rotation matrix for an angle of 45° clockwise.

c By multiplying these two matrices, find the rotation matrix for an angle of 15° anticlockwise.

d Hence write down the exact values of $\sin 15°$ and $\cos 15°$.

13 A transformation consists of a reflection in the line $y = x \tan\varphi$ and then in the line $y = x \tan\theta$. Show that this is equivalent to a single rotation.

16E Inverse transformations

If transformation T maps the point (x, y) to the point (x', y'), then the **inverse transformation** T^{-1} maps the point (x', y') to the point (x, y).

For a linear transformation T, we can write

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{X}' = \mathbf{AX}$$

If the inverse matrix $\mathbf{A}^{-1}$ exists, then we have

$$\mathbf{A}^{-1}\mathbf{X}' = \mathbf{A}^{-1}\mathbf{AX}$$

$$\mathbf{A}^{-1}\mathbf{X}' = \mathbf{IX}$$

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{X}'$$

Therefore $\mathbf{A}^{-1}$ is the matrix of the inverse transformation T^{-1}.

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If the matrix of a linear transformation is

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the matrix of the inverse transformation is

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The inverse exists if and only if $\det(\mathbf{A}) = ad - bc \neq 0$.

Example 11

Find the inverse of the transformation with rule $(x, y) \to (3x + 2y, 5x + 4y)$.

Solution

Since the matrix of this linear transformation is

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$$

the inverse transformation will have matrix

$$\begin{aligned} \mathbf{A}^{-1} &= \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{3 \times 4 - 2 \times 5}\begin{bmatrix} 4 & -2 \\ -5 & 3 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 4 & -2 \\ -5 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ -\frac{5}{2} & \frac{3}{2} \end{bmatrix} \end{aligned}$$

Therefore the rule of the inverse transformation is $(x, y) \to (2x - y, -\frac{5}{2}x + \frac{3}{2}y)$.

Example 12

Find the matrix of the linear transformation such that $(4, 3) \to (9, 10)$ and $(2, 1) \to (5, 6)$.

Solution

We need to find a matrix $\mathbf{A}$ such that

$$\mathbf{A}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 10 \end{bmatrix} \quad \text{and} \quad \mathbf{A}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

This can be written as a single equation:

$$\mathbf{A}\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ 10 & 6 \end{bmatrix}$$

Therefore

$$\mathbf{A} = \begin{bmatrix} 9 & 5 \\ 10 & 6 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}$$

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Inverses of important transformations

For important geometric transformations, it is often obvious what the inverse transformation should be.

Example 13

Let **R** be the matrix corresponding to a rotation of the plane by angle θ anticlockwise. Show that $\mathbf{R}^{-1}$ corresponds to a rotation by angle θ clockwise.

Solution

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

This matrix corresponds to a rotation of the plane by angle θ clockwise.

Explanation

We find the inverse matrix using the formula

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We also use the symmetry properties:

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

The following table summarises the important geometric transformations along with their inverses. You will demonstrate some of these results in the exercises.

Transformation	Matrix **A**	Inverse matrix $\mathbf{A}^{-1}$	Inverse transformation
Dilation from the y-axis	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{bmatrix}$	Dilation from the y-axis
Dilation from the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$	Dilation from the x-axis
Shear parallel to the x-axis	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$	Shear parallel to the x-axis
Rotation by θ anticlockwise	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$	$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$	Rotation by θ clockwise
Reflection in the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Reflection in the x-axis
Reflection in the y-axis	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	Reflection in the y-axis
Reflection in the line $y = mx$	$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$	$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$	Reflection in the line $y = mx$

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Section summary

If the matrix of a linear transformation is

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the matrix of the inverse transformation is

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Exercise 16E

1 Find the inverse matrix of each of the following transformation matrices:

a $\begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 1 \\ 2 & -4 \end{bmatrix}$

c $\begin{bmatrix} 0 & 3 \\ -2 & 4 \end{bmatrix}$ **d** $\begin{bmatrix} -1 & 3 \\ -4 & 5 \end{bmatrix}$

Example 11

2 For each of the following transformations, find the rule for their inverse:

a $(x, y) \to (5x - 2y, 2x - y)$ **b** $(x, y) \to (x - y, x)$

3 Find the point (x, y) that is mapped to $(1, 1)$ by the transformation with matrix:

a $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$

Example 12

4 Find the matrix of the linear transformation such that $(1, 2) \to (2, 1)$ and $(2, 3) \to (1, 1)$.

5 Find the vertices of the rectangle that is mapped to the unit square by the transformation with matrix $\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$.

Example 13

6 Consider a dilation of factor k from the y-axis, where $k > 0$.

a Write down the matrix of this transformation.

b Show that the inverse matrix corresponds to a dilation of factor $\frac{1}{k}$ from the y-axis.

7 Consider a shear of factor k parallel to the x-axis.

a Write down the matrix of this transformation.

b Show that the inverse matrix corresponds to a shear of factor $-k$ parallel to the x-axis.

8 Consider the transformation that reflects each point in the x-axis.

a Write down the matrix $\mathbf{A}$ of this transformation.

b Show that $\mathbf{A}^{-1} = \mathbf{A}$, and explain why you should expect this result.

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9 Consider the transformation that reflects each point in the line $y = mx = x\tan\theta$.

a Write down the matrix **B** of this transformation.

b Show that $\mathbf{B}^{-1} = \mathbf{B}$, and explain why you should expect this result.

16F Transformations of straight lines and other graphs

We have considered the effect of various transformations on points and figures in the plane. We will now turn our attention to graphs.

Here, we will aim to find the equations of transformed graphs. We will also investigate the effects of linear transformations on straight lines. You will study this application in much greater detail in Mathematical Methods.

► Linear transformations of straight lines

We will first investigate the effect of linear transformations on straight lines.

Example 14

Find the equation of the image of the line $y = 2x + 3$ under a reflection in the x-axis followed by a dilation of factor 2 from the y-axis.

Solution

The matrix of the combined transformation is

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

If (x', y') are the coordinates of the image of (x, y), then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ -y \end{bmatrix}$$

Therefore

$$x' = 2x \quad \text{and} \quad y' = -y$$

Rearranging gives

$$x = \frac{x'}{2} \quad \text{and} \quad y = -y'$$

Therefore the equation $y = 2x + 3$ becomes

$$-y' = 2\left(\frac{x'}{2}\right) + 3$$

$$-y' = x' + 3$$

$$y' = -x' - 3$$

We now ignore the dashes, and so the equation of the image is simply

$$y = -x - 3$$

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Example 15

Consider the graph of $y = x + 1$. Find the equation of its image under the linear transformation $(x, y) \to (x + 2y, y)$.

Solution

Let (x', y') be the coordinates of the image of (x, y). Then this transformation can be written in matrix form as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Therefore

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x' - 2y' \\ y' \end{bmatrix}$$

and so $x = x' - 2y'$ and $y = y'$.

The equation $y = x + 1$ becomes

$$\begin{aligned} y' &= x' - 2y' + 1 \\ 3y' &= x' + 1 \\ y' &= \frac{x'}{3} + \frac{1}{3} \end{aligned}$$

The equation of the image is $y = \dfrac{x}{3} + \dfrac{1}{3}$.

In the previous two examples, you will have noticed that the image of each straight line was another straight line. In fact, linear transformations get their name in part from the following fact, which is proved in the exercises.

The image of any straight line under an invertible linear transformation is a straight line.

Example 16

Find a matrix that transforms the line $y = x + 2$ to the line $y = -2x + 4$.

Solution

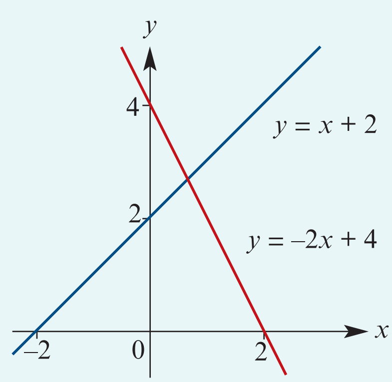
Let's find the matrix that maps the x-axis intercept of the first line to the x-axis intercept of the second line, and likewise for the y-axis intercepts.

We want

$$(-2, 0) \to (2, 0) \quad \text{and} \quad (0, 2) \to (0, 4)$$

This can be achieved by a reflection in the y-axis and then a dilation of factor 2 from the x-axis.

This transformation has the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$.



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Transformations of other graphs

The method for finding the image of a straight line can be used for other graphs.

Example 17

Find the image of the graph of $y = x^2 + 1$ under a translation by the vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ followed by a reflection in the y-axis.

Solution

Let (x', y') be the image of (x, y). Then the transformation is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x+2 \\ y-1 \end{bmatrix} = \begin{bmatrix} -x-2 \\ y-1 \end{bmatrix}$$

Therefore $x' = -x - 2$ and $y' = y - 1$.

This gives $x = -x' - 2$ and $y = y' + 1$.

The equation $y = x^2 + 1$ becomes

$$\begin{aligned} y' + 1 &= (-x' - 2)^2 + 1 \\ y' &= (-x' - 2)^2 \\ &= (x' + 2)^2 \end{aligned}$$

The equation of the image is $y = (x + 2)^2$.

Example 18

Find the image of the unit circle, $x^2 + y^2 = 1$, under a dilation of factor 2 from the y-axis and then a rotation about the origin by 90° anticlockwise. Sketch the circle and its image.

Solution

The dilation matrix is $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

The rotation matrix is $\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

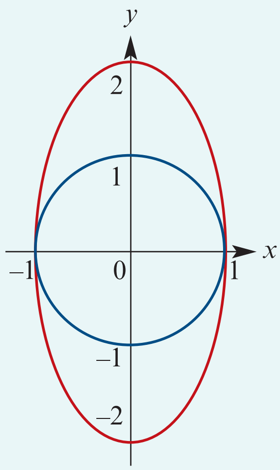
Let (x', y') be the image of (x, y). Then the transformation is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ 2x \end{bmatrix}$$

Thus $x' = -y$ and $y' = 2x$, giving $y = -x'$ and $x = \dfrac{y'}{2}$.

The equation $x^2 + y^2 = 1$ becomes $\left(\dfrac{y'}{2}\right)^2 + (-x')^2 = 1$.

Hence the image is the ellipse with equation $x^2 + \dfrac{y^2}{2^2} = 1$.



ISBN 978-1-316-63608-4

Cambridge University Press

Exercise 16F

Example 14

1 Find the equation of the image of the graph of $y = 3x + 1$ under:

a a reflection in the x-axis

b a dilation of factor 2 from the y-axis

c a dilation of factor 3 from the x-axis and factor 2 from the y-axis

d a reflection in the x-axis and then in the y-axis

e a reflection in the y-axis and then a dilation of factor 3 from the x-axis

f a rotation about the origin by 90° anticlockwise

g a rotation about the origin by 90° clockwise and then a reflection in the x-axis.

Example 15

2 Find the image of $y = 2 - 3x$ under each of the following transformations:

a $(x, y) \to (2x, 3y)$

b $(x, y) \to (-y, x)$

c $(x, y) \to (x - 2y, y)$

d $(x, y) \to (3x + 5y, x + 2y)$

Example 16

3 Find a matrix that transforms the line $x + y = 1$ to the line $x + y = 2$.

4 Find a matrix that transforms the line $y = x + 1$ to the line $y = 6 - 2x$.

Example 17

5 Find the equation of the image of the graph of $y = x^2 - 1$ under a translation by the vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and then a reflection in the x-axis.

6 Find the equation of the image of the graph of $y = (x - 1)^2$ under a reflection in the y-axis and then a translation by the vector $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

Example 18

7 Find the image of the unit circle, $x^2 + y^2 = 1$, under a dilation of factor 3 from the x-axis and then a rotation about the origin by 90° anticlockwise. Sketch the circle and its image.

8 Consider any invertible linear transformation

$$(x, y) \to (ax + by, cx + dy)$$

Show that the image of the straight line $px + qy = r$ is a straight line.

9 Rotate the graph of $y = \dfrac{1}{x}$ by 45° anticlockwise. Show that the equation of the image is $y^2 - x^2 = 2$.

Note: This shows that the two curves are congruent hyperbolas.

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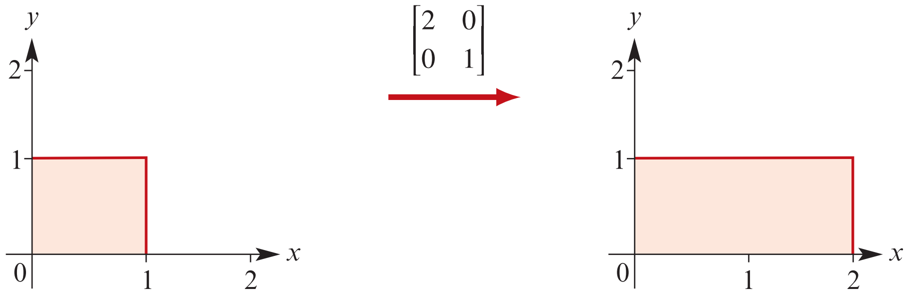
16G Area and determinant

If we apply a linear transformation to some region of the plane, then the area may change.

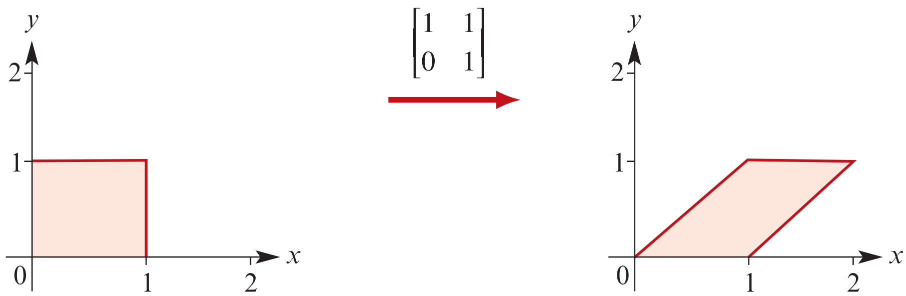
For example, if we dilate the unit square by a factor of 2 from the y-axis, then the area increases by a factor of 2.



Notice that this increase corresponds to the determinant of the transformation matrix:

$$\det\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = 2$$

On the other hand, if we shear the unit square by a factor of 1 parallel to the x-axis, then the area is unchanged.



Notice that the determinant of this transformation matrix is

$$\det\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 1$$

More generally, we can prove the following remarkable result.

> If a region of the plane is transformed by matrix **B**, then
>
> Area of image = $|\det(\mathbf{B})| \times$ Area of region

Proof We will prove the result when the unit square is transformed by matrix

$$\mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The result can be extended to other regions by approximating them by squares.

We will assume that a, b, c and d are all positive and that $\det(\mathbf{B}) > 0$. The proof can easily be adapted if we relax these assumptions.

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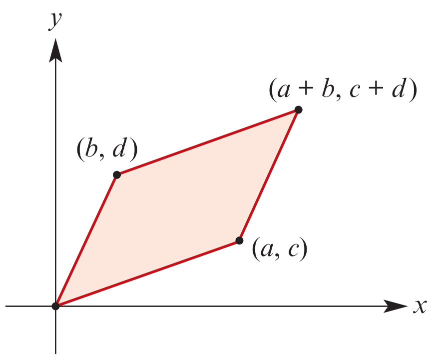
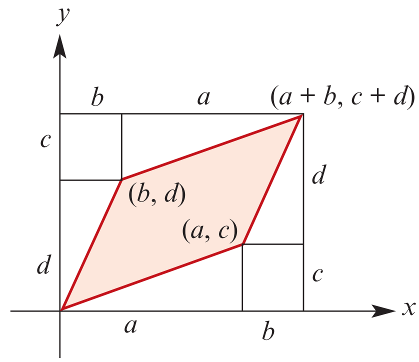
The image of the unit square under transformation **B** is a parallelogram.





To find the area of the image, we draw a rectangle around it as shown, and subtract the area of the two small rectangles and four triangles from the total area:

$$\begin{aligned}\text{Area of image} &= (a+b)(c+d) - bc - bc - \frac{ac}{2} - \frac{ac}{2} - \frac{bd}{2} - \frac{bd}{2}\\ &= (a+b)(c+d) - 2bc - ac - bd\\ &= ac + ad + bc + bd - 2bc - ac - bd\\ &= ad - bc\end{aligned}$$

This is equal to the determinant of matrix **B**.

Example 19

The triangular region with vertices (1, 1), (2, 1) and (1, 2) is transformed by the rule $(x, y) \to (-x + 2y, 2x + y)$.

a Find the matrix of the linear transformation.

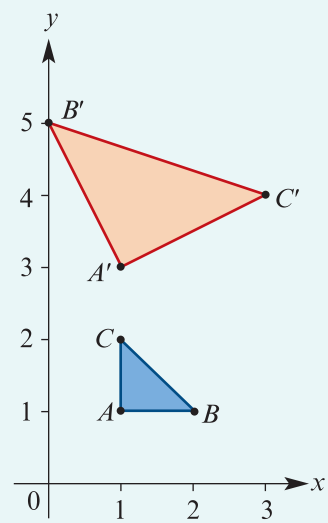
b On the same set of axes, sketch the region and its image.

c Find the area of the image.

Solution

a The matrix is given by $\mathbf{B} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$.

b The region is shown in blue and its image in red.



c The area of the original region is $\frac{1}{2}$.

The determinant of the transformation matrix is

$$\det\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} = (-1)\times 1 - 2\times 2 = -5$$

Therefore

$$\begin{aligned}\text{Area of image} &= |\det(\mathbf{B})| \times \text{Area of region}\\ &= |-5| \times \frac{1}{2} = \frac{5}{2}\end{aligned}$$

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Example 20

The unit square is mapped to a parallelogram of area 3 by the matrix

$$\mathbf{B} = \begin{bmatrix} m & 2 \\ m & m \end{bmatrix}$$

Find the possible values of m.

Solution

The original area is 1. Therefore

$$\text{Area of image} = |\det(\mathbf{B})| \times \text{Area of region}$$
$$3 = |ad - bc| \times 1$$
$$3 = |m^2 - 2m|$$

Therefore either $m^2 - 2m = 3$ or $m^2 - 2m = -3$.

Case 1:

$$m^2 - 2m = 3$$
$$m^2 - 2m - 3 = 0$$
$$(m + 1)(m - 3) = 0$$
$$m = -1 \text{ or } m = 3$$

Case 2:

$$m^2 - 2m = -3$$
$$m^2 - 2m + 3 = 0$$

This quadratic equation has no solutions, since the discriminant is

$$\Delta = b^2 - 4ac$$
$$= (-2)^2 - 4(1)(3) = 4 - 12 < 0$$

The connection between area and determinant has many important applications. In the next example, we see how it can be used to find the area of an ellipse. Alternative approaches to finding this area are much more sophisticated.

Example 21

The circle with equation $x^2 + y^2 = 1$ is mapped to an ellipse by the rule $(x, y) \to (ax, by)$, where both a and b are positive.

a Find the equation of the ellipse and sketch its graph.

b Find the area of the ellipse.

Solution

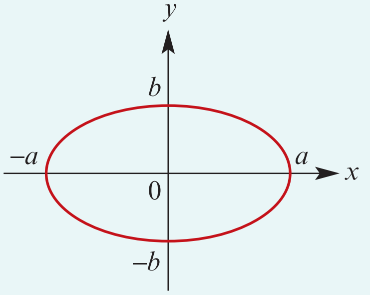
a We have $x' = ax$ and $y' = by$.

This gives $x = \dfrac{x'}{a}$ and $y = \dfrac{y'}{b}$.

The equation $x^2 + y^2 = 1$ becomes

$$\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = 1$$

Hence the equation of the ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.



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b The area of the original circle of radius 1 is π.

The determinant of the transformation matrix is

$$\det\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = a \times b - 0 \times 0 = ab$$

Therefore the area of the ellipse is πab.

Note: When $a = b = r$, this formula gives the area of a circle of radius r.

Section summary

If a region of the plane is transformed by matrix **B**, then

Area of image = |det(**B**)| × Area of region

Exercise 16G

1 Each of the following matrices maps the unit square to a parallelogram. Sketch each parallelogram and find its area.

a $\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$ **d** $\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$

Example 19

2 The matrix $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ maps the triangle with vertices (0, 1), (1, 1) and (0, 0) to a new triangle.

a Sketch the original triangle and its image.

b Find the areas of both triangles.

3 The matrix $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ maps the triangle with vertices (−1, 1), (1, 1) and (1, 0) to a new triangle.

a Sketch the original triangle and its image.

b Find the areas of both triangles.

Example 20

4 The matrix $\begin{bmatrix} m & 2 \\ -1 & m \end{bmatrix}$ maps the unit square to a parallelogram of area 6 square units. Find the value(s) of m.

5 The matrix $\begin{bmatrix} m & m \\ 1 & m \end{bmatrix}$ maps the unit square to a parallelogram of area 2 square units. Find the value(s) of m.

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6 **a** By evaluating a determinant, show that each of the following transformations will not change the area of any region:

i a shear of factor k parallel to the x-axis

ii an anticlockwise rotation about the origin by angle θ

iii a reflection in any straight line through the origin

Note: We say that each of these transformations **preserves area**.

b Let $k > 0$. A linear transformation has matrix $\begin{bmatrix} k & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$.

i Describe the geometric effect of the transformation.

ii Show that this transformation preserves area.

7 For each $x \in \mathbb{R}$, the matrix $\begin{bmatrix} x & 1 \\ -2 & x+2 \end{bmatrix}$ maps the unit square to a parallelogram.

a Show that the area of the parallelogram is $(x + 1)^2 + 1$.

b For what value of x is the area of the parallelogram a minimum?

8 For what values of m does the matrix $\begin{bmatrix} m & 2 \\ 3 & 4 \end{bmatrix}$ map the unit square to a parallelogram of area greater than 2?

9 Find all matrices that will map the unit square to rhombus of area $\frac{1}{2}$ with one vertex at $(0, 0)$ and another at $(1, 0)$.

10 **a** Find a matrix that transforms the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$ to the triangle with vertices $(0, 0)$, (a, c) and (b, d).

b Hence show that the area of the triangle with vertices $(0, 0)$, (a, c) and (b, d) is given by the formula

$$A = \frac{1}{2}|ad - bc|$$

c Hence prove that if a, b, c and d are rational numbers, then the area of this triangle is rational.

d A **rational point** has coordinates (x, y) such that both x and y are rational numbers. Prove that no equilateral triangle can be drawn in the Cartesian plane so that all three of its vertices are rational points.

Hint: You can assume that the vertices of the triangle are $(0, 0)$, (a, c) and (b, d). Find another expression for the area of the triangle using Pythagoras' theorem. You can also assume that $\sqrt{3}$ is irrational.

ISBN 978-1-316-63608-4

16H General transformations

Earlier in this chapter we considered rotations about the origin. But what if we want to rotate a figure about a point that is not the origin? In this section we will see how a more complicated transformation can be achieved by a sequence of simpler transformations.

▶ Rotation about the point (a, b)

If we want to rotate the plane about the point (a, b) by θ degrees anticlockwise, we can do this in a sequence of three steps:

Step 1 Translate the plane so that the centre of rotation is now the origin, by adding $\begin{bmatrix} -a \\ -b \end{bmatrix}$.

Step 2 Rotate the plane through angle θ anticlockwise, by multiplying by $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Step 3 Translate the plane back to its original position, by adding $\begin{bmatrix} a \\ b \end{bmatrix}$.

Chaining these three transformations together gives the overall transformation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

Example 22

a Find the transformation that rotates the plane by 90° anticlockwise about the point (1, 1).

b Check your answer by showing that (0, 1) is mapped to the correct point.

Solution

a We do this in a sequence of three steps, starting with the initial point (x, y):

Initial point	Translate	Rotate 90° anticlockwise	Translate back
$\begin{bmatrix} x \\ y \end{bmatrix}$	$\begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x-1 \\ y-1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

This gives the overall transformation

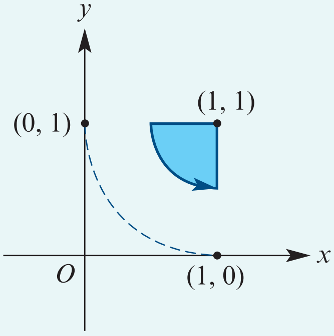
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x-1 \\ y-1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -y+2 \\ x \end{bmatrix}$$

b We check our answer by finding the image of (0, 1).

Let $x = 0$ and $y = 1$. Then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1+2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Therefore $(0, 1) \to (1, 0)$, as expected.



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▶ Reflection in the line $y = x \tan\theta + c$

To reflect the plane in a line $y = x\tan\theta + c$ that does not go through the origin, we can also do this in a sequence of three steps:

Step 1 Translate the plane so that the line passes through the origin, by adding $\begin{bmatrix} 0 \\ -c \end{bmatrix}$.

Step 2 Reflect the plane in the line $y = x\tan\theta$, by multiplying by $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$.

Step 3 Translate the plane back to its original position, by adding $\begin{bmatrix} 0 \\ c \end{bmatrix}$.

Example 23

a Find the transformation that reflects the plane in the line $y = -x + 1$.

b Check your answer by finding the image of the point $(1, 1)$.

Solution

a We do this in a sequence of three steps, starting with the initial point (x, y). The first step translates the line $y = -x + 1$ so that it passes through the origin.

Initial point	Translate	Reflect in line $y = -x$	Translate back
$\begin{bmatrix} x \\ y \end{bmatrix}$	$\begin{bmatrix} x \\ y-1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y-1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y-1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

This gives the overall transformation

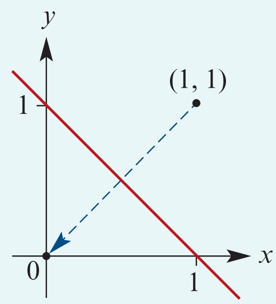
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y-1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -y+1 \\ -x+1 \end{bmatrix}$$

b We check our answer by finding the image of $(1, 1)$.

Let $x = 1$ and $y = 1$. Then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1+1 \\ -1+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore $(1, 1) \to (0, 0)$, as expected.



Section summary

More difficult transformations can be achieved by combining simpler transformations.

- To rotate the plane about the point (a, b):
  1. Translate the plane so that the origin is the centre of rotation.
  2. Rotate the plane about the origin.
  3. Translate the plane back to its original position.

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- To reflect the plane in the line $y = mx + c$:
  1. Translate the plane so that the line passes through the origin.
  2. Reflect the plane in the line $y = mx$.
  3. Translate the plane back to its original position.

Exercise 16H

Example 22

1 Find the transformation that rotates the plane by 90° clockwise about the point $(2, 2)$. Check your answer by showing that the point $(2, 1)$ is mapped to the correct point.

2 Find the transformation that rotates the plane by 180° anticlockwise about the point $(-1, 1)$. Check your answer by showing that the point $(-1, 0)$ is mapped to the correct point.

Example 23

3 Find the transformation that reflects the plane in each of the following lines. Check your answer by showing that the point $(0, 0)$ is mapped to the correct point.

a $y = x - 1$
b $y = -x - 1$
c $y = 1$
d $x = -2$

4 **a** Write down the matrix **A** for a rotation about the origin by angle θ clockwise.
b Write down the matrix **B** for a dilation of factor k from the x-axis.
c Write down the matrix **C** for a rotation about the origin by angle θ anticlockwise.
d Hence find the matrix that increases the perpendicular distance from the line $y = x\tan\theta$ by a factor of k.

5 Find the transformation matrix that projects the point (x, y) onto the line $y = x\tan\theta$.
Hint: First rotate the plane clockwise by angle θ.

6 Consider these two transformations:

- T_1: A reflection in the line $y = x + 1$.
- T_2: A reflection in the line $y = x$.

Show that T_1 followed by T_2 is a translation.

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Chapter summary

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- A **linear transformation** is defined by a rule of the form $(x, y) \to (ax + by, cx + dy)$.
- Linear transformations can be represented using matrix multiplication:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

  The point (x', y') is called the **image** of the point (x, y).
- The matrix of a composition of two linear transformations can be found by multiplying the two transformation matrices in the correct order.
- If $\mathbf{A}$ is the matrix of a linear transformation, then $\mathbf{A}^{-1}$ is the matrix of the inverse transformation.
- If a region of the plane is transformed by matrix $\mathbf{B}$, then

$$\text{Area of image} = |\det(\mathbf{B})| \times \text{Area of region}$$

- Difficult transformations can be achieved by combining simpler transformations.

Transformation	Matrix	Transformation	Matrix
Reflection in the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Reflection in the y-axis	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection in the line $y = x$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	Reflection in the line $y = -x$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
Dilation from the y-axis	$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$	Dilation from the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$
Shear parallel to the x-axis	$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$	Shear parallel to the y-axis	$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$
Projection onto the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	Projection onto the y-axis	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
Rotation by θ anticlockwise	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$	Reflection in the line $y = x\tan\theta$	$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$

Short-answer questions

1 The rule for a transformation is $(x, y) \to (2x + y, -x + 2y)$.

a Find the image of the point $(2, 3)$.

b Find the matrix of this transformation.

c Sketch the image of the unit square and find its area.

d Find the rule for the inverse transformation.

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Review

2 Find the matrix corresponding to each of the following linear transformations:

a reflection in the y-axis
b dilation of factor 5 from the x-axis
c shear of factor -3 parallel to the x-axis
d projection onto the x-axis
e rotation by 30° anticlockwise
f reflection in the line $y = x$

3 **a** Find the matrix that will reflect the plane in the line $y = 3x$.
b Find the image of the point $(2, 4)$ under this transformation.

4 Find the transformation matrix that corresponds to:

a a reflection in the x-axis and then a reflection in the line $y = -x$
b a rotation about the origin by 90° anticlockwise and then a dilation of factor 2 from the x-axis
c a reflection in the line $y = x$ and then a shear of factor 2 parallel to the y-axis.

5 **a** Find the rule for the transformation that will reflect (x, y) in the x-axis and then translate the result by the vector $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.
b Find the rule for the transformation if the translation takes place before the reflection.

6 **a** Write down the matrix for a shear of factor k parallel to the y-axis.
b Show that the inverse matrix corresponds to a shear of factor $-k$ parallel to the y-axis.

7 Each of the following matrices maps the unit square to a parallelogram. Sketch each parallelogram and find its area.

a $\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$
b $\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

8 **a** Find the rule for the transformation that rotates the plane about the point $(1, -1)$ by 90° anticlockwise. (Hint: Translate the point $(1, -1)$ to the origin, rotate the plane, and then translate the point back to its original position.)
b Find the image of the point $(2, -1)$ under this transformation.
c Sketch the unit square and its image under this transformation.

Multiple-choice questions

1 The image of the point $(2, -1)$ under the transformation $(x, y) \to (2x - 3y, -x + 4y)$ is

A $(1, -6)$ **B** $(7, -6)$ **C** $(7, 6)$ **D** $(7, 2)$ **E** $(1, 2)$

2 The matrix that will reflect the plane in the line $y = -x$ is

A $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

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3 The matrix that corresponds to a dilation of factor 2 from the y-axis followed by a reflection in the x-axis is

A $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ **B** $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$

4 The matrix $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{bmatrix}$ will

A rotate the plane by 30°

B rotate the plane by −30°

C rotate the plane by −60°

D reflect the plane in the line $y = x \tan 30°$

E reflect the plane in the line $y = x \tan 60°$

5 The transformation that translates (x, y) by the vector $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and then rotates the result about the origin by 90° anticlockwise is given by

A $(x, y) \to (y - 3, x + 2)$

B $(x, y) \to (-y + 3, -x - 2)$

C $(x, y) \to (-y + 3, x + 2)$

D $(x, y) \to (-x - 2, -y + 3)$

E $(x, y) \to (x + 2, -y - 3)$

6 The matrix $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ corresponds to

A a rotation by 180° and then a dilation of factor 2 from the x-axis

B a rotation by 90° clockwise and then a dilation of factor 2 from the x-axis

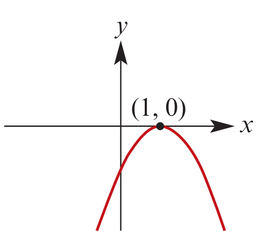
C a rotation by 180° and then a dilation of factor 2 from the y-axis

D a reflection in the y-axis and then a dilation of factor 2 from the y-axis

E a reflection in the x-axis and then a dilation of factor 2 from the x-axis

7 Which of the following shows the image of the graph of $y = (x - 1)^2$ under the linear transformation with matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$?

A



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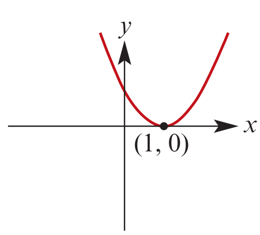

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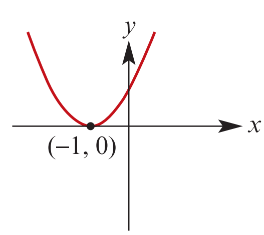
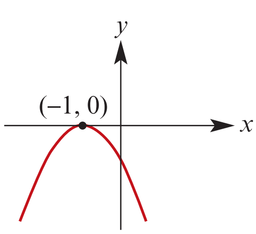
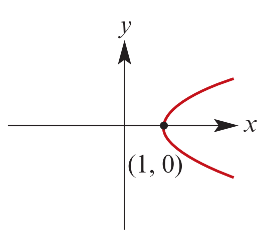


D



E



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Review

8 Which of these matrices maps the unit square to a parallelogram of area 2 square units?

A $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ **C** $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

9 The matrix $\mathbf{R}$ will rotate the plane through angle $40°$. The smallest value of m such that $\mathbf{R}^m = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix, is

A 6 **B** 7 **C** 8 **D** 9 **E** 10

Extended-response questions

1 **a** Find the matrix that will rotate the plane by $45°$ anticlockwise.

b Find the matrix that will rotate the plane by $30°$ anticlockwise.

c Hence find the matrix that will rotate the plane by $75°$ anticlockwise.

d Hence deduce exact values for $\cos 75°$ and $\sin 75°$.

2 The triangle with vertices $(0, 0)$, $(2, 0)$ and $(0, 2)$ is transformed by the matrix $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$.

a Sketch the triangle and its image on the same set of axes.

b Find the area of the triangle and its image.

c The image of the triangle is revolved around the y-axis to create a three-dimensional solid. Find the volume of this solid.

3 Consider the transformation with rule $(x, y) \to (x + y, y)$.

a Write down the matrix of this transformation.

b What name is given to this type of transformation?

c Find the images of the points $(-1, 1)$, $(0, 0)$ and $(1, 1)$ under this transformation.

d Hence sketch the graph of $y = x^2$ and its image under this transformation.

4 A square with vertices $(\pm 1, \pm 1)$ is rotated about the origin by $45°$ anticlockwise.

a Find the coordinates of the vertices of its image.

b Sketch the square and its image on the same set of axes.

c When these two squares are combined, the resulting figure is called a Star of Lakshmi. Find its area.

5 In this chapter we investigated two important transformation matrices. These were the rotation and reflection matrices, which we will now denote by

$$\text{Rot}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{and} \quad \text{Ref}(\theta) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

a Using matrix multiplication and an application of trigonometric identities, prove the following four matrix equations:

i $\text{Rot}(\theta)\,\text{Rot}(\varphi) = \text{Rot}(\theta + \varphi)$ **ii** $\text{Ref}(\theta)\,\text{Ref}(\varphi) = \text{Rot}(2\theta - 2\varphi)$

iii $\text{Rot}(\theta)\,\text{Ref}(\varphi) = \text{Ref}(\varphi + \frac{1}{2}\theta)$ **iv** $\text{Ref}(\theta)\,\text{Rot}(\varphi) = \text{Ref}(\varphi - \frac{1}{2}\theta)$

b Explain in words what each of the above four equations shows.

c Using these identities, find the matrix $\text{Rot}(60°)\,\text{Ref}(60°)\,\text{Ref}(60°)\,\text{Rot}(60°)$.

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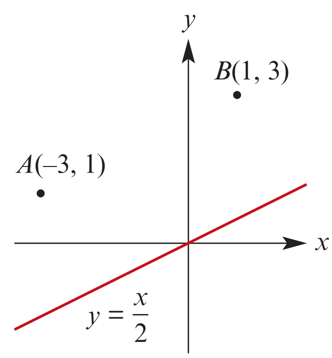
6 An ant is at point $A(-3, 1)$. His friend is at point $B(1, 3)$. The ant wants to walk from A to B, but first wants to visit the straight line $y = \frac{1}{2}x$. Being an economical ant, he wants the total length of his path to be as short as possible.



a Find the matrix that will reflect the plane in the line $y = \frac{1}{2}x$.

b Find the image A' of the point A when reflected in the line $y = \frac{1}{2}x$.

c Find the distance from point A' to point B.

d The straight line $A'B$ intersects the line $y = \frac{1}{2}x$ at the point C. What type of triangle is ACA'?

e Suppose that D is any other point on the line $y = \frac{1}{2}x$. Show that

$$AD + DB \geq AC + CB$$

f Hence find the shortest possible distance travelled by the ant.

7 A rectangle R_1 has vertices $(0, 0)$, $(a, 0)$, $(0, b)$ and (a, b), where a and b are positive real numbers.

a Sketch the rectangle R_1.

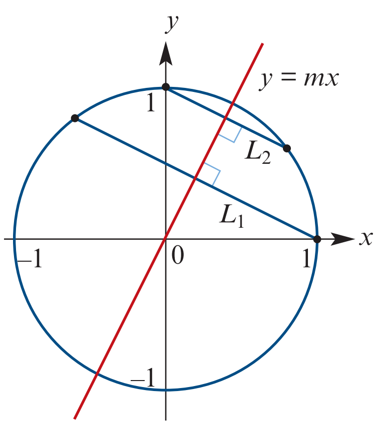
b The rectangle R_1 is rotated about the origin by angle θ anticlockwise, where $0° \leq \theta \leq 90°$. The image is another rectangle R_2. Find the coordinates of the vertices of rectangle R_2 in terms of a, b and θ.

c The vertices of R_2 lie on another rectangle R_3 that has edges parallel to the coordinate axes. Show that the area of rectangle R_3 is

$$A = \frac{1}{2}(a^2 + b^2)\sin(2\theta) + ab$$

d Hence show that the maximum area of rectangle R_3 is $\frac{1}{2}(a + b)^2$, which occurs when $\theta = 45°$.

8 The graphs of the unit circle $x^2 + y^2 = 1$ and the line $y = mx$ are shown. Lines L_1 and L_2 are perpendicular to the line $y = mx$ and go through the points $(1, 0)$ and $(0, 1)$ respectively.



a Find the equation of the line L_1, and find where it intersects the unit circle in terms of m.

b Find the equation of the line L_2, and find where it intersects the unit circle in terms of m.

c Hence deduce the formula for the matrix that reflects the point (x, y) in the line $y = mx$.

Hint: Recall that the columns of the matrix will be the images of the standard unit vectors.

ISBN 978-1-316-63608-4

Chapter 17

17 Vectors

Objectives

- To understand the concept of a **vector** and to apply the basic operations on vectors.
- To recognise when two vectors are **parallel**.
- To use the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ to represent vectors in two dimensions.
- To find the **scalar product** of two vectors.
- To use the scalar product to find the magnitude of the angle between two vectors.
- To use the scalar product to recognise when two vectors are **perpendicular**.
- To resolve a vector into **rectangular components**, where one component is parallel to a given vector and the other component is perpendicular.
- To use the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ to represent vectors in three dimensions.

In scientific experiments, some of the things that are measured are completely determined by their magnitude. Mass, length and time are determined by a number and an appropriate unit of measurement.

length	30 cm is the length of the page of a particular book
time	10 s is the time for one athlete to run 100 m

More is required to describe velocity, displacement or force. The direction must be recorded as well as the magnitude.

displacement	30 km in a direction north
velocity	60 km/h in a direction south-east

A quantity that has both a magnitude and a direction is called a **vector**. Our study of vectors will tie together different ideas from previous chapters, including geometry, trigonometry, complex numbers and transformations.

ISBN 978-1-316-63608-4

Cambridge University Press

17A Introduction to vectors

Suppose that you are asked: ‘Where is your school in relation to your house?’

It is not enough to give an answer such as ‘four kilometres’. You need to specify a direction as well as a distance. You could give the answer ‘four kilometres north-east’.

Position is an example of a vector quantity.

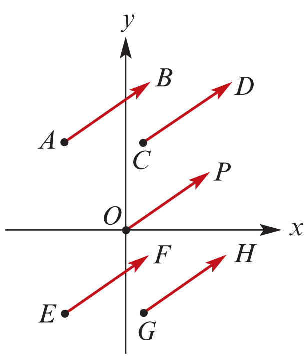
▶ Directed line segments

A quantity that has a direction as well as a magnitude can be represented by an arrow:

- the arrow points in the direction of the action
- the length of the arrow gives the magnitude of the quantity in terms of a suitably chosen unit.

Arrows with the same length and direction are regarded as equivalent. These arrows are **directed line segments** and the sets of equivalent segments are called **vectors**.

The five directed line segments shown all have the same length and direction, and so they are equivalent.



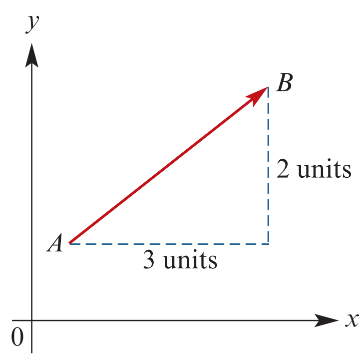
A directed line segment from a point A to a point B is denoted by $\overrightarrow{AB}$.

For simplicity of language, this is also called vector $\overrightarrow{AB}$. That is, the set of equivalent segments can be named through one member of the set.

Note: The five directed line segments in the diagram all name the same vector: $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF} = \overrightarrow{GH}$.

▶ Column vectors

In Chapter 16, we introduced vectors in the context of translations of the plane. We represented each translation by a column of numbers, which was called a vector.



This is consistent with the approach here, as the column of numbers corresponds to a set of equivalent directed line segments.

For example, the column $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ corresponds to the directed line segments which go 3 across and 2 up.

▶ Vector notation

A vector is often denoted by a single bold lowercase letter. The vector from A to B can be denoted by $\overrightarrow{AB}$ or by a single letter $\boldsymbol{v}$. That is, $\boldsymbol{v} = \overrightarrow{AB}$.

When a vector is handwritten, the notation is $\underset{\sim}{v}$.

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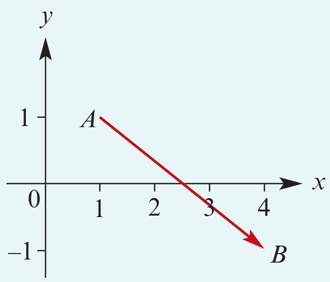
Example 1

Draw a directed line segment corresponding to $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Solution



Explanation

The vector $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ is '3 across to the right and 2 down'.

Note: Here the segment starts at $(1, 1)$ and goes to $(4, -1)$. It can start at any point.

Example 2

The vector $\boldsymbol{u}$ is defined by the directed line segment from $(2, 6)$ to $(3, 1)$.

If $\boldsymbol{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, find a and b.

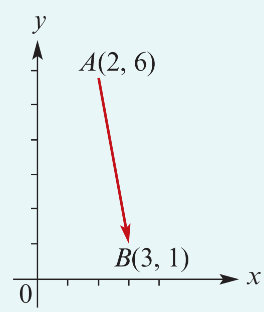
Solution

The vector is

$$\boldsymbol{u} = \begin{bmatrix} 3 - 2 \\ 1 - 6 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

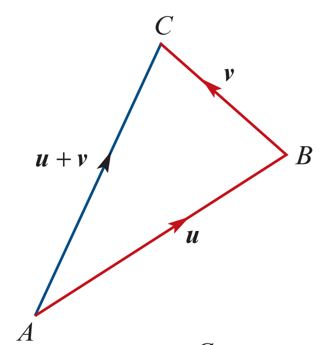
Hence $a = 1$ and $b = -5$.

Explanation



Addition of vectors

Adding vectors geometrically



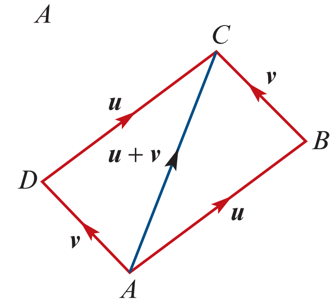
Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ can be added geometrically by drawing a line segment representing $\boldsymbol{u}$ from A to B and then a line segment representing $\boldsymbol{v}$ from B to C.

The sum $\boldsymbol{u} + \boldsymbol{v}$ is the vector from A to C. That is,

$$\boldsymbol{u} + \boldsymbol{v} = \overrightarrow{AC}$$

The same result is achieved if the order is reversed. This is represented in the diagram on the right:

$$\begin{aligned} \boldsymbol{u} + \boldsymbol{v} &= \overrightarrow{AC} \\ &= \boldsymbol{v} + \boldsymbol{u} \end{aligned}$$



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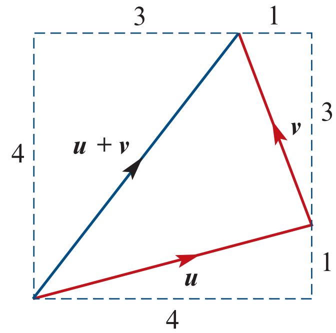
Adding column vectors



Two vectors can be added using column-vector notation.

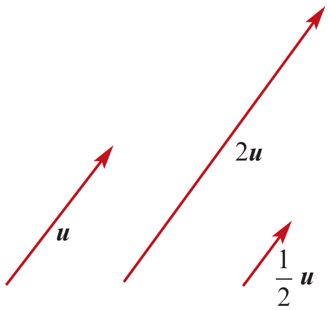
For example, if $\boldsymbol{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, then

$$\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

▶ Scalar multiplication

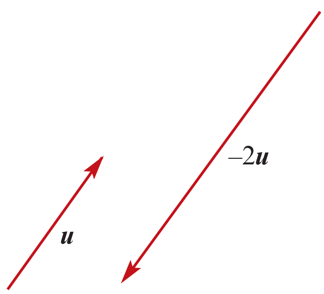
Multiplication by a real number (scalar) changes the length of the vector. For example:

- $2\boldsymbol{u}$ is twice the length of $\boldsymbol{u}$
- $\frac{1}{2}\boldsymbol{u}$ is half the length of $\boldsymbol{u}$



We have $2\boldsymbol{u} = \boldsymbol{u} + \boldsymbol{u}$ and $\frac{1}{2}\boldsymbol{u} + \frac{1}{2}\boldsymbol{u} = \boldsymbol{u}$.

In general, for $k \in \mathbb{R}^+$, the vector $k\boldsymbol{u}$ has the same direction as $\boldsymbol{u}$, but its length is multiplied by a factor of k.



When a vector is multiplied by −2, the vector's direction is reversed and the length is doubled.

When a vector is multiplied by −1, the vector's direction is reversed and the length remains the same.

If $\boldsymbol{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, then $-\boldsymbol{u} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$, $2\boldsymbol{u} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ and $-2\boldsymbol{u} = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$.

If $\boldsymbol{u} = \overrightarrow{AB}$, then

$$-\boldsymbol{u} = -\overrightarrow{AB} = \overrightarrow{BA}$$

The directed line segment $-\overrightarrow{AB}$ goes from B to A.

▶ Zero vector

The **zero vector** is denoted by $\mathbf{0}$ and represents a line segment of zero length. The zero vector has no direction.

▶ Subtraction of vectors

To find $\boldsymbol{u} - \boldsymbol{v}$, we add $-\boldsymbol{v}$ to $\boldsymbol{u}$.

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Example 3

For the vectors $\boldsymbol{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, find $2\boldsymbol{u} + 3\boldsymbol{v}$.

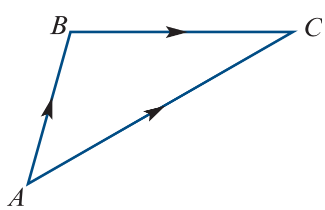
Solution

$$\begin{aligned} 2\boldsymbol{u} + 3\boldsymbol{v} &= 2\begin{bmatrix} 3 \\ -1 \end{bmatrix} + 3\begin{bmatrix} -2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ -2 \end{bmatrix} + \begin{bmatrix} -6 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 4 \end{bmatrix} \end{aligned}$$

▶ Polygons of vectors

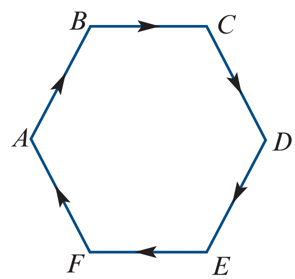
- For two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$, we have

  $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$



- For a polygon $ABCDEF$, we have

  $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} + \overrightarrow{FA} = \mathbf{0}$

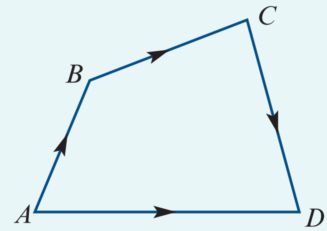


Example 4

Illustrate the vector sum $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$, where A, B, C and D are points in the plane.

Solution

$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}$



▶ Parallel vectors

Two parallel vectors have the same direction or opposite directions.

Two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are **parallel** if there is some $k \in \mathbb{R} \setminus \{0\}$ such that $\boldsymbol{u} = k\boldsymbol{v}$.

For example, if $\boldsymbol{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -6 \\ 9 \end{bmatrix}$, then the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel as $\boldsymbol{v} = 3\boldsymbol{u}$.

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Position vectors

We can use a point O, the origin, as a starting point for a vector to indicate the position of a point A in space relative to O.

For most of this chapter, we study vectors in two dimensions and the point O is the origin of the Cartesian plane. (Vectors in three dimensions are studied in Section 17F.)

For a point A, the **position vector** is $\overrightarrow{OA}$.

Linear combinations of non-parallel vectors

If two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel, then

$$m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b} \quad \text{implies} \quad m = p \text{ and } n = q$$

Proof Assume that $m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b}$. Then

$$m\boldsymbol{a} - p\boldsymbol{a} = q\boldsymbol{b} - n\boldsymbol{b}$$

$$\therefore \quad (m - p)\boldsymbol{a} = (q - n)\boldsymbol{b}$$

If $m \neq p$ or $n \neq q$, we could therefore write

$$\boldsymbol{a} = \frac{q - n}{m - p}\boldsymbol{b} \quad \text{or} \quad \boldsymbol{b} = \frac{m - p}{q - n}\boldsymbol{a}$$

But this is not possible, as $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors that are not parallel. Therefore $m = p$ and $n = q$.

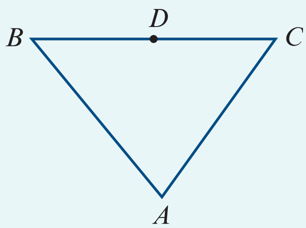
Example 5

Let A, B and C be the vertices of a triangle, and let D be the midpoint of BC.

Let $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{BC}$.

Find each of the following in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

a $\overrightarrow{BD}$ **b** $\overrightarrow{DC}$ **c** $\overrightarrow{AC}$
d $\overrightarrow{AD}$ **e** $\overrightarrow{CA}$



Solution

a $\overrightarrow{BD} = \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}\boldsymbol{b}$

b $\overrightarrow{DC} = \overrightarrow{BD} = \frac{1}{2}\boldsymbol{b}$

c $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \boldsymbol{a} + \boldsymbol{b}$

d $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = \boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$

e $\overrightarrow{CA} = -\overrightarrow{AC} = -(\boldsymbol{a} + \boldsymbol{b})$

Explanation

a: same direction and half the length

b: equivalent vectors

e: since $\overrightarrow{CA} + \overrightarrow{AC} = \boldsymbol{0}$

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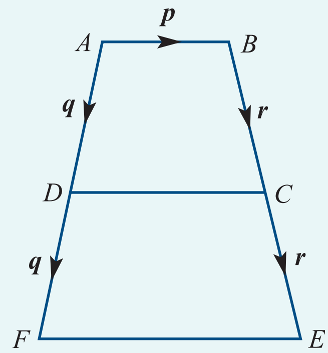
Example 6

In the figure, $\overrightarrow{DC} = k\boldsymbol{p}$ where $k \in \mathbb{R} \setminus \{0\}$.

a Express $\boldsymbol{p}$ in terms of k, $\boldsymbol{q}$ and $\boldsymbol{r}$.

b Express $\overrightarrow{FE}$ in terms of k and $\boldsymbol{p}$ to show that FE is parallel to DC.

c If $\overrightarrow{FE} = 4\overrightarrow{AB}$, find the value of k.



Solution

a $\boldsymbol{p} = \overrightarrow{AB}$

$= \overrightarrow{AD} + \overrightarrow{DC} + \overrightarrow{CB}$

$= \boldsymbol{q} + k\boldsymbol{p} - \boldsymbol{r}$

Therefore

$(1 - k)\boldsymbol{p} = \boldsymbol{q} - \boldsymbol{r}$

b $\overrightarrow{FE} = -2\boldsymbol{q} + \boldsymbol{p} + 2\boldsymbol{r}$

$= 2(\boldsymbol{r} - \boldsymbol{q}) + \boldsymbol{p}$

From part **a**, we have

$\boldsymbol{r} - \boldsymbol{q} = k\boldsymbol{p} - \boldsymbol{p}$

$= (k - 1)\boldsymbol{p}$

Therefore

$\overrightarrow{FE} = 2(k - 1)\boldsymbol{p} + \boldsymbol{p}$

$= 2k\boldsymbol{p} - 2\boldsymbol{p} + \boldsymbol{p}$

$= (2k - 1)\boldsymbol{p}$

c $\overrightarrow{FE} = 4\overrightarrow{AB}$

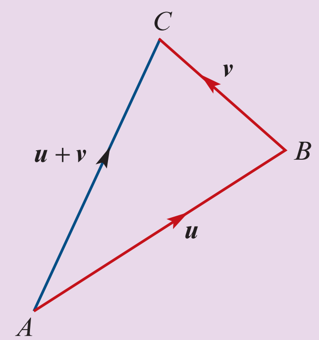
$(2k - 1)\boldsymbol{p} = 4\boldsymbol{p}$

$2k - 1 = 4$

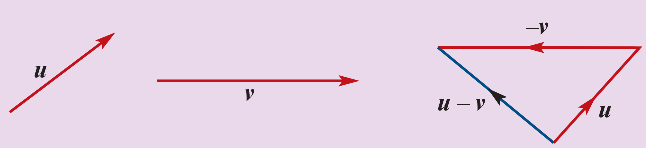
$\therefore \quad k = \dfrac{5}{2}$

Section summary

- A **vector** is a set of equivalent **directed line segments**.
- Addition of vectors
  If $\boldsymbol{u} = \overrightarrow{AB}$ and $\boldsymbol{v} = \overrightarrow{BC}$, then $\boldsymbol{u} + \boldsymbol{v} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.



- Scalar multiplication
  - For $k \in \mathbb{R}^+$, the vector $k\boldsymbol{u}$ has the same direction as $\boldsymbol{u}$, but its length is multiplied by a factor of k.
  - If $\boldsymbol{u} = \overrightarrow{AB}$, then $-\boldsymbol{u} = -\overrightarrow{AB} = \overrightarrow{BA}$.
- Zero vector
  The **zero vector**, denoted by $\boldsymbol{0}$, has zero length and has no direction.
- Subtraction of vectors
  $\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$



- Parallel vectors
  Two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are **parallel** if there is some $k \in \mathbb{R} \setminus \{0\}$ such that $\boldsymbol{u} = k\boldsymbol{v}$.

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Exercise 17A

Skillsheet

Example 1

1 On the same graph, draw arrows which represent the following vectors:

a $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ **b** $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$ **c** $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$ **d** $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$

Example 2

2 The vector $\boldsymbol{u}$ is defined by the directed line segment from $(1, 5)$ to $(6, 6)$.
If $\boldsymbol{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, find a and b.

3 The vector $\boldsymbol{v}$ is defined by the directed line segment from $(-1, 5)$ to $(2, -10)$.
If $\boldsymbol{v} = \begin{bmatrix} a \\ b \end{bmatrix}$, find a and b.

4 Let $A = (1, -2)$, $B = (3, 0)$ and $C = (2, -3)$ and let O be the origin.
Express each of the following vectors in the form $\begin{bmatrix} a \\ b \end{bmatrix}$:

a $\overrightarrow{OA}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{BC}$ **d** $\overrightarrow{CO}$ **e** $\overrightarrow{CB}$

Example 3

5 Let $\boldsymbol{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\boldsymbol{c} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

a Find:

i $\boldsymbol{a} + \boldsymbol{b}$ **ii** $2\boldsymbol{c} - \boldsymbol{a}$ **iii** $\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{c}$

b Show that $\boldsymbol{a} + \boldsymbol{b}$ is parallel to $\boldsymbol{c}$.

Example 4

6 If $A = (2, -3)$, $B = (4, 0)$, $C = (1, -4)$ and O is the origin, sketch the following vectors:

a $\overrightarrow{OA}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{BC}$ **d** $\overrightarrow{CO}$ **e** $\overrightarrow{CB}$

7 On graph paper, sketch the vectors joining the following pairs of points in the direction indicated:

a $(0, 0) \to (2, 1)$ **b** $(3, 4) \to (0, 0)$ **c** $(1, 3) \to (3, 4)$
d $(2, 4) \to (4, 3)$ **e** $(-2, 2) \to (5, -1)$ **f** $(-1, -3) \to (3, 0)$

8 Identify vectors from Question 7 which are parallel to each other.

9 **a** Plot the points $A(-1, 0)$, $B(1, 4)$, $C(4, 3)$ and $D(2, -1)$ on a set of coordinate axes.
b Sketch the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{AD}$ and $\overrightarrow{DC}$.
c Show that:

i $\overrightarrow{AB} = \overrightarrow{DC}$ **ii** $\overrightarrow{BC} = \overrightarrow{AD}$

d Describe the shape of the quadrilateral $ABCD$.

10 Find the values of m and n such that $m\begin{bmatrix} 3 \\ -3 \end{bmatrix} + n\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -19 \\ 61 \end{bmatrix}$

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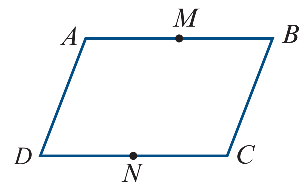
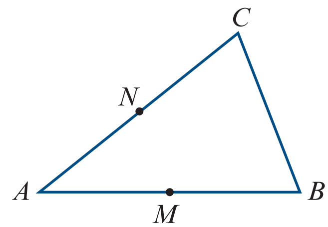
11 Points A, B, C, D are the vertices of a parallelogram, and M and N are the midpoints of AB and DC respectively. Let $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{AD}$.



a Express the following in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:
i $\overrightarrow{MD}$ **ii** $\overrightarrow{MN}$

b Find the relationship between $\overrightarrow{MN}$ and $\overrightarrow{AD}$.

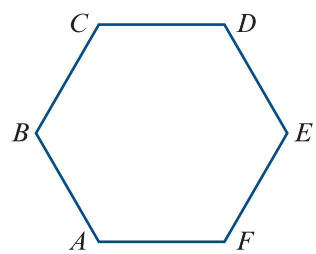
Example 5 **12** The figure represents the triangle ABC, where M and N are the midpoints of AB and AC respectively. Let $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{AC}$.



a Express $\overrightarrow{CB}$ and $\overrightarrow{MN}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

b Hence describe the relation between the two vectors (or directed line segments).

Example 6 **13** The figure shows a regular hexagon $ABCDEF$. Let $\boldsymbol{a} = \overrightarrow{AF}$ and $\boldsymbol{b} = \overrightarrow{AB}$. Express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:



a $\overrightarrow{CD}$ **b** $\overrightarrow{ED}$ **c** $\overrightarrow{BE}$ **d** $\overrightarrow{FC}$
e $\overrightarrow{FA}$ **f** $\overrightarrow{FB}$ **g** $\overrightarrow{FE}$

14 In parallelogram $ABCD$, let $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{BC}$. Express each of the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

a $\overrightarrow{DC}$ **b** $\overrightarrow{DA}$ **c** $\overrightarrow{AC}$ **d** $\overrightarrow{CA}$ **e** $\overrightarrow{BD}$

15 In triangle OAB, let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$. The point P on AB is such that $\overrightarrow{AP} = 2\overrightarrow{PB}$ and the point Q is such that $\overrightarrow{OP} = 3\overrightarrow{PQ}$. Express each of the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

a $\overrightarrow{BA}$ **b** $\overrightarrow{PB}$ **c** $\overrightarrow{OP}$ **d** $\overrightarrow{PQ}$ **e** $\overrightarrow{BQ}$

16 $PQRS$ is a quadrilateral in which $\overrightarrow{PQ} = \boldsymbol{u}$, $\overrightarrow{QR} = \boldsymbol{v}$ and $\overrightarrow{RS} = \boldsymbol{w}$. Express each of the following vectors in terms of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$:

a $\overrightarrow{PR}$ **b** $\overrightarrow{QS}$ **c** $\overrightarrow{PS}$

17 $OABC$ is a parallelogram. Let $\boldsymbol{u} = \overrightarrow{OA}$ and $\boldsymbol{v} = \overrightarrow{OC}$. Let M be the midpoint of AB.

a Express $\overrightarrow{OB}$ and $\overrightarrow{OM}$ in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$.

b Express $\overrightarrow{CM}$ in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$.

c If P is a point on CM and $\overrightarrow{CP} = \frac{2}{3}\overrightarrow{CM}$, express $\overrightarrow{CP}$ in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$.

d Find $\overrightarrow{OP}$ and hence show that P lies on the line segment OB.

e Find the ratio $OP : PB$.

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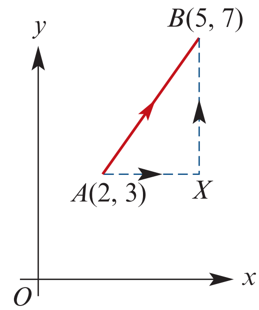
17B Components of vectors

The vector $\overrightarrow{AB}$ in the diagram is described by the column vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.



From the diagram, we see that the vector $\overrightarrow{AB}$ can also be expressed as the sum

$$\overrightarrow{AB} = \overrightarrow{AX} + \overrightarrow{XB}$$

Using column-vector notation:

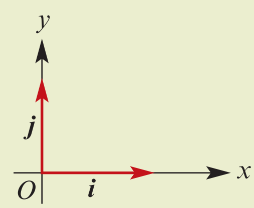
$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

This suggests the introduction of two important vectors.

Standard unit vectors in two dimensions

- Let $\boldsymbol{i}$ be the vector of unit length in the positive direction of the x-axis.
- Let $\boldsymbol{j}$ be the vector of unit length in the positive direction of the y-axis.

Using column-vector notation, we have $\boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



Note: These two vectors also played an important role in our study of linear transformations using matrices in Chapter 16.

For the example above, we have $\overrightarrow{AX} = 3\boldsymbol{i}$ and $\overrightarrow{XB} = 4\boldsymbol{j}$. Therefore

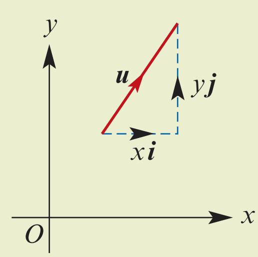
$$\overrightarrow{AB} = 3\boldsymbol{i} + 4\boldsymbol{j}$$

It is possible to describe any two-dimensional vector in this way.

Component form

- We can write the vector $\boldsymbol{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ as $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$.

  We say that $\boldsymbol{u}$ is the sum of the two **components** $x\boldsymbol{i}$ and $y\boldsymbol{j}$.
- The **magnitude** of vector $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$ is denoted by $|\boldsymbol{u}|$ and is given by $|\boldsymbol{u}| = \sqrt{x^2 + y^2}$.



Operations with vectors now look more like basic algebra:

- $(x\boldsymbol{i} + y\boldsymbol{j}) + (m\boldsymbol{i} + n\boldsymbol{j}) = (x + m)\boldsymbol{i} + (y + n)\boldsymbol{j}$
- $k(x\boldsymbol{i} + y\boldsymbol{j}) = kx\boldsymbol{i} + ky\boldsymbol{j}$

Two vectors are equal if and only if their components are equal:

$$x\boldsymbol{i} + y\boldsymbol{j} = m\boldsymbol{i} + n\boldsymbol{j} \quad \text{if and only if} \quad x = m \text{ and } y = n$$

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Example 7

a Find $\overrightarrow{AB}$ if $\overrightarrow{OA} = 3\boldsymbol{i}$ and $\overrightarrow{OB} = 2\boldsymbol{i} - \boldsymbol{j}$.

b Find $|2\boldsymbol{i} - 3\boldsymbol{j}|$.

Solution

a
$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} \\ &= -3\boldsymbol{i} + (2\boldsymbol{i} - \boldsymbol{j}) \\ &= -\boldsymbol{i} - \boldsymbol{j}\end{aligned}$$

b
$$\begin{aligned}|2\boldsymbol{i} - 3\boldsymbol{j}| &= \sqrt{2^2 + (-3)^2} \\ &= \sqrt{4 + 9} \\ &= \sqrt{13}\end{aligned}$$

Example 8

Let A and B be points on the Cartesian plane such that $\overrightarrow{OA} = 2\boldsymbol{i} + \boldsymbol{j}$ and $\overrightarrow{OB} = \boldsymbol{i} - 3\boldsymbol{j}$.

Find $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$.

Solution

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB}\end{aligned}$$

$$\therefore \quad \begin{aligned}\overrightarrow{AB} &= -(2\boldsymbol{i} + \boldsymbol{j}) + \boldsymbol{i} - 3\boldsymbol{j} \\ &= -\boldsymbol{i} - 4\boldsymbol{j}\end{aligned}$$

$$\therefore \quad |\overrightarrow{AB}| = \sqrt{1 + 16} = \sqrt{17}$$

Unit vectors

A **unit vector** is a vector of length one unit. For example, both $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors.

The unit vector in the direction of $\boldsymbol{a}$ is denoted by $\hat{\boldsymbol{a}}$. (We say 'a hat'.)

Since $|\hat{\boldsymbol{a}}| = 1$, we have

$$|\boldsymbol{a}|\,\hat{\boldsymbol{a}} = \boldsymbol{a}$$

$$\therefore \quad \hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$$

Example 9

Let $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j}$.

Find $|\boldsymbol{a}|$, the magnitude of $\boldsymbol{a}$, and hence find the unit vector in the direction of $\boldsymbol{a}$.

Solution

$$|\boldsymbol{a}| = \sqrt{9 + 16} = 5$$

$$\therefore \quad \hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a} = \frac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$$

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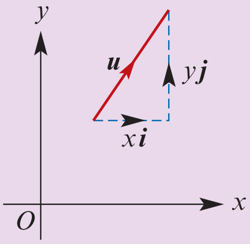
Section summary

- A **unit vector** is a vector of length one unit.
- Each vector $\boldsymbol{u}$ in the plane can be written in **component form** as $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$, where
  - $\boldsymbol{i}$ is the unit vector in the positive direction of the x-axis
  - $\boldsymbol{j}$ is the unit vector in the positive direction of the y-axis.



- The **magnitude** of vector $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$ is given by $|\boldsymbol{u}| = \sqrt{x^2 + y^2}$.
- The unit vector in the direction of vector $\boldsymbol{a}$ is given by $\hat{\boldsymbol{a}} = \dfrac{1}{|\boldsymbol{a}|}\boldsymbol{a}$.

Exercise 17B

Skillsheet

Example 7a

1 If A and B are points in the plane such that $\overrightarrow{OA} = \boldsymbol{i} + 2\boldsymbol{j}$ and $\overrightarrow{OB} = 3\boldsymbol{i} - 5\boldsymbol{j}$, find $\overrightarrow{AB}$.

2 $OAPB$ is a rectangle with $\overrightarrow{OA} = 5\boldsymbol{i}$ and $\overrightarrow{OB} = 6\boldsymbol{j}$. Express each of the following vectors in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$:

a $\overrightarrow{OP}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{BA}$

Example 7b

3 Determine the magnitude of each of the following vectors:

a $5\boldsymbol{i}$ **b** $-2\boldsymbol{j}$ **c** $3\boldsymbol{i} + 4\boldsymbol{j}$ **d** $-5\boldsymbol{i} + 12\boldsymbol{j}$

4 The vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are given by $\boldsymbol{u} = 7\boldsymbol{i} + 8\boldsymbol{j}$ and $\boldsymbol{v} = 2\boldsymbol{i} - 4\boldsymbol{j}$.

a Find $|\boldsymbol{u} - \boldsymbol{v}|$. **b** Find constants x and y such that $x\boldsymbol{u} + y\boldsymbol{v} = 44\boldsymbol{j}$.

5 Points A and B have position vectors $\overrightarrow{OA} = 10\boldsymbol{i}$ and $\overrightarrow{OB} = 4\boldsymbol{i} + 5\boldsymbol{j}$. If M is the midpoint of AB, find $\overrightarrow{OM}$ in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$.

6 $OPAQ$ is a rectangle with $\overrightarrow{OP} = 2\boldsymbol{i}$ and $\overrightarrow{OQ} = \boldsymbol{j}$. Let M be the point on OP such that $OM = \frac{1}{5}OP$ and let N be the point on MQ such that $MN = \frac{1}{6}MQ$.

a Find each of the following vectors in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$:

i $\overrightarrow{OM}$ **ii** $\overrightarrow{MQ}$ **iii** $\overrightarrow{MN}$ **iv** $\overrightarrow{ON}$ **v** $\overrightarrow{OA}$

b **i** Hence show that N is on the diagonal OA.

ii State the ratio of the lengths $ON : NA$.

7 The position vectors of A and B are given by $\overrightarrow{OA} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\overrightarrow{OB} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$. Find the distance between A and B.

8 Find the pronumerals in the following equations:

a $\boldsymbol{i} + 3\boldsymbol{j} = 2(\ell\boldsymbol{i} + k\boldsymbol{j})$ **b** $(x - 1)\boldsymbol{i} + y\boldsymbol{j} = 5\boldsymbol{i} + (x - 4)\boldsymbol{j}$

c $(x + y)\boldsymbol{i} + (x - y)\boldsymbol{j} = 6\boldsymbol{i}$ **d** $k(\boldsymbol{i} + \boldsymbol{j}) = 3\boldsymbol{i} - 2\boldsymbol{j} + \ell(2\boldsymbol{i} - \boldsymbol{j})$

Example 8

9 Let $A = (2, 3)$ and $B = (5, 1)$. Find $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$.

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10 Let $\overrightarrow{OA} = 3\boldsymbol{i}$, $\overrightarrow{OB} = \boldsymbol{i} + 4\boldsymbol{j}$ and $\overrightarrow{OC} = -3\boldsymbol{i} + \boldsymbol{j}$. Find:

a $\overrightarrow{AB}$ **b** $\overrightarrow{AC}$ **c** $|\overrightarrow{BC}|$

11 Let $A = (5, 1)$, $B = (0, 4)$ and $C = (-1, 0)$. Find:

a D such that $\overrightarrow{AB} = \overrightarrow{CD}$ **b** F such that $\overrightarrow{AF} = \overrightarrow{BC}$ **c** G such that $\overrightarrow{AB} = 2\overrightarrow{GC}$

12 Let $\boldsymbol{a} = \boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{b} = -2\boldsymbol{i} + 2\boldsymbol{j}$. Points A, B and C are such that $\overrightarrow{AO} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{BC} = 2\boldsymbol{a}$, where O is the origin. Find the coordinates of A, B and C.

13 A, B, C and D are the vertices of a parallelogram and O is the origin. $A = (2, -1)$, $B = (-5, 4)$ and $C = (1, 7)$.

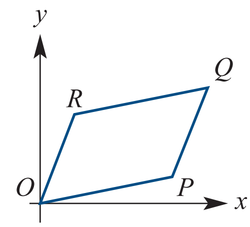
a Find:

i $\overrightarrow{OA}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{OC}$ **iv** $\overrightarrow{BC}$ **v** $\overrightarrow{AD}$

b Hence find the coordinates of D.

14 The diagram shows a parallelogram $OPQR$. The points P and Q have coordinates $(12, 5)$ and $(18, 13)$ respectively. Find:

a $\overrightarrow{OP}$ and $\overrightarrow{PQ}$ **b** $|\overrightarrow{RQ}|$ and $|\overrightarrow{OR}|$



15 $A(1, 6)$, $B(3, 1)$ and $C(13, 5)$ are the vertices of a triangle ABC.

a Find:

i $|\overrightarrow{AB}|$ **ii** $|\overrightarrow{BC}|$ **iii** $|\overrightarrow{CA}|$

b Hence show that ABC is a right-angled triangle.

16 $A(4, 4)$, $B(3, 1)$ and $C(7, 3)$ are the vertices of a triangle ABC.

a Find the vectors:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{BC}$ **iii** $\overrightarrow{CA}$

b Find:

i $|\overrightarrow{AB}|$ **ii** $|\overrightarrow{BC}|$ **iii** $|\overrightarrow{CA}|$

c Hence show that triangle ABC is a right-angled isosceles triangle.

17 $A(-3, 2)$ and $B(0, 7)$ are points on the Cartesian plane, O is the origin and M is the midpoint of AB.

a Find:

i $\overrightarrow{OA}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{BA}$ **iv** $\overrightarrow{BM}$

b Hence find the coordinates of M. (Hint: $\overrightarrow{OM} = \overrightarrow{OB} + \overrightarrow{BM}$.)

Example 9 **18** Find the unit vector in the direction of each of the following vectors:

a $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j}$ **b** $\boldsymbol{b} = 3\boldsymbol{i} - \boldsymbol{j}$ **c** $\boldsymbol{c} = -\boldsymbol{i} + \boldsymbol{j}$

d $\boldsymbol{d} = \boldsymbol{i} - \boldsymbol{j}$ **e** $\boldsymbol{e} = \frac{1}{2}\boldsymbol{i} + \frac{1}{3}\boldsymbol{j}$ **f** $\boldsymbol{f} = 6\boldsymbol{i} - 4\boldsymbol{j}$

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17C Scalar product of vectors

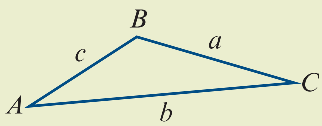
▶ The sine and cosine rules

In our study of vectors and their applications, we will make use of the sine and cosine rules, which are introduced in Mathematical Methods Year 11.

Sine rule

For triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



The sine rule is used to find unknown quantities in a triangle in the following cases:

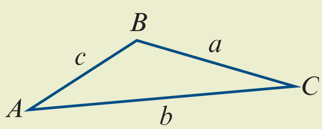
- one side and two angles are given
- two sides and a non-included angle are given.

In the first case, the triangle is uniquely defined. But in the second case, there may be two triangles.

Cosine rule

For triangle ABC:

$$a^2 = b^2 + c^2 - 2bc\cos A$$



The cosine rule is used to find unknown quantities in a triangle in the following cases:

- two sides and the included angle are given
- three sides are given.

▶ Defining the scalar product

The scalar product is an operation that takes two vectors and gives a real number.

Definition of the scalar product

We define the **scalar product** of two vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$ by

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2$$

For example:

$$\begin{aligned}(2\boldsymbol{i} + 3\boldsymbol{j}) \cdot (\boldsymbol{i} - 4\boldsymbol{j}) &= 2 \times 1 + 3 \times (-4) \\ &= -10\end{aligned}$$

The scalar product is often called the **dot product**.

Note: If $\boldsymbol{a} = \boldsymbol{0}$ or $\boldsymbol{b} = \boldsymbol{0}$, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

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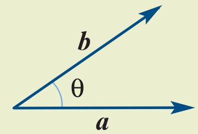
Geometric description of the scalar product

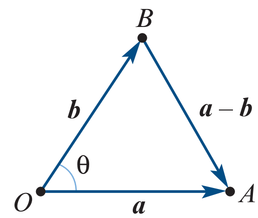
For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we have

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.



Proof Let $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$. Then using the cosine rule in $\triangle OAB$ gives



$$|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta = |\boldsymbol{a} - \boldsymbol{b}|^2$$

$$(a_1^2 + a_2^2) + (b_1^2 + b_2^2) - 2|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta = (a_1 - b_1)^2 + (a_2 - b_2)^2$$

$$2(a_1b_1 + a_2b_2) = 2|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$$

$$a_1b_1 + a_2b_2 = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$$

$$\therefore \quad \boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$$

Example 10

a If $|\boldsymbol{a}| = 4$, $|\boldsymbol{b}| = 5$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $30°$, find $\boldsymbol{a} \cdot \boldsymbol{b}$.

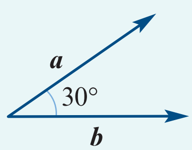
b If $|\boldsymbol{a}| = 4$, $|\boldsymbol{b}| = 5$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $150°$, find $\boldsymbol{a} \cdot \boldsymbol{b}$.

Solution

a $\boldsymbol{a} \cdot \boldsymbol{b} = 4 \times 5 \times \cos 30°$

$= 20 \times \dfrac{\sqrt{3}}{2}$

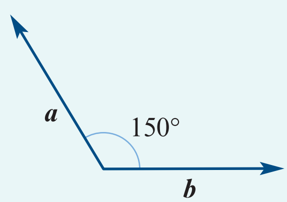
$= 10\sqrt{3}$



b $\boldsymbol{a} \cdot \boldsymbol{b} = 4 \times 5 \times \cos 150°$

$= 20 \times \dfrac{-\sqrt{3}}{2}$

$= -10\sqrt{3}$



Properties of the scalar product

- $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
- $k(\boldsymbol{a} \cdot \boldsymbol{b}) = (k\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (k\boldsymbol{b})$
- $\boldsymbol{a} \cdot \boldsymbol{0} = 0$
- $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$
- If the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- If $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ for non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular.
- For parallel vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we have

$$\boldsymbol{a} \cdot \boldsymbol{b} = \begin{cases} |\boldsymbol{a}|\,|\boldsymbol{b}| & \text{if } \boldsymbol{a} \text{ and } \boldsymbol{b} \text{ are parallel and in the same direction} \\ -|\boldsymbol{a}|\,|\boldsymbol{b}| & \text{if } \boldsymbol{a} \text{ and } \boldsymbol{b} \text{ are parallel and in opposite directions} \end{cases}$$

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▶ Finding the magnitude of the angle between two vectors

To find the angle θ between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we can use the two different forms of the scalar product:

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta \quad \text{and} \quad \boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$$

Therefore

$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|} = \frac{a_1 b_1 + a_2 b_2}{|\boldsymbol{a}|\,|\boldsymbol{b}|}$$

Example 11

A, B and C are points defined by the position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively, where

$$\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}, \quad \boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} \quad \text{and} \quad \boldsymbol{c} = \boldsymbol{i} - 2\boldsymbol{j}$$

Find the magnitude of $\angle ABC$.

Solution

$\angle ABC$ is the angle between vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

$$\overrightarrow{BA} = \boldsymbol{a} - \boldsymbol{b} = -\boldsymbol{i} + 2\boldsymbol{j}$$

$$\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b} = -\boldsymbol{i} - 3\boldsymbol{j}$$

We will apply the scalar product:

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = |\overrightarrow{BA}|\,|\overrightarrow{BC}|\cos(\angle ABC)$$

We have

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = (-\boldsymbol{i} + 2\boldsymbol{j}) \cdot (-\boldsymbol{i} - 3\boldsymbol{j}) = 1 - 6 = -5$$

$$|\overrightarrow{BA}| = \sqrt{1+4} = \sqrt{5}$$

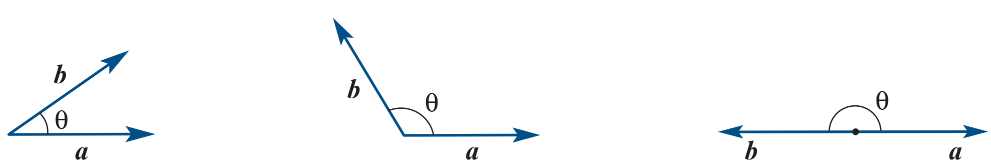
$$|\overrightarrow{BC}| = \sqrt{1+9} = \sqrt{10}$$

Therefore

$$\cos(\angle ABC) = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}|\,|\overrightarrow{BC}|} = \frac{-5}{\sqrt{5}\sqrt{10}} = \frac{-1}{\sqrt{2}}$$

Hence $\angle ABC = \dfrac{3\pi}{4}$.

Note: When two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are placed so that their initial points coincide, the angle θ between $\boldsymbol{a}$ and $\boldsymbol{b}$ is chosen as shown in the diagrams. Note that $0 \le \theta \le \pi$.



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Section summary

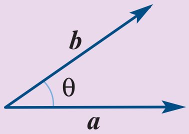
- The **scalar product** of vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$ is given by

  $$\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2$$

- The scalar product can be described geometrically by

  $$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$$

  where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.



- Therefore $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$.
- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are **perpendicular** if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

Exercise 17C

Skillsheet

1 Let $\boldsymbol{a} = \boldsymbol{i} - 4\boldsymbol{j}$, $\boldsymbol{b} = 2\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{c} = -2\boldsymbol{i} - 2\boldsymbol{j}$. Find:

a $\boldsymbol{a} \cdot \boldsymbol{a}$ **b** $\boldsymbol{b} \cdot \boldsymbol{b}$ **c** $\boldsymbol{c} \cdot \boldsymbol{c}$ **d** $\boldsymbol{a} \cdot \boldsymbol{b}$

e $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c})$ **f** $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{c})$ **g** $(\boldsymbol{a} + 2\boldsymbol{b}) \cdot (3\boldsymbol{c} - \boldsymbol{b})$

2 Let $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j}$, $\boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{j}$ and $\boldsymbol{c} = -\boldsymbol{i} + 3\boldsymbol{j}$. Find:

a $\boldsymbol{a} \cdot \boldsymbol{a}$ **b** $\boldsymbol{b} \cdot \boldsymbol{b}$ **c** $\boldsymbol{a} \cdot \boldsymbol{b}$

d $\boldsymbol{a} \cdot \boldsymbol{c}$ **e** $\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b})$

Example 10

3 **a** If $|\boldsymbol{a}| = 5$, $|\boldsymbol{b}| = 6$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 45°, find $\boldsymbol{a} \cdot \boldsymbol{b}$.

b If $|\boldsymbol{a}| = 5$, $|\boldsymbol{b}| = 6$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 135°, find $\boldsymbol{a} \cdot \boldsymbol{b}$.

4 Expand and simplify:

a $(\boldsymbol{a} + 2\boldsymbol{b}) \cdot (\boldsymbol{a} + 2\boldsymbol{b})$ **b** $|\boldsymbol{a} + \boldsymbol{b}|^2 - |\boldsymbol{a} - \boldsymbol{b}|^2$

c $\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b}) - \boldsymbol{b} \cdot (\boldsymbol{a} + \boldsymbol{b})$ **d** $\dfrac{\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b}) - \boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|}$

5 If A and B are points defined by the position vectors $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{b} = -\boldsymbol{i} + 3\boldsymbol{j}$ respectively, find:

a $\overrightarrow{AB}$

b $|\overrightarrow{AB}|$

c the magnitude of the angle between vectors $\overrightarrow{AB}$ and $\boldsymbol{a}$.

6 Let C and D be points with position vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ respectively. If $|\boldsymbol{c}| = 5$, $|\boldsymbol{d}| = 7$ and $\boldsymbol{c} \cdot \boldsymbol{d} = 4$, find $|\overrightarrow{CD}|$.

7 Solve each of the following equations:

a $(\boldsymbol{i} + 2\boldsymbol{j}) \cdot (5\boldsymbol{i} + x\boldsymbol{j}) = -6$ **b** $(x\boldsymbol{i} + 7\boldsymbol{j}) \cdot (-4\boldsymbol{i} + x\boldsymbol{j}) = 10$

c $(x\boldsymbol{i} + \boldsymbol{j}) \cdot (-2\boldsymbol{i} - 3\boldsymbol{j}) = x$ **d** $x(2\boldsymbol{i} + 3\boldsymbol{j}) \cdot (\boldsymbol{i} + x\boldsymbol{j}) = 6$

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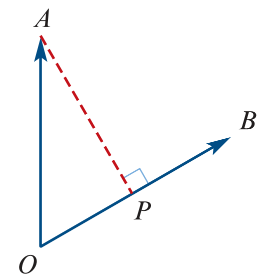
8 Points A and B are defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{b} = 2\boldsymbol{i} + 5\boldsymbol{j}$. Let P be the point on OB such that AP is perpendicular to OB. Then $\overrightarrow{OP} = q\boldsymbol{b}$, for a constant q.



a Express $\overrightarrow{AP}$ in terms of q, $\boldsymbol{a}$ and $\boldsymbol{b}$.
b Use the fact that $\overrightarrow{AP} \cdot \overrightarrow{OB} = 0$ to find the value of q.
c Find the coordinates of the point P.

Example 11

9 Find the angle, in degrees, between each of the following pairs of vectors, correct to two decimal places:

a $\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{i} - 4\boldsymbol{j}$
b $-2\boldsymbol{i} + \boldsymbol{j}$ and $-2\boldsymbol{i} - 2\boldsymbol{j}$
c $2\boldsymbol{i} - \boldsymbol{j}$ and $4\boldsymbol{i}$
d $7\boldsymbol{i} + \boldsymbol{j}$ and $-\boldsymbol{i} + \boldsymbol{j}$

10 Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be non-zero vectors such that $\boldsymbol{a} \cdot \boldsymbol{b} = 0$. Use the geometric description of the scalar product to show that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular vectors.

For Questions 11–12, find the angles in degrees correct to two decimal places.

11 Let A and B be the points defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j}$ respectively. Let M be the midpoint of AB. Find:

a $\overrightarrow{OM}$
b $\angle AOM$
c $\angle BMO$

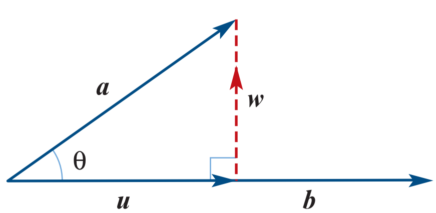
12 Let A, B and C be the points defined by the position vectors $3\boldsymbol{i}$, $4\boldsymbol{j}$ and $-2\boldsymbol{i} + 6\boldsymbol{j}$ respectively. Let M and N be the midpoints of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ respectively. Find:

a **i** $\overrightarrow{OM}$ **ii** $\overrightarrow{ON}$
b $\angle MON$
c $\angle MOC$

17D Vector projections

It is often useful to decompose a vector $\boldsymbol{a}$ into a sum of two vectors, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.

From the diagram, it can be seen that



$$\boldsymbol{a} = \boldsymbol{u} + \boldsymbol{w}$$

where $\boldsymbol{u} = k\boldsymbol{b}$ and so $\boldsymbol{w} = \boldsymbol{a} - \boldsymbol{u} = \boldsymbol{a} - k\boldsymbol{b}$.

For $\boldsymbol{w}$ to be perpendicular to $\boldsymbol{b}$, we must have

$$\begin{aligned} \boldsymbol{w} \cdot \boldsymbol{b} &= 0 \\ (\boldsymbol{a} - k\boldsymbol{b}) \cdot \boldsymbol{b} &= 0 \\ \boldsymbol{a} \cdot \boldsymbol{b} - k(\boldsymbol{b} \cdot \boldsymbol{b}) &= 0 \end{aligned}$$

Hence $k = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}$ and therefore $\boldsymbol{u} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$.

This vector $\boldsymbol{u}$ is called the **vector projection** (or **vector resolute**) of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$.

ISBN 978-1-316-63608-4

Cambridge University Press

Vector resolute

The **vector resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ can be expressed in any one of the following equivalent forms:

$$\boldsymbol{u} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\,\boldsymbol{b} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|^2}\,\boldsymbol{b} = \left(\boldsymbol{a} \cdot \frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right)\left(\frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right) = (\boldsymbol{a} \cdot \hat{\boldsymbol{b}})\,\hat{\boldsymbol{b}}$$

Note: The quantity $\boldsymbol{a} \cdot \hat{\boldsymbol{b}} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$ is the 'signed length' of the vector resolute $\boldsymbol{u}$ and is called the **scalar resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$.

Note that, from our previous calculation, we have $\boldsymbol{w} = \boldsymbol{a} - \boldsymbol{u} = \boldsymbol{a} - \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\,\boldsymbol{b}$.

Expressing $\boldsymbol{a}$ as the sum of the two components, the first parallel to $\boldsymbol{b}$ and the second perpendicular to $\boldsymbol{b}$, gives

$$\boldsymbol{a} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\,\boldsymbol{b} + \left(\boldsymbol{a} - \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\,\boldsymbol{b}\right)$$

This is sometimes described as resolving the vector $\boldsymbol{a}$ into **rectangular components**, one parallel to $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.

Example 12

Let $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j}$. Find the vector resolute of:

a $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$

b $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$.

Solution

a $\boldsymbol{a} \cdot \boldsymbol{b} = 1 - 3 = -2$

$\boldsymbol{b} \cdot \boldsymbol{b} = 1 + 1 = 2$

The vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is

$$\begin{aligned}\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\,\boldsymbol{b} &= \frac{-2}{2}(\boldsymbol{i} - \boldsymbol{j})\\ &= -1(\boldsymbol{i} - \boldsymbol{j})\\ &= -\boldsymbol{i} + \boldsymbol{j}\end{aligned}$$

b $\boldsymbol{b} \cdot \boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{b} = -2$

$\boldsymbol{a} \cdot \boldsymbol{a} = 1 + 9 = 10$

The vector resolute of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ is

$$\begin{aligned}\frac{\boldsymbol{b} \cdot \boldsymbol{a}}{\boldsymbol{a} \cdot \boldsymbol{a}}\,\boldsymbol{a} &= \frac{-2}{10}(\boldsymbol{i} + 3\boldsymbol{j})\\ &= -\frac{1}{5}(\boldsymbol{i} + 3\boldsymbol{j})\end{aligned}$$

Example 13

Find the scalar resolute of $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j}$ in the direction of $\boldsymbol{b} = -\boldsymbol{i} + 3\boldsymbol{j}$.

Solution

$\boldsymbol{a} \cdot \boldsymbol{b} = -2 + 6 = 4$

$|\boldsymbol{b}| = \sqrt{1 + 9} = \sqrt{10}$

The scalar resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is

$$\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|} = \frac{4}{\sqrt{10}} = \frac{2\sqrt{10}}{5}$$

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Example 14

Resolve $\boldsymbol{i} + 3\boldsymbol{j}$ into rectangular components, one of which is parallel to $2\boldsymbol{i} - 2\boldsymbol{j}$.

Solution

Let $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = 2\boldsymbol{i} - 2\boldsymbol{j}$.

The vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is given by $\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}} \boldsymbol{b}$.

We have

$$\boldsymbol{a} \cdot \boldsymbol{b} = 2 - 6 = -4$$

$$\boldsymbol{b} \cdot \boldsymbol{b} = 4 + 4 = 8$$

Therefore the vector resolute is

$$\frac{-4}{8}(2\boldsymbol{i} - 2\boldsymbol{j}) = -\frac{1}{2}(2\boldsymbol{i} - 2\boldsymbol{j}) = -\boldsymbol{i} + \boldsymbol{j}$$

The perpendicular component is

$$\boldsymbol{a} - (-\boldsymbol{i} + \boldsymbol{j}) = (\boldsymbol{i} + 3\boldsymbol{j}) - (-\boldsymbol{i} + \boldsymbol{j}) = 2\boldsymbol{i} + 2\boldsymbol{j}$$

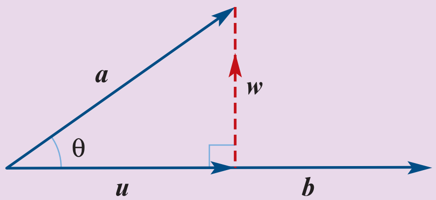
Hence we can write

$$\boldsymbol{i} + 3\boldsymbol{j} = (-\boldsymbol{i} + \boldsymbol{j}) + (2\boldsymbol{i} + 2\boldsymbol{j})$$

Check: We can check our calculation by verifying that the second component is indeed perpendicular to $\boldsymbol{b}$. We have $(2\boldsymbol{i} + 2\boldsymbol{j}) \cdot (2\boldsymbol{i} - 2\boldsymbol{j}) = 4 - 4 = 0$, as expected.

Section summary

- Resolving a vector $\boldsymbol{a}$ into rectangular components is expressing the vector $\boldsymbol{a}$ as a sum of two vectors, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.
- The **vector resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is given by $\boldsymbol{u} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}} \boldsymbol{b}$.
- The **scalar resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is the 'signed length' of the vector resolute $\boldsymbol{u}$ and is given by $\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$.



Exercise 17D

1 Points A and B are defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = 2\boldsymbol{i} + 2\boldsymbol{j}$.

a Find $\hat{\boldsymbol{a}}$. **b** Find $\hat{\boldsymbol{b}}$. **c** Find $\hat{\boldsymbol{c}}$, where $\boldsymbol{c} = \overrightarrow{AB}$.

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2 Let $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j}$.

a Find:

i $\hat{\boldsymbol{a}}$ **ii** $|\boldsymbol{b}|$

b Find the vector with the same magnitude as $\boldsymbol{b}$ and with the same direction as $\boldsymbol{a}$.

3 Points A and B are defined by the position vectors $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{b} = 5\boldsymbol{i} + 12\boldsymbol{j}$.

a Find:

i $\hat{\boldsymbol{a}}$ **ii** $\hat{\boldsymbol{b}}$

b Find the unit vector which bisects $\angle AOB$.

Example 12

4 For each pair of vectors, find the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$:

a $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - 4\boldsymbol{j}$

b $\boldsymbol{a} = \boldsymbol{i} - 3\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - 4\boldsymbol{j}$

c $\boldsymbol{a} = 4\boldsymbol{i} - \boldsymbol{j}$ and $\boldsymbol{b} = 4\boldsymbol{i}$

Example 13

5 For each of the following pairs of vectors, find the scalar resolute of the first vector in the direction of the second vector:

a $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i}$

b $\boldsymbol{a} = 3\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{c} = \boldsymbol{i} - 2\boldsymbol{j}$

c $\boldsymbol{b} = 2\boldsymbol{j}$ and $\boldsymbol{a} = 2\boldsymbol{i} + \sqrt{3}\boldsymbol{j}$

d $\boldsymbol{b} = \boldsymbol{i} - \sqrt{5}\boldsymbol{j}$ and $\boldsymbol{c} = -\boldsymbol{i} + 4\boldsymbol{j}$

Example 14

6 For each of the following pairs of vectors, find the resolution of the vector $\boldsymbol{a}$ into rectangular components, one of which is parallel to $\boldsymbol{b}$:

a $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j},\ \boldsymbol{b} = 5\boldsymbol{i}$

b $\boldsymbol{a} = 3\boldsymbol{i} + \boldsymbol{j},\ \boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j}$

c $\boldsymbol{a} = -\boldsymbol{i} + \boldsymbol{j},\ \boldsymbol{b} = 2\boldsymbol{i} + 2\boldsymbol{j}$

7 Let A and B be the points defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j}$ respectively. Find:

a the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$

b a unit vector through A perpendicular to OB

8 Let A and B be the points defined by the position vectors $\boldsymbol{a} = 4\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j}$ respectively. Find:

a the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$

b the vector component of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$

c the shortest distance from A to line OB

9 Points A, B and C have position vectors $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j}$, $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{c} = 2\boldsymbol{i} - 3\boldsymbol{j}$. Find:

a **i** $\overrightarrow{AB}$ **ii** $\overrightarrow{AC}$

b the vector resolute of $\overrightarrow{AB}$ in the direction of $\overrightarrow{AC}$

c the shortest distance from B to line AC

d the area of triangle ABC

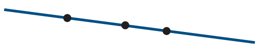
ISBN 978-1-316-63608-4

17E Geometric proofs

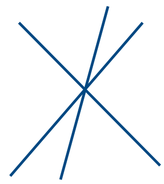
In this section we see how vectors can be used as a tool for proving geometric results.

We require the following two definitions.

Collinear points Three or more points are collinear if they all lie on a single line.



Concurrent lines Three or more lines are concurrent if they all pass through a single point.



Here are some properties of vectors that will be useful:

- For $k \in \mathbb{R}^+$, the vector $k\boldsymbol{a}$ is in the same direction as $\boldsymbol{a}$ and has magnitude $k|\boldsymbol{a}|$, and the vector $-k\boldsymbol{a}$ is in the opposite direction to $\boldsymbol{a}$ and has magnitude $k|\boldsymbol{a}|$.
- If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, then $\boldsymbol{b} = k\boldsymbol{a}$ for some $k \in \mathbb{R} \setminus \{0\}$. Conversely, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors such that $\boldsymbol{b} = k\boldsymbol{a}$ for some $k \in \mathbb{R} \setminus \{0\}$, then $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.
- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel with at least one point in common, then $\boldsymbol{a}$ and $\boldsymbol{b}$ lie on the same straight line. For example, if $\overrightarrow{AB} = k\overrightarrow{BC}$ for some $k \in \mathbb{R} \setminus \{0\}$, then A, B and C are collinear.
- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$

Example 15

Three points P, Q and R have position vectors $\boldsymbol{p}$, $\boldsymbol{q}$ and $k(2\boldsymbol{p} + \boldsymbol{q})$ respectively, relative to a fixed origin O. The points O, P and Q are not collinear.

Find the value of k if:

a $\overrightarrow{QR}$ is parallel to $\boldsymbol{p}$ **b** $\overrightarrow{PR}$ is parallel to $\boldsymbol{q}$ **c** P, Q and R are collinear.

Solution

a
$$\begin{aligned}\overrightarrow{QR} &= \overrightarrow{QO} + \overrightarrow{OR}\\ &= -\boldsymbol{q} + k(2\boldsymbol{p} + \boldsymbol{q})\\ &= 2k\boldsymbol{p} + (k-1)\boldsymbol{q}\end{aligned}$$

If $\overrightarrow{QR}$ is parallel to $\boldsymbol{p}$, then there is some $\lambda \in \mathbb{R} \setminus \{0\}$ such that

$$2k\boldsymbol{p} + (k-1)\boldsymbol{q} = \lambda\boldsymbol{p}$$

This implies that

$$2k = \lambda \quad \text{and} \quad k - 1 = 0$$

Hence $k = 1$.

b
$$\begin{aligned}\overrightarrow{PR} &= \overrightarrow{PO} + \overrightarrow{OR}\\ &= -\boldsymbol{p} + k(2\boldsymbol{p} + \boldsymbol{q})\\ &= (2k-1)\boldsymbol{p} + k\boldsymbol{q}\end{aligned}$$

If $\overrightarrow{PR}$ is parallel to $\boldsymbol{q}$, then there is some $m \in \mathbb{R} \setminus \{0\}$ such that

$$(2k-1)\boldsymbol{p} + k\boldsymbol{q} = m\boldsymbol{q}$$

This implies that

$$2k - 1 = 0 \quad \text{and} \quad k = m$$

Hence $k = \frac{1}{2}$.

Note: Since points O, P and Q are not collinear, the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ are not parallel.

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c If points P, Q and R are collinear, then there exists $n \in \mathbb{R} \setminus \{0\}$ such that

$$n\overrightarrow{PQ} = \overrightarrow{QR}$$

$$\therefore \quad n(-\boldsymbol{p} + \boldsymbol{q}) = 2k\boldsymbol{p} + (k-1)\boldsymbol{q}$$

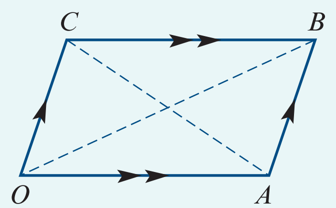
This implies that

$$-n = 2k \quad \text{and} \quad n = k - 1$$

Therefore $3k - 1 = 0$ and so $k = \frac{1}{3}$.

Example 16

Suppose that $OABC$ is a parallelogram. Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{c} = \overrightarrow{OC}$.



a Express each of the following in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$:

i $\overrightarrow{OB}$ **ii** $\overrightarrow{CA}$

b Find in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$:

i $|\overrightarrow{OB}|^2$ **ii** $|\overrightarrow{CA}|^2$

c Hence, prove that if the diagonals of a parallelogram are of equal length, then the parallelogram is a rectangle.

Solution

a **i**
$$\begin{aligned}\overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{AB}\\ &= \overrightarrow{OA} + \overrightarrow{OC}\\ &= \boldsymbol{a} + \boldsymbol{c}\end{aligned}$$

ii
$$\begin{aligned}\overrightarrow{CA} &= \overrightarrow{CB} + \overrightarrow{BA}\\ &= \overrightarrow{OA} - \overrightarrow{OC}\\ &= \boldsymbol{a} - \boldsymbol{c}\end{aligned}$$

b **i**
$$\begin{aligned}|\overrightarrow{OB}|^2 &= \overrightarrow{OB} \cdot \overrightarrow{OB}\\ &= (\boldsymbol{a} + \boldsymbol{c}) \cdot (\boldsymbol{a} + \boldsymbol{c})\\ &= \boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{c} \cdot \boldsymbol{a} + \boldsymbol{c} \cdot \boldsymbol{c}\\ &= |\boldsymbol{a}|^2 + 2\boldsymbol{a} \cdot \boldsymbol{c} + |\boldsymbol{c}|^2\end{aligned}$$

ii
$$\begin{aligned}|\overrightarrow{CA}|^2 &= \overrightarrow{CA} \cdot \overrightarrow{CA}\\ &= (\boldsymbol{a} - \boldsymbol{c}) \cdot (\boldsymbol{a} - \boldsymbol{c})\\ &= \boldsymbol{a} \cdot \boldsymbol{a} - \boldsymbol{a} \cdot \boldsymbol{c} - \boldsymbol{c} \cdot \boldsymbol{a} + \boldsymbol{c} \cdot \boldsymbol{c}\\ &= |\boldsymbol{a}|^2 - 2\boldsymbol{a} \cdot \boldsymbol{c} + |\boldsymbol{c}|^2\end{aligned}$$

c Assume that the diagonals of the parallelogram $OABC$ are of equal length. Then $|\overrightarrow{OB}| = |\overrightarrow{CA}|$. This implies that

$$\begin{aligned}|\overrightarrow{OB}|^2 &= |\overrightarrow{CA}|^2\\ |\boldsymbol{a}|^2 + 2\boldsymbol{a} \cdot \boldsymbol{c} + |\boldsymbol{c}|^2 &= |\boldsymbol{a}|^2 - 2\boldsymbol{a} \cdot \boldsymbol{c} + |\boldsymbol{c}|^2\\ 4\boldsymbol{a} \cdot \boldsymbol{c} &= 0\\ \therefore \quad \boldsymbol{a} \cdot \boldsymbol{c} &= 0\end{aligned}$$

We have shown that $\overrightarrow{OA} \cdot \overrightarrow{OC} = 0$. So $\angle COA = 90°$. Hence the parallelogram $OABC$ is a rectangle.

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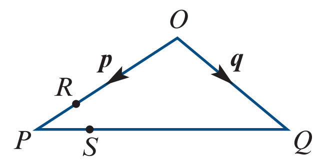
Exercise 17E

1 In the diagram, $OR = \frac{4}{5}OP$, $\boldsymbol{p} = \overrightarrow{OP}$, $\boldsymbol{q} = \overrightarrow{OQ}$ and $PS : SQ = 1 : 4$.



- **a** Express each of the following in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$:
  - **i** $\overrightarrow{OR}$ **ii** $\overrightarrow{RP}$ **iii** $\overrightarrow{PO}$ **iv** $\overrightarrow{PS}$ **v** $\overrightarrow{RS}$
- **b** What can be said about line segments RS and OQ?
- **c** What type of quadrilateral is $ORSQ$?
- **d** The area of triangle PRS is 5 cm^2. What is the area of $ORSQ$?

2 The position vectors of three points A, B and C relative to an origin O are $\boldsymbol{a}$, $\boldsymbol{b}$ and $k\boldsymbol{a}$ respectively. The point P lies on AB and is such that $AP = 2PB$. The point Q lies on BC and is such that $CQ = 6QB$.

- **a** Find in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:
  - **i** the position vector of P **ii** the position vector of Q
- **b** Given that OPQ is a straight line, find:
  - **i** the value of k **ii** the ratio $\dfrac{OP}{PQ}$
- **c** The position vector of a point R is $\frac{7}{3}\boldsymbol{a}$. Show that PR is parallel to BC.

Example 15

3 The position vectors of two points A and B relative to an origin O are $3\boldsymbol{i} + 3.5\boldsymbol{j}$ and $6\boldsymbol{i} - 1.5\boldsymbol{j}$ respectively.

- **a** **i** Given that $\overrightarrow{OD} = \frac{1}{3}\overrightarrow{OB}$ and $\overrightarrow{AE} = \frac{1}{4}\overrightarrow{AB}$, write down the position vectors of D and E.
  - **ii** Hence find $|\overrightarrow{ED}|$.
- **b** Given that OE and AD intersect at X and that $\overrightarrow{OX} = p\overrightarrow{OE}$ and $\overrightarrow{XD} = q\overrightarrow{AD}$, find the position vector of X in terms of:
  - **i** p **ii** q
- **c** Hence determine the values of p and q.

4 Points P and Q have position vectors $\boldsymbol{p}$ and $\boldsymbol{q}$, with reference to an origin O, and M is the point on PQ such that

$$\beta\overrightarrow{PM} = \alpha\overrightarrow{MQ}$$

- **a** Prove that the position vector of M is given by $\boldsymbol{m} = \dfrac{\beta\boldsymbol{p} + \alpha\boldsymbol{q}}{\alpha + \beta}$.
- **b** Write the position vectors of P and Q as $\boldsymbol{p} = k\boldsymbol{a}$ and $\boldsymbol{q} = \ell\boldsymbol{b}$, where k and ℓ are positive real numbers and $\boldsymbol{a}$ and $\boldsymbol{b}$ are unit vectors.
  - **i** Prove that the position vector of any point on the internal bisector of $\angle POQ$ has the form $\lambda(\boldsymbol{a} + \boldsymbol{b})$.
  - **ii** If M is the point where the internal bisector of $\angle POQ$ meets PQ, show that

$$\frac{\alpha}{\beta} = \frac{k}{\ell}$$

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Example 16

5 Suppose that $OABC$ is a parallelogram. Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{c} = \overrightarrow{OC}$.

a Express each of the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{AC}$

b Find $\overrightarrow{OB} \cdot \overrightarrow{AC}$.

c Hence, prove that the diagonals of a parallelogram intersect at right angles if and only if it is a rhombus.

6 Suppose that $ORST$ is a parallelogram, where O is the origin. Let U be the midpoint of RS and let V be the midpoint of ST. Denote the position vectors of R, S, T, U and V by $\boldsymbol{r}$, $\boldsymbol{s}$, $\boldsymbol{t}$, $\boldsymbol{u}$ and $\boldsymbol{v}$ respectively.

a Express $\boldsymbol{s}$ in terms of $\boldsymbol{r}$ and $\boldsymbol{t}$.

b Express $\boldsymbol{v}$ in terms of $\boldsymbol{s}$ and $\boldsymbol{t}$.

c Hence, or otherwise, show that $4(\boldsymbol{u} + \boldsymbol{v}) = 3(\boldsymbol{r} + \boldsymbol{s} + \boldsymbol{t})$.

7 Prove that the midpoints of any quadrilateral are the vertices of a parallelogram.

8 Prove that the diagonals of a square are of equal length and bisect each other.

9 Prove that the diagonals of a parallelogram bisect each other.

10 Prove that the altitudes of a triangle are concurrent. That is, they meet at a point.

11 Apollonius' theorem

For $\triangle OAB$, the point C is the midpoint of side AB. Prove that:

a $4\overrightarrow{OC} \cdot \overrightarrow{OC} = OA^2 + OB^2 + 2\overrightarrow{OA} \cdot \overrightarrow{OB}$

b $4\overrightarrow{AC} \cdot \overrightarrow{AC} = OA^2 + OB^2 - 2\overrightarrow{OA} \cdot \overrightarrow{OB}$

c $2OC^2 + 2AC^2 = OA^2 + OB^2$

12 If P is any point in the plane of rectangle $ABCD$, prove that

$$PA^2 + PC^2 = PB^2 + PD^2$$

13 Prove that the medians bisecting the equal sides of an isosceles triangle are equal.

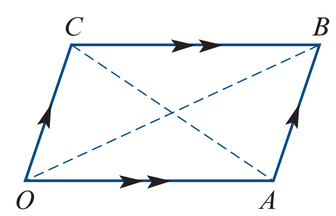
14 **a** Prove that if $(\boldsymbol{c} - \boldsymbol{b}) \cdot \boldsymbol{a} = 0$ and $(\boldsymbol{c} - \boldsymbol{a}) \cdot \boldsymbol{b} = 0$, then $(\boldsymbol{b} - \boldsymbol{a}) \cdot \boldsymbol{c} = 0$.

b Use part **a** to prove that the altitudes of a triangle meet at a point.

15 For a parallelogram $OABC$, prove that

$$OB^2 + AC^2 = 2OA^2 + 2OC^2$$

That is, prove that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.



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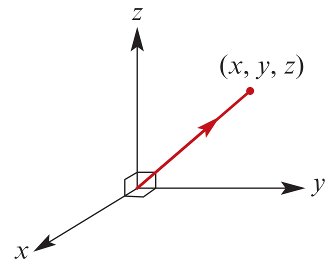
17F Vectors in three dimensions

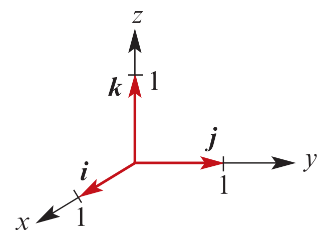
Points in three dimensions are represented using three perpendicular axes as shown.



Vectors in three dimensions are of the form

$$\boldsymbol{a} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

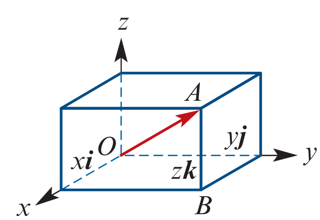
where $\boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$



The vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are the standard unit vectors for three dimensions.

The position vector for point $A(x, y, z)$ is

$$\overrightarrow{OA} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$



Using Pythagoras' theorem twice:

$$\begin{aligned} OA^2 &= OB^2 + BA^2 \\ &= OB^2 + z^2 \\ &= x^2 + y^2 + z^2 \end{aligned}$$

$$\therefore \quad |\overrightarrow{OA}| = \sqrt{x^2 + y^2 + z^2}$$

Example 17

Let $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} + 7\boldsymbol{k}$.

Find:

a $\boldsymbol{a} + \boldsymbol{b}$ **b** $\boldsymbol{b} - 3\boldsymbol{a}$ **c** $|\boldsymbol{a}|$

Solution

a $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} + \boldsymbol{i} + 7\boldsymbol{k}$

$= 2\boldsymbol{i} + \boldsymbol{j} + 6\boldsymbol{k}$

b $\boldsymbol{b} - 3\boldsymbol{a} = \boldsymbol{i} + 7\boldsymbol{k} - 3(\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$

$= -2\boldsymbol{i} - 3\boldsymbol{j} + 10\boldsymbol{k}$

c $|\boldsymbol{a}| = \sqrt{1^2 + 1^2 + (-1)^2}$

$= \sqrt{3}$

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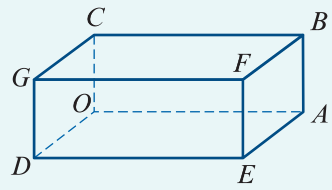
Example 18

$OABCDEFG$ is a cuboid such that $\overrightarrow{OA} = 3\boldsymbol{j}$, $\overrightarrow{OC} = \boldsymbol{k}$ and $\overrightarrow{OD} = \boldsymbol{i}$.



a Express each of the following in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$:

i $\overrightarrow{OE}$ **ii** $\overrightarrow{OF}$ **iii** $\overrightarrow{GF}$ **iv** $\overrightarrow{GB}$

b Let M and N be the midpoints of OD and GF respectively. Find MN.

Solution

a **i** $\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE} = 3\boldsymbol{j} + \boldsymbol{i}$ (as $\overrightarrow{AE} = \overrightarrow{OD}$)

ii $\overrightarrow{OF} = \overrightarrow{OE} + \overrightarrow{EF} = 3\boldsymbol{j} + \boldsymbol{i} + \boldsymbol{k}$ (as $\overrightarrow{EF} = \overrightarrow{OC}$)

iii $\overrightarrow{GF} = \overrightarrow{OA} = 3\boldsymbol{j}$

iv $\overrightarrow{GB} = \overrightarrow{DA} = \overrightarrow{DO} + \overrightarrow{OA} = -\boldsymbol{i} + 3\boldsymbol{j}$

b
$$\begin{aligned}\overrightarrow{MN} &= \overrightarrow{MD} + \overrightarrow{DG} + \overrightarrow{GN}\\ &= \frac{1}{2}\overrightarrow{OD} + \overrightarrow{OC} + \frac{1}{2}\overrightarrow{OA}\\ &= \frac{1}{2}\boldsymbol{i} + \boldsymbol{k} + \frac{3}{2}\boldsymbol{j}\end{aligned}$$

$$|\overrightarrow{MN}| = \sqrt{\frac{1}{4} + 1 + \frac{9}{4}} = \frac{\sqrt{14}}{2}$$

Example 19

If $\boldsymbol{a} = 3\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$, find $\hat{\boldsymbol{a}}$.

Solution

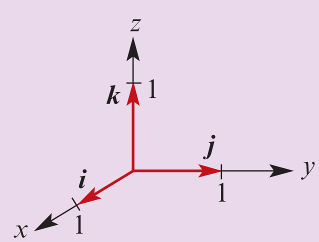
$$|\boldsymbol{a}| = \sqrt{9 + 4 + 4} = \sqrt{17}$$

$$\therefore \quad \hat{\boldsymbol{a}} = \frac{1}{\sqrt{17}}(3\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k})$$

Section summary

In three dimensions:

- The standard unit vectors are $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$.
- Each vector can be written in the form $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$.
- If $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, then $|\boldsymbol{u}| = \sqrt{x^2 + y^2 + z^2}$.



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Exercise 17F

Skillsheet
Example 17

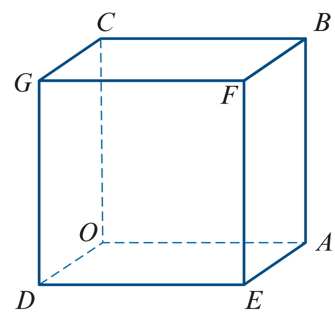
1 Let $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{c} = -\boldsymbol{i} + \boldsymbol{k}$. Find:

- **a** $\boldsymbol{a} - \boldsymbol{b}$
- **b** $3\boldsymbol{b} - 2\boldsymbol{a} + \boldsymbol{c}$
- **c** $|\boldsymbol{b}|$
- **d** $|\boldsymbol{b} + \boldsymbol{c}|$
- **e** $3(\boldsymbol{a} - \boldsymbol{b}) + 2\boldsymbol{c}$

Example 18

2 $OABCDEFG$ is a cuboid such that

$$\overrightarrow{OA} = 2\boldsymbol{j}, \quad \overrightarrow{OC} = 2\boldsymbol{k} \quad \text{and} \quad \overrightarrow{OD} = \boldsymbol{i}$$



Express the following vectors in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$:

- **a** $\overrightarrow{OB}$
- **b** $\overrightarrow{OE}$
- **c** $\overrightarrow{OG}$
- **d** $\overrightarrow{OF}$
- **e** $\overrightarrow{ED}$
- **f** $\overrightarrow{EG}$
- **g** $\overrightarrow{CE}$
- **h** $\overrightarrow{BD}$

Example 19

3 Let $\boldsymbol{a} = 3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$.

- **a** **i** Find $\hat{\boldsymbol{a}}$.
  - **ii** Find $-2\hat{\boldsymbol{a}}$.
- **b** Find the vector $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ such that $|\boldsymbol{b}| = 5$.

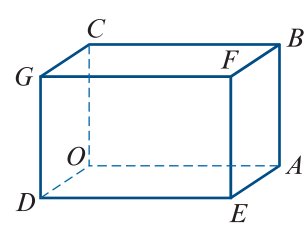
4 If $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} - 3\boldsymbol{k}$, find the vector $\boldsymbol{c}$ in the direction of $\boldsymbol{a}$ such that $|\boldsymbol{c}| = |\boldsymbol{b}|$.

5 Let P and Q be the points defined by the position vectors $\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ respectively. Let M be the midpoint of PQ. Find:

- **a** $\overrightarrow{PQ}$
- **b** $|\overrightarrow{PQ}|$
- **c** $\overrightarrow{OM}$

6 $OABCDEFG$ is a cuboid such that

$$\overrightarrow{OA} = 3\boldsymbol{j}, \quad \overrightarrow{OC} = 2\boldsymbol{k} \quad \text{and} \quad \overrightarrow{OD} = \boldsymbol{i}$$



The point M is such that $\overrightarrow{OM} = \frac{1}{3}\overrightarrow{OE}$, and N is the midpoint of BF. Find:

- **a** $\overrightarrow{MN}$
- **b** $|\overrightarrow{MN}|$

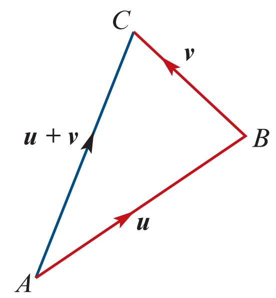
ISBN 978-1-316-63608-4

Review

Chapter summary

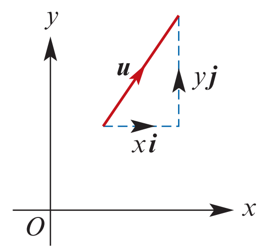
- A **vector** is a set of equivalent **directed line segments**.
- A directed line segment from a point A to a point B is denoted by $\overrightarrow{AB}$.
- The **position vector** of a point A is the vector $\overrightarrow{OA}$, where O is the origin.
- A vector can be written as a column of numbers. The vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is '2 across and 3 up'.

Basic operations on vectors



- Addition
  - If $\boldsymbol{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} c \\ d \end{bmatrix}$, then $\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} a + c \\ b + d \end{bmatrix}$.
  - The sum $\boldsymbol{u} + \boldsymbol{v}$ can also be obtained geometrically as shown.
- Scalar multiplication
  - For $k \in \mathbb{R}^+$, the vector $k\boldsymbol{u}$ has the same direction as $\boldsymbol{u}$, but its length is multiplied by a factor of k.
  - The vector $-\boldsymbol{v}$ has the same length as $\boldsymbol{v}$, but the opposite direction.
  - Two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are **parallel** if there exists $k \in \mathbb{R} \setminus \{0\}$ such that $\boldsymbol{u} = k\boldsymbol{v}$.
- Subtraction $\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$

Component form

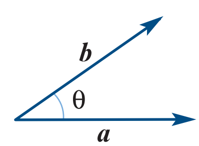


- In two dimensions, each vector $\boldsymbol{u}$ can be written in the form $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$, where
  - $\boldsymbol{i}$ is the unit vector in the positive direction of the x-axis
  - $\boldsymbol{j}$ is the unit vector in the positive direction of the y-axis.
- The **magnitude** of vector $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$ is given by $|\boldsymbol{u}| = \sqrt{x^2 + y^2}$.
- The unit vector in the direction of vector $\boldsymbol{a}$ is given by $\hat{\boldsymbol{a}} = \dfrac{1}{|\boldsymbol{a}|}\,\boldsymbol{a}$.

Scalar product and vector projections

- The **scalar product** of vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$ is given by

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$$

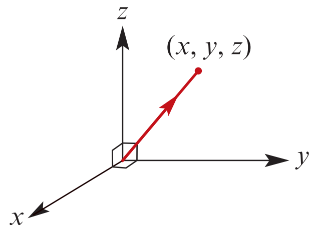


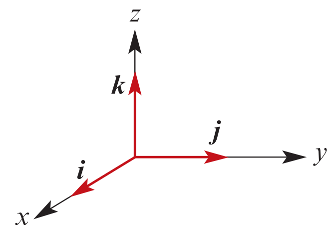
- The scalar product is described geometrically by $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.
- Therefore $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$.
- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are **perpendicular** if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- Resolving a vector $\boldsymbol{a}$ into rectangular components is expressing the vector $\boldsymbol{a}$ as a sum of two vectors, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.
- The **vector resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\,\boldsymbol{b}$.
- The **scalar resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$.

ISBN 978-1-316-63608-4

Vectors in three dimensions

- In three dimensions, each vector $\boldsymbol{u}$ can be written in the form $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors as shown.





- If $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, then $|\boldsymbol{u}| = \sqrt{x^2 + y^2 + z^2}$.

Short-answer questions

1 Given that $\boldsymbol{a} = 7\boldsymbol{i} + 6\boldsymbol{j}$ and $\boldsymbol{b} = 2\boldsymbol{i} + x\boldsymbol{j}$, find the values of x for which:

a $\boldsymbol{a}$ is parallel to $\boldsymbol{b}$ **b** $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same magnitude.

2 $ABCD$ is a parallelogram where $\overrightarrow{OA} = 2\boldsymbol{i} - \boldsymbol{j}$, $\overrightarrow{AB} = 3\boldsymbol{i} + 4\boldsymbol{j}$ and $\overrightarrow{AD} = -2\boldsymbol{i} + 5\boldsymbol{j}$. Find the coordinates of the four vertices of the parallelogram.

3 Let $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k}$ and $\boldsymbol{c} = -\boldsymbol{i} - 4\boldsymbol{j} + 2\boldsymbol{k}$. Find the values of p and q such that $\boldsymbol{a} + p\boldsymbol{b} + q\boldsymbol{c}$ is parallel to the x-axis.

4 The position vectors of P and Q are $2\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}$ and $3\boldsymbol{i} - 7\boldsymbol{j} + 12\boldsymbol{k}$ respectively.

a Find $|\overrightarrow{PQ}|$. **b** Find the unit vector in the direction of $\overrightarrow{PQ}$.

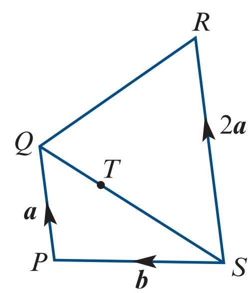
5 The position vectors of A, B and C are $2\boldsymbol{j} + 2\boldsymbol{k}$, $4\boldsymbol{i} + 10\boldsymbol{j} + 18\boldsymbol{k}$ and $x\boldsymbol{i} + 14\boldsymbol{j} + 26\boldsymbol{k}$ respectively. Find x if A, B and C are collinear.

6 $\overrightarrow{OA} = 4\boldsymbol{i} + 3\boldsymbol{j}$ and C is a point on OA such that $|\overrightarrow{OC}| = \dfrac{16}{5}$.

a Find the unit vector in the direction of $\overrightarrow{OA}$.

b Hence find $\overrightarrow{OC}$.

7 In the diagram, $ST = 2TQ$, $\overrightarrow{PQ} = \boldsymbol{a}$, $\overrightarrow{SR} = 2\boldsymbol{a}$ and $\overrightarrow{SP} = \boldsymbol{b}$.



a Find each of the following in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

i $\overrightarrow{SQ}$ **ii** $\overrightarrow{TQ}$ **iii** $\overrightarrow{RQ}$ **iv** $\overrightarrow{PT}$ **v** $\overrightarrow{TR}$

b Show that P, T and R are collinear.

8 If $\boldsymbol{a} = 5\boldsymbol{i} - s\boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{b} = t\boldsymbol{i} + 2\boldsymbol{j} + u\boldsymbol{k}$ are equal vectors.

a Find s, t and u. **b** Find $|\boldsymbol{a}|$.

9 The vector $\boldsymbol{p}$ has magnitude 7 units and bearing 050° and the vector $\boldsymbol{q}$ has magnitude 12 units and bearing 170°. (These are compass bearings on the horizontal plane.) Draw a diagram (not to scale) showing $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{p} + \boldsymbol{q}$. Calculate the magnitude of $\boldsymbol{p} + \boldsymbol{q}$.

10 If $\boldsymbol{a} = 5\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$, find:

a $\boldsymbol{a} + 2\boldsymbol{b}$ **b** $|\boldsymbol{a}|$ **c** $\hat{\boldsymbol{a}}$ **d** $\boldsymbol{a} - \boldsymbol{b}$

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Review

11 Let O, A and B be the points $(0, 0)$, $(3, 4)$ and $(4, -6)$ respectively.

a If C is the point such that $\overrightarrow{OA} = \overrightarrow{OC} + \overrightarrow{OB}$, find the coordinates of C.

b If D is the point $(1, 24)$ and $\overrightarrow{OD} = h\overrightarrow{OA} + k\overrightarrow{OB}$, find the values of h and k.

12 Let $\boldsymbol{p} = 3\boldsymbol{i} + 7\boldsymbol{j}$ and $\boldsymbol{q} = 2\boldsymbol{i} - 5\boldsymbol{j}$. Find the values of m and n such that $m\boldsymbol{p} + n\boldsymbol{q} = 8\boldsymbol{i} + 9\boldsymbol{j}$.

13 The points A, B and C have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ relative to an origin O. Write down an equation connecting $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ for each of the following cases:

a $OABC$ is a parallelogram

b B divides AC in the ratio $3 : 2$. That is, $AB : BC = 3 : 2$.

14 Let $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j}$, $\boldsymbol{b} = -\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{c} = -2\boldsymbol{i} - 2\boldsymbol{j}$. Find:

a $\boldsymbol{a} \cdot \boldsymbol{a}$ **b** $\boldsymbol{b} \cdot \boldsymbol{b}$ **c** $\boldsymbol{c} \cdot \boldsymbol{c}$ **d** $\boldsymbol{a} \cdot \boldsymbol{b}$

e $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c})$ **f** $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{c})$ **g** $(\boldsymbol{a} + 2\boldsymbol{b}) \cdot (3\boldsymbol{c} - \boldsymbol{b})$

15 Points A, B and C have position vectors $\boldsymbol{a} = 4\boldsymbol{i} + \boldsymbol{j}$, $\boldsymbol{b} = 3\boldsymbol{i} + 5\boldsymbol{j}$ and $\boldsymbol{c} = -5\boldsymbol{i} + 3\boldsymbol{j}$ respectively. Evaluate $\overrightarrow{AB} \cdot \overrightarrow{BC}$ and hence show that $\triangle ABC$ is right-angled at B.

16 Given the vectors $\boldsymbol{p} = 5\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{q} = 2\boldsymbol{i} + t\boldsymbol{j}$, find the values of t for which:

a $\boldsymbol{p} + \boldsymbol{q}$ is parallel to $\boldsymbol{p} - \boldsymbol{q}$ **b** $\boldsymbol{p} - 2\boldsymbol{q}$ is perpendicular to $\boldsymbol{p} + 2\boldsymbol{q}$ **c** $|\boldsymbol{p} - \boldsymbol{q}| = |\boldsymbol{q}|$

17 Points A, B and C have position vectors $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j}$, $\boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{c} = 2\boldsymbol{i} - 3\boldsymbol{j}$. Find:

a **i** $\overrightarrow{AB}$ **ii** $\overrightarrow{AC}$

b the vector resolute of $\overrightarrow{AB}$ in the direction of $\overrightarrow{AC}$

c the shortest distance from B to the line AC.

Multiple-choice questions

1 The vector $\boldsymbol{v}$ is defined by the directed line segment from $(1, 1)$ to $(3, 5)$.
If $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$, then

A $a = 3$ and $b = 5$ **B** $a = -2$ and $b = -4$ **C** $a = 2$ and $b = 4$

D $a = 2$ and $b = 3$ **E** $a = 4$ and $b = 2$

2 If vector $\overrightarrow{AB} = \boldsymbol{u}$ and vector $\overrightarrow{AC} = \boldsymbol{v}$, then vector $\overrightarrow{CB}$ is equal to

A $\boldsymbol{u} + \boldsymbol{v}$ **B** $\boldsymbol{v} - \boldsymbol{u}$ **C** $\boldsymbol{u} - \boldsymbol{v}$ **D** $\boldsymbol{u} \times \boldsymbol{v}$ **E** $\boldsymbol{v} + \boldsymbol{u}$

3 If vector $\boldsymbol{a} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and vector $\boldsymbol{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then $\boldsymbol{a} + \boldsymbol{b} =$

A $\begin{bmatrix} 1 \\ -5 \end{bmatrix}$ **B** $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ **C** $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ **D** $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ **E** $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

4 If vector $\boldsymbol{a} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and vector $\boldsymbol{b} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, then $2\boldsymbol{a} - 3\boldsymbol{b} =$

A $\begin{bmatrix} 9 \\ -13 \end{bmatrix}$ **B** $\begin{bmatrix} 9 \\ 7 \end{bmatrix}$ **C** $\begin{bmatrix} 9 \\ -7 \end{bmatrix}$ **D** $\begin{bmatrix} 3 \\ -13 \end{bmatrix}$ **E** $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$

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5 $PQRS$ is a parallelogram. If $\overrightarrow{PQ} = \boldsymbol{p}$ and $\overrightarrow{QR} = \boldsymbol{q}$, then $\overrightarrow{SQ}$ is equal to

A $\boldsymbol{p} + \boldsymbol{q}$ **B** $\boldsymbol{p} - \boldsymbol{q}$ **C** $\boldsymbol{q} - \boldsymbol{p}$ **D** $2\boldsymbol{q}$ **E** $2\boldsymbol{p}$

6 $|3\boldsymbol{i} - 5\boldsymbol{j}| =$

A 2 **B** $\sqrt{34}$ **C** 34 **D** 8 **E** -16

7 If $\overrightarrow{OA} = 2\boldsymbol{i} + 3\boldsymbol{j}$ and $\overrightarrow{OB} = \boldsymbol{i} - 2\boldsymbol{j}$, then $\overrightarrow{AB}$ equals

A $-\boldsymbol{i} - 5\boldsymbol{j}$ **B** $-\boldsymbol{i} + 5\boldsymbol{j}$ **C** $-\boldsymbol{i} - \boldsymbol{j}$ **D** $-\boldsymbol{i} + \boldsymbol{j}$ **E** $\boldsymbol{i} + \boldsymbol{j}$

8 If $\overrightarrow{OA} = 2\boldsymbol{i} + 3\boldsymbol{j}$ and $\overrightarrow{OB} = \boldsymbol{i} - 2\boldsymbol{j}$, then $|\overrightarrow{AB}|$ equals

A 6 **B** 26 **C** $\sqrt{26}$ **D** $\sqrt{24}$ **E** 36

9 If $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j}$, then the unit vector in the direction of $\boldsymbol{a}$ is

A $2\boldsymbol{i} + 3\boldsymbol{j}$ **B** $\frac{1}{13}(2\boldsymbol{i} + 3\boldsymbol{j})$ **C** $\frac{1}{\sqrt{5}}(2\boldsymbol{i} + 3\boldsymbol{j})$

D $\frac{1}{\sqrt{13}}(2\boldsymbol{i} + 3\boldsymbol{j})$ **E** $\sqrt{13}(2\boldsymbol{i} + 3\boldsymbol{j})$

10 If $\boldsymbol{a} = -3\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$, then $\hat{\boldsymbol{a}}$ is

A $\frac{1}{7}(-3\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k})$ **B** $\frac{1}{\sqrt{7}}(-3\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k})$ **C** $\frac{1}{\sqrt{19}}(-3\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k})$

D $\frac{1}{19}(-3\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k})$ **E** $-3\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$

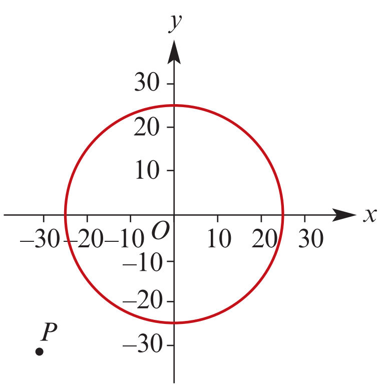
Extended-response questions

1 Let $\begin{bmatrix}1\\0\end{bmatrix}$ represent a displacement 1 km due east.

Let $\begin{bmatrix}0\\1\end{bmatrix}$ represent a displacement 1 km due north.

The diagram shows a circle of radius 25 km with centre at $O(0, 0)$. A lighthouse entirely surrounded by sea is located at O. The lighthouse is not visible from points outside the circle.

A ship is initially at point P, which 31 km west and 32 km south of the lighthouse.



a Write down the vector $\overrightarrow{OP}$.

The ship is travelling in the direction of vector $\boldsymbol{u} = \begin{bmatrix}4\\3\end{bmatrix}$ with speed 20 km/h.

An hour after leaving P, the ship is at point R.

b Show that $\overrightarrow{PR} = \begin{bmatrix}16\\12\end{bmatrix}$ and hence find the vector $\overrightarrow{OR}$.

c Show that the lighthouse first becomes visible when the ship reaches R.

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Review

2 Given that $\boldsymbol{p} = 3\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{q} = -2\boldsymbol{i} + 4\boldsymbol{j}$, find:

a $|\boldsymbol{p} - \boldsymbol{q}|$

b $|\boldsymbol{p}| - |\boldsymbol{q}|$

c $\boldsymbol{r}$ such that $\boldsymbol{p} + 2\boldsymbol{q} + \boldsymbol{r} = \boldsymbol{0}$

3 Let $\boldsymbol{a} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 11 \\ 7 \\ 3 \end{bmatrix}$, $\boldsymbol{c} = \begin{bmatrix} 7 \\ 9 \\ 7 \end{bmatrix}$ and $\boldsymbol{d} = \begin{bmatrix} 26 \\ 12 \\ 2 \end{bmatrix}$.

a Find the value of the scalar k such that $\boldsymbol{a} + 2\boldsymbol{b} - \boldsymbol{c} = k\boldsymbol{d}$.

b Find the scalars x and y such that $x\boldsymbol{a} + y\boldsymbol{b} = \boldsymbol{d}$.

c Use your answers to **a** and **b** to find scalars p, q and r (not all zero) such that $p\boldsymbol{a} + q\boldsymbol{b} - r\boldsymbol{c} = \boldsymbol{0}$.

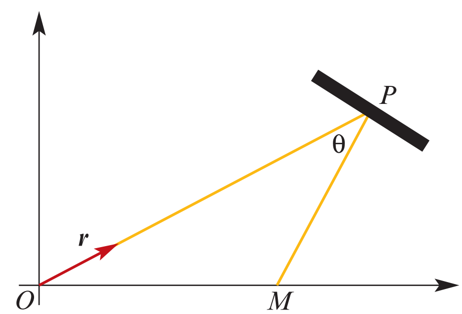
4 The quadrilateral $PQRS$ is a parallelogram. The point P has coordinates $(5, 8)$, the point R has coordinates $(32, 17)$ and the vector $\overrightarrow{PQ}$ is given by $\overrightarrow{PQ} = \begin{bmatrix} 20 \\ -15 \end{bmatrix}$.

a Find the coordinates of Q and write down the vector $\overrightarrow{QR}$.

b Write down the vector $\overrightarrow{RS}$ and show that the coordinates of S are $(12, 32)$.

5 The diagram shows the path of a light beam from its source at O in the direction of the vector $\boldsymbol{r} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

At point P, the beam is reflected by an adjustable mirror and meets the x-axis at M. The position of M varies, depending on the adjustment of the mirror at P.



a Given that $\overrightarrow{OP} = 4\boldsymbol{r}$, find the coordinates of P.

b The point M has coordinates $(k, 0)$. Find an expression, in terms of k, for vector $\overrightarrow{PM}$.

c Find the magnitudes of vectors $\overrightarrow{OP}$, $\overrightarrow{OM}$ and $\overrightarrow{PM}$, and hence find the value of k for which θ is equal to $90°$.

d Find the value θ for which M has coordinates $(9, 0)$.

ISBN 978-1-316-63608-4

18 Revision of Chapters 15–17

Revision

18A Short-answer questions

1 Let $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & -1 & -2 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} -2 & 4 \end{bmatrix}$ and $\mathbf{E} = \begin{bmatrix} -5 \\ 5 \\ 0 \end{bmatrix}$.

a State whether or not each of the following products exists: **AB**, **AC**, **CD**, **BE**

b Evaluate **DA** and $\mathbf{A}^{-1}$.

2 If $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 & 3 \\ 1 & 5 \end{bmatrix}$, find:

a $(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B})$

b $\mathbf{A}^2 - \mathbf{B}^2$

3 If the matrix $\begin{bmatrix} 1 & 2 \\ 4 & x \end{bmatrix}$ is non-invertible, find the value of x.

4 Find all possible matrices **A** which satisfy the equation $\begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}\mathbf{A} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$.

5 If $\mathbf{A} = \begin{bmatrix} -1 & -2 & 3 \\ -5 & -1 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & -4 \\ -1 & -6 \\ -3 & -8 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$, evaluate **AB** and $\mathbf{C}^{-1}$.

6 A transformation has rule $(x, y) \to (2x + y, -x - 2y)$.

a Find the image of the point $(2, 3)$.

b Find the matrix of this transformation.

c Sketch the image of the unit square and find its area.

d Find the rule for the inverse transformation.

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Revision

7 Find the matrix corresponding to each of the following linear transformations:

a reflection in the x-axis

b dilation of factor 3 from the y-axis

c shear of factor 2 parallel to the y-axis

d projection onto the y-axis

e rotation by 45° anticlockwise

f rotation by 30° clockwise

g reflection in the line $y = -x$

h reflection in the line $y = x \tan 30°$

8 **a** Find the matrix that will reflect the plane in the line $y = 4x$.

b Find the image of the point $(2, 4)$ under this transformation.

9 Find the transformation matrix that corresponds to:

a a reflection in the y-axis and then a dilation of factor 2 from the x-axis

b a rotation by 90° anticlockwise and then a reflection in the line $y = x$

c a reflection in the line $y = -x$ and then a shear of factor 2 parallel to the x-axis

10 **a** Find the rule for the transformation that will reflect (x, y) in the y-axis then translate the result by the vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

b Find the rule for the transformation if the translation takes place before the reflection.

11 Each of the following matrices maps the unit square to a parallelogram. Sketch each parallelogram and find its area.

a $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

b $\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

12 **a** Find the rule for the transformation that will reflect the plane in the line $y = x - 1$.
Hint: Translate the plane 1 unit in the y-direction, reflect in the line $y = x$, and then translate the plane back to its original position.

b Find the image of the point $(0, 0)$ under this transformation.

c Sketch the unit square and its image under this transformation.

13 Let $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j}$, $\boldsymbol{b} = -2\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{c} = -3\boldsymbol{i} - 2\boldsymbol{j}$. Find:

a $\boldsymbol{a} \cdot \boldsymbol{a}$

b $\boldsymbol{b} \cdot \boldsymbol{b}$

c $\boldsymbol{c} \cdot \boldsymbol{c}$

d $\boldsymbol{a} \cdot \boldsymbol{b}$

e $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c})$

f $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{c})$

g $(\boldsymbol{a} + 2\boldsymbol{b}) \cdot (3\boldsymbol{c} - \boldsymbol{b})$

14 The points A, B, C and D have position vectors $\overrightarrow{OA} = 4\boldsymbol{i} + 2\boldsymbol{j}$, $\overrightarrow{OB} = -\boldsymbol{i} + 7\boldsymbol{j}$, $\overrightarrow{OC} = 8\boldsymbol{i} + 6\boldsymbol{j}$ and $\overrightarrow{OD} = p\boldsymbol{i} - 2\boldsymbol{j}$.

a Find the values of m and n such that $m\overrightarrow{OA} + n\overrightarrow{BC} = 2\boldsymbol{i} + 10\boldsymbol{j}$.

b Find the value of p such that $\overrightarrow{OB}$ is perpendicular to $\overrightarrow{CD}$.

c Find the value of p such that $|\overrightarrow{AD}| = \sqrt{17}$.

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18B Multiple-choice questions

1 If $\mathbf{P}^2 = 4\mathbf{I}$, then $\mathbf{P}^{-1}$ equals

A $\frac{1}{4}\mathbf{P}$ **B** $\frac{1}{2}\mathbf{P}$ **C** $\frac{1}{2}\mathbf{I}$ **D** $2\mathbf{P}$ **E** $4\mathbf{P}$

2 If $\mathbf{R} = \begin{bmatrix} 5 & 3 & 1 \end{bmatrix}$ and $\mathbf{S} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$, then $\mathbf{RS}$ is

A undefined **B** $\begin{bmatrix} -1 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 0 & 0 \\ -5 & -3 & -1 \\ 10 & 6 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & -3 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}$

3 If $\mathbf{A} = \begin{bmatrix} 9 & 8 \\ -11 & 5 \end{bmatrix}$, then $\det(\mathbf{A})$ equals

A -43 **B** $-\frac{1}{43}$ **C** $\frac{1}{333}$ **D** 17 **E** 133

4 If $\mathbf{A} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 & 6 & 4 \end{bmatrix}$, then $\mathbf{BA}$ has size

A 1×1 **B** 3×1 **C** 1×3 **D** 3×3 **E** 3×2

5 Let $\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & -1 \\ 6 & 7 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 5 & 4 \\ 8 & 9 \end{bmatrix}$. If $\mathbf{AX} + \mathbf{B} = \mathbf{C}$, then $\mathbf{X}$ equals

A $\frac{1}{20}\begin{bmatrix} -2 & 19 \\ -2 & 6 \end{bmatrix}$ **B** $\begin{bmatrix} -1 & 1 \\ 4 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} -2 & 19 \\ -2 & 6 \end{bmatrix}$

D $\begin{bmatrix} 3 & -10 \\ -4 & 10 \end{bmatrix}$ **E** $\frac{1}{20}\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$

6 Let $\mathbf{P} = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} 4 & 2 \\ 6 & 5 \end{bmatrix}$ and $\mathbf{R} = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$.

If $\mathbf{X} = \mathbf{PQR}$, then the number of zero entries of $\mathbf{X}$ is

A 0 **B** 1 **C** 2 **D** 3 **E** 4

7 If $\mathbf{X} = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}$, then $\mathbf{X}^{-1}$ is

A $\begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}$ **B** $\begin{bmatrix} -2 & -5 \\ 1 & 3 \end{bmatrix}$ **C** $\begin{bmatrix} \frac{1}{3} & \frac{1}{5} \\ -1 & -\frac{1}{2} \end{bmatrix}$ **D** $\begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} 3 & -1 \\ -5 & -2 \end{bmatrix}$

8 The determinant of the matrix $\begin{bmatrix} 4 & 6 \\ 2 & 4 \end{bmatrix}$ is

A 16 **B** 4 **C** -16 **D** $\frac{1}{4}$ **E** -4

ISBN 978-1-316-63608-4

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9 If $\mathbf{S} = \begin{bmatrix} 5 & 7 \\ 2 & 2 \end{bmatrix}$, then $\mathbf{S}^{-1}$ is

A $-\begin{bmatrix} 5 & 7 \\ 2 & 2 \end{bmatrix}$ **B** $\begin{bmatrix} 5 & -7 \\ -2 & 5 \end{bmatrix}$ **C** $-\frac{1}{4}\begin{bmatrix} -2 & 7 \\ 2 & -5 \end{bmatrix}$

D $\frac{1}{4}\begin{bmatrix} -2 & 7 \\ 2 & -5 \end{bmatrix}$ **E** $\frac{1}{4}\begin{bmatrix} -2 & -7 \\ -2 & -5 \end{bmatrix}$

10 The point $(5, -2)$ is reflected in the line $y = x$. The coordinates of its image are

A $(5, -2)$ **B** $(-5, 2)$ **C** $(2, -5)$ **D** $(-2, 5)$ **E** $(-5, -2)$

11 The point $(2, -6)$ is reflected in the line $y = -x$. The coordinates of its image are

A $(2, -6)$ **B** $(-2, 6)$ **C** $(6, -2)$ **D** $(-6, 2)$ **E** $(-2, -6)$

12 Let P be the point $(5, -4)$. After a translation by the vector $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and then a reflection in the line $y = 1$, the coordinates of the image of P are

A $(7, 7)$ **B** $(7, 9)$ **C** $(-5, -7)$ **D** $(7, 11)$ **E** $(7, 10)$

13 The point (a, b) is reflected in the line with equation $x = m$. The image has coordinates

A $(2m - a, b)$ **B** $(a, 2m - b)$ **C** $(a - m, b)$ **D** $(a, b - m)$ **E** $(2m + a, b)$

14 The image of the line $\{ (x, y) : x + y = 4 \}$ under a dilation of factor $\frac{1}{2}$ from the y-axis followed by a reflection in the line $x = 4$ is

A $\{ (x, y) : y = 2x \}$ **B** $\{ (x, y) : y + 2 = 0 \}$ **C** $\{ (x, y) : y + 2x - 16 = 0 \}$

D $\{ (x, y) : x + y = 0 \}$ **E** $\{ (x, y) : y = 2x - 12 \}$

15 The image of $\{ (x, y) : y = x^2 \}$ under a translation by the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ followed by a reflection in the x-axis is

A $\{ (x, y) : y = (x - 3)^2 + 2 \}$ **B** $\{ (x, y) : -(x - 3)^2 = y + 2 \}$

C $\{ (x, y) : y = (x + 3)^2 + 2 \}$ **D** $\{ (x, y) : -y + 2 = (x - 3)^2 \}$

E none of these

16 The image of the graph of $y = 2^x$ under a dilation of factor 2 from the x-axis followed by a dilation of factor $\frac{1}{3}$ from the y-axis has the equation

A $y = \frac{1}{3} \times 2^{3x}$ **B** $y = 3 \times 2^{\frac{x}{2}}$ **C** $y = 2 \times 2^{3x}$ **D** $y = 2 \times 2^{\frac{x}{3}}$ **E** none of these

17 Consider these two transformations:

- T_1: A reflection in the line with equation $x = 2$.
- T_2: A translation by the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

The rule for T_1 followed by T_2 is given by

A $(x, y) \to (2 - x, y + 3)$ **B** $(x, y) \to (-x, y + 3)$ **C** $(x, y) \to (x + 2, y + 3)$

D $(x, y) \to (6 - x, y + 3)$ **E** none of these

 ISBN 978-1-316-63608-4

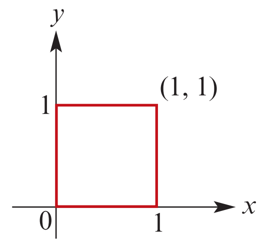
18 A transformation has rule $(x, y) \to (4x + 3y, 5x + 4y)$. The rule for the inverse transformation is

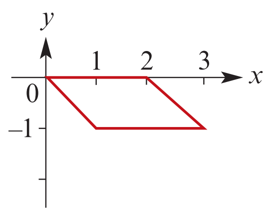
A $(x, y) \to (3x + 4y, 5x + 4y)$
B $(x, y) \to (3x - 4y, 5x - 4y)$
C $(x, y) \to (4x + 3y, 5x + 4y)$
D $(x, y) \to (4x - 3y, -5x + 4y)$
E $(x, y) \to (-4x + 3y, 5x - 4y)$

19 The unit square is subject to two successive linear transformations:

- the first transformation has matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
- the second transformation has matrix $\begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}$.

Which one of the following shows the image of the unit square under these two transformations?



A


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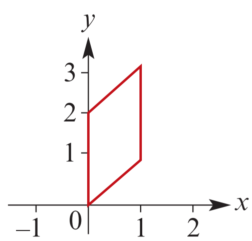

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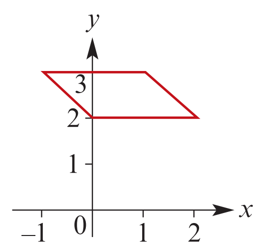

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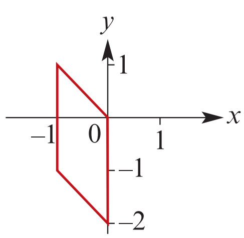
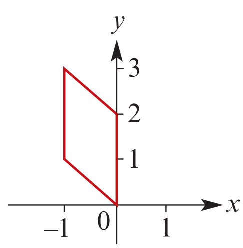


E


20 Transformation T rotates the plane about the origin by 35° clockwise. Transformation S rotates the plane about the origin by 15° anticlockwise. The matrix of T followed by S is

A $\begin{bmatrix} \cos 50^\circ & -\sin 50^\circ \\ \sin 50^\circ & \cos 50^\circ \end{bmatrix}$
B $\begin{bmatrix} \cos 20^\circ & -\sin 20^\circ \\ \sin 20^\circ & \cos 20^\circ \end{bmatrix}$
C $\begin{bmatrix} \cos 50^\circ & \sin 50^\circ \\ -\sin 50^\circ & \cos 50^\circ \end{bmatrix}$
D $\begin{bmatrix} \cos 20^\circ & \sin 20^\circ \\ -\sin 20^\circ & \cos 20^\circ \end{bmatrix}$
E $\begin{bmatrix} \cos 50^\circ & \sin 50^\circ \\ \sin 50^\circ & -\cos 50^\circ \end{bmatrix}$

21 The linear transformation determined by the matrix $\begin{bmatrix} \cos 40^\circ & -\sin 40^\circ \\ \sin 40^\circ & \cos 40^\circ \end{bmatrix}$ is

A clockwise rotation by 40°
B anticlockwise rotation by 40°
C reflection in the line $y = x \tan 40^\circ$
D reflection in the line $y = x \tan 20^\circ$
E anticlockwise rotation by 20°

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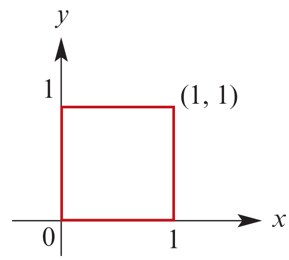
22 The unit square is transformed by the linear transformation with matrix $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$. The area of the image is

A 1 **B** 2 **C** 3 **D** 4 **E** 5



23 The unit vector in the direction of vector $\boldsymbol{a} = 3\boldsymbol{i} - 4\boldsymbol{j}$ is

A $\boldsymbol{i} - \boldsymbol{j}$ **B** $\frac{1}{5}(3\boldsymbol{i} - 4\boldsymbol{j})$ **C** $\boldsymbol{i} + \boldsymbol{j}$ **D** $\frac{1}{25}(3\boldsymbol{i} - 4\boldsymbol{j})$ **E** $\frac{3}{5}\boldsymbol{i} + \frac{4}{5}\boldsymbol{j}$

24 If $\overrightarrow{OA} = 2\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$ and $\overrightarrow{OB} = 3\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$, then $\overrightarrow{AB}$ equals

A $5\boldsymbol{i} + 2\boldsymbol{k}$ **B** $-\boldsymbol{i} - 8\boldsymbol{j}$ **C** $\boldsymbol{i} + 8\boldsymbol{j} + 2\boldsymbol{k}$ **D** $\boldsymbol{i} + 8\boldsymbol{j}$ **E** $\boldsymbol{i}$

25 If $\boldsymbol{a} = 2\boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{j}$, then $\boldsymbol{a} - \boldsymbol{b}$ equals

A $5\boldsymbol{i} - 6\boldsymbol{j}$ **B** $-\boldsymbol{i} + 6\boldsymbol{j}$ **C** $5\boldsymbol{i} - 2\boldsymbol{j}$ **D** $5\boldsymbol{i} + 2\boldsymbol{j}$ **E** $\boldsymbol{i} - 6\boldsymbol{j}$

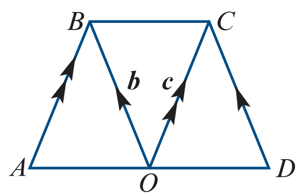
26 The magnitude of vector $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$ is

A $\sqrt{21}$ **B** 21 **C** 19 **D** $\sqrt{19}$ **E** 7

27 In the diagram, AB is parallel to OC, DC is parallel to OB, $\boldsymbol{b} = \overrightarrow{OB}$, $\boldsymbol{c} = \overrightarrow{OC}$ and $AB = OB = OC = DC$.

Vector $\overrightarrow{AD}$ is equal to

A $\boldsymbol{b} + \boldsymbol{c}$ **B** $2(\boldsymbol{c} - \boldsymbol{b})$ **C** $2(\boldsymbol{b} - \boldsymbol{c})$
D $2\boldsymbol{b} + 2\boldsymbol{c}$ **E** $|\boldsymbol{b} + \boldsymbol{c}|$

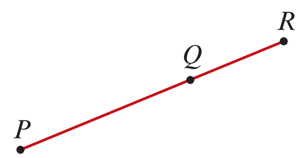


28 If $\boldsymbol{r} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{s} = -\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$, then $2\boldsymbol{r} - \boldsymbol{s}$ equals

A $3\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k}$ **B** $3\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k}$ **C** $5\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k}$ **D** $5\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k}$ **E** $6\boldsymbol{i} - 4\boldsymbol{j} - 4\boldsymbol{k}$

29 PQR is a straight line and $PQ = 2QR$. If $\overrightarrow{OQ} = 2\boldsymbol{i} - 3\boldsymbol{j}$ and $\overrightarrow{OR} = \boldsymbol{i} + 2\boldsymbol{j}$, then $\overrightarrow{OP}$ could be equal to

A $4\boldsymbol{i} - 13\boldsymbol{j}$ **B** $3\boldsymbol{i} - \boldsymbol{j}$ **C** $2\boldsymbol{i} - 10\boldsymbol{j}$
D $3\boldsymbol{i} + \boldsymbol{j}$ **E** $\boldsymbol{i} - 5\boldsymbol{j}$



30 Let $\boldsymbol{u} = \boldsymbol{i} + a\boldsymbol{j} - 5\boldsymbol{k}$ and $\boldsymbol{v} = b\boldsymbol{i} - 3\boldsymbol{j} + 6\boldsymbol{k}$. Vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel when

A $a = -3$ and $b = -1$ **B** $a = \frac{5}{2}$ and $b = -\frac{6}{5}$ **C** $a = 3$ and $b = -1$

D $a = -\frac{5}{6}$ and $b = \frac{6}{5}$ **E** $a = \frac{2}{5}$ and $b = \frac{5}{6}$

31 Let $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j}$, $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j}$ and $\boldsymbol{x} = \boldsymbol{i} + 5\boldsymbol{j}$. If $\boldsymbol{x} = s\boldsymbol{a} + t\boldsymbol{b}$, then the scalars s and t are given by

A $s = -1$ and $t = -1$ **B** $s = -1$ and $t = 1$ **C** $s = 1$ and $t = -1$
D $s = 1$ and $t = 1$ **E** $s = 5$ and $t = 5$

ISBN 978-1-316-63608-4

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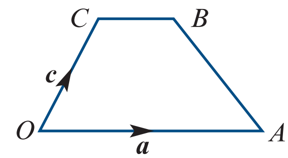
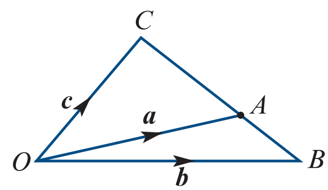
32 In this diagram, $OABC$ is a trapezium.
If $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{c} = \overrightarrow{OC}$ and $\overrightarrow{OA} = 3\overrightarrow{CB}$, then $\overrightarrow{AB}$ equals



A $3\boldsymbol{c}$ **B** $\boldsymbol{c} - \frac{2}{3}\boldsymbol{a}$ **C** $3\boldsymbol{c} - 2\boldsymbol{a}$

D $\frac{2}{3}\boldsymbol{a} - \boldsymbol{c}$ **E** $\frac{4}{3}\boldsymbol{a} + \boldsymbol{c}$

33 In this diagram, $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$, $\boldsymbol{c} = \overrightarrow{OC}$ and $AC : AB = 2 : 1$. The vector $\boldsymbol{c}$ is equal to



A $\boldsymbol{a} + 2\boldsymbol{b}$ **B** $3\boldsymbol{a} - 2\boldsymbol{b}$ **C** $2\boldsymbol{a} + \boldsymbol{b}$

D $2\boldsymbol{a} - \boldsymbol{b}$ **E** $3\boldsymbol{a} + \boldsymbol{b}$

18C Extended-response questions

1 Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $b \neq 0$ and $c \neq 0$.

a **i** Find $\mathbf{A}^2$. **ii** Find $3\mathbf{A}$.

b If $\mathbf{A}^2 = 3\mathbf{A} - \mathbf{I}$, show that:

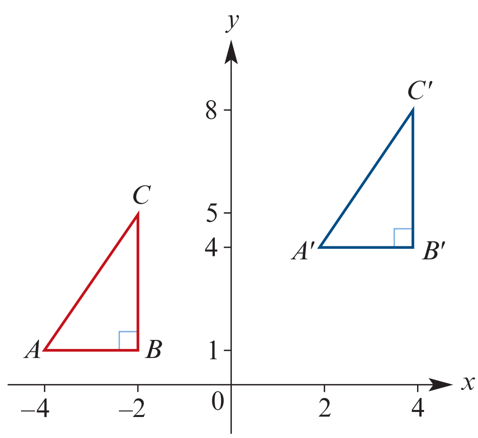
i $a + d = 3$ **ii** $\det(\mathbf{A}) = 1$

c Assume that $\mathbf{A}$ has the properties:

- $a + d = 3$
- $\det(\mathbf{A}) = 1$

Show that $\mathbf{A}^2 = 3\mathbf{A} - \mathbf{I}$.

2 The coordinates of A, B and C are $(-4, 1)$, $(-2, 1)$ and $(-2, 5)$ respectively.



a Find the rule of the transformation that maps triangle ABC to triangle $A'B'C'$.

b On graph paper, draw triangle ABC and its image under a reflection in the x-axis.

c On the same set of axes, draw the image of ABC under a dilation of factor 2 from the y-axis.

d Find the image of the parabola $y = x^2$ under a dilation of factor 2 from the x-axis followed by a translation by the vector $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

e Find the rule for the transformation that maps the graph of $y = x^2$ to the graph of $y = -2(x - 3)^2 + 4$.

f Let $f(x) = x^3 - 2x$. Use a calculator to help sketch the graph of $y = 3f(x - 2) + 4$.

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Revision

3 Let transformation D be a dilation of factor 4 from the y-axis.

a Find the image of the point $(1, 1)$ under dilation D.

b **i** Describe the image of the square with vertices $A(0, 0)$, $B(0, 1)$, $C(1, 1)$, $E(1, 0)$ under the dilation D.

ii Find the area of the square $ABCE$.

iii Find the area of the image of $ABCE$.

iv If the dilation had been of factor k, what would be the area of the image?

c State the rule for the dilation D.

d **i** Find the equation of the image of the graph of $y = x^2$ under the dilation D.

ii Find the equation of the image of the graph of $y = x^2$ under the dilation D followed by the translation by the vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

iii Sketch the graph of $y = x^2$ and its image defined in **d ii** on the one set of axes. State the coordinates of the vertex and the axis intercepts of the image.

e State the rule for the transformation that maps the graph of $y = 5(x + 2)^2 - 3$ to the graph of $y = x^2$.

4 A linear transformation is represented by the matrix

$$\mathbf{M} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

a Show that this transformation is a rotation.

b Let C be the circle that passes through the origin and has its centre at $(0, 1)$.

i Find the equation of C.

ii Find the equation of C', the image of C under the transformation defined by $\mathbf{M}$.

c Find the coordinates of the points of intersection of C and C'.

5 A linear transformation is represented by the matrix

$$\mathbf{M} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

a Find the image of the point $(-2, 5)$ under this transformation.

b Find the inverse of $\mathbf{M}$.

c Given that the point $(11, 13)$ is the image of the point (a, b), find the values of a and b.

d Find the coordinates of the image of the point (a, a) in terms of a.

e Given that $a \neq 0$ and $b \neq 0$ with

$$\mathbf{M}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \lambda a \\ \lambda b \end{bmatrix}$$

find the possible values of λ and the relationship between a and b for each of these values.

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6 Let **R** be the transformation matrix for a rotation about the origin by $\frac{\pi}{4}$ anticlockwise.

a Give the 2×2 matrix **R**.

b Find the inverse of this matrix.

c If the image of (a, b) is $(1, 1)$, find the values of a and b.

d If the image of (c, d) is $(1, 2)$, find the values of c and d.

e **i** If $(x, y) \to (x', y')$ under this transformation, use the result of **b** to find x and y in terms of x' and y'.

ii Find the image of $y = x^2$ under this transformation.

7 Consider lines $y = x$ and $y = 2x$.

a Sketch these two lines on the same set of axes.

b The acute angle between the two lines, θ radians, can be written in the form $\theta = \tan^{-1}(a) - b$. What are the values of a and b?

c Hence find a rotation matrix that will rotate the line $y = x$ to the line $y = 2x$. You will need to use the addition formulas for sine and cosine.

8 Let M be the transformation that reflects the plane in the line $y = x$.

a **i** Find the image of the point $A(1, 3)$ under this transformation.

ii The image of the triangle with vertices $A(1, 3)$, $B(1, 5)$ and $C(3, 3)$ is another triangle. Find the coordinates of the vertices of the image.

iii Sketch triangle ABC and its image on a set of axes, with both axes from -5 to 5.

b **i** Show that the equation of the image of the graph of $y = x^2 - 2$ under the transformation M is $x = y^2 - 2$.

ii Find the coordinates of the points of intersection of $y = x^2 - 2$ and the line $y = x$.

iii Show that the x-coordinates of the points of intersection of $y = x^2 - 2$ and its image may be determined by the equation $x^4 - 4x^2 - x + 2 = 0$.

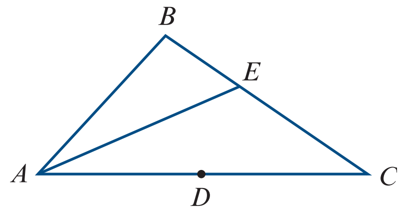
iv Two solutions of the equation $x^4 - 4x^2 - x + 2 = 0$ are

$$x = \frac{1}{2}(-1 + \sqrt{5}) \quad \text{and} \quad x = \frac{1}{2}(-1 - \sqrt{5})$$

Use this result and the result of **b ii** to find the coordinates of the points of intersection of $y = x^2 - 2$ and its image under M.

9 In the diagram, D is the midpoint of AC and E is the point on BC such that $BE : EC = 1 : t$, where $t > 0$. Suppose that DE is extended to a point F such that $DE : EF = 1 : 7$.

Let $\boldsymbol{a} = \overrightarrow{AD}$ and $\boldsymbol{b} = \overrightarrow{AB}$.



a Express $\overrightarrow{AE}$ in terms of t, $\boldsymbol{a}$ and $\boldsymbol{b}$.

b Express $\overrightarrow{AE}$ in terms of $\boldsymbol{a}$ and $\overrightarrow{AF}$.

c Show that $\overrightarrow{AF} = \frac{9 - 7t}{1 + t}\boldsymbol{a} + \frac{8t}{1 + t}\boldsymbol{b}$.

d If A, B and F are collinear, find the value of t.

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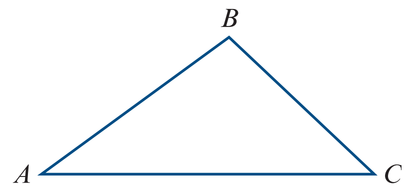
10 The vertices A, B and C of a triangle have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively, relative to an origin in the plane ABC.



a Let P be an arbitrary point on the line segment AB. Show that the position vector of P can be written in the form

$$m\boldsymbol{a} + n\boldsymbol{b}, \quad \text{where } m \geq 0,\ n \geq 0 \text{ and } m + n = 1$$

Hint: Assume that P divides AB in the ratio $x : y$.

b Find $\overrightarrow{PC}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

c Let Q be an arbitrary point on the line segment PC. Show that the position vector of Q can be written in the form

$$\lambda\boldsymbol{a} + \mu\boldsymbol{b} + \gamma\boldsymbol{c}, \quad \text{where } \lambda \geq 0,\ \mu \geq 0,\ \gamma \geq 0 \text{ and } \lambda + \mu + \gamma = 1$$

Note: The triple of numbers (λ, μ, γ) are known as the **barycentric coordinates** of the point Q in the triangle ABC.

11 $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$, $\overrightarrow{OP} = \frac{4}{5}\overrightarrow{OA}$ and Q is the midpoint of AB.

a Express $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

b The line segments PQ and OB are extended to meet at R, with $\overrightarrow{QR} = n\overrightarrow{PQ}$ and $\overrightarrow{BR} = k\boldsymbol{b}$. Express the vector $\overrightarrow{QR}$ in terms of:

- **i** n, $\boldsymbol{a}$ and $\boldsymbol{b}$
- **ii** k, $\boldsymbol{a}$ and $\boldsymbol{b}$

c Find the values of n and k.

12 **a** A man walks north at a rate of 4 km/h and notices that the wind *appears* to blow from the west. He doubles his speed and now the wind appears to blow from the north-west. What is the velocity of the wind?

Note: Both the direction and the magnitude must be given.

b A river 400 m wide flows from east to west at a steady speed of 1 km/h. A swimmer, whose speed in still water is 2 km/h, starts from the south bank and heads north across the river. Find the swimmer's speed over the river bed and how far downstream he is when he reaches the north bank.

c To a motorcyclist travelling due north at 50 km/h, the wind appears to come from the north-west at 60 km/h. What is the true velocity of the wind?

d A dinghy in distress is 6 km on a bearing of 230° from a lifeboat and is drifting in a direction of 150° at 5 km/h. In what direction should the lifeboat travel to reach the dinghy as quickly as possible if the maximum speed of the lifeboat is 35 km/h?

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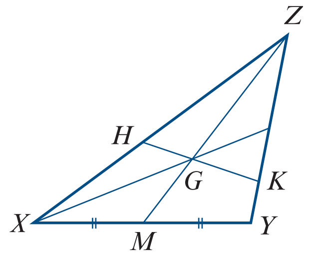
13 **a** Let points O, A, B and C be coplanar and let $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$ and $\boldsymbol{c} = \overrightarrow{OC}$. Assume that $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel. If points A, B and C are collinear with

$$\boldsymbol{c} = \alpha\boldsymbol{a} + \beta\boldsymbol{b} \quad \text{where } \alpha, \beta \in \mathbb{R}$$

show that $\alpha + \beta = 1$.

b In the figure, the point G is the centroid of a triangle (i.e. the point where the lines joining each vertex to the midpoint of the opposite side meet).
A line passing through G meets ZX and ZY at points H and K respectively, with $ZH = hZX$ and $ZK = kZY$.



i Prove that $\overrightarrow{ZG} = \frac{2}{3}\overrightarrow{ZM}$.

ii Express $\overrightarrow{ZG}$ in terms of h, k, $\overrightarrow{ZH}$ and $\overrightarrow{ZK}$.

iii Find the value of $\frac{1}{h} + \frac{1}{k}$. (Use the result from **a**.)

iv If $h = k$, find the value of h and describe geometrically what this implies.

v If the area of triangle XYZ is 1 cm^2, what is the area of triangle HKZ when $h = k$?

vi If $k = 2h$, find the value of h and describe geometrically what this implies.

vii Describe the restrictions on h and k, and sketch the graph of h against k for suitable values of k.

viii Investigate the area, A cm^2, of triangle HKZ as a ratio with respect to the area of triangle XYZ, as k varies. Sketch the graph of A against k. Be careful with the domain.

14 The **trace** of a square matrix $\mathbf{A}$ is defined to be the sum of the entries along the main diagonal of $\mathbf{A}$ (from top-left to bottom-right) and is denoted by $\text{Tr}(\mathbf{A})$.

For example, if $\mathbf{A} = \begin{bmatrix} 6 & -3 \\ 2 & 2 \end{bmatrix}$, then $\text{Tr}(\mathbf{A}) = 6 + 2 = 8$.

a Prove each of the following for all 2×2 matrices $\mathbf{X}$ and $\mathbf{Y}$:

i $\text{Tr}(\mathbf{X} + \mathbf{Y}) = \text{Tr}(\mathbf{X}) + \text{Tr}(\mathbf{Y})$

ii $\text{Tr}(-\mathbf{X}) = -\text{Tr}(\mathbf{X})$

iii $\text{Tr}(\mathbf{XY}) = \text{Tr}(\mathbf{YX})$

b Use the results of **a** to show that $\mathbf{XY} - \mathbf{YX} \neq \mathbf{I}$ for all 2×2 matrices $\mathbf{X}$ and $\mathbf{Y}$.

ISBN 978-1-316-63608-4

Chapter 19

19 Kinematics

Objectives

- To model **motion in a straight line**.
- To use **differential calculus** to solve problems involving motion in a straight line.
- To apply the formulas for motion with **constant acceleration**.
- To use **graphical methods** to solve problems involving motion in a straight line.

Kinematics is the study of motion without reference to the cause of the motion.

In this chapter, we will consider the motion of a particle in a straight line only. This simple model can be applied in various real-life situations. For example:

- finding the braking distance of a car travelling at 60 km/h
- finding the maximum height reached by a stone thrown into the air
- finding the time required for a train to travel between two stations.

When studying motion, it is important to make a distinction between vector quantities and scalar quantities:

Vector quantities Position, displacement, velocity and acceleration must be specified by both magnitude and direction.

Scalar quantities Distance, speed and time are specified by their magnitude only.

Since we are considering movement in a straight line, the **direction** of each vector quantity is simply specified by the **sign** of the numerical value.

This chapter uses your knowledge of differential calculus from Mathematical Methods Year 11.

ISBN 978-1-316-63608-4

Cambridge University Press

19A Position, velocity and acceleration

▶ Position

The **position** of a particle moving in a straight line is determined by its distance from a fixed point O on the line, called the **origin**, and whether it is to the right or left of O. By convention, the direction to the right of the origin is considered to be positive.

Consider a particle which starts at O and begins to move. The position of the particle at any instant can be specified by a real number x. For example, if the unit is metres and if $x = -3$, the position is 3 m to the left of O; while if $x = 3$, the position is 3 m to the right of O.

Sometimes there is a rule that enables the position at any instant to be calculated. In this case, we can view x as being a function of t. Hence $x(t)$ is the position at time t.

For example, imagine that a stone is dropped from the top of a vertical cliff 45 metres high. Assume that the stone is a particle travelling in a straight line. Let $x(t)$ metres be the downwards position of the particle from O, the top of the cliff, t seconds after the particle is dropped. If air resistance is neglected, then an approximate model for the position is

$$x(t) = 5t^2 \quad \text{for } 0 \le t \le 3$$

Example 1

A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = t^2 - 7t + 6$, $t \ge 0$.

a Find its initial position. **b** Find its position at $t = 4$.

Solution

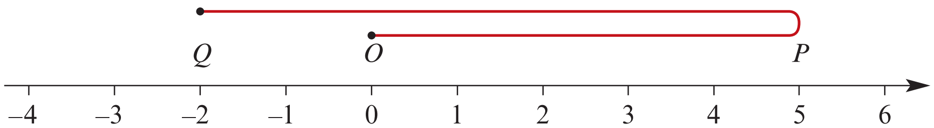
a At $t = 0$, $x = +6$, i.e. the particle is 6 cm to the right of O.

b At $t = 4$, $x = (4)^2 - 7(4) + 6 = -6$, i.e. the particle is 6 cm to the left of O.

▶ Displacement and distance

The **displacement** of a particle is defined as the change in position of the particle.

It is important to distinguish between the scalar quantity **distance** and the vector quantity displacement (which has a direction). For example, consider a particle that starts at O and moves first 5 units to the right to point P, and then 7 units to the left to point Q.



The difference between its final position and its initial position is -2. So the displacement of the particle is -2 units. However, the distance it has travelled is 12 units.

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Velocity and speed

You are already familiar with rates of change through your studies in Mathematical Methods.

Average velocity

The average rate of change of position with respect to time is **average velocity**.

A particle's average velocity for a time interval $[t_1, t_2]$ is given by

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}} = \frac{x_2 - x_1}{t_2 - t_1}$$

where x_1 is the position at time t_1 and x_2 is the position at time t_2.

Instantaneous velocity

The instantaneous rate of change of position with respect to time is **instantaneous velocity**. We will refer to the instantaneous velocity as simply the **velocity**.

If a particle's position, x, at time t is given as a function of t, then the velocity of the particle at time t is determined by differentiating the rule for position with respect to time.

If x is the position of a particle at time t, then

$$\text{velocity } v = \frac{dx}{dt}$$

Velocity may be positive, negative or zero. If the velocity is positive, the particle is moving to the right, and if it is negative, the particle is moving to the left. A velocity of zero means the particle is instantaneously at rest.

Speed and average speed

- **Speed** is the magnitude of the velocity.
- **Average speed** for a time interval $[t_1, t_2]$ is given by $\dfrac{\text{distance travelled}}{t_2 - t_1}$

Units of measurement

Common units for velocity (and speed) are:

$$1 \text{ metre per second} = 1 \text{ m/s} = 1 \text{ m s}^{-1}$$
$$1 \text{ centimetre per second} = 1 \text{ cm/s} = 1 \text{ cm s}^{-1}$$
$$1 \text{ kilometre per hour} = 1 \text{ km/h} = 1 \text{ km h}^{-1}$$

The first and third units are connected in the following way:

$$1 \text{ km/h} = 1000 \text{ m/h} = \frac{1000}{60 \times 60} \text{ m/s} = \frac{5}{18} \text{ m/s}$$

$$\therefore \quad 1 \text{ m/s} = \frac{18}{5} \text{ km/h}$$

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Example 2

A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = t^2 - 7t + 6$, $t \geq 0$.

a Find its initial velocity.

b When does its velocity equal zero, and what is its position at this time?

c What is its average velocity for the first 4 seconds?

d Determine its average speed for the first 4 seconds.

Solution

a $x = t^2 - 7t + 6$

$v = \dfrac{dx}{dt} = 2t - 7$

At $t = 0$, $v = -7$. The particle is initially moving to the left at 7 cm/s.

b $\dfrac{dx}{dt} = 0$ implies $2t - 7 = 0$, i.e. $t = 3.5$

When $t = 3.5$, $x = (3.5)^2 - 7(3.5) + 6$

$= -6.25$

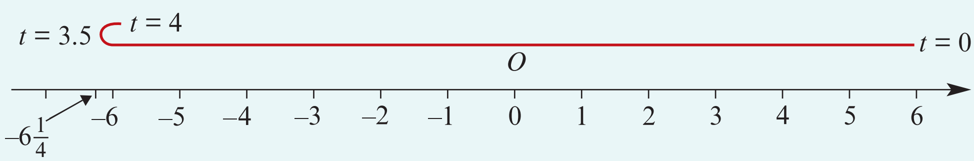
So, at $t = 3.5$ seconds, the particle is at rest 6.25 cm to the left of O.

c Average velocity $= \dfrac{\text{change in position}}{\text{change in time}}$

Position is given by $x = t^2 - 7t + 6$. So at $t = 4$, $x = -6$, and at $t = 0$, $x = 6$.

$\therefore$ Average velocity $= \dfrac{-6 - 6}{4} = -3$ cm/s

d Average speed $= \dfrac{\text{distance travelled}}{\text{change in time}}$



The particle stopped at $t = 3.5$ and began to move in the opposite direction. So we must consider the distance travelled in the first 3.5 seconds (from $x = 6$ to $x = -6.25$) and then the distance travelled in the final 0.5 seconds (from $x = -6.25$ to $x = -6$).

Total distance travelled $= 12.25 + 0.25 = 12.5$

$\therefore$ Average speed $= \dfrac{12.5}{4} = 3.125$ cm/s

Note: Remember that speed is the magnitude of the velocity. However, we can see from this example that average speed is *not* the magnitude of the average velocity.

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Acceleration

The acceleration of a particle is the rate of change of its velocity with respect to time.

- **Average acceleration** for the time interval $[t_1, t_2]$ is given by $\dfrac{v_2 - v_1}{t_2 - t_1}$, where v_2 is the velocity at time t_2 and v_1 is the velocity at time t_1.
- **Instantaneous acceleration** $a = \dfrac{dv}{dt} = \dfrac{d}{dt}\left(\dfrac{dx}{dt}\right) = \dfrac{d^2x}{dt^2}$

The second derivative $\dfrac{d^2x}{dt^2}$ is just the derivative of the derivative.

Acceleration may be positive, negative or zero. Zero acceleration means the particle is moving at a constant velocity.

The direction of motion and the acceleration need not coincide. For example, a particle may have a positive velocity, indicating it is moving to the right, but a negative acceleration, indicating it is slowing down.

Also, although a particle may be instantaneously at rest, its acceleration at that instant need not be zero. If acceleration has the same sign as velocity, then the particle is 'speeding up'. If the sign is opposite, the particle is 'slowing down'.

The most commonly used units for acceleration are cm/s^2 and m/s^2.

Example 3

A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = t^3 - 6t^2 + 5$, $t \geq 0$.

a Find its initial position, velocity and acceleration, and hence describe its motion.

b Find the times when it is instantaneously at rest and determine its position and acceleration at those times.

Solution

a $x = t^3 - 6t^2 + 5$

$$v = \frac{dx}{dt} = 3t^2 - 12t$$

$$a = \frac{dv}{dt} = 6t - 12$$

So when $t = 0$, we have $x = 5$, $v = 0$ and $a = -12$.

Initially, the particle is instantaneously at rest 5 cm to the right of O, with an acceleration of $-12\ \text{cm/s}^2$.

b $v = 0$ implies
$$3t^2 - 12t = 0$$
$$3t(t - 4) = 0$$
$$\therefore\quad t = 0 \text{ or } t = 4$$

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The particle is initially at rest and stops again after 4 seconds.

At $t = 0$, $x = 5$ and $a = -12$.
At $t = 4$, $x = (4)^3 - 6(4)^2 + 5 = -27$ and $a = 6(4) - 12 = 12$.

After 4 seconds, the particle's position is 27 cm to the left of O, and its acceleration is 12 cm/s^2.

Section summary

- The **position** of a particle moving in a straight line is determined by its distance from a fixed point O on the line, called the **origin**, and whether it is to the right or left of O. By convention, the direction to the right of the origin is positive.
- **Average velocity** for a time interval $[t_1, t_2]$ is given by

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}} = \frac{x_2 - x_1}{t_2 - t_1}$$

  where x_2 is the position at time t_2 and x_1 is the position at time t_1.
- The instantaneous rate of change of position with respect to time is called the **instantaneous velocity**, or simply the **velocity**.
  If x is the position of the particle at time t, then its velocity is $v = \frac{dx}{dt}$
- **Speed** is the magnitude of the velocity.
- **Average speed** for a time interval $[t_1, t_2]$ is $\frac{\text{distance travelled}}{t_2 - t_1}$
- **Average acceleration** for a time interval $[t_1, t_2]$ is given by $\frac{v_2 - v_1}{t_2 - t_1}$, where v_2 is the velocity at time t_2 and v_1 is the velocity at time t_1.
- **Instantaneous acceleration** $a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$

Exercise 19A

Skillsheet
Example 1, 2

1 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds ($t \geq 0$) is given by $x = t^2 - 7t + 12$. Find:

a its initial position
b its position at $t = 5$
c its initial velocity
d when and where its velocity equals zero
e its average velocity in the first 5 s
f its average speed in the first 5 s.

Example 3

2 The position, x metres, at time t seconds ($t \geq 0$) of a particle moving in a straight line is given by $x = t^2 - 7t + 10$. Find:

a when its velocity equals zero
b its acceleration at this time
c the distance travelled in the first 5 s
d when and where its velocity is -2 m/s.

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3 A particle moving in a straight line has position x cm relative to the point O at time t seconds ($t \geq 0$), where $x = t^3 - 11t^2 + 24t - 3$. Find:

- **a** its initial position and velocity
- **b** its velocity at any time t
- **c** at what times the particle is stationary
- **d** where the particle is stationary
- **e** for how long the particle's velocity is negative
- **f** its acceleration at any time t
- **g** when the particle's acceleration is zero and its velocity and position at that time.

4 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds ($t \geq 0$) is given by $x = 2t^3 - 5t^2 + 4t - 5$. Find:

- **a** when its velocity is zero and its acceleration at that time
- **b** when its acceleration is zero and its velocity at that time.

5 A particle is moving in a straight line in such a way that its position, x cm, relative to the point O at time t seconds ($t \geq 0$) satisfies $x = t^3 - 13t^2 + 46t - 48$. When does the particle pass through O, and what is its velocity and acceleration at those times?

6 Two particles are moving along a straight path so that their positions, x_1 cm and x_2 cm, relative to a fixed point P at any time t seconds are given by $x_1 = t + 2$ and $x_2 = t^2 - 2t - 2$. Find:

- **a** the time when the particles are at the same position
- **b** the time when they are moving with the same velocity.

19B Applications of antidifferentiation to kinematics

In the previous section we considered examples in which we were given a rule for the position of a particle in terms of time, and from it we derived rules for the velocity and the acceleration by differentiation.

We may be given a rule for the acceleration at time t and, by using antidifferentiation with respect to t and some additional information, we can deduce rules for both velocity and position.

Example 4

A body starts from O and moves in a straight line. After t seconds ($t \geq 0$) its velocity, v m/s, is given by $v = 2t - 4$.

- **a** Find its position x in terms of t.
- **b** Find its position after 3 seconds.
- **c** What is the distance travelled in the first 3 seconds?
- **d** Find its average velocity in the first 3 seconds.
- **e** Find its average speed in the first 3 seconds.

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Solution

a Antidifferentiate v to find the expression for position, x m, at time t seconds:

$$x = t^2 - 4t + c$$

When $t = 0$, $x = 0$, and so $c = 0$.

$$\therefore \quad x = t^2 - 4t$$

b When $t = 3$, $x = -3$. The body is 3 m to the left of O.

c First find when the body is at rest: $v = 0$ implies $2t - 4 = 0$, i.e. $t = 2$.

When $t = 2$, $x = -4$.

Therefore the body goes from $x = 0$ to $x = -4$ in the first 2 seconds, and then back to $x = -3$ in the next second.

Thus it has travelled 5 m in the first 3 seconds.

d $$\text{Average velocity} = \frac{-3 - 0}{3} = -1 \text{ m/s}$$

e From part **c**, the distance travelled is 5 m.

$$\therefore \text{Average speed} = \frac{5}{3} \text{ m/s}$$

Example 5

A particle starts from rest 3 metres from a fixed point and moves in a straight line with an acceleration of $a = 6t + 8$. Find its position and velocity at any time t seconds.

Solution

We are given the acceleration:

$$a = \frac{dv}{dt} = 6t + 8$$

Find the velocity by antidifferentiating:

$$v = 3t^2 + 8t + c$$

At $t = 0$, $v = 0$, and so $c = 0$.

$$\therefore \quad v = 3t^2 + 8t$$

Find the position by antidifferentiating again:

$$x = t^3 + 4t^2 + d$$

At $t = 0$, $x = 3$, and so $d = 3$.

$$\therefore \quad x = t^3 + 4t^2 + 3$$

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Example 6

A stone is projected vertically upwards from the top of a 20 m high building with an initial velocity of 15 m/s.

a Find the time taken for the stone to reach its maximum height.

b Find the maximum height reached by the stone.

c What is the time taken for the stone to reach the ground?

d What is the velocity of the stone as it hits the ground?

In this case we only consider the stone's motion in a vertical direction, so we can treat it as motion in a straight line. Also we will assume that the acceleration due to gravity is approximately -10 m/s^2. (Note that downwards is considered the negative direction.)

Solution

We have

$$a = -10$$
$$v = -10t + c$$

At $t = 0$, $v = 15$, so $c = 15$.

$$\therefore \quad v = -10t + 15$$
$$x = -5t^2 + 15t + d$$

At $t = 0$, $x = 20$, so $d = 20$.

$$\therefore \quad x = -5t^2 + 15t + 20$$

a The stone will reach its maximum height when $v = 0$, i.e. when $-10t + 15 = 0$, which implies $t = 1.5$.

The stone reaches its maximum height when $t = 1.5$ seconds.

b At $t = 1.5$, $\quad x = -5(1.5)^2 + 15(1.5) + 20$

$$= 31.25$$

The maximum height reached by the stone is 31.25 metres.

c The stone reaches the ground when $x = 0$:

$$-5t^2 + 15t + 20 = 0$$
$$-5(t^2 - 3t - 4) = 0$$
$$-5(t - 4)(t + 1) = 0$$

Thus $t = 4$. (The solution of $t = -1$ is rejected, since $t \geq 0$.)

The stone takes 4 seconds to reach the ground.

d At $t = 4$, $\quad v = -10(4) + 15$

$$= -25$$

Thus its velocity on impact is -25 m/s.

ISBN 978-1-316-63608-4

Section summary

Antidifferentiation may be used to go from acceleration to velocity, and from velocity to position.

Exercise 19B

Skillsheet
Example 4

1 A body starts from O and moves in a straight line. After t seconds ($t \geq 0$) its velocity, v cm/s, is given by $v = 4t - 6$. Find:

- **a** its position x in terms of t
- **b** its position after 3 s
- **c** the distance travelled in the first 3 s
- **d** its average velocity in the first 3 s
- **e** its average speed in the first 3 s.

2 The velocity of a particle, v m/s, at time t seconds ($t \geq 0$) is given by $v = 3t^2 - 8t + 5$. It is initially 4 m to the right of a point O. Find:

- **a** its position and acceleration at any time t
- **b** its position when the velocity is zero
- **c** its acceleration when the velocity is zero.

Example 5

3 A body moves in a straight line with an acceleration of 10 m/s^2. If after 2 s it passes through O and after 3 s it is 25 m from O, find its initial position relative to O.

4 A body moves in a straight line so that its acceleration, a m/s^2, after time t seconds ($t \geq 0$) is given by $a = 2t - 3$. If the initial position of the body is 2 m to the right of a point O and its velocity is 3 m/s, find the particle's position and velocity after 10 s.

Example 6

5 An object is projected vertically upwards with a velocity of 25 m/s. (Its acceleration due to gravity is -10 m/s^2.) Find:

- **a** the object's velocity at any time t
- **b** its height above the point of projection at any time t
- **c** the time it takes to reach its maximum height
- **d** the maximum height reached
- **e** the time taken to return to the point of projection.

6 The lift in a tall building passes the 50th floor with a velocity of -8 m/s and an acceleration of $\frac{1}{9}(t - 5)$ m/s^2. If each floor spans a height of 6 metres, find at which floor the lift will stop.

ISBN 978-1-316-63608-4

19C Constant acceleration

If an object is moving due to a constant force (for example, gravity), then its acceleration is constant. There are several useful formulas that apply in this situation.

Formulas for constant acceleration

For a particle moving in a straight line with constant acceleration a, we can use the following formulas, where u is the initial velocity, v is the final velocity, s is the displacement and t is the time taken:

1 $v = u + at$ **2** $s = ut + \frac{1}{2}at^2$ **3** $v^2 = u^2 + 2as$ **4** $s = \frac{1}{2}(u + v)t$

Proof **1** We can write

$$\frac{dv}{dt} = a$$

where a is a constant and v is the velocity at time t. By antidifferentiating with respect to t, we obtain

$$v = at + c$$

where the constant c is the initial velocity. We denote the initial velocity by u, and therefore $v = u + at$.

2 We now write

$$\frac{dx}{dt} = v = u + at$$

where x is the position at time t. By antidifferentiating again, we have

$$x = ut + \frac{1}{2}at^2 + d$$

where the constant d is the initial position. The particle's displacement (change in position) is given by $s = x - d$, and so we obtain the second equation.

3 Transform the first equation $v = u + at$ to make t the subject:

$$t = \frac{v - u}{a}$$

Now substitute this into the second equation:

$$s = ut + \frac{1}{2}at^2$$

$$s = \frac{u(v - u)}{a} + \frac{a(v - u)^2}{2a^2}$$

$$\begin{aligned} 2as &= 2u(v - u) + (v - u)^2 \\ &= 2uv - 2u^2 + v^2 - 2uv + u^2 \\ &= v^2 - u^2 \end{aligned}$$

4 Similarly, the fourth equation can be derived from the first and second equations.

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These four formulas are very useful, but it must be remembered that they only apply when the acceleration is constant.

When approaching problems involving constant acceleration, it is a good idea to list the quantities you are given, establish which quantity or quantities you require, and then use the appropriate formula. Ensure that all quantities are converted to compatible units.

Example 7

An object is moving in a straight line with uniform acceleration. Its initial velocity is 12 m/s and after 5 seconds its velocity is 20 m/s. Find:

a the acceleration

b the distance travelled during the first 5 seconds

c the time taken to travel a distance of 200 m.

Solution

We are given $u = 12$, $v = 20$ and $t = 5$.

a Find a using

$$v = u + at$$
$$20 = 12 + 5a$$
$$a = 1.6$$

The acceleration is 1.6 m/s^2.

b Find s using

$$s = ut + \frac{1}{2}at^2$$
$$= 12(5) + \frac{1}{2}(1.6)5^2 = 80$$

The distance travelled is 80 m.

Note: Since the object is moving in one direction, the distance travelled is equal to the displacement.

c We are now given $a = 1.6$, $u = 12$ and $s = 200$.

Find t using

$$s = ut + \frac{1}{2}at^2$$
$$200 = 12t + \frac{1}{2} \times 1.6 \times t^2$$
$$200 = 12t + \frac{4}{5}t^2$$
$$1000 = 60t + 4t^2$$
$$250 = 15t + t^2$$
$$t^2 + 15t - 250 = 0$$
$$(t - 10)(t + 25) = 0$$
$$\therefore \quad t = 10 \text{ or } t = -25$$

As $t \geq 0$, the only allowable solution is $t = 10$.

The object takes 10 s to travel a distance of 200 m.

ISBN 978-1-316-63608-4

Section summary

Constant acceleration

If acceleration is constant, then the following formulas can be used (for acceleration a, initial velocity u, final velocity v, displacement s and time taken t):

1 $v = u + at$ **2** $s = ut + \frac{1}{2}at^2$ **3** $v^2 = u^2 + 2as$ **4** $s = \frac{1}{2}(u + v)t$

Exercise 19C

Skillsheet

1 How long does it take for an object that is initially at rest to travel a distance of 30 m if it is accelerated at 1.5 m/s^2?

2 A car is travelling at 25 m/s when the brakes are applied. It is brought to rest with uniform deceleration in 3 s. How far did it travel after the brakes were applied?

Example 7

3 A motorcycle accelerates uniformly from 3 m/s to 30 m/s in 9 seconds. Find:

- **a** the acceleration
- **b** the time it will take to increase in speed from 30 m/s to 50 m/s
- **c** the distance travelled in the first 15 seconds (assuming it starts from rest)
- **d** the time taken to reach a speed of 200 km/h (assuming it starts from rest).

4 A car accelerating uniformly from rest reaches a speed of 45 km/h in 5 seconds.

- **a** Find its acceleration.
- **b** Find the distance travelled in the 5 seconds.

5 A train starts from rest at a station and accelerates uniformly at 0.5 m/s^2 until it reaches a speed of 90 km/h.

- **a** How long does the train take to reach this speed?
- **b** How far does the train travel in reaching this speed?

6 A train travelling at 54 km/h begins to climb an incline of constant gradient that produces a deceleration of 0.25 m/s^2.

- **a** How long will the train take to travel a distance of 250 m?
- **b** What will the train's speed be then?

For Questions 7–11, assume that the acceleration due to gravity is −9.8 m/s^2 and ignore air resistance. Upward motion is considered to be in the positive direction.

7 A stone is projected vertically upwards from O with a speed of 20 m/s. Find:

- **a** the velocity of the stone after 4 s
- **b** the distance of the stone from O after 4 s.

8 Repeat Question 7 for the stone being projected downwards from O with the same speed.

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9 An object is projected vertically upwards with a velocity of 49 m/s.

a After what time will the object return to the point of projection?

b When will the object be at a height of 102.9 m above the point of projection?

10 A man dives from a springboard where his centre of gravity is initially 3 m above the water and his initial velocity is 4.9 m/s upwards. Regarding the diver as a particle at his centre of gravity and assuming that the diver's motion is vertical, find:

a the diver's velocity after t seconds

b the diver's height above the water after t seconds

c the maximum height of the diver above the water

d the time taken for the diver to reach the water.

11 A stone is thrown vertically upwards from the top edge of a cliff 24.5 m high with a speed of 19.6 m/s. Find:

a the time taken for the stone to reach its maximum height

b the maximum height above the base of the cliff reached by the stone

c the time taken for the stone to return to the point of projection

d the time taken for the stone to reach the base of the cliff.

12 A body is travelling at 20 m/s when it passes point P and 40 m/s when it passes point Q. Find its speed when it is halfway from P to Q, assuming uniform acceleration.

19D Velocity–time graphs

Many kinematics problems can be solved using velocity–time graphs. These are particularly useful if acceleration is constant, but with a broader knowledge of integral calculus they can also be used when acceleration is variable. (Integration will not be used in this book.)

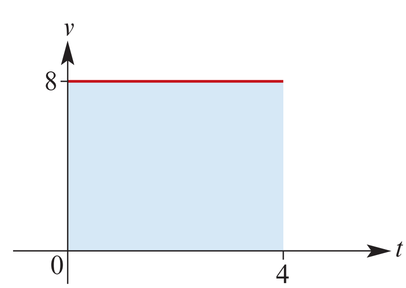
We begin with examples where the velocity is always positive.

Constant velocity

When a particle is moving with constant velocity, the corresponding velocity–time graph (v against t) is a straight line parallel to the t-axis.

The velocity–time graph for a particle moving at 8 m/s for 4 seconds is shown.

The shaded region is a rectangle of area $8 \times 4 = 32$, which is the product of the velocity and the time taken. Therefore this area is equal to the particle's displacement, 32 m, over the 4 seconds.



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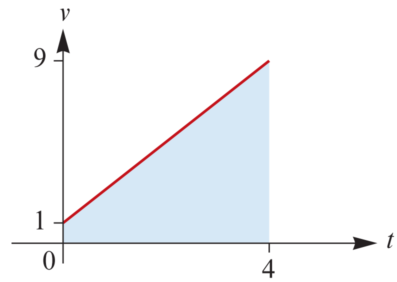
Constant acceleration

If a particle moves with constant acceleration a, its velocity v at time t is given by $v = u + at$, where u is the initial velocity. The velocity–time graph is a straight line with gradient a.

This graph shows the motion of a particle with initial velocity $u = 1$ m/s and acceleration $a = 2$ m/s^2. The equation of the straight line is $v = 1 + 2t$.



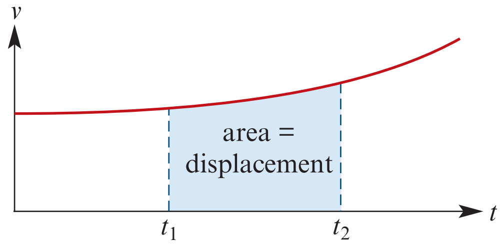
The particle's displacement over the 4 seconds is

$$s = \frac{1}{2}(u + v)t = \frac{1}{2}(1 + 9)4 = 20 \text{ m}$$

This is the area of the shaded trapezium.

Variable acceleration

If the velocity is always positive, then the displacement is equal to the distance travelled.



The total area of the region(s) between the velocity–time graph and the t-axis corresponds to the distance travelled by the particle between times t_1 and t_2.

Note: In Mathematical Methods Year 12, you will meet the fundamental theorem of calculus and see the connection between the area under a graph and antidifferentiation.

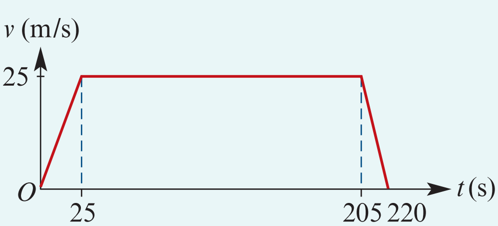
A velocity–time graph is particularly useful in situations where there are several stages to the particle's motion.

Example 8

A car starts from rest and accelerates uniformly for 25 s until it is travelling at 25 m/s. It maintains this velocity for 3 minutes, before decelerating uniformly until it stops in another 15 s. Construct a velocity–time graph and use it to determine the total distance travelled in kilometres.

Solution

From the graph we can calculate the area of the trapezium:



$$\begin{aligned}\text{Area} &= \frac{1}{2}(a + b)h \\ &= \frac{1}{2}(220 + 180)25 \\ &= 5000 \text{ m} \\ &= 5 \text{ km}\end{aligned}$$

The total distance travelled is 5 km.

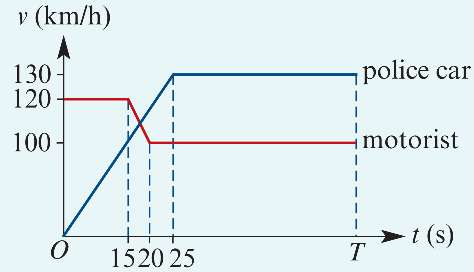
ISBN 978-1-316-63608-4

Example 9

A motorist is travelling at a constant speed of 120 km/h when he passes a stationary police car. He continues at that speed for another 15 s before uniformly decelerating to 100 km/h in 5 s. The police car takes off after the motorist the instant that he passes. It accelerates uniformly for 25 s, by which time it has reached 130 km/h. It continues at that speed until it catches up to the motorist. After how long does the police car catch up to the motorist and how far has he travelled in that time?

Solution

We start by representing the information on a velocity–time graph.



The distances travelled by the motorist and the police car will be the same, so the areas under the two velocity–time graphs will be equal. This fact can be used to find T, the time taken for the police car to catch up to the motorist.

Note: The factor $\frac{5}{18}$ changes velocities from km/h to m/s.

The distances travelled (in metres) after T seconds are given by

$$\begin{aligned}\text{Distance for motorist} &= \frac{5}{18}\left(120 \times 15 + \frac{1}{2}(120 + 100) \times 5 + 100(T - 20)\right)\\ &= \frac{5}{18}(1800 + 550 + 100T - 2000)\\ &= \frac{5}{18}(100T + 350)\end{aligned}$$

$$\begin{aligned}\text{Distance for police car} &= \frac{5}{18}\left(\frac{1}{2} \times 25 \times 130 + 130(T - 25)\right)\\ &= \frac{5}{18}(130T - 1625)\end{aligned}$$

When the police car catches up to the motorist:

$$\begin{aligned}100T + 350 &= 130T - 1625\\ 30T &= 1975\\ T &= \frac{395}{6}\end{aligned}$$

The police car catches up to the motorist after 65.83 s.

$$\begin{aligned}\therefore \quad \text{Distance for motorist} &= \frac{5}{18}(100T + 350) \qquad \text{where } T = \frac{395}{6}\\ &= \frac{52\,000}{27}\text{ m}\\ &= 1.926\text{ km}\end{aligned}$$

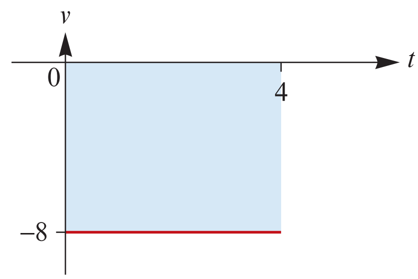
The motorist has travelled 1.926 km when the police car catches up.

ISBN 978-1-316-63608-4

Signed area

This graph shows the motion of a particle with a velocity of −8 m/s for 4 seconds. The shaded region represents a displacement of −32 m. The region has a signed area of −32.

- A region *above* the t-axis has *positive* signed area.
- A region *below* the t-axis has *negative* signed area.



Example 10

A particle is moving in a straight line. The initial velocity of the particle is 10 m/s and it has a constant acceleration of -2 m/s^2.

a Sketch the velocity–time graph for the motion.
b Describe the motion of the particle during the first 8 seconds.
c Find the total distance travelled in the first 8 seconds of motion.
d Find the displacement of the particle after the first 8 seconds of motion.

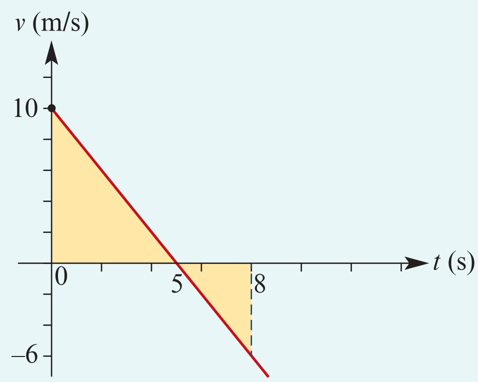
Solution

a We are given $u = 10$ and $a = -2$.

The equation of the line is

$$v = -2t + 10$$

where v m/s is the velocity at time t s.



b From $t = 0$ to $t = 5$, the particle has positive velocity; it is moving to the right.

At $t = 5$, the particle has velocity zero; it is momentarily stationary.

From $t = 5$ to $t = 8$, the particle has negative velocity; it is moving to the left.

c Distance travelled = total area $= \frac{1}{2} \times 5 \times 10 + \frac{1}{2} \times 3 \times 6 = 34$ metres

d Displacement = total signed area $= \frac{1}{2} \times 5 \times 10 - \frac{1}{2} \times 3 \times 6 = 16$ metres

Section summary

- **Distance travelled** is given by the sum of the **areas** of the regions between the velocity–time graph and the t-axis.
- **Displacement** is given by the sum of the **signed areas** of the regions between the velocity–time graph and the t-axis.

ISBN 978-1-316-63608-4

Exercise 19D

Skillsheet *It is suggested that you draw a velocity–time graph for each of the following questions.*

Example 8

1 A particle starts from rest and accelerates uniformly for 5 s until it reaches a speed of 10 m/s. It immediately decelerates uniformly until it comes to rest after a further 8 s. How far did it travel?

2 A car accelerates uniformly from rest for 10 s to a speed of 15 m/s. It maintains this speed for 25 s before decelerating uniformly to rest after a further 15 s. Find:

a the total distance travelled by the car

b the distance it had travelled when it started to decelerate

c the time taken for it to reach the halfway point of its journey.

3 A particle starts from rest and travels 1 km before coming to rest again. For the first 5 s it accelerates uniformly. It next maintains a constant speed for 500 m, and then decelerates uniformly for the last 10 s. Find the maximum speed of the particle.

4 A car passes point P with a speed of 36 km/h and continues at this speed for 12 s before accelerating to a speed of 72 km/h in 6 s. How far from P is the car when it reaches a speed of 72 km/h?

5 A tram decelerates uniformly from a speed of 60 km/h to rest in 60 s. Find:

a the distance travelled by the tram

b how far it had travelled by the time it had reduced its speed by half

c the time taken for it to travel half the total distance.

Example 9

6 A car passes a point A with a speed of 15 m/s and continues travelling at that speed. A second car is stationary at point A. At the moment when the first car passes A, the second car accelerates uniformly until it reaches a speed of 25 m/s in 10 s. Both cars continue with a constant speed on to point B, which they reach at the same time.

a How long does it take for both cars to reach point B?

b How far is it from A to B?

Example 10

7 A particle is moving in a straight line. The initial velocity of the particle is 20 m/s and it has a constant acceleration of -2 m/s^2.

a Sketch the velocity–time graph for the motion.

b Describe the motion of the particle during the first 14 seconds.

c Find the total distance travelled in the first 14 seconds of motion.

d Find the displacement of the particle after the first 14 seconds of motion.

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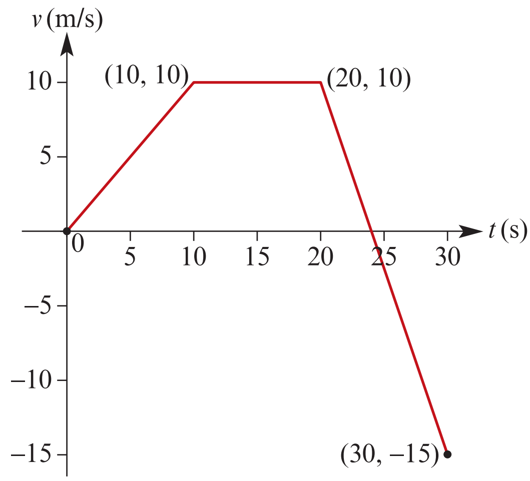
8 The velocity–time graph for the motion of a particle is shown.



a Find the acceleration for the first 10 seconds.

b Find the acceleration for the period from $t = 20$ to $t = 30$, where t is the time in seconds from the beginning of the motion.

c Find the total distance travelled in the first 30 seconds.

d Find the displacement of the particle after 30 seconds.

9 A particle moves in a straight line, starting from rest at a point O. It first moves in a positive direction with an acceleration of 2 m/s^2, until its velocity reaches 10 m/s. It then continues with a constant velocity of 10 m/s for some time, before decelerating to rest after a total time of 20 seconds. The total distance travelled is 160 m.

a Sketch the velocity–time graph.

b Find the magnitude of the deceleration.

10 Two stations A and B are 14 km apart. A train passes through station A, heading towards B, maintaining a constant speed of 60 km/h. At the instant that it passes through A, a second train on the same track leaves station B, heading towards A, and accelerates uniformly. After 5 minutes, the alarm is raised at both stations simultaneously that a collision is imminent. Both trains are radioed and instructed to brake. The first train decelerates uniformly so that it will stop in 2.5 minutes. The second train, which has reached a speed of 80 km/h, will take 4 minutes to stop. Will they collide?

11 Two tram stops are 800 m apart. A tram starts from rest at the first stop and accelerates at a constant rate of a m/s^2 for a certain time and then decelerates at a constant rate of $2a$ m/s^2, before coming to rest at the second stop. The time taken to travel between the two stops is 1 minute 40 seconds. Find:

a the maximum speed reached by the tram (in km/h)

b the time at which the brakes are applied

c the value of a.

ISBN 978-1-316-63608-4

Review

Chapter summary

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- The **position** of a particle moving in a straight line is determined by its distance from a fixed point O on the line, called the origin, and whether it is to the right or left of O. By convention, the direction to the right of the origin is considered to be positive.
- **Average velocity** $= \dfrac{\text{change in position}}{\text{change in time}}$
- For a particle moving in a straight line with position x at time t:
  - **velocity** (v) is the rate of change of position with respect to time
  - **acceleration** (a) is the rate of change of velocity with respect to time

$$v = \frac{dx}{dt}, \qquad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

- **Displacement** is the change in position (i.e. final position minus initial position).
- Scalar quantities:
  - **Distance travelled** means the total distance travelled.
  - **Speed** is the magnitude of the velocity.
  - **Average speed** $= \dfrac{\text{distance travelled}}{\text{change in time}}$
- Constant acceleration

  If acceleration is constant, then the following formulas can be used (for acceleration a, initial velocity u, final velocity v, displacement s and time taken t):

  1 $v = u + at$ **2** $s = ut + \frac{1}{2}at^2$ **3** $v^2 = u^2 + 2as$ **4** $s = \frac{1}{2}(u + v)t$
- Velocity–time graphs
  - *Distance travelled* is given by the sum of the *areas* of the regions between the velocity–time graph and the t-axis.
  - *Displacement* is given by the sum of the *signed areas* of the regions between the velocity–time graph and the t-axis.

Short-answer questions

1 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds ($t \geq 0$) is given by $x = t^2 - 4t - 5$. Find:

a its initial position

b its position at $t = 3$

c its initial velocity

d when and where its velocity equals zero

e its average velocity in the first 3 s

f its average speed in the first 3 s.

2 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds ($t \geq 0$) is given by $x = t^3 - 2t^2 + 8$. Find:

a its initial position, velocity and acceleration and hence describe its motion

b the times when it is stationary and its position and acceleration at those times.

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3 A particle moving in a straight line has position x cm relative to the point O at time t seconds ($t \geq 0$), where $x = -2t^3 + 3t^2 + 12t + 7$. Find:

a when the particle passes through O and its velocity and its acceleration at those times

b when the particle is at rest

c the distance travelled in the first 3 seconds.

4 Two particles A and B are moving in a straight line such that their positions, x_A cm and x_B cm, relative to the point O at time t seconds ($t \geq 0$) are given by

$$x_A(t) = t^3 - t^2 \quad \text{and} \quad x_B(t) = t^2$$

a Find:

i the position of A after $\frac{1}{2}$ s

ii the acceleration of A after $\frac{1}{2}$ s

iii the velocity of B after $\frac{1}{2}$ s.

b Find:

i the times when A and B collide (i.e. have the same position)

ii the maximum distance between A and B during the first 2 s of motion.

5 A particle moving in a straight line has an acceleration of $6t$ m/s^2 at time t seconds ($t \geq 0$). If the particle starts from rest at the origin O, find:

a the velocity after 2 s

b the position at any time t.

6 A particle moving in a straight line has an acceleration of $(3 - 2t)$ m/s^2 at time t seconds ($t \geq 0$). If the particle starts at the origin O with a velocity of 4 m/s, find:

a the time when the particle comes to rest

b the position of the particle at the instant it comes to rest

c the acceleration at this instant

d the time when the acceleration is zero

e the velocity at this time.

7 A particle moves in a straight line and, at time t seconds after it starts from point O, its velocity is $(2t^2 - 3t^3)$ m/s. Find:

a the position after 1 s

b the velocity after 1 s

c the acceleration after 1 s.

8 For a particle moving in a straight line, the velocity function is $v\colon \mathbb{R}^+ \to \mathbb{R}$, $v(t) = \dfrac{1}{2t^2}$. Find:

a the acceleration at time t

b the position at time t, given that the particle is at O when $t = 1$.

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9 The velocity, v m/s, of an object t seconds after it starts moving from O along a straight line is given by $v = t^3 - 11t^2 + 24t$, $t \geq 0$.

a Find the acceleration at time t.

b Find the acceleration at the instant when the object first changes direction.

c Find the displacement of the object from O after 5 s, and the total distance travelled in the first 5 s.

10 A car is travelling at 20 m/s when the brakes are applied. It is brought to rest with uniform deceleration in 4 s. How far did it travel after the brakes were applied?

11 A car accelerates uniformly from 0 m/s to 30 m/s in 12 seconds. Find:

a the acceleration

b the time it will take to increase in speed from 30 m/s to 50 m/s

c the distance travelled in the first 20 seconds

d the time taken to reach a speed of 100 km/h.

12 A train starts from rest at a station and accelerates uniformly at 0.4 m/s^2 until it reaches a speed of 60 km/h.

a How long does the train take to reach this speed?

b How far does the train travel in reaching this speed?

For Questions 13–14, assume that the acceleration due to gravity is −9.8 m/s^2 and ignore air resistance. Upward motion is considered to be in the positive direction.

13 An object is projected vertically upwards with a velocity of 35 m/s.

a After what time will the object return to the point of projection?

b When will the object be at a height of 60 m above the point of projection?

14 A stone is projected vertically upwards from the top of a cliff 20 m high with a speed of 19.6 m/s. Find:

a the time taken for the stone to reach its maximum height

b the maximum height reached with respect to the base of the cliff

c the time taken for the stone to return to the point of projection

d the time taken for the stone to reach the base of the cliff.

It is suggested that you draw a velocity–time graph for each of Questions 15–18.

15 A particle starts from rest and accelerates uniformly for 15 s until it reaches a speed of 25 m/s. It immediately decelerates uniformly until it comes to rest after a further 20 s. How far did it travel?

16 A car accelerates uniformly from rest for 8 s to a speed of 12 m/s. It maintains this speed for 15 s before decelerating uniformly to rest after a further 10 s. Find:

a the total distance travelled by the car

b the time taken for it to reach the halfway point of its journey.

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17 A vehicle starts from rest and travels 1 km before coming to rest again. For the first 15 s it accelerates uniformly, before maintaining a constant speed for 800 m and then finally decelerating uniformly to rest in 10 s. Find the maximum speed of the vehicle.

18 A car travels at a constant speed of 12 m/s along a straight road. It passes a second stationary car, which sets off in pursuit 3 s later. Find the constant acceleration required for the second car so that it catches the first car after a further 27 s has passed.

19 A particle moves in a straight line so that t seconds after passing a fixed point O in the line its velocity, v m/s, is given by $v = \frac{t^2}{4} - 3t + 5$. Calculate:

a the velocity after 10 s

b the acceleration when $t = 0$

c the minimum velocity

d the distance travelled in the first 2 s

e the distance travelled in the 3rd second.

20 A spot of light moves along a straight line so that its acceleration t seconds after passing a fixed point O on the line is $(2 - 2t)$ cm/s^2. Three seconds after passing O, the spot has a velocity of 5 cm/s. Find an expression, in terms of t, for:

a the velocity of the spot of light after t seconds

b the position of the spot relative to O after t seconds.

21 A particle P is moving along a straight line. It passes through a point O with a velocity of 6 m/s. At time t seconds after passing through O, its acceleration is $(4 - 4t)$ m/s^2.

a Show that, at time t seconds, the velocity of P is $(6 + 4t - 2t^2)$ m/s.

b Calculate:

- **i** the maximum velocity of P
- **ii** the value of t when the velocity of P is again 6 m/s
- **iii** the distance OP when the velocity of P is zero.

22 A particle travelling in a straight line passes a fixed point O with velocity 5 m/s. Its acceleration, a m/s^2, is given by $a = 27 - 4t^2$, where t seconds is the time after passing O. Calculate:

a the acceleration of the particle as it passes O

b its velocity when $t = 3$

c the value of t when its velocity is again 5 m/s.

23 A particle passes a fixed point O with a velocity of 2 m/s and moves in a straight line with an acceleration of $3(1 - t)$ m/s^2, where t is the time in seconds after passing O. Calculate:

a the velocity when $t = 4$

b the position of the particle at this instant.

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24 A particle P travels in a straight line starting at a fixed point O so that its velocity, v m/s, is given by $v = t^2 - 10t + 24$, where t is the time in seconds after leaving O. Find:

a the values of t for which P is instantaneously at rest

b the distance OP when $t = 3$

c the range of values of t for which the acceleration is negative.

Review

Multiple-choice questions

1 A particle moves in a straight line so that its position, x cm, relative to a fixed point O at time t seconds ($t \geq 0$) is given by $x = -t^3 + 7t^2 - 12t$. The initial position of the particle relative to O is

A 0 cm **B** −6 cm **C** 12 cm **D** −20 cm **E** 5 cm

2 A particle moves in a straight line so that its position, x cm, relative to a fixed point O at time t seconds ($t \geq 0$) is given by $x = -t^3 + 7t^2 - 12t$. The average velocity of the particle in the first 2 seconds, correct to two decimal places, is

A 4 cm/s **B** −4 cm/s **C** 2 cm/s **D** 4.06 cm/s **E** −2 cm/s

3 A particle moves in a straight line with an acceleration of $4 - 6t$ m/s^2 at time t seconds. The particle has an initial velocity of −1 m/s and an initial position of 4 m relative to a fixed point O. The velocity of the particle when $t = 1$ is

A −1 m/s **B** 6 m/s **C** 0 m/s **D** 4 m/s **E** −2 m/s

4 A body starts from rest with a uniform acceleration of 1.8 m/s^2. The time it will take for the body to travel 90 m is

A 5 s **B** $\sqrt{10}$ s **C** 10 s **D** $\sqrt{10}$ **E** $10\sqrt{2}$ s

5 A car accelerating uniformly from rest reaches a speed of 60 km/h in 4 s. The car's acceleration is

A 15 km/h^2 **B** 15 m/s^2 **C** 54 m/s^2 **D** $\frac{25}{6}$ km/h^2 **E** $\frac{25}{6}$ m/s^2

6 A car accelerating uniformly from rest reaches a speed of 60 km/h in 4 s. The distance travelled by the car in the 4 s is

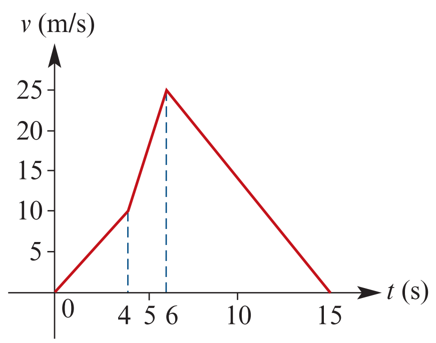
A 200 m **B** 100 km **C** $\frac{100}{3}$ m **D** 100 m **E** 360 m

7 This velocity–time graph shows the motion of a car. The total distance travelled by the car over the 15 s is

A 75 m **B** 315 m

C 182.5 m **D** 167.5 m

E 375 m



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Review

8 A rock falls from the top of a cliff 40 m high. Assuming that $g = 9.8$ m/s^2, the rock's speed (in m/s) just before it hits the ground is

A 20 **B** 22 **C** 24 **D** 26 **E** 28

9 A body initially travelling at 20 m/s is subject to a constant deceleration of 4 m/s^2. The time it takes to come to rest (t seconds) and the distance travelled before it comes to rest (s metres) are given by

A $t = 5, \ s = 50$ **B** $t = 5, \ s = 45$ **C** $t = 4, \ s = 20$
D $t = 5, \ s = 40$ **E** $t = 4, \ s = 35$

10 A particle moves in a straight line with an acceleration of $12t - 5$ m/s^2 at time t seconds. The particle has an initial velocity of 1 m/s and an initial position of 0 m relative to a fixed point O. The velocity of the particle at time $t = 1$ is

A 1 m/s **B** −5 m/s **C** 7 m/s **D** 2 m/s **E** 3 m/s

Extended-response questions

1 A particle moves in a straight line so that its position, x cm, relative to point O at time t seconds is given by

$$x = \frac{1}{3}t^3 - 2t^2 + 4t - 2\tfrac{1}{3}$$

a Find its initial position.
b Find its initial velocity.
c Find its acceleration after 3 seconds.
d When is its velocity zero?
e What is its position when the velocity is zero?
f When is the particle at point O?

2 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds ($t \geq 0$) is given by $x = t^4 + 2t^2 - 8t$. Show that:

a the particle moves first to the left
b the greatest distance of the particle to the left of O occurs after 1 second
c after this time, the particle always moves to the right.

3 A defective rocket rises vertically upwards into the air and then crashes back to the ground. The rocket's height above the ground, h metres, at time t seconds after take-off is given by $h = 6t^2 - t^3$. (This is an approximate model.)

a When does the rocket crash and what is its velocity at this time?
b At what time is the speed of the rocket zero, and what is its maximum height?
c When does the acceleration of the rocket become negative?

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4 An object is projected vertically upwards at 20 m/s from the top of a tower 10 m high on the edge of a vertical cliff. At time t seconds after projection, the object has position $x(t)$ metres relative to the base of the tower, where $x(t) = -4.9t^2 + 20t + 10$ for $t \geq 0$. Use a CAS calculator to evaluate the values

$$x(1) - x(0), \quad x(2) - x(1), \quad x(3) - x(2), \quad \ldots, \quad x(10) - x(9)$$

Analyse your results and draw some inference about the motion of the object.

5 A particle is projected vertically upwards with an initial speed of u m/s. Prove that:

a the time taken by the particle to reach its highest point is $\frac{u}{g}$ seconds

b the total time taken for the particle to return to the point of projection is $\frac{2u}{g}$ seconds

c the particle's speed when returning to the point of projection is u m/s.

6 A stone is projected vertically upwards with a speed of 14 m/s from a point O at the top of a mine shaft. Five seconds earlier, a lift began to descend the mine shaft from O with a constant speed of 3.5 m/s. Find the depth of the lift (to the nearest metre) at the instant when the stone falls on it. (Neglect air resistance and take the acceleration due to gravity to be 9.8 m/s^2.)

7 A car is travelling along a straight road at 90 km/h when the brakes are applied. The car comes to rest in 5 seconds and, during this time, its velocity decreases linearly with time. Find:

a the rule for the velocity function after the brakes are applied

b the distance travelled in the 5 seconds.

8 A particle moves in a straight line so that its position, x cm, relative to point O at time t seconds ($t \geq 0$) is given by $x = 3t^4 - 4t^3 + 24t^2 - 48t$. Show that the particle moves at first to the left, comes to rest at a point A and then moves always to the right. Find the position of A.

9 A particle is projected vertically upwards with a velocity of u m/s from a point O on the ground, and T seconds later a second particle is projected vertically upwards from O with the same velocity.

a Prove that:

i the time taken for the two particles to collide is $\frac{u}{g} + \frac{T}{2}$ seconds after the first particle was launched

ii the height of the particles when they collide is $\frac{4u^2 - g^2T^2}{8g}$ metres above O.

b Interpret the case where $T = \frac{2u}{g}$.

c What happens if $T > \frac{2u}{g}$?

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Chapter 20

20 Statics of a particle

Objectives

- To determine the **resultant force** acting on a particle.
- To identify a **weight force**, a **normal force** and a **tension force**.
- To use a **triangle of forces** to solve problems.
- To **resolve forces** acting in a plane in two directions at right angles.

A force is a vector quantity, so you should have completed some of Chapter 17 before beginning this chapter.

A **force** is a measure of the strength of a **push** or **pull**. Forces can:

- start motion
- stop motion
- make objects move faster or slower
- change the direction of motion.

Force can be defined as the physical quantity that causes a change in motion.

In this chapter, we are only concerned with the situation where the forces 'cancel each other out'. This is the study of **statics**, which is part of the area of mathematical physics called **classical mechanics**.

For example, you may have heard of **Archimedes' principle**, which applies to objects in a fluid. If an inflatable raft is floating in a swimming pool, then the water is exerting an upwards force on the raft (called the buoyant force) that cancels out the downwards force of gravity.

Situations where the forces do not 'cancel out' are studied in Specialist Mathematics Year 12.

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20A Forces and triangle of forces

A force has both magnitude and direction – it may be represented by a vector.

When considering the forces that act on an object, it is convenient to treat the forces as acting on a single particle. The single particle may be thought of as a point at which the entire mass of the object is concentrated.

▶ Weight and units of force

Every object near the surface of the Earth is subject to the force of gravity. We refer to this force as the **weight** of the object. Weight is a force that acts vertically downwards on an object (actually towards the centre of the Earth).

The unit of force used in this chapter is the **kilogram weight** (kg wt). If an object has a **mass** of 1 kg, then the force due to gravity acting on the object is 1 kg wt.

This unit is convenient for objects near the Earth's surface. An object with a mass of 1 kg would have a different weight on the moon.

Note: The standard unit of force is the newton (N). At the Earth's surface, a mass of m kg has a force of m kg wt $= mg$ N acting on it, where g is the acceleration due to gravity.

▶ Resultant force and equilibrium

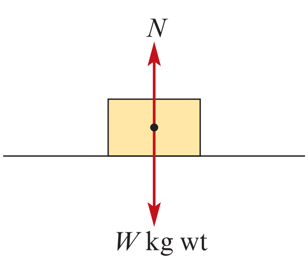
When a number of forces act simultaneously on an object, their combined effect is called the **resultant force**. The resultant force is the vector sum of the forces acting on the particle.

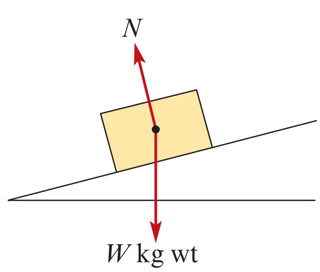
If the resultant force acting on an object is zero, the object will remain at rest or continue moving with constant velocity. The object is said to be in **equilibrium**.

Note: Planet Earth is moving and our galaxy is moving, but we use Earth as our frame of reference and so our observation of an object being at rest is determined in this way.

▶ Normal force

Any mass placed on a surface, either horizontal or inclined, experiences a force perpendicular to the surface. This force is referred to as a **normal force**.





For example, a book sitting on a table is obviously being subjected to a force due to gravity. But the fact that it does not fall to the ground indicates that there must be a second force on the book. The table is exerting a force on the book equal in magnitude to gravity, but in the opposite direction. Hence the book remains at rest; it is in **equilibrium**.

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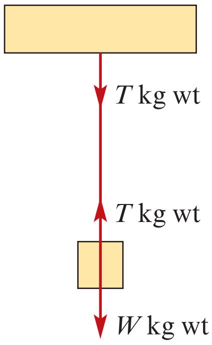
Tension force

The diagram shows a string attached to the ceiling supporting a mass, which is at rest. The force of gravity, W kg wt, acts downwards on the mass and the string exerts an equal force, T kg wt, upwards on the mass. The force exerted by the string is called the **tension force**.

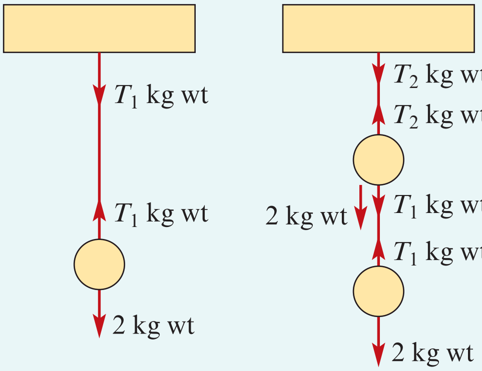


Note that there is a force, equal in magnitude but opposite in direction, acting on the ceiling at the point of contact.

Example 1

a In the diagram on the left, one spherical mass of weight 2 kg wt is attached to the ceiling. Find T_1.

b In the diagram on the right, two equal spherical masses of weight 2 kg wt are attached to the ceiling as shown. Find T_1 and T_2.



Solution

The tensions in the one section of string have the same magnitude.

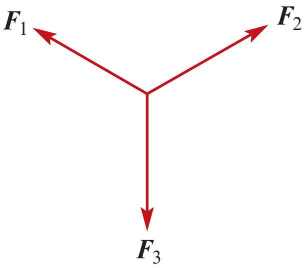
a $T_1 = 2$ kg wt

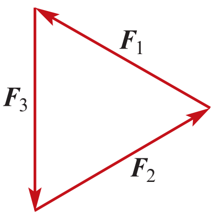
b The forces acting on the lower mass are the same as before, and so $T_1 = 2$ kg wt. For the higher mass, we have $T_2 = 2 + T_1$ and so $T_2 = 4$ kg wt.

Triangle of forces

If three forces are acting on a point in equilibrium, then they can be represented by three vectors forming a triangle.

Suppose that three forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ are acting on a particle in equilibrium, as shown in the diagram on the left. Since the particle is in equilibrium, we must have $\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 = \boldsymbol{0}$. Therefore the three forces can be rearranged into a triangle as shown on the right.





The magnitudes of the forces and the angles between the forces can now be found using trigonometric ratios (if the triangle contains a right angle) or using the sine or cosine rule.

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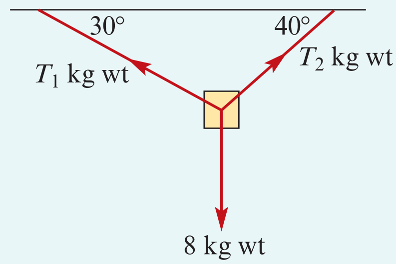
In the following examples and exercises, strings and ropes are considered to have negligible mass. A smooth light pulley is considered to have negligible mass and the friction between a rope and pulley is considered to be negligible.

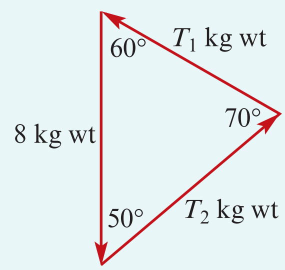
Example 2

A particle of mass 8 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of 30° and 40° to the horizontal, find the tension in each string.



Solution

Represent the forces in a triangle. The sine rule gives



$$\frac{T_1}{\sin 50^\circ} = \frac{T_2}{\sin 60^\circ} = \frac{8}{\sin 70^\circ}$$

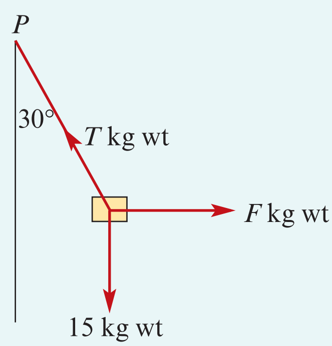
$$T_1 = \frac{8}{\sin 70^\circ} \times \sin 50^\circ \approx 6.52 \text{ kg wt}$$

$$T_2 = \frac{8}{\sin 70^\circ} \times \sin 60^\circ \approx 7.37 \text{ kg wt}$$

Example 3

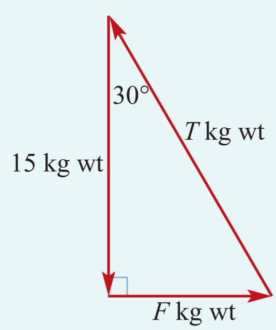
A particle of mass 15 kg is suspended vertically from a point P by a string. The particle is pulled horizontally by a force of F kg wt so that the string makes an angle of 30° with the vertical.

Find the value of F and the tension in the string.



Solution

Representing the forces in a triangle gives



$$\frac{F}{15} = \tan 30^\circ$$

$$F = 15 \tan 30^\circ = 5\sqrt{3}$$

and $$\frac{15}{T} = \cos 30^\circ$$

$$T = \frac{15}{\cos 30^\circ} = 10\sqrt{3}$$

The tension in the string is $10\sqrt{3}$ kg wt.

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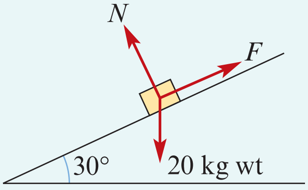
Example 4

A body of mass 20 kg is placed on a smooth plane inclined at 30° to the horizontal. A string is attached to a point further up the plane which prevents the body from moving. Find the tension in the string and the magnitude of the force exerted on the body by the plane.

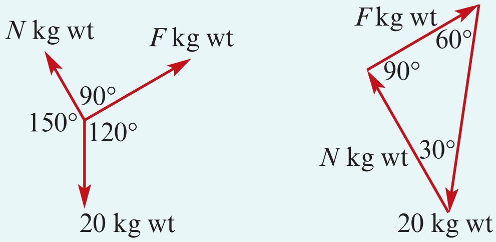


Solution

The three forces form a triangle (as the body is in equilibrium). Therefore

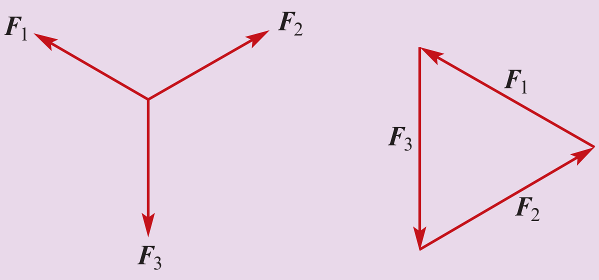
$$F = 20 \sin 30^\circ = 10 \text{ kg wt}$$

$$N = 20 \cos 30^\circ = 10\sqrt{3} \text{ kg wt}$$



Section summary

- **Force** is a vector quantity.
- The magnitude of a force can be measured using **kilogram weight** (kg wt).
  If an object near the surface of the Earth has a mass of 1 kg, then the force due to gravity acting on the object is 1 kg wt.
- Triangle of forces
  If three forces are acting on a point in equilibrium, then they can be represented by three vectors forming a triangle.

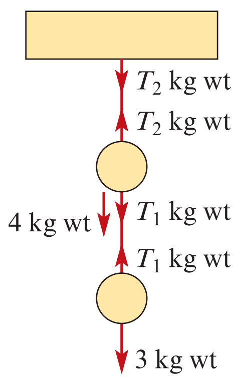


Exercise 20A

Skillsheet

Example 1

1 Two spherical weights of mass 3 kg and 4 kg are attached to the ceiling as shown. Find T_1 and T_2.



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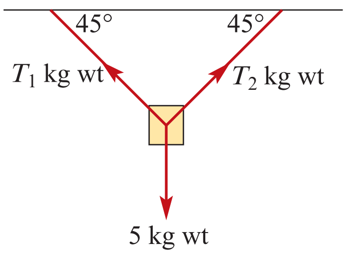
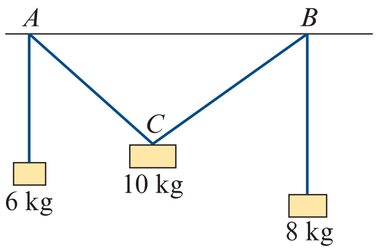
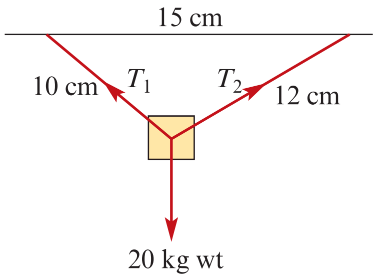
Example 2 **2** A particle of mass 5 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of 45° with the horizontal, find the tension in each string.



3 Using strings and pulleys, three weights of mass 6 kg, 8 kg and 10 kg are suspended in equilibrium as shown. Calculate the magnitude of the angle ACB.

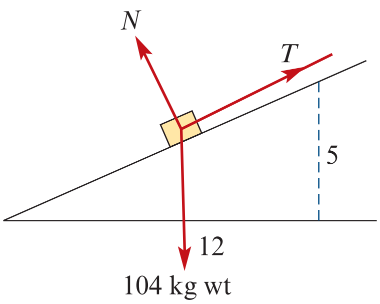


4 A mass of 20 kg is suspended from two strings of length 10 cm and 12 cm, the ends of the strings being attached to two points in a horizontal line, 15 cm apart. Find the tension in each string.

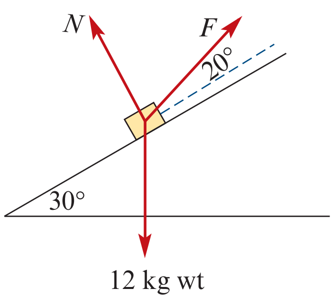


Example 3 **5** A boat is being pulled by a force of 40 kg wt towards the east and by a force of 30 kg wt towards the north-west. What third force must be acting on the boat if it remains stationary? Give the magnitude and direction.

Example 4 **6** A body of mass 104 kg is placed on a smooth inclined plane which rises 5 cm vertically for every 12 cm horizontally. A string is attached to a point further up the plane which prevents the body from moving. Find the tension in the string and the magnitude of the force exerted on the body by the plane.



7 A body of mass 12 kg is kept at rest on a smooth inclined plane of 30° by a force acting at an angle of 20° to the plane. Find the magnitude of the force.



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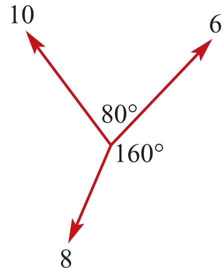
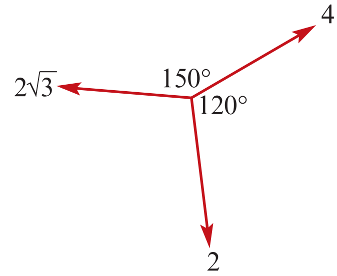
8 In each of the following cases, determine whether the particle is in equilibrium:

a 

b 

9 Three forces of magnitude 4 kg wt, 7 kg wt and 10 kg wt are in equilibrium. Determine the magnitudes of the angles between the forces.

10 A mass of 15 kg is maintained at rest on a smooth inclined plane by a string that is parallel to the plane. Determine the tension in the string if:

a the plane is at 30° to the horizontal

b the plane is at 40° to the horizontal

c the plane is at 30° to the horizontal, but the string is held at an angle of 10° to the plane.

11 A string is connected to two points A and D in a horizontal line and masses of 12 kg and W kg are attached at points B and C. If AB, BC and CD make angles of 40°, 20° and 50° respectively with the horizontal, calculate the tensions in the string and the value of W.

20B Resolution of forces

Obviously there are many situations where more than three forces (or in fact only two forces) will be acting on a body. An alternative method is required to solve such problems.

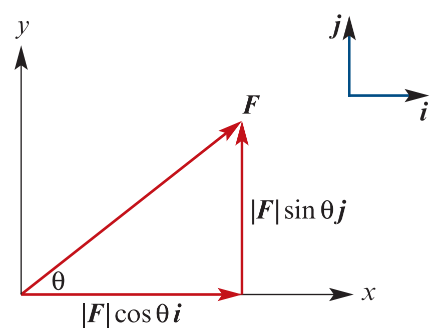
If all forces under consideration are acting in the same plane, then these forces and the resultant force can each be expressed as a sum of its $\boldsymbol{i}$- and $\boldsymbol{j}$-components.

If a force $\boldsymbol{F}$ acts at an angle of θ to the x-axis, then $\boldsymbol{F}$ can be written as the sum of two forces, one 'horizontal' and the other 'vertical':

$$\boldsymbol{F} = |\boldsymbol{F}|\cos\theta\,\boldsymbol{i} + |\boldsymbol{F}|\sin\theta\,\boldsymbol{j}$$

The force $\boldsymbol{F}$ is **resolved** into two components:

- the $\boldsymbol{i}$-component is parallel to the x-axis
- the $\boldsymbol{j}$-component is parallel to the y-axis.



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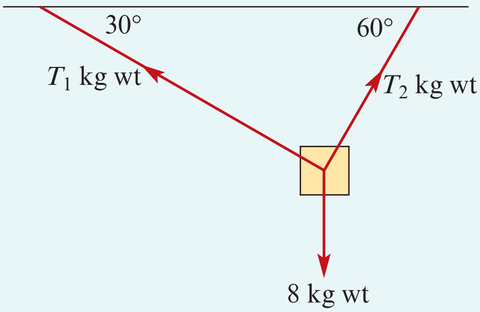
For a particle that is in equilibrium, if all the forces acting on the particle are resolved into their $\boldsymbol{i}$- and $\boldsymbol{j}$-components, then:

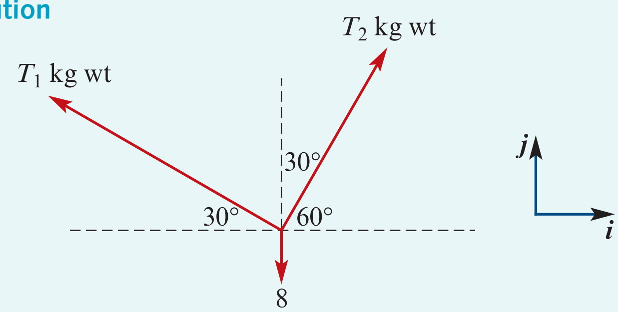
- the sum of all the $\boldsymbol{i}$-components is zero
- the sum of all the $\boldsymbol{j}$-components is zero.

Example 5

A particle of mass 8 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of 30° and 60° to the horizontal, find the tension in each string.



Solution



Resolution in the $\boldsymbol{j}$-direction:

$$T_1 \sin 30^\circ + T_2 \sin 60^\circ - 8 = 0$$

$$T_1\left(\frac{1}{2}\right) + T_2\left(\frac{\sqrt{3}}{2}\right) - 8 = 0 \qquad (1)$$

Resolution in the $\boldsymbol{i}$-direction:

$$-T_1 \cos 30^\circ + T_2 \cos 60^\circ = 0$$

$$-T_1\left(\frac{\sqrt{3}}{2}\right) + T_2\left(\frac{1}{2}\right) = 0 \qquad (2)$$

From (2): $\sqrt{3}\,T_1 = T_2$

Substituting in (1) gives

$$T_1\left(\frac{1}{2}\right) + \sqrt{3}\,T_1\left(\frac{\sqrt{3}}{2}\right) - 8 = 0$$

$$4T_1 = 16$$

$$\therefore \quad T_1 = 4$$

Hence $T_1 = 4$ and $T_2 = 4\sqrt{3}$. The tensions in the strings are 4 kg wt and $4\sqrt{3}$ kg wt.

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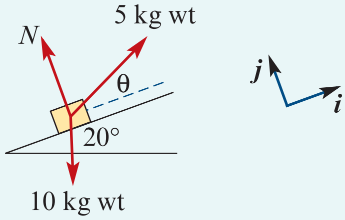
Example 6

A body of mass 10 kg is held at rest on a smooth plane inclined at 20° by a string with tension 5 kg wt as shown.

Find the angle between the string and the inclined plane.



Solution

We resolve the forces parallel and perpendicular to the plane. Then N has no parallel component, since N is perpendicular to the plane.

Resolving in the $\boldsymbol{i}$-direction:

$$5\cos\theta^\circ - 10\sin 20^\circ = 0$$

$$\cos\theta^\circ = \frac{10\sin 20^\circ}{5}$$

$$\therefore \quad \theta = \cos^{-1}(0.684)$$

$$= 46.84^\circ$$

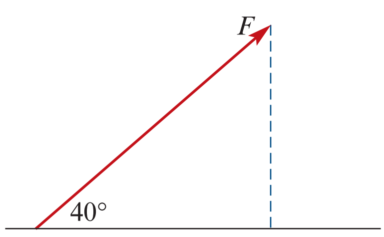
Section summary

- A force $\boldsymbol{F}$ is **resolved** into components by writing it in the form $\boldsymbol{F} = x\boldsymbol{i} + y\boldsymbol{j}$.
- If forces are acting on a particle that is in equilibrium, then:
  - the sum of the $\boldsymbol{i}$-components of all the forces is zero
  - the sum of the $\boldsymbol{j}$-components of all the forces is zero.

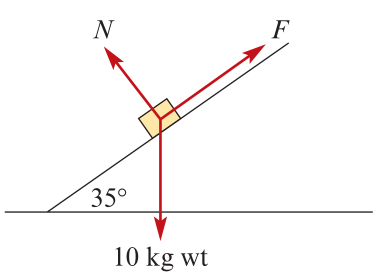
Exercise 20B

Skillsheet *For the following questions, give answers correct to two decimal places.*

1 A force F kg wt makes an angle of 40° with the horizontal. If its horizontal component is a force of 10 kg wt, find the magnitude of F.



Example 5 **2** Find the magnitude of the force, acting on a smooth inclined plane of angle 35°, required to support a mass of 10 kg resting on the plane.



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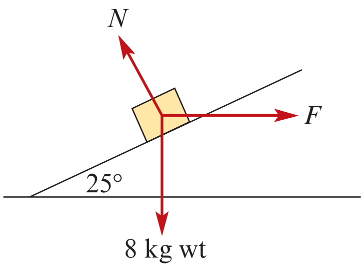
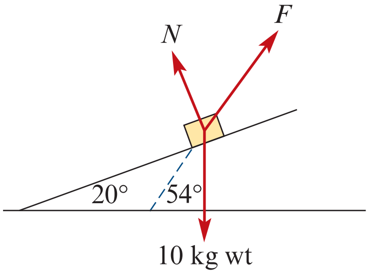
3 A body of mass 8 kg rests on a smooth inclined plane of angle 25° under the action of a horizontal force. Find the magnitude of the force and the reaction of the plane on the body.



Example 6

4 A body of mass 10 kg rests on a smooth inclined plane of angle 20°. Find the force that will keep it in equilibrium when it acts at an angle of 54° with the horizontal.

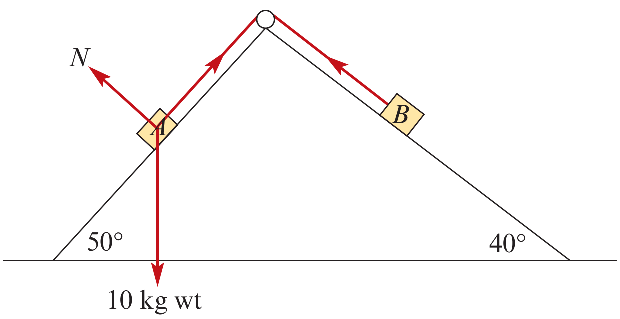


5 If a body of mass 12 kg is suspended by a string, find the horizontal force required to hold it at an angle of 30° from the vertical.

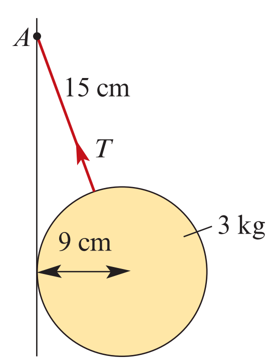
6 A force of 20 kg wt acting directly up a smooth plane inclined at an angle of 40° maintains a body in equilibrium on the plane. Calculate the weight of the body and the pressure it exerts on the plane.

7 Two men are supporting a block by ropes. One exerts a force of 20 kg wt, his rope making an angle of 35° with the vertical, and the other exerts a force of 30 kg wt. Determine the weight of the block and the angle of direction of the second rope.

8 A body A of mass 10 kg is supported against a smooth plane of angle 50°. Find the pressure of the body on the plane and the tension in the string, which is parallel to the slope. A body B on a plane of angle 40° is connected to A by a string passing over a smooth pulley on the ridge. If the system is in equilibrium, what is the mass of B?



9 A sphere of radius 9 cm is attached to a point A on a vertical wall by a string of length 15 cm. If the mass of the sphere is 3 kg, find the tension in the string.



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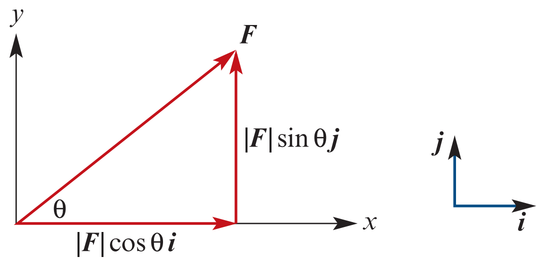
Chapter summary

- **Resultant force** When a number of forces act simultaneously on an object, their combined effect is called the **resultant force**.
- **Equilibrium** If the resultant force acting on an object is zero, then the object is said to be in **equilibrium**; it will remain at rest or continue moving with constant velocity.
- **Triangle of forces** If three forces are acting on a particle in equilibrium, then the vectors representing the forces may be arranged to form a triangle. The magnitudes of the forces and the angles between them can be found using trigonometric ratios (if the triangle contains a right angle) or using the sine or cosine rule.
- **Resolution of forces**
  If all forces on a particle are acting in two dimensions, then each force can be expressed in terms of its components in the $\boldsymbol{i}$- and $\boldsymbol{j}$-directions:

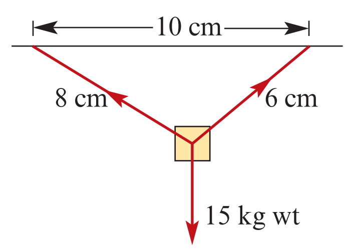
  $$\boldsymbol{F} = |\boldsymbol{F}|\cos\theta\,\boldsymbol{i} + |\boldsymbol{F}|\sin\theta\,\boldsymbol{j}$$



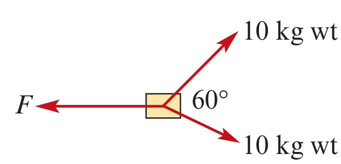
  For the particle to be in equilibrium, the sum of all the $\boldsymbol{i}$-components must be zero and the sum of all the $\boldsymbol{j}$-components must be zero.

Short-answer questions

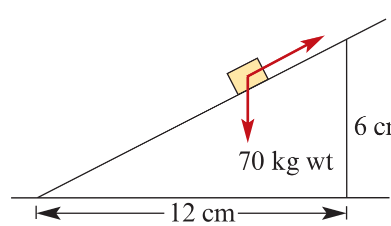
1 A mass of 15 kg is suspended from two strings of length 6 cm and 8 cm, the ends of the strings being attached to two points in a horizontal line, 10 cm apart. Find the tension in each string.



2 An object is being pulled by two forces of 10 kg wt as shown in the diagram. What is the magnitude and direction of the third force acting on the object if it remains stationary?



3 A body of mass 70 kg is placed on a smooth inclined plane which rises 6 cm vertically for every 12 cm horizontally. A string is attached to a point further up the plane which prevents the body from moving. Find the tension in the string and the magnitude of the force exerted on the body by the plane.



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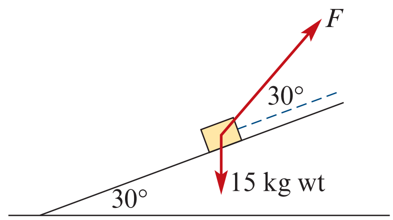
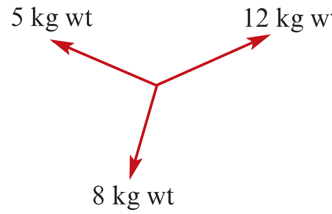
Review

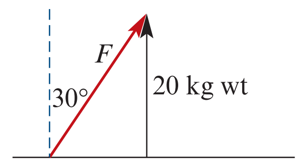
4 A body of mass 15 kg is kept at rest on a smooth inclined plane of 30° by a force acting at an angle of 30° to the plane. Find the magnitude of the force.



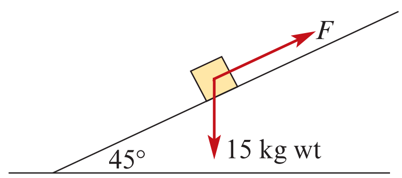
5 Three forces of magnitude 5 kg wt, 8 kg wt and 12 kg wt are in equilibrium. Determine the cosine of the angle between the 5 kg wt and 12 kg wt forces.



6 A force of F kg wt makes an angle of 30° with the vertical. If its vertical component is a force of 20 kg wt, find the magnitude of F.



7 Find the magnitude of the force, acting up a smooth inclined plane of angle 45°, required to support a mass of 15 kg resting on the plane.



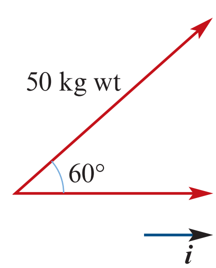
8 A force of 14 kg wt acting directly up a smooth plane inclined at an angle of 30° maintains a body in equilibrium on the plane. Calculate the weight of the body and the force it exerts on the plane.

9 A body of mass 12 kg is kept at rest on a smooth incline of 30° by a horizontal force. Find the magnitude of the force.

Multiple-choice questions

1 The magnitude of the component of force $\boldsymbol{F}$ in the $\boldsymbol{i}$-direction is

A 300 kg wt **B** 50 kg wt **C** 40 kg wt
D 20 kg wt **E** 25 kg wt



2 Two perpendicular forces have magnitudes 5 kg wt and 4 kg wt. The magnitude of the resultant force is

A 3 kg wt **B** $\sqrt{11}$ kg wt **C** $\sqrt{41}$ kg wt **D** 1 kg wt **E** 9 kg wt

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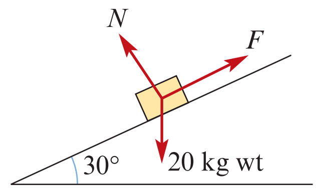
Review

Questions 3–4 refer to the following information:

A 20 kg mass is resting on a smooth plane inclined at 30° to the horizontal and is prevented from slipping down the plane by a string as shown in the diagram.



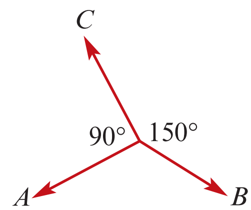
3 The magnitude of N is

A 10 kg wt **B** 20 kg wt **C** $\frac{\sqrt{3}}{2}$ kg wt **D** 60 kg wt **E** $10\sqrt{3}$ kg wt

4 The magnitude of the tension in the string is

A 10 kg wt **B** 20 kg wt **C** $\frac{\sqrt{3}}{2}$ kg wt **D** 60 kg wt **E** $10\sqrt{3}$ kg wt

5 The diagram represents a particle in equilibrium acted on by three forces in a plane of magnitudes A, B and C. Which one of the following statements is not true?



A $A = B\cos 60°$ **B** $A = \dfrac{C\cos 60°}{\cos 30°}$

C $B = A\cos 60°$ **D** $B = A\cos 60° + C\cos 30°$

E $C = B\cos 30°$

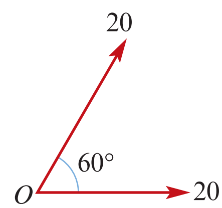
6 A particle is acted on by a force of magnitude 7 kg wt acting on a bearing of 45° and by a force of magnitude a kg wt acting on a bearing of 135°. If the magnitude of the resultant force is 9 kg wt, then the value of a must be

A 2 **B** $4\sqrt{2}$ **C** $\sqrt{130}$ **D** 16 **E** 32

7 Two forces of magnitude 20 kg wt act on a particle at O as shown. The magnitude of the resultant force in kg wt is

A 40 **B** $20\sqrt{3}$ **C** 0

D 20 **E** 10



8 The resultant force when two forces of magnitude 300 kg wt and 200 kg wt act at an angle of 60° to each other is

A $100\sqrt{19}$ kg wt **B** 436 kg wt **C** 100 kg wt

D 350 kg wt **E** 500 kg wt

9 Two perpendicular forces have magnitudes 16 kg wt and 30 kg wt. The magnitude of the resultant force is

A 50 kg wt **B** 10 kg wt **C** 34 kg wt **D** $6\sqrt{35}$ kg wt **E** 2 kg wt

10 A particle is acted on by a force of magnitude 8 kg wt acting on a bearing of 30° and by a force of magnitude a kg wt acting on a bearing of 120°. If the magnitude of the resultant force is 12 kg wt, then the value of a must be

A 2 **B** $4\sqrt{5}$ **C** $\sqrt{130}$ **D** 20 **E** $4\sqrt{13}$

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Revision

21 Revision of Chapters 19–20

21A Short-answer questions

1 A particle starts from rest at a point A and moves in a straight line ABC with constant acceleration. At B its speed is 10 m/s and at C its speed is 15 m/s. Given that $BC = 15$ metres, find:

- **a** the time taken to go from B to C
- **b** the acceleration of the particle
- **c** the time taken to go from A to B
- **d** the distance AB.

2 Two forces of equal magnitude F kg wt act on a particle and they have a resultant force of magnitude 6 kg wt. When one of the forces is doubled in magnitude, the resultant force is 11 kg wt. Find the value of F and the cosine of the angle between the two forces.

3 Two forces $\boldsymbol{P}$ and $\boldsymbol{Q}$ of magnitudes $P = 6$ kg wt and $Q = 2$ kg wt have a resultant force of magnitude 5 kg wt. Find the cosine of the angle between forces $\boldsymbol{P}$ and $\boldsymbol{Q}$.

4 An object is dropped from a point A, which is 100 m above the ground, and at the same instant a second object is projected upwards from the ground at a point vertically below A. If the two objects meet at 50 m above the ground, find the velocity of projection of the second object. (Take the acceleration due to gravity to be 10 m/s^2.)

5 The following forces are acting in the same plane on a particle:

- 6 kg wt in a direction of 045°
- 8 kg wt in a direction of 180°
- Q kg wt

If the particle is in equilibrium, find the square of the magnitude of Q.

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6 A particle starts from rest and moves in a straight line with constant acceleration. If the particle travels 51 metres during the 9th second, find its acceleration.

7 A particle is travelling in a straight line. It passes a certain point at 8 m/s and it accelerates at a constant rate of 2 m/s^2 for a distance of 65 m. Find the time taken to travel this distance.

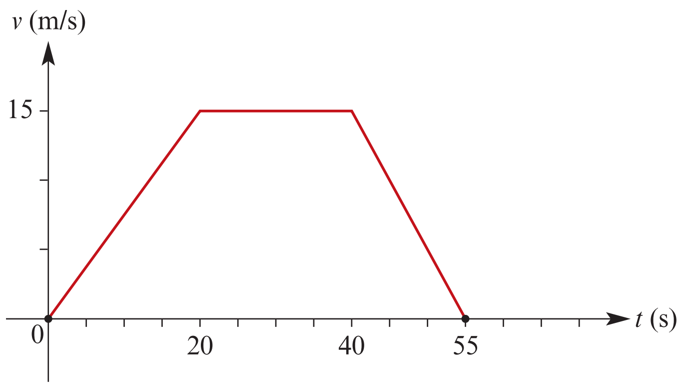
8 A block of mass 10 kg is maintained at rest on a smooth plane inclined at 30° to the horizontal by a string. Calculate the tension in the string and the reaction of the plane if:

a the string is parallel to the plane **b** the string is horizontal.

9 A mass of 8 kg is supported by a string attached to a fixed point and is pulled from the vertical by a horizontal force of 6 kg wt. Find the tension in the string and the tangent of the angle that the string makes with the vertical.

10 A mass of 10 kg is suspended by two strings of lengths 5 cm and 12 cm that are attached to fixed points on the same horizontal level 13 cm apart. Find the tensions in the strings.

11 A delivery van travels in a straight line for 55 seconds. The motion of the delivery van is shown in the velocity–time graph.



Find:

a the distance travelled by the van

b the average speed of the van

c the acceleration in the first 20 seconds

d the two times at which the speed is 10 m/s.

12 An object is projected vertically upwards from ground level. Between 2 seconds and 3 seconds after leaving the ground, it travels 45 metres. (Take the acceleration due to gravity to be 10 m/s^2.) Find:

a the speed of projection

b the maximum height reached

c the time interval during which it is more than 165 metres above ground level.

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21B Multiple-choice questions

Questions 1–9 refer to a particle that moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = 2t^2 - 5t - 12$ *for* $t \geq 0$.

1 The initial position of the particle is

A 12 cm **B** 0 cm **C** −15 cm **D** −12 cm **E** −3 cm

2 The initial velocity of the particle is

A 0 cm/s **B** 1 cm/s **C** −1 cm/s **D** 3 cm/s **E** −5 cm/s

3 The initial acceleration of the particle is

A 0 cm/s^2 **B** 4 cm/s^2 **C** −4 cm/s^2 **D** 3 cm/s^2 **E** −1 cm/s^2

4 The particle is stationary

A after 1 s **B** after 1.25 s **C** after 5 s **D** after 0.8 s **E** never

5 The particle passes through O at time $t =$

A 1 s **B** −1 s **C** 4 s **D** −4 s **E** never

6 The particle's position after 3 s is

A 9 cm **B** 0 cm **C** −9 cm **D** 13 cm **E** 21 cm

7 The average velocity of the particle over the first 3 seconds is

A 1 cm/s **B** −1 cm/s **C** 7 cm/s **D** −7 cm/s **E** 0 cm/s

8 The distance travelled by the particle in the first 3 seconds is

A 3 cm **B** −3 cm **C** −9 cm **D** 9.25 cm **E** 9 cm

9 The average speed of the particle over the first 3 seconds is

A 1 cm/s **B** −1 cm/s **C** $3\frac{1}{12}$ cm/s **D** 3 cm/s **E** −3 cm/s

Questions 10–13 refer to an object projected vertically upwards from the ground with a velocity of 15 m/s. Its acceleration due to gravity is -10 m/s^2.

10 The object's velocity at time $t = 3$ s is

A 15 m/s **B** −15 m/s **C** 0 m/s **D** 30 m/s **E** −30 m/s

11 The object's velocity is 0 m/s at time $t =$

A 1 s **B** 2 s **C** 0 s **D** 1.5 s **E** never

12 The maximum height reached by the object is

A 11 m **B** 15 m **C** 10 m **D** 11.25 m **E** 20 m

13 The object returns to the ground after

A 2 s **B** 4 s **C** 3 s **D** 1.5 s **E** never

Revision

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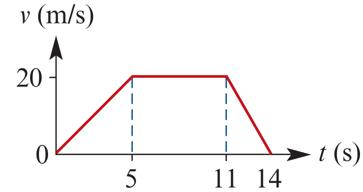
Questions 14–15 refer to this velocity–time graph for a moving vehicle.



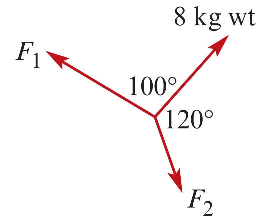
14 The distance travelled by the vehicle over the 14 seconds is

A 100 m **B** 150 m **C** 160 m **D** 180 m **E** 200 m

15 The acceleration of the vehicle over the first 5 seconds is

A 20 m/s^2 **B** 10 m/s^2 **C** 2.5 m/s^2 **D** 4 m/s^2 **E** −4 m/s^2

Questions 16–17 refer to this system of forces, which is in equilibrium.



16 The magnitude of force $\boldsymbol{F}_1$ is approximately

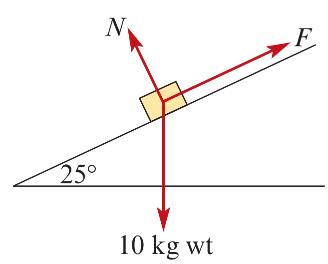
A 10.78 kg wt **B** 5.94 kg wt **C** 9.10 kg wt **D** 12.26 kg wt **E** 7.04 kg wt

17 The magnitude of force $\boldsymbol{F}_2$ is approximately

A 10.78 kg wt **B** 5.94 kg wt **C** 9.10 kg wt **D** 12.26 kg wt **E** 7.04 kg wt

Questions 18–19 refer to the following information:

A 10 kg block is resting on a smooth plane inclined at 25° to the horizontal and is prevented from slipping down the plane by a string, as shown in the diagram.



18 The magnitude of N is approximately

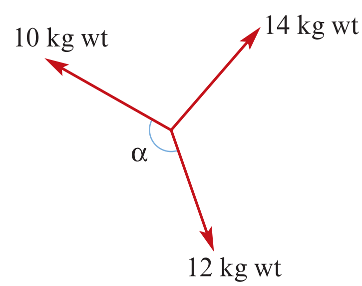
A 4.23 kg wt **B** 9.06 kg wt **C** 8.19 kg wt **D** 2.59 kg wt **E** 10 kg wt

19 The magnitude of the tension in the string is approximately

A 4.23 kg wt **B** 9.06 kg wt **C** 8.19 kg wt **D** 2.59 kg wt **E** 10 kg wt

20 If this system of forces is in equilibrium, then α is approximately

A 120° **B** 136° **C** 102°
D 110° **E** 100°



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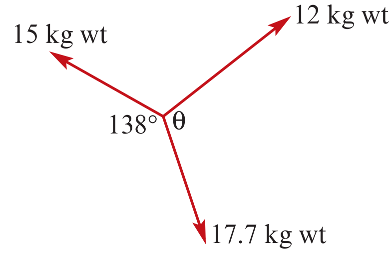
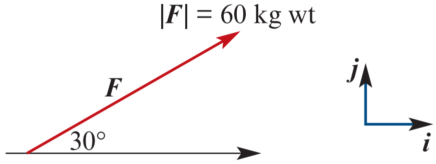
21 If this system of forces is in equilibrium, then θ is approximately

A 138° **B** 130° **C** 123°
D 100° **E** 90°



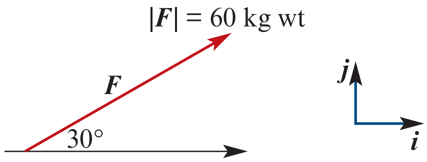
22 The component of force $\boldsymbol{F}$ in the $\boldsymbol{i}$-direction is

A 34.64 kg wt
B 30 kg wt
C 40 kg wt
D 20 kg wt
E 51.96 kg wt



23 The component of force $\boldsymbol{F}$ in the $\boldsymbol{j}$-direction is

A 34.64 kg wt
B 30 kg wt
C 40 kg wt
D 20 kg wt
E 51.96 kg wt



21C Extended-response questions

1 The velocity, v m/s, of a particle moving in a straight line is given by

$$v = 6 + pt + qt^3$$

where t is the time in seconds after the particle passes through a fixed point O. At time $t = 2$ s, the position of the particle relative to O is 16 m and its acceleration is 32 m/s^2. Calculate:

a the values of p and q
b the velocity of the particle at the instant when the acceleration is zero.

2 A stone is projected vertically upwards from the top of a cliff 20 m high. After a time of 3 seconds, it passes the edge of the cliff on its way down. Calculate:

a the speed of projection
b the speed when it hits the ground at the base of the cliff
c the times when it is 10 m above the top of the cliff
d the time when it is 5 m above the ground.

ISBN 978-1-316-63608-4

Revision

3 Trials are being conducted on a horizontal road to test the performance of an electrically powered car. The car has top speed V. In a test run, the car moves from rest with uniform acceleration a and is brought to rest with uniform deceleration r.

a If the car is to achieve top speed during a test run, show that the length of the test run must be at least

$$\frac{V^2(a+r)}{2ar}$$

b Find the least time taken for a test run of length:

i $\dfrac{2V^2(a+r)}{9ar}$ **ii** $\dfrac{2V^2(a+r)}{3ar}$

c Find, in terms of V, the average speed of the car for the test run described in **b ii**.

4 **a** A particle X is projected vertically upwards from the ground with a velocity of 80 m/s. Calculate the maximum height reached by X.

b A particle Y is held at a height of 300 m above the ground. At the moment when X has dropped 80 m from its maximum height, the particle Y is projected downwards with a speed of v m/s. If the two particles reach the ground at the same time, find the value of v.

5 **a** A particle X moves along a horizontal straight line so that its position, s m, relative to a fixed point O, at time t seconds after the motion has begun, is given by

$$s = 28 + 4t - 5t^2 - t^3$$

Find expressions in terms of t for:

i velocity

ii acceleration

b State:

i the initial velocity of X

ii the initial acceleration of X

c A second particle Y moves along the same horizontal straight line as X and starts from O at the same instant that X begins to move. The initial velocity of Y is 2 m/s, and its acceleration, a m/s^2, at time t seconds after motion has begun is given by $a = 2 - 6t$. Find the value of t at the instant when X and Y collide.

d Find the velocity of X and the velocity of Y at this instant and comment on the direction and motion of each of the particles.

ISBN 978-1-316-63608-4

Chapter 22

22 Sampling and sampling distributions

Objectives

- To understand the difference between a **population** and a **sample**.
- To understand **random samples** and how they may be obtained.
- To define **population parameters** and **sample statistics**.
- To introduce **random variables** and discrete probability distributions.
- To use simulation to generate random samples.
- To introduce the concept of sample statistics as random variables which can be described by **sampling distributions**.
- To investigate the sampling distributions of the **sample proportion** and the **sample mean**.
- To investigate the effect of sample size on a sampling distribution.
- To introduce the use of sample statistics as estimates of the associated population parameters.

This chapter is concerned with the collection of data and with the principles which must be adhered to in order to make meaningful generalisations based on these data.

We will also begin to develop an understanding of the key ideas underpinning the study of statistical inference (a topic which you will study in depth in Year 12). We will not delve too deeply into the theoretical basis of these ideas, but instead will use technology to undertake empirical investigations.

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22A Populations and samples

The set of all eligible members of a group which we intend to study is called a **population**. For example, if we are interested in the IQ scores of the Year 12 students at ABC Secondary College, then this group of students could be considered a population; we could collect and analyse all the IQ scores for these students. However, if we are interested in the IQ scores of all Year 12 students across Australia, then this becomes the population.

Often, dealing with an entire population is not practical:

- The population may be too large – for example, all Year 12 students in Australia.
- The population may be hard to access – for example, all blue whales in the Pacific Ocean.
- The data collection process may be destructive – for example, testing every battery to see how long it lasts would mean that there were no batteries left to sell.

Nevertheless, we often wish to make statements about a property of a population when data about the entire population is unavailable.

The solution is to select a subset of the population – called a **sample** – in the hope that what we find out about the sample is also true about the population it comes from. Dealing with a sample is generally quicker and cheaper than dealing with the whole population, and a well-chosen sample will give much useful information about this population. How to select the sample then becomes a very important issue.

► Random samples

Suppose we are interested in investigating the effect of sustained computer use on the eyesight of a group of university students. To do this we go into a lecture theatre containing the students and select all the students sitting in the front two rows as our sample. This sample may be quite inappropriate, as students who already have problems with their eyesight are more likely to be sitting at the front, and so the sample may not be typical of the population. To make valid conclusions about the population from the sample, we would like the sample to have a similar nature to the population.

While there are many sophisticated methods of selecting samples, the general principle of sample selection is that the method of choosing the sample should not favour or disfavour any subgroup of the population. Since it is not always obvious if the method of selection will favour a subgroup or not, we try to choose the sample so that every member of the population has an equal chance of being in the sample. In this way, all subgroups have a chance of being represented. The way we do this is to choose the sample at random.

The simplest way to obtain a valid sample is to choose a **random sample**, where every member of the population has an equal chance of being included in the sample.

To choose a sample from the group of university students, we could put the name of every student in a hat and then draw out, one at a time, the names of the students who will be in the sample.

 ISBN 978-1-316-63608-4 Cambridge University Press

Choosing the sample in an appropriate manner is critical in order to obtain useable results.

Example 1

A researcher wishes to evaluate how well the local library is catering to the needs of a town's residents. To do this she hands out a questionnaire to each person entering the library over the course of a week. Will this method result in a random sample?

Solution

Since the members of the sample are already using the library, they are possibly satisfied with the service available. Additional valuable information might well be obtained by finding out the opinion of those who do not use the library.

A better sample would be obtained by selecting at random from the town's entire population, so the sample contains both people who use the library and people who do not.

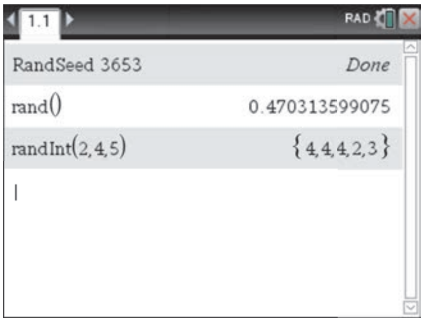
Thus, we have a very important consideration when sampling if we wish to generalise from the results of the sample.

In order to make valid conclusions about a population from a sample, we would like the sample chosen to be representative of the population as a whole. This means that all the different subgroups present in the population appear in the sample in similar proportions as they do in the population.

One very useful method for drawing random samples is to generate random numbers using a calculator or a computer.

Using the TI-Nspire

- In a **Calculator** page, go to (Menu) > **Probability** > **Random** > **Seed** and enter the last 4 digits of your phone number. This ensures that your random-number starting point differs from the calculator default.
- For a random number between 0 and 1, use (Menu) > **Probability** > **Random** > **Number**.
- For a random integer, use (Menu) > **Probability** > **Random** > **Integer**. To obtain five random integers between 2 and 4 inclusive, use the command randInt(2, 4, 5) as shown.



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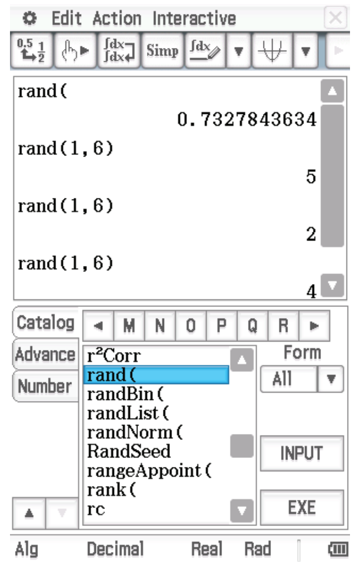
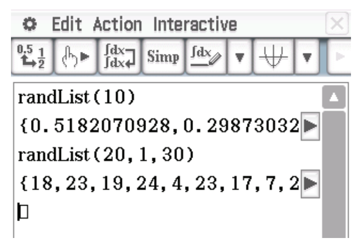
Using the Casio ClassPad

- In $\sqrt{\alpha}$ (Main), press the [Keyboard] button.
- Find and then select [Catalog] by first tapping ▼ at the bottom of the left sidebar.
- Scroll across the alphabet to the letter R.



- To generate a random number between 0 and 1:
  - In [Catalog], select **rand(**.
  - Tap [EXE].
- To generate three random integers between 1 and 6 inclusive:
  - In [Catalog], select **rand(**.
  - Type: 1, 6)
  - Tap [EXE] three times.
- To generate a list of 10 random numbers between 0 and 1:
  - In [Catalog], select **randList(**. Type: 10)
  - Tap [EXE] and then tap ► to view all the numbers.
- To generate a list of 20 random integers between 1 and 30 inclusive:
  - In [Catalog], select **randList(**. Type: 20, 1, 30)
  - Tap [EXE] and then tap ► to view all the integers.



Example 2

Consider the population of Year 12 students at ABC Secondary College given in the table, which shows the sex and IQ score for each of the 50 students (25 males and 25 females). Each student has been given an identity number (Id). Use a random number generator to select a random sample of size 5 from this population.

Id	Sex	IQ	Id	Sex	IQ	Id	Sex	IQ	Id	Sex	IQ	Id	Sex	IQ
1	F	105	11	F	103	21	M	95	31	F	68	41	M	79
2	M	111	12	F	97	22	M	113	32	M	113	42	M	106
3	M	104	13	F	122	23	M	108	33	M	87	43	F	118
4	M	93	14	M	101	24	M	106	34	F	93	44	M	98
5	F	92	15	F	84	25	F	95	35	F	100	45	M	113
6	F	99	16	F	108	26	F	86	36	M	114	46	F	120
7	F	88	17	M	95	27	F	87	37	F	119	47	F	93
8	M	107	18	M	88	28	F	134	38	M	100	48	M	81
9	M	97	19	M	95	29	M	118	39	F	100	49	F	114
10	M	88	20	F	86	30	M	58	40	F	114	50	F	107

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Solution

Generating five random integers from 1 to 50 gives on this occasion: 5, 1, 42, 16, 32. Thus the sample chosen consists of the students listed in the table on the right.

Id	Sex	IQ
1	F	105
5	F	92
16	F	108
32	M	113
42	M	106

Population parameters and sample statistics

There are 25 females and 25 males in the population of Year 12 students at ABC Secondary College, and therefore the proportion of females in the population is 0.5. This is called the **population proportion** and is generally denoted by p.

$$\text{Population proportion } p = \frac{\text{number in population with attribute}}{\text{population size}}$$

For the sample chosen in Example 2, the proportion of females in the sample is $\frac{3}{5} = 0.6$. This value is called the **sample proportion** and is denoted by $\hat{p}$. (We say 'p hat'.)

$$\text{Sample proportion } \hat{p} = \frac{\text{number in sample with attribute}}{\text{sample size}}$$

In this particular case, $\hat{p} = 0.6$, which is not the same as the population proportion $p = 0.5$. This does not mean there is a problem. In fact, each time a sample is selected the number of females in the sample will vary.

Now consider the IQ scores of the Year 12 students at ABC Secondary College. The mean IQ for the whole population is 100.0. This is called the **population mean** and is generally denoted by the Greek letter μ (pronounced mu).

$$\text{Population mean } \mu = \frac{\text{sum of the data values in the population}}{\text{population size}}$$

The mean IQ for the sample chosen in Example 2 is

$$\frac{105 + 92 + 108 + 113 + 106}{5} = 104.8$$

This value is called the **sample mean** and is denoted by $\bar{x}$. (We say 'x bar'.)

$$\text{Sample mean } \bar{x} = \frac{\text{sum of the data values in the sample}}{\text{sample size}}$$

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In this particular case, the value of the sample mean $\bar{x}$ (104.8) is not the same as the value of the population mean μ (100.0). As before, the IQ scores will vary from sample to sample.

- The population proportion p and the population mean μ are **population parameters**; their values are constant for a given population.
- The sample proportion $\hat{p}$ and the sample mean $\bar{x}$ are **sample statistics**; their values are not constant, but vary from sample to sample.

Section summary

- A **population** is the set of all eligible members of a group which we intend to study.
- A **sample** is a subset of the population which we select in order to make inferences about the population. Generalising from the sample to the population will not be useful unless the sample is representative of the population.
- The simplest way to obtain a valid sample is to choose a **random sample**, where every member of the population has an equal chance of being included in the sample.
- The **population proportion** p is the proportion of individuals in the entire population possessing a particular attribute, and is constant for a given population.
- The **sample proportion** $\hat{p}$ is the proportion of individuals in a particular sample possessing this attribute, and varies from sample to sample.
- The **population mean** μ is the mean of all values of a measure in the entire population, and is constant for a given population.
- The **sample mean** $\bar{x}$ is the mean of the values of this measure in a particular sample, and varies from sample to sample.

Exercise 22A

Example 1

1 In order to estimate the proportion of students in the state who use the internet for learning purposes, a researcher conducted an email poll. She found that 83% of those surveyed use the internet for learning purposes. Do you think that this is an appropriate way of selecting a random sample of students? Explain your answer.

2 A market researcher wishes to determine the age profile of the customers of a popular fast-food chain. She positions herself outside one of the restaurants between 4 p.m. and 8 p.m. one weekend, and asks customers to fill out a short questionnaire. Do you think this sample will be representative of the population? Explain your answer.

3 To measure support for the current Prime Minister, a television station conducts a phone-in poll, where viewers are asked to telephone one number if they support the Prime Minister and another number if they do not. Is this an appropriate method of choosing a random sample? Give reasons for your answer.

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Example 2

4 **a** Use a random number generator to select a random sample of size 5 from the population of Year 12 students at ABC Secondary College given in Example 2.

b Determine the proportion of females in your sample.

c Determine the mean IQ of the students in your sample.

5 Of a random sample of 100 homes, 48 were found to have one or more pet dogs.

a What proportion of these homes have one or more pet dogs?

b Is this the value of the population proportion p or the sample proportion $\hat{p}$?

6 In a certain school, 42% of the students travel to school by public transport. A group of 100 students were selected in a random sample, and 37 of them travel to school by public transport. In this example:

a What is the population?

b What is the value of the population proportion p?

c What is the value of the sample proportion $\hat{p}$?

7 Recent research has established that Australian adults spend on average four hours per day on sedentary leisure activities such as watching television. A group of 100 people were selected at random and found to spend an average of 3.5 hours per day on sedentary leisure activities. In this example:

a What is the population?

b What is the value of the population mean μ?

c What is the value of the sample mean $\bar{x}$?

22B The distribution of the sample proportion

We have seen that the sample proportion is not constant, but varies from sample to sample. In this section we will look more closely at this variation in the sample proportion and, in particular, at the values which it might be expected to take. In order to do this, we need to introduce some further concepts in probability.

► Random variables

Consider the sample space obtained when a coin is tossed three times:

$$\varepsilon = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Suppose we are particularly interested in the number of heads associated with each outcome. We let X represent the number of heads observed when a coin is tossed three times. Then each outcome in the sample space can be associated with a value of X, as shown in the following table.

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Outcome	Number of heads
HHH	$X = 3$
HHT	$X = 2$
HTH	$X = 2$
THH	$X = 2$
HTT	$X = 1$
THT	$X = 1$
TTH	$X = 1$
TTT	$X = 0$

From the table we can see that the possible values of X are 0, 1, 2 and 3. Since the actual value that X will take is the result of a random experiment, X is called a **random variable**.

A random variable can be discrete or continuous:

- A **discrete random variable** is one which may take on only a countable number of distinct values, such as 0, 1, 2, 3, 4. Discrete random variables are usually (but not necessarily) generated by counting. The number of children in a family, the number of brown eggs in a carton of a dozen eggs, and the number of times we roll a die before we observe a 'six' are all examples of discrete random variables.
- A **continuous random variable** is one that can take any value in an interval of the real number line, and is usually (but not always) generated by measuring. Height, weight, and the time taken to complete a puzzle are all examples of continuous random variables.

Discrete probability distributions

Because the values of a random variable are associated with outcomes in the sample space, we can determine the probability of each value of the random variable occurring.

Let's look again at the results obtained when a coin is tossed three times. Assuming that the coin is fair, we can add probabilities to the previous table.

Outcome	Number of heads	Probability	
HHH	$X = 3$	$\frac{1}{8}$	$\Pr(X = 3) = \dfrac{1}{8}$
HHT	$X = 2$	$\frac{1}{8}$	
HTH	$X = 2$	$\frac{1}{8}$	$\Pr(X = 2) = \dfrac{3}{8}$
THH	$X = 2$	$\frac{1}{8}$	
HTT	$X = 1$	$\frac{1}{8}$	
THT	$X = 1$	$\frac{1}{8}$	$\Pr(X = 1) = \dfrac{3}{8}$
TTH	$X = 1$	$\frac{1}{8}$	
TTT	$X = 0$	$\frac{1}{8}$	$\Pr(X = 0) = \dfrac{1}{8}$

ISBN 978-1-316-63608-4

Cambridge University Press

The list of all possible values of the random variable X, together with the probability associated with each value, is known as the probability distribution of X. More usually, we would summarise the probability distribution associated with the number of heads observed when a fair coin is tossed three times in a table as follows.

x	0	1	2	3
$\Pr(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Note that, since every possible value of the random variable is included, the probabilities must add to 1.

The **probability distribution** of a discrete random variable X is a function

$$p(x) = \Pr(X = x)$$

that assigns a probability to each value of X. It can be represented by a rule, a table or a graph, and must give a probability $p(x)$ for every value x that X can take.

For any discrete probability distribution, the following must be true:

- $0 \le p(x) \le 1$, for every value x that X can take.
- The sum of the values of $p(x)$ must be 1.

To determine the probability that X lies in a given interval, we add together the probabilities that X takes each value included in that interval, as shown in the following example.

Example 3

Consider the probability distribution:

x	1	2	3	4	5	6
$\Pr(X = x)$	0.2	0.3	0.1	0.2	0.15	0.05

Use the table to find:

a $\Pr(X = 3)$ **b** $\Pr(X < 3)$ **c** $\Pr(X \ge 4)$ **d** $\Pr(3 \le X \le 5)$ **e** $\Pr(X \ne 5)$

Solution

a $\Pr(X = 3) = 0.1$

b $\Pr(X < 3) = 0.2 + 0.3 = 0.5$

Explanation: If X is less than 3, then from the table we see that X can take the value 1 or 2.

c $\Pr(X \ge 4) = 0.2 + 0.15 + 0.05 = 0.4$

Explanation: If X is greater than or equal to 4, then X can take the value 4, 5 or 6.

d $\Pr(3 \le X \le 5) = 0.1 + 0.2 + 0.15 = 0.45$

Explanation: Here X can take the value 3, 4 or 5.

e $\Pr(X \ne 5) = 1 - \Pr(X = 5)$
$= 1 - 0.15 = 0.85$

Explanation: The sum of all the probabilities must be 1.

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▶ The sample proportion as a random variable

Since $\hat{p}$ varies according to the contents of the random samples, we can consider the sample proportions $\hat{p}$ as being the values of a random variable, which we will denote by $\hat{P}$.

As $\hat{P}$ is a random variable, it can be described by a probability distribution. The distribution of a statistic which is calculated from a sample (such as the sample proportion) has a special name – it is called a **sampling distribution**.

Sampling from a small population

Suppose we have a bag containing six blue balls and four red balls, and from the bag we take a sample of size 4. We are interested in the proportion of blue balls in the sample. We know that the population proportion is equal to $\frac{6}{10}$. That is,

$$p = 0.6$$

The probabilities associated with the possible values of the sample proportion $\hat{p}$ can be calculated using our knowledge of combinations. Recall that

$$\binom{n}{x} = \frac{n!}{x!\,(n-x)!}$$

is the number of different ways to choose x objects from n objects. This notation is read as 'n choose x'.

Example 4

A bag contains six blue balls and four red balls, and a random sample of size 4 is drawn. Find the probability that there is one blue ball in the sample. That is, find $\Pr(\hat{P} = 0.25)$.

Solution

In total, there are $\binom{10}{4} = 210$ ways to select 4 balls from 10 balls.

There are $\binom{6}{1} = 6$ ways to select 1 blue ball from 6 blue balls, and there are $\binom{4}{3} = 4$ ways to select 3 red balls from 4 red balls.

Thus the probability of obtaining 1 blue ball and 3 red balls is

$$\Pr(\hat{P} = 0.25) = \frac{\binom{6}{1} \times \binom{4}{3}}{\binom{10}{4}} = \frac{24}{210}$$

This is an example of the **hypergeometric distribution**, which applies in the following circumstances:

- We have a population of N objects, and each object can be considered as either a *defective* or a *non-defective*. There is a total of D defectives in the population.
- We will choose, without replacement, a sample of size n from the population, and the random variable of interest X is the number of defectives in the sample.

Note: There does not have to be anything wrong with an object classified as 'defective' – this distribution just happened to arise in a context where this terminology made sense.

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Hypergeometric distribution

The probability of obtaining x defectives in a sample of size n for a hypergeometric random variable X is

$$\Pr(X = x) = \frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}} \qquad \text{for } x = 0, 1, 2, \dots, \min(n, D)$$

where N is the size of the population and D is the number of defectives in the population.

Note: There cannot be more defectives in the sample than in the population, nor more than the sample size, and so the upper limit on the values of X is determined by D and n.

Continuing with our example, we let X denote the number of blue balls in a sample of size 4. We can now use the formula for the hypergeometric distribution to find the probabilities of values of $\hat{P}$. For example:

$$\Pr(\hat{P} = 0.5) = \Pr(X = 2) = \frac{\binom{6}{2}\binom{4}{2}}{\binom{10}{4}} = \frac{90}{210}$$

The following table gives the probability of obtaining each possible sample proportion $\hat{p}$ when selecting a random sample of size 4 from the bag.

Number of blue balls in the sample, x	0	1	2	3	4
Proportion of blue balls in the sample, $\hat{p}$	0	0.25	0.5	0.75	1
Probability	$\frac{1}{210}$	$\frac{24}{210}$	$\frac{90}{210}$	$\frac{80}{210}$	$\frac{15}{210}$

The possible values of $\hat{p}$ and their associated probabilities together form a probability distribution for the random variable $\hat{P}$, which can summarised as follows:

$\hat{p}$	0	0.25	0.5	0.75	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{210}$	$\frac{24}{210}$	$\frac{90}{210}$	$\frac{80}{210}$	$\frac{15}{210}$

Example 5

A bag contains six blue balls and four red balls, and a random sample of size 4 is drawn (without replacement). Use the sampling distribution in the previous table to determine the probability that the proportion of blue balls in the sample is more than 0.25.

Solution

$$\Pr(\hat{P} > 0.25) = \Pr(\hat{P} = 0.5) + \Pr(\hat{P} = 0.75) + \Pr(\hat{P} = 1)$$

$$= \frac{90}{210} + \frac{80}{210} + \frac{15}{210} = \frac{185}{210}$$

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Example 6

It is known that 80% of a group of 40 students have been immunised against measles. Use the hypergeometric distribution to find the sampling distribution of the sample proportion of immunised students if a random sample of 5 students is selected from this group.

Solution

Since 80% of the students have been immunised, there are 32 immunised students and 8 non-immunised students in the population.

Let X be the number of immunised students in the sample. Then

$$\Pr(\hat{P} = 0) = \Pr(X = 0) = \frac{\binom{32}{0}\binom{8}{5}}{\binom{40}{5}} = 0.000085$$

$$\Pr(\hat{P} = 0.2) = \Pr(X = 1) = \frac{\binom{32}{1}\binom{8}{4}}{\binom{40}{5}} = 0.0034$$

$$\Pr(\hat{P} = 0.4) = \Pr(X = 2) = \frac{\binom{32}{2}\binom{8}{3}}{\binom{40}{5}} = 0.0422$$

Continuing in this way, we obtain the following table for the sampling distribution (with probabilities given to four decimal places).

$\hat{p}$	0	0.2	0.4	0.6	0.8	1
$\Pr(\hat{P} = \hat{p})$	0.0001	0.0034	0.0422	0.2111	0.4372	0.3060

Sampling from a large population

Generally, when we select a sample, it is from a population which is too large or too difficult to enumerate or even count – populations such as all the people in Australia, or all the cows in Texas, or all the people who will ever have asthma. When the population is so large, we assume that the probability of observing the attribute we are interested in remains constant with each selection, irrespective of prior selections for the sample.

In such cases, we cannot use the hypergeometric distribution to determine the sampling distribution of $\hat{P}$. Instead, we use another well-known discrete probability distribution – the **binomial distribution** – which applies in the following circumstances:

- The experiment consists of a number, n, of identical trials.
- Each trial results in one of two outcomes, either a *success* or a *failure*.
- The probability of success on a single trial, p, is constant for all trials.
- The trials are independent (so that the outcome of any trial is not affected by the outcome of any previous trial).

Trials which meet these conditions are called **Bernoulli trials**, and the number of successes observed is then called a **binomial random variable**.

ISBN 978-1-316-63608-4

Binomial distribution

The probability of achieving x successes in a sequence of n trials for a binomial random variable X is

$$\Pr(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \qquad \text{for } x = 0, 1, 2, \dots, n$$

where p is the probability of success on each trial.

Proof There are $\binom{n}{x}$ different ways that x successes and $n - x$ failures can be ordered within the n trials, since we just have to choose x trials from n trials to be the successes.

In each trial, the probability of success is p and the probability of failure is $1 - p$. Thus the probability of obtaining x successes and $n - x$ failures in a given order is $p^x(1 - p)^{n-x}$ by the multiplication rule, since the trials are independent.

The $\binom{n}{x}$ different orderings are mutually exclusive, and so we obtain the formula for $\Pr(X = x)$ using the addition rule.

Suppose we know that 70% of all 17-year-olds in Australia attend school. That is, $p = 0.7$. We will assume that this probability remains constant for all selections for the sample.

Now consider selecting a random sample of size 4 from the population of all 17-year-olds in Australia. The probabilities associated with each value of the sample proportion $\hat{p}$ can be calculated using the binomial distribution:

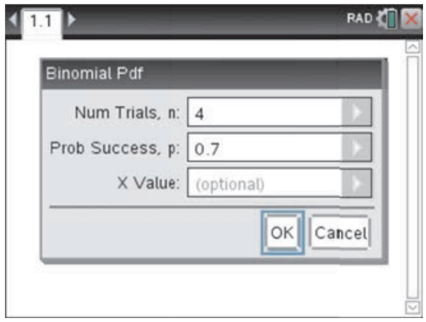
$$\Pr(X = x) = \binom{4}{x} 0.7^x 0.3^{4-x} \qquad \text{for } x = 0, 1, 2, 3, 4$$

These probabilities can be found easily using your calculator.

Using the TI-Nspire

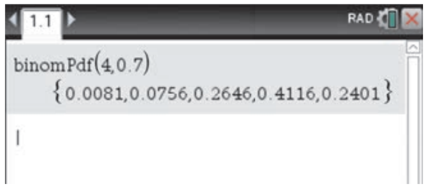
To find the values of $\Pr(X = x)$ for a binomial random variable X with $n = 4$ and $p = 0.7$:

- Use (menu) > **Probability** > **Distributions** > **Binomial Pdf** and complete as shown.
- Use (tab) or ▼ to move between cells.



- The result is as shown.

Note: You can also type in the command and the parameter values directly if preferred.



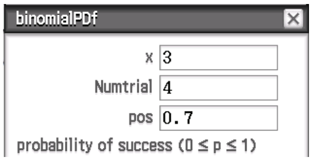
ISBN 978-1-316-63608-4

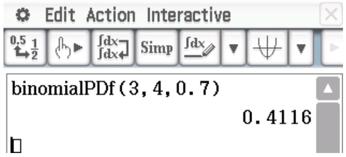
Cambridge University Press

Using the Casio ClassPad

To find $\Pr(X = 3)$ for a binomial random variable X with $n = 4$ and $p = 0.7$:

- In Main $\sqrt{\alpha}$, go to **Interactive** > **Distribution** > **Discrete** > **binomialPDf**.
- Enter the number of successes and the parameters as shown. Tap OK.





The following table gives the probability of obtaining each possible sample proportion $\hat{p}$ when selecting a random sample of four 17-year-olds.

Number at school in the sample, x	0	1	2	3	4
Proportion at school in the sample, $\hat{p}$	0	0.25	0.5	0.75	1
Probability	0.0081	0.0756	0.2646	0.4116	0.2401

Once again, we can summarise the sampling distribution of the sample proportion as follows:

$\hat{p}$	0	0.25	0.5	0.75	1
$\Pr(\hat{P} = \hat{p})$	0.0081	0.0756	0.2646	0.4116	0.2401

Note that the probabilities for the sample proportions, $\hat{p}$, correspond to the probabilities for the numbers of successes, x. That is:

$$\Pr(\hat{P} = 0) = \Pr(X = 0) \qquad \Pr(\hat{P} = \tfrac{1}{4}) = \Pr(X = 1) \qquad \Pr(\hat{P} = \tfrac{2}{4}) = \Pr(X = 2)$$
$$\Pr(\hat{P} = \tfrac{3}{4}) = \Pr(X = 3) \qquad \Pr(\hat{P} = 1) = \Pr(X = 4)$$

Example 7

Use the sampling distribution in the previous table to determine the probability that, in a random sample of four Australian 17-year-olds, the proportion attending school is less than 50%.

Solution

$$\Pr(\hat{P} < 0.5) = \Pr(\hat{P} = 0) + \Pr(\hat{P} = 0.25)$$
$$= 0.0081 + 0.0756 = 0.0837$$

Example 8

Suppose that 10% of the batteries produced by a particular production line are faulty. Use the binomial distribution to find the sampling distribution of the sample proportion of faulty batteries when a random sample of 3 batteries is selected from this production line.

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Solution

Using the binomial distribution with $n = 3$ and $p = 0.1$:

$$\Pr(\hat{P} = 0) = \Pr(X = 0) = \binom{3}{0}(0.1)^0(0.9)^3 = 0.729$$

$$\Pr(\hat{P} = \tfrac{1}{3}) = \Pr(X = 1) = \binom{3}{1}(0.1)^1(0.9)^2 = 0.243$$

$$\Pr(\hat{P} = \tfrac{2}{3}) = \Pr(X = 2) = \binom{3}{2}(0.1)^2(0.9)^1 = 0.027$$

$$\Pr(\hat{P} = 1) = \Pr(X = 3) = \binom{3}{3}(0.1)^3(0.9)^0 = 0.001$$

We can summarise the sampling distribution of the sample proportion as follows:

$\hat{p}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\Pr(\hat{P} = \hat{p})$	0.729	0.243	0.027	0.001

Comparing the hypergeometric and binomial distributions

Both the hypergeometric and the binomial distributions can be considered to arise from the process of sampling.

- **Hypergeometric distribution:** describes situations where the sampling is taking place without replacement, so that the probability of observing a particular outcome on any trial depends on the results of previous trials.
- **Binomial distribution:** applies to sampling with replacement, or sampling from a population that is so large that it does not matter if it is with or without replacement.

How large should the population be for sampling to be considered binomial?

Consider a jar containing 50 black balls and 50 white balls (that is, a population of size 100). Suppose that three balls are selected at random from the jar, and define X to be the number of black balls in the sample. Then X is a binomial random variable if the selected ball is replaced between consecutive selections, and a hypergeometric random variable if it is not.

Let's look at the probability distribution for each of these situations.

Without replacement: Hypergeometric ($N = 100$, $D = 50$, $n = 3$)

x	0	1	2	3
$\Pr(X = x)$	0.121	0.379	0.379	0.121

With replacement: Binomial ($n = 3$, $p = 0.5$)

x	0	1	2	3
$\Pr(X = x)$	0.125	0.375	0.375	0.125

We can see that the two probability distributions are very similar for this example.

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Whether it is appropriate to use the binomial distribution when sampling without replacement from a population this small (size 100) will also depend on the size of the sample. We have seen that when the sample is small (only 3), the approximation is quite good.

Example 9

Suppose that five students are selected at random from a population containing 50 male students and 50 female students, and that $\hat{P}$ is the random variable representing the proportion of females in the sample. Find the probability distribution of $\hat{P}$:

a exactly

b assuming that the proportion of female students remains unchanged throughout the sampling process.

Solution

a Use the hypergeometric distribution with $N = 100$, $D = 50$ and $n = 5$:

x	0	1	2	3	4	5
$\hat{p}$	0	0.2	0.4	0.6	0.8	1
$\Pr(\hat{P} = \hat{p})$	0.0281	0.1529	0.3189	0.3189	0.1529	0.0281

b Use the binomial distribution with $n = 5$ and $p = 0.5$:

x	0	1	2	3	4	5
$\hat{p}$	0	0.2	0.4	0.6	0.8	1
$\Pr(\hat{P} = \hat{p})$	0.03125	0.15625	0.3125	0.3125	0.15625	0.03125

Section summary

- For a discrete random variable X, the **probability distribution** of X is a function $p(x) = \Pr(X = x)$ that assigns a probability to each value of X.
- The **sample proportion** $\hat{P} = \dfrac{X}{n}$ is a random variable, where X is the number of favourable outcomes in a sample of size n.
- The distribution of $\hat{P}$ is known as the **sampling distribution** of the sample proportion.
- When the population is *small*, the sampling distribution of the sample proportion $\hat{P}$ can be determined using the **hypergeometric distribution**: The probability of obtaining x defectives in a sample of size n is given by

$$\Pr(X = x) = \frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}} \qquad \text{for } x = 0, 1, 2, \dots, \min(n, D)$$

where N is the size of the population and D is the number of defectives in the population.

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- When the population is *large*, the sampling distribution of the sample proportion $\hat{P}$ can be determined using the **binomial distribution**: The probability of achieving x successes in a sequence of n trials is given by

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad \text{for } x = 0, 1, 2, \dots, n$$

where p is the probability of success on each trial.

Exercise 22B

Example 3

1 A biased coin is such that the probability of obtaining a head on any toss is 0.6.

a Find the probability distribution of X, the number of heads observed when the coin is tossed twice.

b Find $\Pr(X \geq 1)$.

2 Samar has determined the following probability distribution for the number of cups of coffee, X, that he drinks in a day.

x	1	2	3	4	5	6
$p(x)$	0.05	0.15	0.35	0.25	0.15	0.05

Use the table to find:

a $\Pr(X = 3)$ **b** $\Pr(X < 3)$ **c** $\Pr(X \geq 4)$

d $\Pr(1 < X < 5)$ **e** $\Pr(X \neq 5)$ **f** $\Pr(1 < X < 5 \mid X > 1)$

Example 5

3 The following table gives the sampling distribution of the sample proportion when a sample of size 5 is selected from a group of 40 students, 80% of whom have been immunised against measles.

$\hat{p}$	0	0.2	0.4	0.6	0.8	1
$\Pr(\hat{P} = \hat{p})$	0.0001	0.0034	0.0422	0.2111	0.4372	0.3060

Use the table to find:

a $\Pr(\hat{P} = 0.2)$ **b** $\Pr(\hat{P} < 0.4)$ **c** $\Pr(\hat{P} \geq 0.8)$

d $\Pr(0.2 < \hat{P} < 0.8)$ **e** $\Pr(\hat{P} < 0.8 \mid \hat{P} > 0)$ **f** $\Pr(0.2 < \hat{P} < 0.8 \mid \hat{P} > 0.4)$

Example 6

4 A chocolate box contains eight soft-centred and eight hard-centred chocolates.

a What is p, the proportion of soft-centred chocolates in the box?

b Three chocolates are to be selected at random. What are the possible values of the sample proportion $\hat{p}$ of soft-centred chocolates in the sample?

c Use the hypergeometric distribution to construct a probability distribution table which summarises the sampling distribution of the sample proportion of soft-centred chocolates when samples of size 3 are selected from the box.

d Use the sampling distribution from **c** to determine the probability that the proportion of soft-centred chocolates in the sample is more than 0.25.

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5 A swimming club has 20 members: 12 males and 8 females.

a What is p, the proportion of males in the swimming club?

b A team of five swimmers is to be selected from the club at random. What are the possible values of the sample proportion $\hat{p}$ of males on the team?

c Use the hypergeometric distribution to construct a probability distribution table which summarises the sampling distribution of the sample proportion of males on the team.

d Use the sampling distribution from **c** to determine the probability that the proportion of males on the team is more than 0.7.

e Find $\Pr(0 < \hat{P} < 0.8)$ and hence find $\Pr(\hat{P} < 0.8 \mid \hat{P} > 0)$.

6 A random sample of four items is selected from a batch of 50 items which contains 15 defectives.

a What is p, the proportion of defectives in the batch?

b What are the possible values of the sample proportion $\hat{p}$ of defectives in the sample?

c Use the hypergeometric distribution to construct a probability distribution table which summarises the sampling distribution of the sample proportion of defectives in the sample.

d Use the sampling distribution from **c** to determine the probability that the proportion of defectives in the sample is more than 0.5.

e Find $\Pr(0 < \hat{P} < 0.5)$ and hence find $\Pr(\hat{P} < 0.5 \mid \hat{P} > 0)$.

Example 7

7 The following table gives the sampling distribution of the sample proportion of defectives in a random sample of 3 batteries from a particular production line.

$\hat{p}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\Pr(\hat{P} = \hat{p})$	0.729	0.243	0.027	0.001

a Use the table to determine the probability that the proportion of defectives in the sample is more than 0.6.

b Find $\Pr(0 < \hat{P} < 0.6)$ and hence find $\Pr(\hat{P} < 0.6 \mid \hat{P} > 0)$.

Example 8

8 Suppose that a fair coin is tossed 10 times and the number of heads observed.

a What is p, the probability that a head is observed when a fair coin is tossed?

b What are the possible values of the sample proportion $\hat{p}$ of heads in the sample?

c Use the binomial distribution to construct a probability distribution table which summarises the sampling distribution of the sample proportion of heads in the sample.

d Use the sampling distribution from **c** to determine the probability that the proportion of heads in the sample is more than 0.5.

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9 Suppose that the probability of a male child being born is 0.52. Of the next six children born at a maternity hospital:

a What are the possible values of the sample proportion $\hat{p}$ of male children born?

b Use the binomial distribution to construct a probability distribution table which summarises the sampling distribution of the sample proportion of male children born.

c Use the sampling distribution from **b** to determine the probability that the proportion of male children born is less than 0.4.

d Find $\Pr(\hat{P} < 0.3 \mid \hat{P} < 0.8)$.

10 Suppose that, in a certain country, the probability that a person is right-handed is 0.8. If eight people are selected at random from that country:

a What are the possible values of the sample proportion $\hat{p}$ of right-handed people in the sample?

b Construct a probability distribution table which summarises the sampling distribution of the sample proportion of right-handed people in the sample.

c Use the sampling distribution from **b** to determine the probability that the proportion of right-handed people in the sample is more than 0.6.

d Find $\Pr(\hat{P} > 0.6 \mid \hat{P} > 0.25)$.

Example 9 **11** Suppose that a population consists of 50 male and 50 female students, and that $\hat{P}$ is the random variable representing the proportion of females in a sample randomly chosen from this population.

a If the sample size is 4, find the probability distribution of $\hat{P}$:

i exactly (using the hypergeometric distribution)

ii assuming that the proportion of female students remains unchanged throughout the sampling process (using the binomial distribution).

b If the sample size is 10, find the probability distribution of $\hat{P}$:

i exactly (using the hypergeometric distribution)

ii assuming that the proportion of female students remains unchanged throughout the sampling process (using the binomial distribution).

c Compare your answers to parts **i** and **ii** for each of the two sample sizes. What is the effect of the increased sample size on the similarity of the answers obtained using the hypergeometric and binomial distributions?

ISBN 978-1-316-63608-4

22C Investigating the distribution of the sample proportion using simulation

In the previous section, we used our knowledge of probability to investigate the distribution of the sample proportion for small samples. In this section we are going to use simulation to investigate the distribution of the sample proportion for larger samples.

► The shape of the distribution of the sample proportion

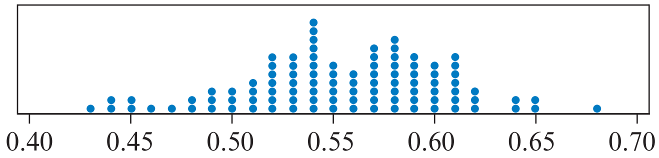
Suppose, for example, we know that 55% of people in Australia have blue eyes ($p = 0.55$) and we are interested in the values of the sample proportion $\hat{p}$ which might be observed when samples of size 100 are drawn at random from the population.

If we select one sample of 100 people and find that 58 people have blue eyes, then the value of the sample proportion is $\hat{p} = \frac{58}{100} = 0.58$.

If a second sample of 100 people is selected and this time 63 people have blue eyes, then the value of the sample proportion for this second sample is $\hat{p} = \frac{63}{100} = 0.63$.

Continuing in this way, the values of $\hat{p}$ obtained from 10 samples might look like those in the following dotplot. The proportion of people with blue eyes in the sample, $\hat{p}$, is varying from sample to sample: from as low as 0.53 to as high as 0.65 for these particular 10 samples.

What does the distribution of the sample proportions look like if we continue with this sampling process? The following dotplot summarises the values of $\hat{p}$ from 100 samples (each of size 100). We can see from the dotplot that the distribution is reasonably symmetric, centred at 0.55, and has values ranging from 0.43 to 0.68.



Example 10

A random sample of 100 people is drawn from a population in which 55% of people have blue eyes. Use the previous dotplot to estimate the probability that 65% or more of the people in the sample have blue eyes.

Solution

From the dotplot we can count 3 out of 100 samples where the sample proportion is 0.65 or more. Thus we can estimate

$$\Pr(\hat{P} \geq 0.65) \approx \frac{3}{100} = 0.03$$

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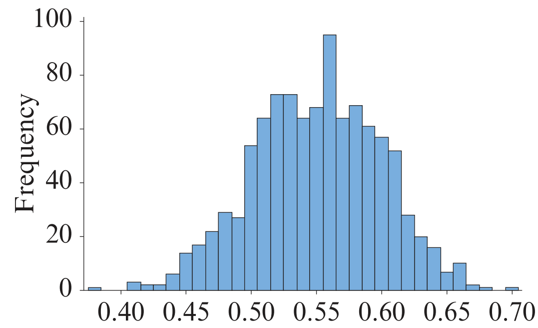
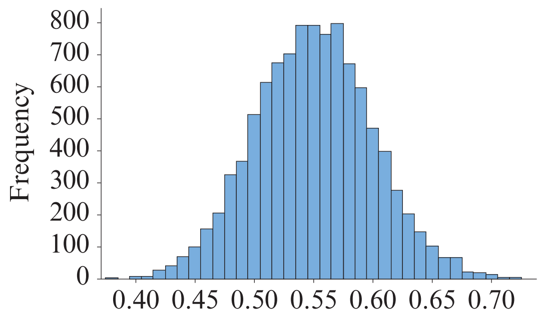
What does the distribution look like when 1000 samples (each of size 100) are chosen?



This time, because of the amount of data, the distribution is presented as a histogram. We can see that the distribution is becoming smoother and more clearly symmetric about 0.55, with a similar spread as before.

By the time we have taken 10 000 samples (each of size 100), the distribution is quite smooth and clearly symmetric about 0.55.



We can see that the majority of values are concentrated in the central region close to 0.55, with relatively few values of the sample proportion $\hat{p}$ that are less than 0.45 or more than 0.65.

The number of simulations to use for a good picture of the distribution is somewhat arbitrary. We have seen here that more is better. But generally, repetitions of 1000 or more are not necessary. Around 100 to 200 simulations are usually sufficient.

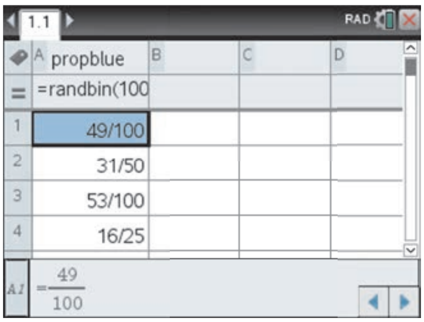
We saw in the previous section that the sample proportion $\hat{P}$ follows a binomial distribution, and so we can use a calculator to investigate repeated sampling.

Example 11

Assume that 55% of people in Australia have blue eyes. Use your calculator to illustrate a possible distribution of sample proportions $\hat{p}$ that may be obtained when 200 different samples (each of size 100) are selected from the population.

Using the TI-Nspire

- To generate the sample proportions:
  - Start from a **Lists & Spreadsheet** page.
  - Name the list 'propblue' in Column A.
  - In the formula cell of Column A, enter the formula using (Menu) > **Data** > **Random** > **Binomial** and complete as: $= \text{randbin}(100, 0.55, 200)/100$



Note: The syntax is: randbin(*sample size*, *population proportion*, *number of samples*)
To calculate as a proportion, divide by the sample size.

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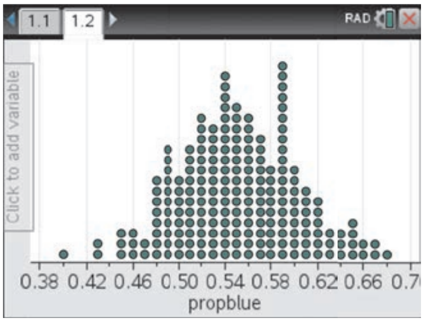
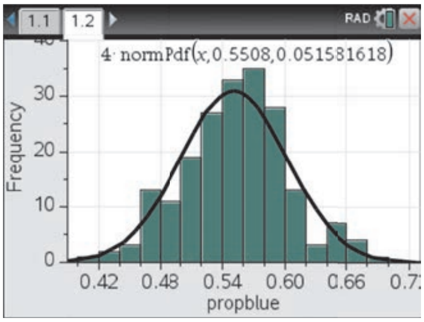
- To display the distribution of sample proportions:
  - Insert a **Data & Statistics** page (ctrl I or ctrl doc ▾).
  - Click on 'Click to add variable' on the x-axis and select 'propblue'. A dotplot is displayed.

Note: You can recalculate the random sample proportions by using ctrl R while in the **Lists & Spreadsheet** page.



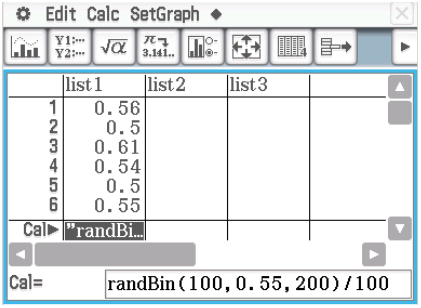
- To fit a normal curve to the distribution:
  - Menu > **Plot Type** > **Histogram**
  - Menu > **Analyze** > **Show Normal PDF**

Note: The normal curve is superimposed on the plot, showing the mean and standard deviation of the sample proportion. (Normal distributions are discussed in Section 22D.)



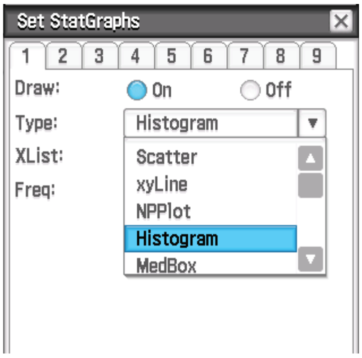
Using the Casio ClassPad

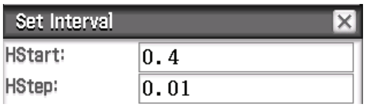
- To generate the sample proportions:
  - Open the **Statistics** application.
  - Tap the 'Calculation' cell at the bottom of list1.
  - Type: randBin(100, 0.55, 200)/100
  - Tap Set.



Note: The syntax is: randBin(*sample size*, *population proportion*, *number of samples*)
To calculate as a proportion, divide by the sample size.

- To display the distribution of sample proportions:
  - Tap on the **Set StatGraphs** icon, select the type 'Histogram' and tap Set.
  - Tap on the graph icon in the toolbar.
  - In the **Set Interval** window, enter the values shown below and tap OK.





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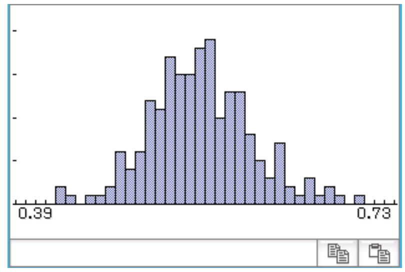
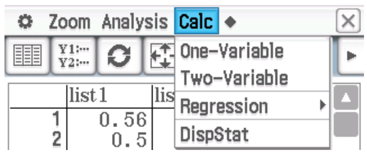
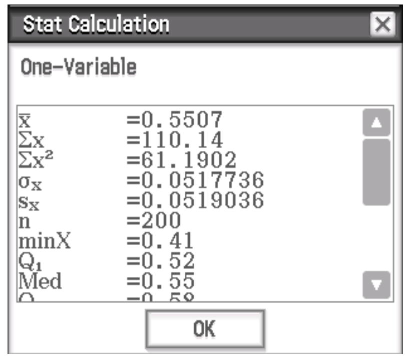
- The histogram of sample proportions is shown.



- To obtain statistics from the distribution, select **Calc** > **One–Variable**. Tap OK.





Note: The mean of the sample proportions, $\bar{x}$, estimates the population proportion.

Example 12

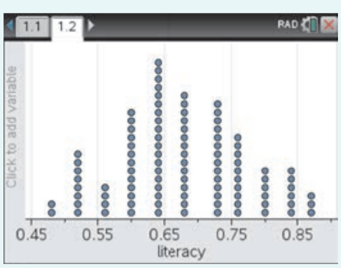
In a certain country, the literacy rate for adults is known to be 68%.

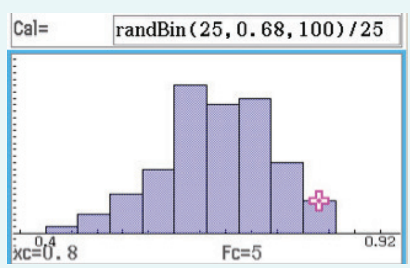
a Use your calculator to simulate 100 samples, each of size 25, drawn at random from this population. Construct a dotplot of the proportion of people in each sample who are literate.

b Use your dotplot to estimate the probability that the proportion of people in a sample of size 25 who are literate is 80% or more.

Solution

a





b

- In the dotplot on the left, there are 15 out of 100 samples where the sample proportion is 0.8 or more. This gives $\Pr(\hat{P} \geq 0.8) \approx \dfrac{15}{100} = 0.15$.
- In the histogram on the right, there are 5 out of 100 samples where the sample proportion is 0.8 or more. This gives $\Pr(\hat{P} \geq 0.8) \approx \dfrac{5}{100} = 0.05$.

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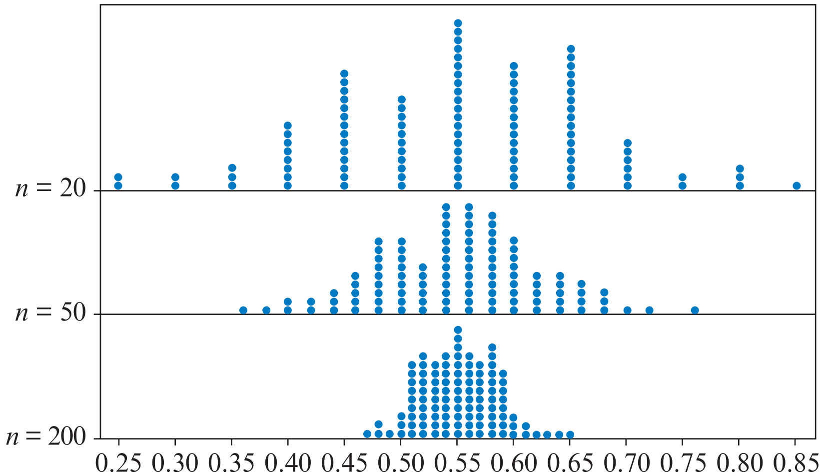
▶ The effect of sample size on the distribution of the sample proportion

Using simulation, we have seen that the sample proportion $\hat{P}$ has a symmetric bell-shaped distribution. We can also use simulation to explore how the distribution of the sample proportion is affected by the size of the sample chosen.

Again, we suppose that 55% of people in Australia have blue eyes ($p = 0.55$). The following dotplots show the sample proportions $\hat{p}$ obtained when 200 samples of size 20, then size 50 and then size 200 were chosen from this population.

Note: It is important not to confuse the *size* of each sample with the *number* of samples used for the simulation, which is quite arbitrary. We used 200 simulations because this is sufficient to illustrate the features of the sampling distribution.



Each symbol represents up to 2 observations.

We can see from the dotplots that all three sampling distributions are symmetric and centred at 0.55, the value of the population proportion p. Furthermore, as the sample size increases, the values of the sample proportion $\hat{p}$ are more tightly clustered around that value.

For example, we can see that for the particular 200 simulations used to construct each dotplot, a sample proportion $\hat{p}$ of 0.75 or more was observed:

- a few times when the sample size was 20
- only once when the sample size was 50
- never when the sample size was 200.

When the sample size is larger, we are less likely to get a value of the sample proportion that is very different from the population proportion.

These observations are confirmed in following table, which gives the mean and standard deviation for each of the three simulated sampling distributions shown in the dotplots.

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Sample size	20	50	200
Population proportion p	0.55	0.55	0.55
Mean of the values of $\hat{p}$	0.5475	0.5489	0.5492
Standard deviation of the values of $\hat{p}$	0.1164	0.0704	0.0331

The sampling distribution of $\hat{P}$ is symmetric and centred at the value of the population proportion p. The variation in the sampling distribution decreases as the size of the sample increases.

When the population proportion p is not known, the sample proportion $\hat{p}$ can be used as an estimate of this parameter. The larger the sample used to calculate the sample proportion, the more confident we can be that this is a good estimate of the population proportion.

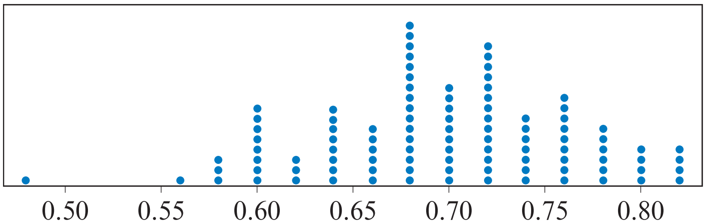
Section summary

- We can use technology (calculators or computers) to repeat a random sampling process many times. This is known as simulation.
- Through simulation, we can see that the sampling distribution of $\hat{P}$ is symmetric and centred at the value of the population proportion p.
- The variation in the sampling distribution decreases as the size of the sample increases.
- When the population proportion p is not known, we can use the sample proportion $\hat{p}$ as an estimate of this parameter. The larger the sample size, the more confident we can be that $\hat{p}$ is a good estimate of the population proportion p.

Exercise 22C

Example 10

1 Researchers believe that 70% of people prefer brewed coffee to instant coffee. The following dotplot shows the values of the sample proportions obtained when 100 random samples of size 50 were generated from this population.



Use the dotplot to estimate:

a $\Pr(\hat{P} \geq 0.8)$

b $\Pr(\hat{P} \leq 0.5)$

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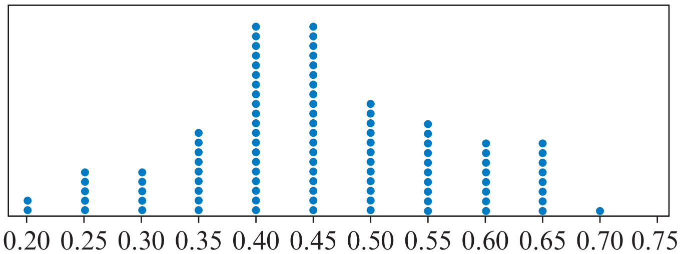
2 It is known that 45% of fish in a certain lake are underweight. The following dotplot shows the sample proportions of underweight fish obtained when 100 samples of size 20 were drawn from the lake.



Use the dotplot to estimate:

a $\Pr(\hat{P} \geq 0.7)$ **b** $\Pr(\hat{P} \leq 0.25)$

Example 11, 12

3 A politician believes that 55% of her electorate will vote for her in the next election.

a Use your calculator to simulate 100 values of the sample proportion of people who will vote for this politician in a random sample of size 100.

b Summarise the values obtained in part **a** in a dotplot.

c Use your dotplot to estimate:

i $\Pr(\hat{P} \geq 0.64)$ **ii** $\Pr(\hat{P} \leq 0.44)$

4 A shooter claims that he hits the target with 80% of his shots.

a Assuming that his claim is correct, use your calculator to simulate 50 values of the sample proportion of targets hit when the shooter takes 50 shots.

b Summarise the values obtained in part **a** in a dotplot.

c Use your dotplot to estimate:

i $\Pr(\hat{P} \geq 0.9)$ **ii** $\Pr(\hat{P} \leq 0.7)$

5 One in three workers spend at least three-quarters of their work time sitting.

a Use your calculator to simulate 100 values of the sample proportion of workers who spend at least three-quarters of their work time sitting, when a sample of 40 workers is drawn from this population.

b Summarise the values obtained in part **a** in a dotplot.

c Use your dotplot to estimate:

i $\Pr(\hat{P} \geq 0.45)$ **ii** $\Pr(\hat{P} \leq 0.25)$

6 **a** Repeat the simulation carried out in part **a** of Question 5, but this time using a sample of 80 workers.

b Summarise the values obtained in part **a** in a dotplot.

c Use your dotplot to estimate:

i $\Pr(\hat{P} \geq 0.45)$ **ii** $\Pr(\hat{P} \leq 0.25)$

d Compare your answers to part **c** with those from Question 5.

ISBN 978-1-316-63608-4

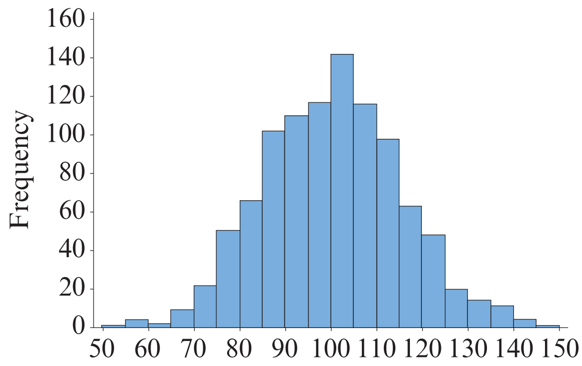
22D Investigating the distribution of the sample mean using simulation

In Section 22A, we saw that while the population mean μ is constant for a given population, the sample mean $\bar{x}$ is not constant, but varies from sample to sample. In this section we use simulation to investigate this variation in the sample mean.

► The sample mean as a random variable

Consider the random variable IQ, which has a mean of 100 and a standard deviation of 15 in the population. In order to use technology to investigate this random variable, we use a distribution that you may not have met before, called the **normal distribution**. You will study this distribution in Year 12, but for now it is enough to know that many commonly occurring random variables – such as height, weight and IQ – follow this distribution.

This histogram shows the IQ scores of 1000 people randomly drawn from the population.

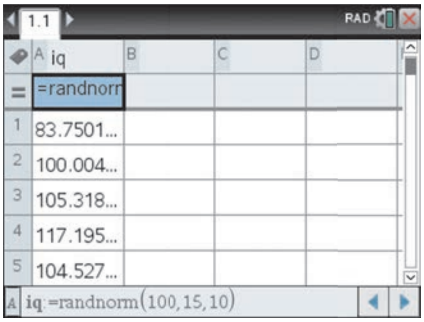


You can see that the distribution is symmetric and bell-shaped, with its centre of symmetry at the population mean. The normal distribution is fully defined by its mean and standard deviation. If we know these values, then we can use technology to generate random samples.

Using the TI-Nspire

To generate a random sample of size 10 from a normal population with mean 100 and standard deviation 15:

- Start from a **Lists & Spreadsheet** page.
- Name the list 'iq' in Column A.
- In the formula cell of Column A, enter the formula using (Menu) > **Data** > **Random** > **Normal** and complete as: = randnorm(100, 15, 10)



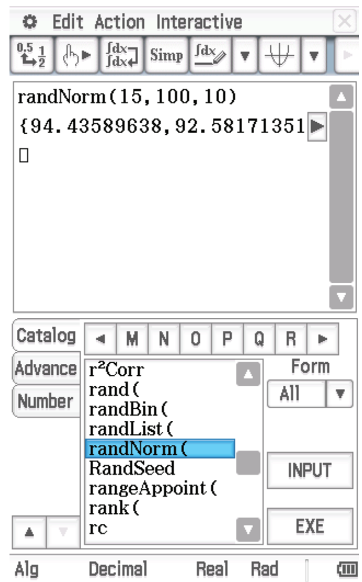
Note: The syntax is: randnorm(*mean*, *standard deviation*, *sample size*)

ISBN 978-1-316-63608-4

Using the Casio ClassPad

To generate a random sample of size 10 from a normal population with mean 100 and standard deviation 15:

- In Main $\sqrt{\alpha}$, press the Keyboard button.
- Find and then select Catalog by first tapping ▼ at the bottom of the left sidebar.
- Scroll across the alphabet to the letter R.
- Select **randNorm(** and type: 15, 100, 10)
- Tap ► to view all the values.



Notes:

- The syntax is: randNorm(*standard deviation*, *mean*, *sample size*)
- Alternatively, the random sample can be generated in the **Statistics** application.

One random sample of 10 scores, obtained by simulation, is

105, 109, 104, 86, 118, 100, 81, 94, 70, 88

Recall that the sample mean is denoted by $\bar{x}$ and that

$$\bar{x} = \frac{\sum x}{n}$$

where $\sum$ means ‘sum’ and n is the size of the sample.

Here the sample mean is

$$\bar{x} = \frac{105 + 109 + 104 + 86 + 118 + 100 + 81 + 94 + 70 + 88}{10} = 95.5$$

A second sample, also obtained by simulation, is

114, 124, 128, 133, 95, 107, 117, 91, 115, 104

with sample mean

$$\bar{x} = \frac{114 + 124 + 128 + 133 + 95 + 107 + 117 + 91 + 115 + 104}{10} = 112.8$$

Since $\bar{x}$ varies according to the contents of the random samples, we can consider the sample means $\bar{x}$ as being the values of a random variable, which we will denote by $\bar{X}$.

Since $\bar{x}$ is a statistic which is calculated from a sample, the probability distribution of the random variable $\bar{X}$ is again called a **sampling distribution**.

► The sampling distribution of the sample mean

Generating random samples and then calculating the mean from the sample is quite a tedious process if we wish to investigate the sampling distribution of $\bar{X}$ empirically. Luckily, we can also use technology to simulate values of the sample mean.

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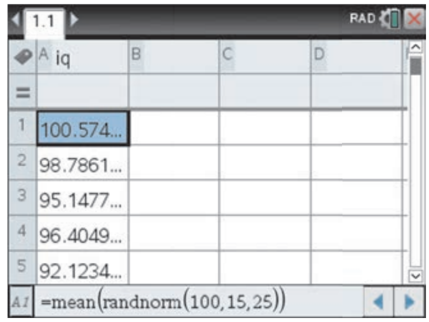
Using the TI-Nspire

To generate the sample means for 10 random samples of size 25 from a normal population with mean 100 and standard deviation 15:



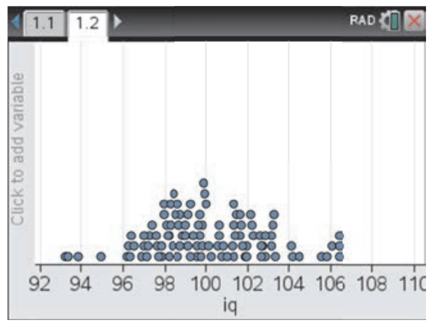
- Start from a **Lists & Spreadsheet** page.
- Name the list 'iq' in Column A.
- In cell A1, enter the formula using (Menu) > **Data** > **Random** > **Normal** and complete as: $= \text{mean}(\text{randnorm}(100, 15, 25))$
- Fill down to obtain the sample means for 10 random samples.

For a large number of simulations, an alternative method is easier.

To generate the sample means for 500 random samples of size 25, enter the following formula in the formula cell of Column A:

$$= \text{seq}(\text{mean}(\text{randnorm}(100, 15, 25)), k, 1, 500)$$

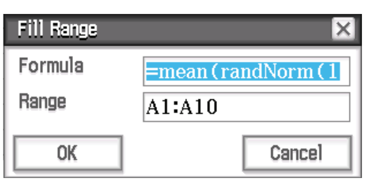
The dotplot on the right was created this way.

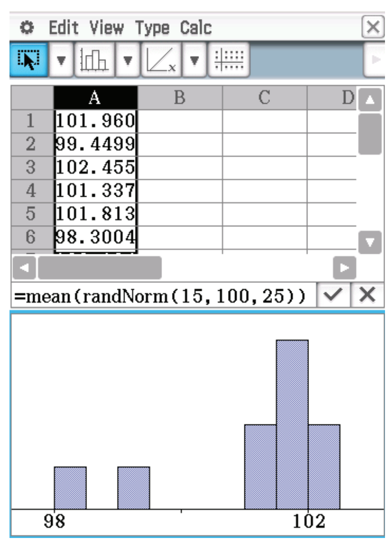


Using the Casio ClassPad

To generate the sample means for 10 random samples of size 25 from a normal population with mean 100 and standard deviation 15:

- Open the **Spreadsheet** application.
- Tap in cell A1.
- Type: $= \text{mean}(\text{randNorm}(15, 100, 25))$
- Go to **Edit** > **Fill** > **Fill Range**.
- Type A1:A10 for the range and tap OK.





To sketch a histogram of these sample means:

- Go to **Edit** > **Select** > **Select Range**.
- Type A1:A10 for the range and tap OK.
- Select **Graph** and tap **Histogram**.

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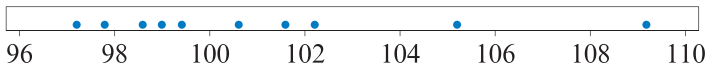
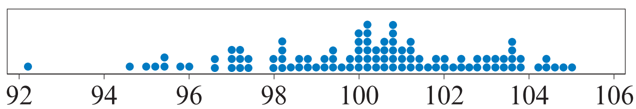
Cambridge University Press

Suppose that 10 random samples (each of size 25) are selected from a population with mean 100 and standard deviation 15. The values of $\bar{x}$ obtained might look like those in the following dotplot. The values look to be centred around 100, ranging from 97.3 to 109.2.



To better investigate the distribution requires more sample means. The following dotplot summarises the values of $\bar{x}$ observed for 100 samples (each of size 25).



Example 13

The IQ scores for a population have mean 100 and standard deviation 15. Use the previous dotplot to estimate the probability that, for a random sample of 25 people drawn from this population, the sample mean $\bar{x}$ is 104 or more.

Solution

From the dotplot we can count 6 out of 100 samples where the sample mean is 104 or more. Thus we can estimate

$$\Pr(\bar{X} \geq 104) \approx \frac{6}{100} = 0.06$$

Example 14

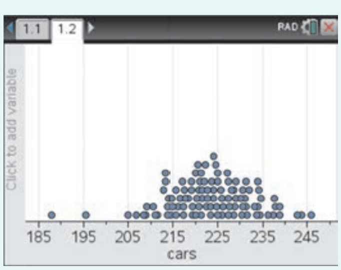
Suppose it is known that parking times in a large city car park are normally distributed, with mean $\mu = 223$ minutes and standard deviation $\sigma = 48$ minutes.

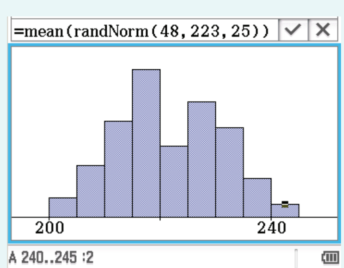
a Use your calculator to generate the sample means for 100 samples, each of size 25, drawn at random from this population. Summarise these values in a dotplot.

b Use your dotplot to estimate the probability that, in a random sample of 25 cars, the mean parking time is greater than or equal to 240 minutes.

Solution

a





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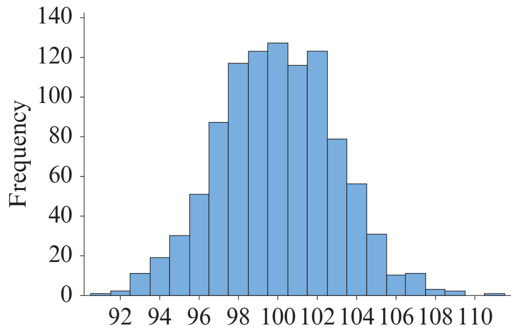
b ■ In the dotplot on the left, there are 3 out of 100 samples where the sample mean is 240 or more. This gives $\Pr(\bar{X} \geq 240) \approx \dfrac{3}{100} = 0.03$.

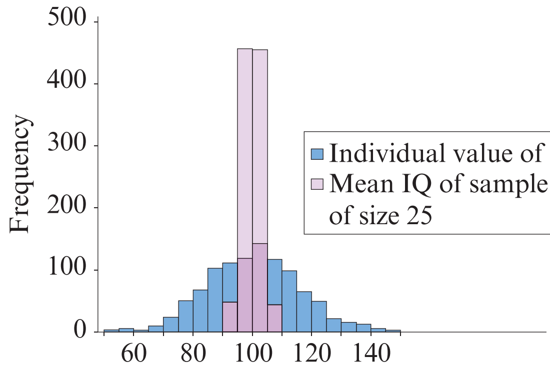
■ In the histogram on the right, there are 2 out of 100 samples where the sample mean is 240 or more. This gives $\Pr(\bar{X} \geq 240) \approx \dfrac{2}{100} = 0.02$.

This histogram shows the sampling distribution of the sample mean when 1000 samples (each of size 25) were selected from a population with mean 100 and standard deviation 15.



We can see from this plot that the distribution of sample means is symmetric and bell-shaped, suggesting that the sampling distribution of the sample mean may also be described by the normal distribution. This is an area of study which will be further explored in Year 12.

How does the sampling distribution of the values of $\bar{X}$ (where each value is the mean of a sample of size 25) compare to the distribution of the individual values of X? The following plot shows the two previous histograms together with the same scale.



We can see that while both distributions are symmetric and bell-shaped, and centred at the value of the population mean, they exhibit very different variation. The individual IQ scores are clustered between 70 and 130, but the sample means are almost all between 90 and 110.

On reflection, this seems quite reasonable. While it would not be very unusual to find an individual person in the population with an IQ over 130, for example, it would seem highly unlikely that we would select a random sample of 25 people from the population and find that their mean IQ was over 130.

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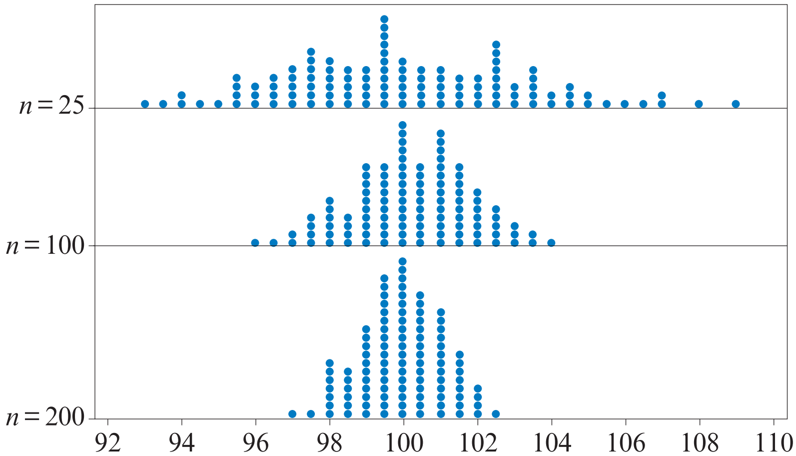
► The effect of sample size on the distribution of the sample mean

Using simulation, we have seen that the sample mean $\bar{X}$ has a symmetric bell-shaped distribution. We can also use simulation to explore how the distribution of the sample mean is affected by the size of the sample chosen.

The following dotplots show the sample means $\bar{x}$ obtained when 200 samples of size 25, then size 100 and then size 200 were chosen from a population. Again, it is important not to confuse the *size* of each sample with the *number* of samples used for the simulation, which is quite arbitrary.



Each symbol represents up to 2 observations

We can see from the dotplots that all three sampling distributions appear to be centred at 100, the value of the population mean μ. Furthermore, as the sample size increases, the values of the sample mean $\bar{x}$ are more tightly clustered around that value.

These observations are confirmed in following table, which gives the mean and standard deviation for each of the three simulated sampling distributions shown in the dotplots.

Sample size	25	100	200
Population mean μ	100	100	100
Mean of the values of $\bar{x}$	99.24	100.24	100.03
Standard deviation of the values of $\bar{x}$	3.05	1.59	1.06

The sampling distribution of $\bar{X}$ is symmetric and centred at the value of the population mean μ. The variation in the sampling distribution decreases as the size of the sample increases.

When the population mean μ is not known, the sample mean $\bar{x}$ can be used as an estimate of this parameter. The larger the sample used to calculate the sample mean, the more confident we can be that this is a good estimate of the population mean.

ISBN 978-1-316-63608-4

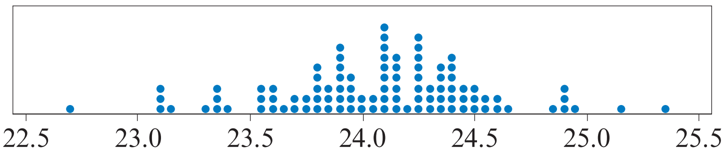
Section summary

- The **sample mean** $\bar{X}$ is a random variable and so can be described by a probability distribution, called the sampling distribution of the sample mean.
- Through simulation, we can see that the sampling distribution of $\bar{X}$ is symmetric and centred at the value of the population mean μ.
- The variation in the sampling distribution decreases as the size of the sample increases.
- When the population mean μ is not known, the sample mean $\bar{x}$ can be used as an estimate of this parameter. The larger the sample size, the more confident we can be that $\bar{x}$ is a good estimate of the population mean μ.

Exercise 22D

Example 13

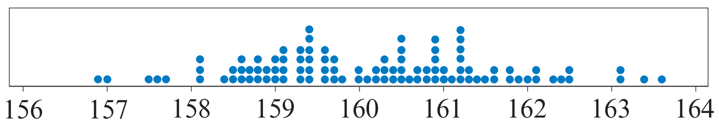
1 In a certain city, the average size of a kindergarten class is $\mu = 24$ children, with a standard deviation of $\sigma = 2$. The following dotplot shows the sample means $\bar{x}$ for 100 random samples of 20 classes.



Use the dotplot to estimate:

a $\Pr(\bar{X} \geq 25)$ **b** $\Pr(\bar{X} \leq 23)$

2 The mean height of women in a certain country is $\mu = 160$ cm, with a standard deviation of $\sigma = 8$ cm. The following dotplot shows the sample means $\bar{x}$ for 100 random samples of 30 women.



Use the dotplot to estimate:

a $\Pr(\bar{X} \geq 163)$ **b** $\Pr(\bar{X} \leq 158)$

Example 14

3 The lengths of a species of fish are normally distributed with mean length $\mu = 40$ cm and standard deviation $\sigma = 4$ cm.

a Use your calculator to simulate 100 values of the sample mean calculated from a sample of size 50 drawn from this population of fish.

b Summarise the values obtained in part **a** in a dotplot.

c Use your dotplot to estimate:

i $\Pr(\bar{X} \geq 41)$ **ii** $\Pr(\bar{X} \leq 39)$

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4 The marks in a statistics examination in a certain university are normally distributed with a mean of $\mu = 48$ marks and a standard deviation of $\sigma = 15$ marks.

a Use your calculator to simulate 100 values of the sample mean calculated from a sample of size 20 drawn from the students at this university.

b Summarise the values obtained in part **a** in a dotplot.

c Use your dotplot to estimate:

i $\Pr(\bar{X} \geq 55)$

ii $\Pr(\bar{X} \leq 40)$

5 At the Fizzy Drinks Company, the volume of soft drink in a 1 litre bottle is normally distributed with mean $\mu = 1$ litre and standard deviation $\sigma = 0.01$ litres.

a Use your calculator to simulate 100 values of the sample mean calculated from a sample of 25 bottles from this company.

b Summarise the values obtained in part **a** in a dotplot.

c Use your dotplot to estimate:

i $\Pr(\bar{X} \geq 1.003)$

ii $\Pr(\bar{X} \leq 0.995)$

6 **a** Repeat the simulation carried out in part **a** of Question 5, but this time using samples of 50 bottles.

b Summarise the values obtained in part **a** in a dotplot.

c Use your dotplot to estimate:

i $\Pr(\bar{X} \geq 1.003)$

ii $\Pr(\bar{X} \leq 0.995)$

d Compare your answers in part **c** to those from Question 5.

ISBN 978-1-316-63608-4

Review

Chapter summary

AS

- A **population** is the set of all eligible members of a group which we intend to study.
- A **sample** is a subset of the population which we select in order to make inferences about the population. Generalising from the sample to the population will not be useful unless the sample is representative of the population.
- The simplest way to obtain a valid sample is to choose a **random sample**, where every member of the population has an equal chance of being included in the sample.
- The **population proportion** p is the proportion of individuals in the entire population possessing a particular attribute; the **sample proportion** $\hat{p}$ is the proportion of individuals in a particular sample possessing this attribute.
- The **population mean** μ is the mean of all values of a measure in the entire population; the **sample mean** $\bar{x}$ is the mean of these values in a particular sample.
- Both the population proportion p and the population mean μ are **population parameters**; their values are constant for a given population.
- Both the sample proportion $\hat{p}$ and the sample mean $\bar{x}$ are **sample statistics**; their values are not constant, but vary from sample to sample.
- For a discrete random variable X, the **probability distribution** of X is a function $p(x) = \Pr(X = x)$ that assigns a probability to each value of X.
- Both the sample proportion $\hat{P}$ and the sample mean $\bar{X}$ can be viewed as random variables, and their distributions are called **sampling distributions**.
- When the population is *small*, the sampling distribution of the sample proportion $\hat{P}$ can be determined using the **hypergeometric distribution**: The probability of obtaining x defectives in a sample of size n is given by

$$\Pr(X = x) = \frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}} \qquad \text{for } x = 0, 1, 2, \dots, \min(n, D)$$

  where N is the size of the population and D is the number of defectives in the population.
- When the population is *large*, the sampling distribution of the sample proportion $\hat{P}$ can be determined using the **binomial distribution**: The probability of achieving x successes in a sequence of n trials is given by

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad \text{for } x = 0, 1, 2, \dots, n$$

  where p is the probability of success on each trial.
- We can use technology (calculators or computers) to repeat a random sampling process many times. This is known as simulation.
- Through simulation, we see that the sampling distribution of the sample proportion $\hat{P}$ is symmetric and centred at the value of the population proportion p, and that the sampling distribution of the sample mean $\bar{X}$ is symmetric and centred at the value of the population mean μ.

ISBN 978-1-316-63608-4

Review

- The variation in the sampling distribution decreases as the size of the sample increases.
- When a population parameter (such as p or μ) is not known, we can use the associated sample statistic (such as $\hat{p}$ or $\bar{x}$) as an estimate of this parameter. The larger the sample size, the more confident we can be that the sample statistic gives a good estimate of the population parameter.

Short-answer questions

1 A company has 2000 employees, 700 of whom are female. A random sample of 100 employees was selected, and 40 of them were female. In this example:

a What is the population?

b What is the value of the population proportion p?

c What is the value of the sample proportion $\hat{p}$?

2 To study the effectiveness of yoga for reducing stress, a researcher measured the stress levels of 50 people who had just enrolled in a 10-week introductory yoga course, and then measured their stress at the end the course. Do you think that this sample will be representative of the general population? Explain your answer.

3 Social researchers suggested that people who spend more time gardening also tend to spend more time on housework. They randomly selected 120 people who lived in houses with a garden and over the course of one month, measured the amount of time they spent gardening and the amount of time they spent on housework. Do you think this sample will be representative of the general population? Explain your answer.

4 Medical researchers were interested in the amount of water consumed by people with Type II diabetes, which they suspect may be more than the 1 litre per day average observed in the general population. They randomly selected a sample of 50 people with Type II diabetes and found their average daily water consumption was 1.5 litres per day.

a What is the population of interest here?

b Why did the researchers select a sample rather than studying the entire population?

c What is the value of the population mean μ?

d What is the value of the sample mean $\bar{x}$?

5 A company has 5400 employees, 1080 of whom have a tertiary qualification. A group of 200 employees were selected to complete a survey, and 44 of them were tertiary qualified. In this example:

a What is the population?

b What is the value of the population proportion p?

c What is the value of the sample proportion $\hat{p}$?

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6 A tennis team has five members: two males and three females.

a What is p, the proportion of females in the tennis team?

b Three players are to be selected at random to play in a tournament. What are the possible values of the sample proportion $\hat{p}$ of females in the selected group?

c Use the hypergeometric distribution to construct a probability distribution table which summarises the sampling distribution of the sample proportion of females when samples of size 3 are selected from the tennis team.

d Use the sampling distribution from **c** to determine the probability that the proportion of females in the selected group is more than 0.5.

e Find $\Pr(0 < \hat{P} < 0.5)$ and hence find $\Pr(\hat{P} < 0.5 \mid \hat{P} > 0)$.

7 Suppose that the probability of a male child being born is 0.5. Of the next four children born at a maternity hospital:

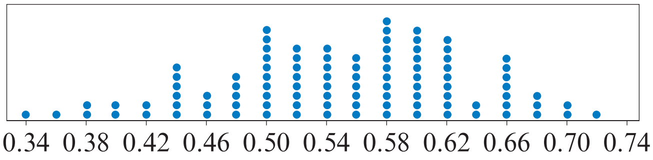
a What are the possible values of the sample proportion $\hat{p}$ of male children born?

b Use the binomial distribution to construct a probability distribution table which summarises the sampling distribution of the sample proportion of male children born.

c Use the sampling distribution from **b** to determine the probability that the proportion of male children born is less than 0.5.

d Find $\Pr(\hat{P} < 0.5 \mid \hat{P} < 0.8)$.

8 It is known that 55% of people in a certain electorate voted for Bill Bloggs in the last election. The following dotplot shows the distribution of sample proportions obtained when 100 samples of size 50 were drawn from a population with population proportion 0.55.



a Use the dotplot to estimate:

i $\Pr(\hat{P} \geq 0.70)$

ii $\Pr(\hat{P} \leq 0.38)$

b One of Bill Bloggs's team members selects a random sample of 50 people from the electorate and finds that only 21 intend to vote for Bill Bloggs at the next election.

i What is the value of the sample proportion $\hat{p}$?

ii Use the dotplot to determine how likely it is that, if the proportion of people supporting Bill Bloggs is still 55%, we would find a value of $\hat{p}$ as low as or lower than this value.

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Review

Multiple-choice questions

1 In order to estimate the ratio of males to females at a school, a teacher determines the number of males and the number of females in a particular class. The ratio that he then calculates is called a

A sample **B** sample statistic **C** population parameter
D population **E** sample parameter

2 In a complete census of the population of a particular community, it is found that 59% of families have two or more children. Here '59%' represents the value of a

A sample **B** sample statistic **C** population parameter
D population **E** sample parameter

3 Which of the following statements is true?

A We use sample statistics to estimate population parameters.
B We use sample parameters to estimate population statistics.
C We use population parameters to estimate sample statistics.
D We use population statistics to estimate sample parameters.
E None of the above.

4 A sampling distribution can best be described as a distribution which

A gives the most likely value of the sample statistic
B describes how a statistic's value will change from sample to sample
C describes how samples do not give reliable estimates
D gives the distribution of the values observed in a particular sample
E None of the above.

5 Which of the following statements is generally correct?

A μ is an estimate of $\bar{x}$, and $\hat{p}$ is an estimate of p
B $\bar{x}$ is an estimate of μ, and $\hat{p}$ is an estimate of p
C μ is an estimate of $\hat{p}$, and $\bar{x}$ is an estimate of p
D $\bar{x}$ is an estimate of μ, and p is an estimate of $\hat{p}$
E μ is an estimate of $\bar{x}$, and p is an estimate of $\hat{p}$

6

$\hat{p}$	0	0.2	0.4	0.6	0.8	1
$\Pr(\hat{P} = \hat{p})$	0.116	0.312	0.336	0.181	0.048	0.005

Using this table, the value of $\Pr(\hat{P} \geq 0.7 \mid \hat{P} > 0.2)$ is

A 0.053 **B** 0.143 **C** 0.060 **D** 0.572 **E** 0.093

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7 Noah has invited 10 friends to a dinner party, and four of them are vegetarians. He has only six china plates, and so four guests will have to use plastic plates. If the plastic plates are distributed among the guests at random, then the probability that the proportion of vegetarians using plastic plates is more than 0.5 is closest to

A 0.119 **B** 0.114 **C** 0.154 **D** 0.179 **E** 0.881

8 Ethan puts 20 fish in his new fishpond: 12 gold and 8 black. If he catches a random sample of five fish the next day to check their health, what is the probability that 80% or more of the sample are gold fish?

A 0.0063 **B** 0.2592 **C** 0.0778 **D** 0.3065 **E** 0.3370

9 It is known that 20% of students in a very large school study Chinese. The probability that, in a random sample of 20 students, the proportion of students who study Chinese is less than 10% is closest to

A 0.0160 **B** 0.0115 **C** 0.0054 **D** 0.1369 **E** 0.0692

10 If a fair coin is tossed 10 times, then the probability that the proportion of heads obtained is less than 20% or more than 80% is

A close to zero **B** about 2% **C** about 5%
D about 10% **E** none of these

11 When is it appropriate to use the binomial distribution to calculate the probabilities related to the sampling distribution of the sample proportion?

A when the population is small and the sample size is small
B whenever the sample size is small
C when the sample size is small compared to the population size
D when the sample size is large and the population size is small
E never appropriate

12 A market research company has decided to increase the size of the random sample of Australians that it will select for a survey, from about 1000 people to about 1500 people. What is the effect of this increase in sample size?

A The increase will ensure that the sampling distribution is symmetric.
B The effect cannot be predicted without knowing the population size.
C There will be no effect as the population size is the same.
D The variability of the sample estimate will increase, as more people are involved.
E The variability of the sample estimate will decrease.

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Review

Extended-response questions

1 **a** Use simulation to generate sampling distributions for the sample proportion obtained when samples of size 50 are drawn from populations with values of the population proportion $p = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$. (Generate 100 samples for each value of p.)

Construct dotplots of the sampling distributions and, by counting five values in from each end, use them to find the lower limit a and upper limit b such that $\Pr(a \le \hat{P} \le b) \approx 0.90$.

p	a	b
0.1		
0.2		
0.3		
0.4		
0.5		
0.6		

b Miller wishes to estimate the proportion of people in a certain city who are left-handed. He selects a random sample of 50 people, and finds that 17 are left-handed.

i What is the value of the sample proportion here?

ii Refer to the table you constructed in part **a**. If a possible value of the population proportion is one where the observed value of the sample proportion lies within the interval $[a, b]$ such that $\Pr(a \le \hat{P} \le b) \approx 0.90$, which values of p are consistent with Miller's sample?

2 For a certain type of mobile phone, the length of time between charges of the battery is normally distributed with a mean of 50 hours and a standard deviation of 5 hours.

a **i** Use your calculator to simulate 100 values of the sample mean calculated from a sample of 20 phones.

ii Summarise the values obtained in part **i** in a dotplot.

iii Determine the mean and standard deviation of this sampling distribution.

b **i** Use your calculator to simulate 100 values of the sample mean calculated from a sample of 50 phones.

ii Summarise the values obtained in part **i** in a dotplot.

iii Determine the mean and standard deviation of this sampling distribution.

c **i** Use your calculator to simulate 100 values of the sample mean calculated from a sample of 100 phones.

ii Summarise the values obtained in part **i** in a dotplot.

iii Determine the mean and standard deviation of this sampling distribution.

d It can be shown theoretically that the standard deviation of the sampling distribution is inversely proportional to $\sqrt{n}$, where n is the sample size. Use your answers to parts **a**–**c** to demonstrate this relationship empirically.

ISBN 978-1-316-63608-4

Chapter 23

23 Logic and algebra

Objectives

- To understand the Boolean operations $\vee$, $\wedge$, $'$ and the axioms of Boolean algebra.
- To understand and apply the concepts of **proposition** and **truth value**.
- To construct **truth tables** for compound statements and to recognise **tautologies**.
- To represent circuits using **logic gates**.
- To use **Karnaugh maps** to simplify Boolean expressions.

The words ‘or’, ‘and’, ‘not’, ‘true’ and ‘false’ are central to this chapter. You may have already met these words in your studies of sets, probability and proofs. In this chapter, these ideas are brought together in a formal way as **Boolean algebra**.

In Chapter 2, we looked at sets and operations on sets, including union $\cup$, intersection $\cap$ and complementation $'$. This chapter begins by studying these operations on the set of all subsets of a given set. We have seen in Chapter 5 that, if a finite set has n elements, then there are 2^n subsets. Such structures are examples of Boolean algebras.

We will see that the ideas of Boolean algebra flow through into a formal study of logic. Some of the concepts introduced in Chapter 6 will reappear in this chapter, including negation, De Morgan’s laws, implication, converse and contrapositive.

Early work in logic was carried out by George Boole (1815–1864) in his book *An Investigation of the Laws of Thought*, published in 1854. His aims in the book were to

> . . . investigate the fundamental laws of those operations of the mind by which reasoning is performed; to give expression to them in the symbolical language of a Calculus, and upon this foundation to establish the science of Logic and construct its method.

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The ideas of Boolean algebra were first applied to electronic circuits in the 1930s, by Claude Shannon (1916–2001). Today, Boolean algebra forms the foundation of computer science, and is central to electronics and programming. You can use Boolean logic when you submit a query to a search engine, such as Google or Yahoo.

23A Sets and circuits

▶ Basic set notation

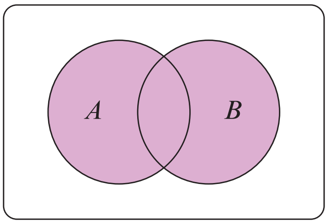
A set is any collection of objects where order is not important. We first recall some of the basic notation from Section 2A.

- The set of all the elements being considered in a given context is called the **universal set** and is denoted by ξ.
- The set with no elements is called the **empty set** and is denoted by $\varnothing$.
- We say that a set B is a **subset** of a set A if each element of B is also in A. In this case, we write $B \subseteq A$. Note that $\varnothing \subseteq A$ and $A \subseteq A$.

The following three operations on sets play an important role in this chapter.

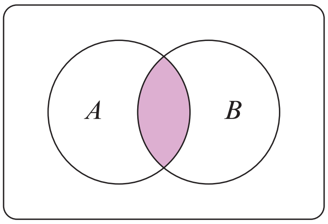
The **union** of sets A and B is denoted by $A \cup B$ and consists of all elements that belong to A or B:

$$A \cup B = \{ x : x \in A \text{ or } x \in B \}$$



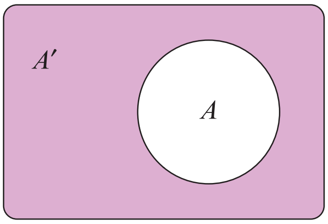
The **intersection** of sets A and B is denoted by $A \cap B$ and consists of all elements that belong to both A and B:

$$A \cap B = \{ x : x \in A \text{ and } x \in B \}$$



The **complement** of a set A is denoted by A' and consists of all elements of the universal set ξ that are not in A:

$$A' = \{ x \in \xi : x \notin A \}$$



In this chapter, you will see how these ideas can be extended and applied in analogous situations. We begin by looking at the 'structure' of the set of all subsets of a set.

 ISBN 978-1-316-63608-4

▶ The set of all subsets of a set

We consider the operations $\cup$, $\cap$ and $'$ on the set of all subsets of a set. We have proved in Section 5G that a set with n elements has 2^n subsets.

For example, consider the universal set $\xi = \{a, b, c, d\}$. There are 16 subsets:

- the empty set: $\varnothing$
- the subsets of size 1: $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$
- the subsets of size 2: $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$
- the subsets of size 3: $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$
- the universal set: ξ.

We can make the following observations for these sets and, more generally, for any such collection of subsets. We will illustrate how to prove these laws in Example 1. You will see similar laws reoccurring throughout this chapter.

Algebra of sets

For $A, B, C \subseteq \xi$, the following are true:

Primary
- $A \cup A = A$
- $A \cap A = A$
- $A \cup \varnothing = A$
- $A \cap \xi = A$
- $A \cup \xi = \xi$
- $A \cap \varnothing = \varnothing$

Associativity
- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \cap B) \cap C = A \cap (B \cap C)$

Commutativity
- $A \cup B = B \cup A$
- $A \cap B = B \cap A$

Distributivity
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Absorption
- $A \cup (A \cap B) = A$
- $A \cap (A \cup B) = A$

Complements
- $A \cup A' = \xi$
- $A \cap A' = \varnothing$
- $\varnothing' = \xi$
- $\xi' = \varnothing$
- $(A \cup B)' = A' \cap B'$
- $(A \cap B)' = A' \cup B'$
- $(A')' = A$

You may notice that some of these laws are similar to laws of arithmetic involving $+$, $\times$, 0 and 1. For example, the laws $A \cup \varnothing = A$ and $A \cap \xi = A$ are analogous to the arithmetic statements $a + 0 = a$ and $a \times 1 = a$.

You may also notice that all but one of these laws occurs as a member of a pair. The two laws in each pair are called **dual statements**.

Dual statements

For a given statement about sets, the dual statement is obtained by interchanging:

$\cup$ with $\cap$, $\quad\varnothing$ with ξ, $\quad\subseteq$ with $\supseteq$

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In the following example, we show a method for proving a result about sets. We also show how to illustrate the result with a Venn diagram. But note that drawing a Venn diagram is not a proof, just as in geometry drawing a diagram is not a proof.

Equality for sets

When proving that two sets are equal, we can use the following equivalence:

$$X \subseteq Y \text{ and } Y \subseteq X \Leftrightarrow X = Y$$

Example 1

a Illustrate $(A \cup B)' = A' \cap B'$ with Venn diagrams.

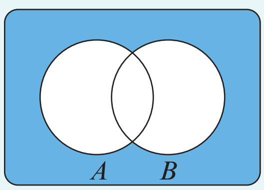
b Prove that $(A \cup B)' = A' \cap B'$.

Solution

a Note: Required regions are shaded.

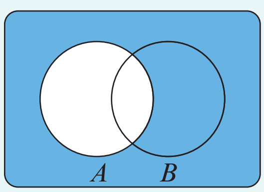
We first draw the diagram for $(A \cup B)'$.

$(A \cup B)'$

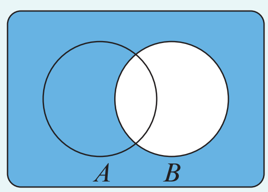


To help draw the diagram for $A' \cap B'$, we draw diagrams for A' and B'.

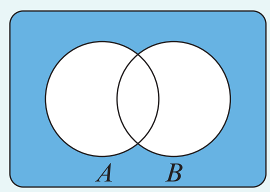
A'



B'



$A' \cap B'$



The diagrams for $(A \cup B)'$ and $A' \cap B'$ are the same.

b We must show that $(A \cup B)' \subseteq A' \cap B'$ and $A' \cap B' \subseteq (A \cup B)'$.

Let $x \in \xi$.

i $x \in (A \cup B)' \Rightarrow x \notin A \cup B$
$\Rightarrow x \notin A$ and $x \notin B$
$\Rightarrow x \in A'$ and $x \in B'$
$\Rightarrow x \in A' \cap B'$

Hence $(A \cup B)' \subseteq A' \cap B'$.

ii $x \in A' \cap B' \Rightarrow x \in A'$ and $x \in B'$
$\Rightarrow x \notin A$ and $x \notin B$
$\Rightarrow x \notin A \cup B$
$\Rightarrow x \in (A \cup B)'$

Hence $A' \cap B' \subseteq (A \cup B)'$.

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Example 2

Write the dual of $(A \cap B') \cap B = \varnothing$.

Solution

$(A \cup B') \cup B = \xi$

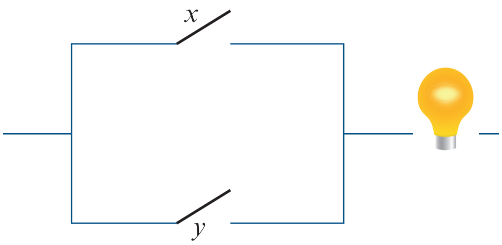
Light switches

Claude Shannon was first to realise that the ideas of logic could be applied to electronic circuits. 'On' and 'off' can be used in a way analogous to 'true' and 'false'. In this chapter, we follow the opposite path. We first introduce switching circuits, and then see in the next section how they can be interpreted using Boolean algebra.

Switches in parallel

The following diagram shows two switches x and y in **parallel**. If at least one of the switches is closed, then current flows and the light is on. In this case, we say that the system is closed.

The four possible situations for two switches in parallel are summarised in the table.



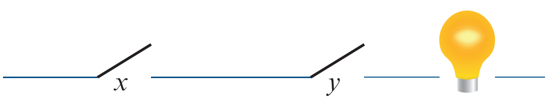
Switches x and y in parallel

x	y	State of system
open	open	open
open	closed	closed
closed	open	closed
closed	closed	closed

Switches in series

The following diagram shows two switches x and y in **series**. If both switches are closed, then current flows and the light is on. In this case, we say that the system is closed.

The four possible situations for two switches in series are summarised in the table.



Switches x and y in series

x	y	State of system
open	open	open
open	closed	open
closed	open	open
closed	closed	closed

Complementary switches

The **complement switch** x' is always in the opposite state to x.

x	x'
open	closed
closed	open

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New notation

We now introduce a new notation that will be used throughout this chapter in several different contexts.

- Use 0 for open.
- Use 1 for closed.
- Use the notation $x \vee y$, read as 'x or y', for two switches x and y connected in parallel.
- Use the notation $x \wedge y$, read as 'x and y', for two switches x and y connected in series.
- Use x' for the complement of x.

We repeat the three tables above with the new notation.

Or (parallel)

x	y	$x \vee y$
0	0	0
0	1	1
1	0	1
1	1	1

And (series)

x	y	$x \wedge y$
0	0	0
0	1	0
1	0	0
1	1	1

Complement

x	x'
0	1
1	0

We now have three operations $\vee$, $\wedge$ and $'$ acting on the set $\{0, 1\}$.

Note: The operations $\vee$ and $\wedge$ are analogous to union $\cup$ and intersection $\cap$. For this reason, the symbols $\vee$ and $\wedge$ are also read as 'join' and 'meet'.

This new notation allows us to represent more complicated switching circuits. You will see the notation used again in the next section in a more general context.

Example 3

Consider the expression

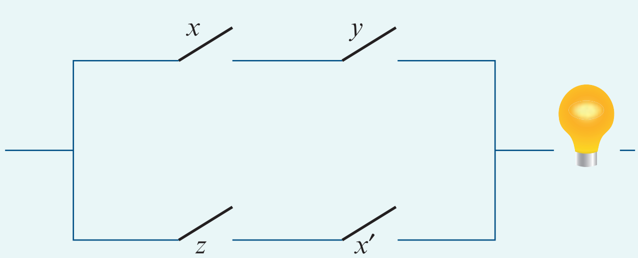
$$(x \wedge y) \vee (z \wedge x')$$

a Draw the switching circuit that is represented by this expression.

b Give the table with entries 0s and 1s describing the operation of this circuit.

Solution

a



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b For three variables x, y and z, there are $2^3 = 8$ possible combinations of 0s and 1s. This gives the first three columns of the table. To find the value of $(x \wedge y) \vee (z \wedge x')$ in each case, we start by finding the values of the simpler expressions $x \wedge y$ and $z \wedge x'$.

x	y	z	x'	$x \wedge y$	$z \wedge x'$	$(x \wedge y) \vee (z \wedge x')$
0	0	0	1	0	0	0
0	0	1	1	0	1	1
0	1	0	1	0	0	0
0	1	1	1	0	1	1
1	0	0	0	0	0	0
1	0	1	0	0	0	0
1	1	0	0	1	0	1
1	1	1	0	1	0	1

Exercise 23A

Example 1b

1 Prove each of the following results:

a $(A \cap B)' = A' \cup B'$

b $(A \cup B) \cap (A \cup B') = A$

c $A = (A \cap B) \cup (A \cap B')$

d $(P \cup Q)' \cup (P \cap Q) = (P' \cup Q) \cap (Q' \cup P)$

2 Set difference is defined as

$$A \setminus B = A \cap B' = \{ x : x \in A \text{ and } x \notin B \}$$

Prove the following results:

a $P \setminus (Q \setminus R) = (P \setminus Q) \cup (P \cap R)$

b $P \cap (Q \setminus R) = (P \cap Q) \setminus (P \cap R)$

Example 2

3 Write the dual of each of the following:

a $(A \cup \xi) \cap (A \cap \varnothing) = \varnothing$

b If $A \cap B = \varnothing$, then $A' \cup B = A'$.

c $A \cap B \subseteq A \cup B$

Example 3

4 For each of the following expressions:

i draw the switching circuit that is represented by the expression

ii give the table with entries 0s and 1s describing the operation of this circuit.

a $(x \wedge y) \vee z$

b $(x \vee y) \wedge (x \wedge y)$

c $x \wedge (y \vee z)$

d $(x \vee y') \wedge (y \vee z)$

5 Consider the expression $(x \wedge y) \vee ((z \vee x) \wedge y')$.

a Draw the switching circuit that is represented by this expression.

b Give the table with entries 0s and 1s describing the operation of this circuit.

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23B Boolean algebra

We have now seen two different examples of a Boolean algebra:

- the set of all subsets of a set with the operations $\cup$, $\cap$ and $'$
- the set $\{0, 1\}$ with the operations $\vee$, $\wedge$ and $'$.

In general, a **Boolean algebra** is a set B with operations $\vee$, $\wedge$, $'$ and distinguished elements 0, 1 such that the following axioms are satisfied, for all $x, y, z \in B$:

Axiom 1	$x \vee y = y \vee x$	($\vee$ is commutative)
	$x \wedge y = y \wedge x$	($\wedge$ is commutative)
Axiom 2	$(x \vee y) \vee z = x \vee (y \vee z)$	($\vee$ is associative)
	$(x \wedge y) \wedge z = x \wedge (y \wedge z)$	($\wedge$ is associative)
Axiom 3	$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$	($\vee$ distributes over $\wedge$)
	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	($\wedge$ distributes over $\vee$)
Axiom 4	$x \vee 0 = x$	(0 is the identity for $\vee$)
	$x \wedge 1 = x$	(1 is the identity for $\wedge$)
Axiom 5	$x \vee x' = 1$	($'$ is complementation)
	$x \wedge x' = 0$	

Given these rules, we can prove new results, as shown in the following example.

Example 4

Prove that, for each Boolean algebra B and all $x, y, z \in B$:

a $x \vee 1 = 1$

b $x \wedge 0 = 0$

c If $x \vee z = 1$ and $x \wedge z = 0$, then $z = x'$.

d $(x \vee y)' = x' \wedge y'$

Solution

a
$$\begin{aligned} x \vee 1 &= (x \vee 1) \wedge 1 && \text{(axiom 4)} \\ &= (x \vee 1) \wedge (x \vee x') && \text{(axiom 5)} \\ &= x \vee (1 \wedge x') && \text{(axiom 3)} \\ &= x \vee (x' \wedge 1) && \text{(axiom 1)} \\ &= x \vee x' && \text{(axiom 4)} \\ &= 1 && \text{(axiom 5)} \end{aligned}$$

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b

$$\begin{aligned} x \wedge 0 &= (x \wedge 0) \vee 0 && \text{(axiom 4)} \\ &= (x \wedge 0) \vee (x \wedge x') && \text{(axiom 5)} \\ &= x \wedge (0 \vee x') && \text{(axiom 3)} \\ &= x \wedge (x' \vee 0) && \text{(axiom 1)} \\ &= x \wedge x' && \text{(axiom 4)} \\ &= 0 && \text{(axiom 5)} \end{aligned}$$

Note: Part **b** is the dual of part **a**.

c Let $x \in B$. We know that $x \vee x' = 1$ and $x \wedge x' = 0$, by axiom 5. Assume that there is another element z which satisfies $x \vee z = 1$ and $x \wedge z = 0$. Then

$$\begin{aligned} x' &= x' \wedge 1 && \text{(axiom 4)} \\ &= x' \wedge (x \vee z) && \text{since } x \vee z = 1 \\ &= (x' \wedge x) \vee (x' \wedge z) && \text{(axiom 3)} \\ &= (x \wedge x') \vee (x' \wedge z) && \text{(axiom 1)} \\ &= 0 \vee (x' \wedge z) && \text{(axiom 5)} \\ &= (x \wedge z) \vee (x' \wedge z) && \text{since } x \wedge z = 0 \\ &= (z \wedge x) \vee (z \wedge x') && \text{(axiom 1)} \\ &= z \wedge (x \vee x') && \text{(axiom 3)} \\ &= z \wedge 1 && \text{(axiom 5)} \\ &= z && \text{(axiom 4)} \end{aligned}$$

d We will show that $x' \wedge y'$ is the complement of $x \vee y$ using part **c**.

First consider $(x \vee y) \vee (x' \wedge y')$:

$$\begin{aligned} (x \vee y) \vee (x' \wedge y') &= ((x \vee y) \vee x') \wedge ((x \vee y) \vee y') && \text{(axiom 3)} \\ &= (y \vee (x \vee x')) \wedge (x \vee (y \vee y')) && \text{(axioms 1 and 2)} \\ &= (y \vee 1) \wedge (x \vee 1) && \text{(axiom 5)} \\ &= 1 \wedge 1 && \text{by part } \mathbf{a} \\ &= 1 && \text{(axiom 4)} \end{aligned}$$

Now consider $(x \vee y) \wedge (x' \wedge y')$:

$$\begin{aligned} (x \vee y) \wedge (x' \wedge y') &= (x' \wedge y') \wedge (x \vee y) && \text{(axiom 1)} \\ &= ((x' \wedge y') \wedge x) \vee ((x' \wedge y') \wedge y) && \text{(axiom 3)} \\ &= (y' \wedge (x \wedge x')) \vee (x' \wedge (y \wedge y')) && \text{(axioms 1 and 2)} \\ &= (y' \wedge 0) \vee (x' \wedge 0) && \text{(axiom 5)} \\ &= 0 \vee 0 && \text{by part } \mathbf{b} \\ &= 0 && \text{(axiom 4)} \end{aligned}$$

Hence $x' \wedge y'$ is the complement of $x \vee y$ by part **c**. That is, $(x \vee y)' = x' \wedge y'$.

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The set of all subsets of a set is a Boolean algebra with the operations $\cup$, $\cap$, $'$ and the identity elements $\varnothing$, ξ. All the laws for sets given in Section 23A have parallel results for Boolean algebras in general. Some are axioms, and the others can be proved from the axioms (see Example 4 and Exercise 23B).

Properties of Boolean algebras

Primary
- $x \vee x = x$
- $x \wedge x = x$
- $x \vee 0 = x$ (A4)
- $x \wedge 1 = x$ (A4)
- $x \vee 1 = 1$
- $x \wedge 0 = 0$

Associativity (A2)
- $(x \vee y) \vee z = x \vee (y \vee z)$
- $(x \wedge y) \wedge z = x \wedge (y \wedge z)$

Commutativity (A1)
- $x \vee y = y \vee x$
- $x \wedge y = y \wedge x$

Distributivity (A3)
- $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Absorption
- $x \vee (x \wedge y) = x$
- $x \wedge (x \vee y) = x$

Complements
- $x \vee x' = 1$ (A5)
- $x \wedge x' = 0$ (A5)
- $0' = 1$
- $1' = 0$
- $(x \vee y)' = x' \wedge y'$
- $(x \wedge y)' = x' \vee y'$
- $(x')' = x$

Note: The properties $(x \vee y)' = x' \wedge y'$ and $(x \wedge y)' = x' \vee y'$ are called **De Morgan's laws**. You have seen them used in the context of logic in Section 6B.

Boolean functions and expressions

You have met functions in other areas of mathematics and we can think about the expressions that we have met above in this way. A simple example of a Boolean function is

$$f\colon \{0, 1\} \to \{0, 1\}, \quad f(x) = x \vee 1$$

In this case, we have $f(0) = 0 \vee 1 = 1$ and $f(1) = 1 \vee 1 = 1$.

In general, a **Boolean function** has one or more arguments from $\{0, 1\}$ and values in $\{0, 1\}$.

Example 5

Give the table of values for the Boolean function $f(x, y) = (x \wedge y) \vee y'$.

Solution

x	y	y'	$x \wedge y$	$f(x, y) = (x \wedge y) \vee y'$
0	0	1	0	1
0	1	0	0	0
1	0	1	0	1
1	1	0	1	1

ISBN 978-1-316-63608-4

The next example shows how to find a Boolean expression for a Boolean function given in table form. We don't give a formal proof, but you can easily verify the result by forming the table of values. In Section 23D, we will see the importance of this process in electronics.

Example 6

Find a Boolean expression for the Boolean function given by the following table.

	x	y	z	$f(x, y, z)$
1	0	0	0	0
2	0	0	1	0
3	0	1	0	1
4	0	1	1	0
5	1	0	0	1
6	1	0	1	0
7	1	1	0	1
8	1	1	1	1

Solution

We look at the rows where a 1 occurs in the rightmost column: rows 3, 5, 7 and 8.

Each row corresponds to some combination using $\wedge$ of either x or x', either y or y', and either z or z'.

For example, row 3 corresponds to $x' \wedge y \wedge z'$. We use x for a 1-entry in the x-column and use x' for a 0-entry. The same rule is followed for the y- and z-columns.

- Row 3 $x' \wedge y \wedge z'$
- Row 5 $x \wedge y' \wedge z'$
- Row 7 $x \wedge y \wedge z'$
- Row 8 $x \wedge y \wedge z$

Now combine these terms using $\vee$. We obtain the Boolean expression

$$(x' \wedge y \wedge z') \vee (x \wedge y' \wedge z') \vee (x \wedge y \wedge z') \vee (x \wedge y \wedge z)$$

We can write the Boolean function as

$$f(x, y, z) = (x' \wedge y \wedge z') \vee (x \wedge y' \wedge z') \vee (x \wedge y \wedge z') \vee (x \wedge y \wedge z)$$

Notes:

- We will see in Example 20 that this expression simplifies to $(x \wedge y) \vee (x \wedge z') \vee (y \wedge z')$.
- It follows from the associativity of $\vee$ and $\wedge$ that we can write expressions such as $x \wedge y \wedge z$ and $a \vee b \vee c \vee d$ unambiguously without using brackets.

ISBN 978-1-316-63608-4

Equivalent Boolean expressions

Two Boolean expressions are **equivalent** if they represent the same Boolean function.

If two Boolean expressions represent the same Boolean function, then it is possible to derive one expression from the other using the axioms of Boolean algebras.

Example 7

Consider the two Boolean expressions

$((x \vee y) \wedge (x' \vee y)) \vee x$ and $x \vee y$

Show that these two expressions are equivalent by:

a showing that they represent the same Boolean function

b using the axioms and properties of Boolean algebras.

Solution

a

x	y	$x \vee y$	x'	$x' \vee y$	$(x \vee y) \wedge (x' \vee y)$	$((x \vee y) \wedge (x' \vee y)) \vee x$
0	0	0	1	1	0	0
0	1	1	1	1	1	1
1	0	1	0	0	0	1
1	1	1	0	1	1	1

Columns 3 and 7 of the table are the same, and therefore the two expressions determine the same Boolean function.

b
$$\begin{aligned}((x \vee y) \wedge (x' \vee y)) \vee x &= ((y \vee x) \wedge (y \vee x')) \vee x && \text{(axiom 1)}\\ &= (y \vee (x \wedge x')) \vee x && \text{(axiom 3)}\\ &= (y \vee 0) \vee x && \text{(axiom 5)}\\ &= y \vee x && \text{(axiom 4)}\\ &= x \vee y && \text{(axiom 1)}\end{aligned}$$

Exercise 23B

Example 4

1 Prove each of the following, where x and y are elements of a Boolean algebra B:

a $x \wedge x = x$ **b** $x \vee x = x$ **c** $(x')' = x$

d $x = (x \wedge y) \vee (x \wedge y')$ **e** $x \vee (x \wedge y) = x$

2 Show that $a \vee [(b \wedge c') \wedge (d \wedge b')] = a$.

ISBN 978-1-316-63608-4

Cambridge University Press

Example 5

3 For each of the following Boolean functions, produce a table of values:

a $f(x, y) = (x \wedge y') \wedge (x \wedge y')$

b $f(x, y, z) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$

4 Draw a switching circuit for each of the following expressions. Try to simplify your circuit by first simplifying the expression using the properties of Boolean algebras.

a $(x \wedge y') \vee (x' \wedge y')$

b $(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y')$

Example 6

5 For each of the following, find a Boolean expression for the given Boolean function:

a

x	y	$f(x, y)$
0	0	1
0	1	1
1	0	0
1	1	1

b

x	y	z	$f(x, y, z)$
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

Example 7

6 Show that the expressions $(x' \vee y) \wedge (x \vee y')$ and $(x \wedge y) \vee (x' \wedge y')$ are equivalent by:

a showing that they represent the same Boolean function

b using the axioms and properties of Boolean algebras.

23C Logical connectives and truth tables

A **proposition** or **statement** is a sentence which is either true or false. Examples of statements are:

- The boy plays tennis.
- $5 + 7 = 12$
- $5 + 7 = 10$

Note that '5 + 7' is not a statement.

A statement can be assigned a **truth value**: T if it is true, or F if it is false. For example, the statement '5 + 7 = 12' is true (T), and the statement '5 + 7 = 10' is false (F).

A statement is often denoted by a capital letter $A, B, C, \ldots$.

ISBN 978-1-316-63608-4

▶ The logical connectives ‘or’, ‘and’, ‘not’

Logical connectives enable statements to be combined together to form new statements.

In English, two sentences may be combined by a grammatical connective to form a new compound sentence. For example, consider the following sentences:

A Gary went to the cinema.

B Gary did **not** go to the cinema.

C Kay went to the cinema.

D Gary **and** Kay went to the cinema.

Statement B is the negation (not) of statement A. Statement D is the conjunction (and) of statements A and C. The same can be done using logical connectives.

We will consider two statements about an integer n.

- Let G be the statement ‘n is odd’.
- Let H be the statement ‘$n > 10$’.

Given two statements, there are four possible combinations of truth values, as shown in the table on the right.

	G	H
1	T	T
2	T	F
3	F	T
4	F	F

Or

The symbol ∨ is used for ‘or’.

- The statement $G \vee H$ is ‘n is odd or $n > 10$’.

The statement $G \vee H$ is known as the **disjunction** of G and H. If either or both of the statements are true, then the compound statement is true. If both are false, then the compound statement is false. This is shown in the table on the right, which is called a **truth table**.

Truth table for ‘or’

G	H	$G \vee H$
T	T	T
T	F	T
F	T	T
F	F	F

And

The symbol ∧ is used for ‘and’.

- The statement $G \wedge H$ is ‘n is odd and $n > 10$’.

The statement $G \wedge H$ is known as the **conjunction** of G and H. If both statements are true, then the compound statement is true. If either or both are false, then the compound statement is false. This can be shown conveniently in a truth table.

Truth table for ‘and’

G	H	$G \wedge H$
T	T	T
T	F	F
F	T	F
F	F	F

Not

The symbol ¬ is used for ‘not’.

- The statement $\neg G$ is ‘n is even’.
- The statement $\neg H$ is ‘$n \leq 10$’.

In general, the statement $\neg A$ is called the **negation** of A and has the opposite truth value to A.

Truth table for ‘not’

A	$\neg A$
T	F
F	T

Note: The negation operation ¬ corresponds to complementation in Boolean algebra. But in propositional logic, it is common to use the notation $\neg A$ instead of A'.

ISBN 978-1-316-63608-4

Truth tables for compound statements

More complicated compound statements can be built up using these three connectives. For example, the statement $\neg G \wedge \neg H$ is 'n is even and $n \leq 10$'.

We can use truth tables to find the truth values of compound statements.

To construct the truth table for $\neg G \wedge \neg H$, we first find the truth values for the simpler statements $\neg G$ and $\neg H$, and then use the truth table for $\wedge$. Note that $\neg G \wedge \neg H$ is true if and only if both $\neg G$ and $\neg H$ are true.

G	H	$\neg G$	$\neg H$	$\neg G \wedge \neg H$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Example 8

Write the truth table for $\neg(A \vee B)$.

Solution

The truth values for the simpler statement $A \vee B$ are found first.

A	B	$A \vee B$	$\neg(A \vee B)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Example 9

Write the truth table for $A \wedge B \wedge (\neg A)$.

Solution

A	B	$A \wedge B$	$\neg A$	$A \wedge B \wedge (\neg A)$
T	T	T	F	F
T	F	F	F	F
F	T	F	T	F
F	F	F	T	F

ISBN 978-1-316-63608-4

Equivalent statements, tautologies and contradictions

Two statements that have the same truth values are **logically equivalent**.

Example 10

Show that $\neg(A \wedge B)$ is logically equivalent to $\neg A \vee \neg B$.

Solution

A	B	$\neg A$	$\neg B$	$A \wedge B$	$\neg(A \wedge B)$	$\neg A \vee \neg B$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

The truth values for $\neg(A \wedge B)$ and $\neg A \vee \neg B$ are the same, so the statements are equivalent.

- A **tautology** is a statement which is true under all circumstances.
- A **contradiction** is a statement which is false under all circumstances.

Example 11

Show that $(\neg A) \vee (A \vee B)$ is a tautology.

Solution

A	B	$\neg A$	$A \vee B$	$(\neg A) \vee (A \vee B)$
T	T	F	T	T
T	F	F	T	T
F	T	T	T	T
F	F	T	F	T

Example 12

Show that $(A \vee B) \wedge (\neg A \wedge \neg B)$ is a contradiction.

Solution

A	B	$\neg A$	$\neg B$	$A \vee B$	$\neg A \wedge \neg B$	$(A \vee B) \wedge (\neg A \wedge \neg B)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	F

ISBN 978-1-316-63608-4

▶ Implication

We now consider another logical connective, called **implication**.

For statements A and B, we can form the compound statement $A \Rightarrow B$, which is read as 'A implies B' or as 'If A, then B'.

The truth table for $\Rightarrow$ is shown on the right.

It is useful to consider whether this truth table is consistent with a 'common sense' view of implication.

- Let A be the statement 'I am elected'.
- Let B be the statement 'I will make public transport free'.

Truth table for 'implies'

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

Therefore $A \Rightarrow B$ is the statement 'If I am elected, then I will make public transport free'.

Now consider each row of the truth table:

Row 1 I am elected (A is true) and public transport is made free (B is true). I have kept my election promise. The statement $A \Rightarrow B$ is true.

Row 2 I am elected (A is true) but public transport is not made free (B is false). I have broken my election promise. The statement $A \Rightarrow B$ is false.

Rows 3 & 4 I am not elected (A is false). Whether or not public transport is made free, I have not broken my promise, as I was not elected. The statement $A \Rightarrow B$ is true.

Note that the only possible way that $A \Rightarrow B$ could be false is if I am elected but do not make public transport free. Otherwise, the statement is not false, and therefore must be true.

In general, the statement $A \Rightarrow B$ is true, except when A is true and B is false.

The statement $A \Rightarrow B$ is logically equivalent to $\neg A \vee B$, as shown in the truth table on the right.

A	B	$\neg A$	$\neg A \vee B$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Example 13

Give the truth table for $B \Rightarrow (A \wedge \neg B)$.

Solution

A	B	$\neg B$	$A \wedge \neg B$	$B \Rightarrow (A \wedge \neg B)$
T	T	F	F	F
T	F	T	T	T
F	T	F	F	F
F	F	T	F	T

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Example 14

Give the truth table for $[(A \vee B) \wedge \neg B] \Rightarrow A$.

Solution

A	B	$\neg B$	$A \vee B$	$(A \vee B) \wedge \neg B$	$[(A \vee B) \wedge \neg B] \Rightarrow A$
T	T	F	T	F	T
T	F	T	T	T	T
F	T	F	T	F	T
F	F	T	F	F	T

Note: The statement $[(A \vee B) \wedge \neg B] \Rightarrow A$ is a tautology.

Equivalence

Another logical connective is **equivalence**, which is represented by the symbol $\Leftrightarrow$. It has the truth table shown.

You can check using truth tables that the statement $A \Leftrightarrow B$ is logically equivalent to $(A \Rightarrow B) \wedge (B \Rightarrow A)$ and also to $(A \wedge B) \vee (\neg A \wedge \neg B)$.

Truth table for equivalence

A	B	$A \Leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

Example 15

Give the truth table for $(\neg A \vee \neg B) \Leftrightarrow \neg(A \wedge B)$.

Solution

A	B	$\neg A$	$\neg B$	$A \wedge B$	$\neg(A \wedge B)$	$\neg A \vee \neg B$	$(\neg A \vee \neg B) \Leftrightarrow \neg(A \wedge B)$
T	T	F	F	T	F	F	T
T	F	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	F	T	T	F	T	T	T

Note: It can be seen that this is a tautology. The two statements $\neg A \vee \neg B$ and $\neg(A \wedge B)$ are logically equivalent.

Converse and contrapositive

Let A be the statement 'You study Mathematics' and let B be the statement 'You study Physics'. The following two statements form a pair of **converse statements**:

- 'If you study Mathematics, then you study Physics.' $(A \Rightarrow B)$
- 'If you study Physics, then you study Mathematics.' $(B \Rightarrow A)$

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The following two statements form a pair of **contrapositive statements**:

- 'If you study Mathematics, then you study Physics.' ($A \Rightarrow B$)
- 'If you do not study Physics, then you do not study Mathematics.' ($\neg B \Rightarrow \neg A$)

In general, for a conditional statement $A \Rightarrow B$:

- the **converse** statement is $B \Rightarrow A$
- the **contrapositive** statement is $\neg B \Rightarrow \neg A$.

Note: Using truth tables, you can check that a statement $A \Rightarrow B$ is equivalent to its contrapositive $\neg B \Rightarrow \neg A$, but is not equivalent to its converse $B \Rightarrow A$.

Deductions

We can use truth tables to check the validity of logical deductions.

Example 16

Use truth tables to investigate the validity of each of the following deductions:

a All kangaroos jump. Jumping needs strength. So kangaroos need strength.

b In March, there are strong winds every day. The wind is not strong today. Therefore it is not March.

c On Mondays I go swimming. Today is not Monday. Therefore I do not swim today.

Solution

a
- Let A be the statement 'It is a kangaroo'.
- Let B be the statement 'It jumps'.
- Let C be the statement 'It needs strength'.

The compound statement to be considered is

$$[(A \Rightarrow B) \wedge (B \Rightarrow C)] \Rightarrow (A \Rightarrow C)$$

A	B	C	$A \Rightarrow B$	$B \Rightarrow C$	$A \Rightarrow C$	$(A \Rightarrow B) \wedge (B \Rightarrow C)$	$[(A \Rightarrow B) \wedge (B \Rightarrow C)] \Rightarrow (A \Rightarrow C)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Therefore $[(A \Rightarrow B) \wedge (B \Rightarrow C)] \Rightarrow (A \Rightarrow C)$ is a tautology, so the deduction is correct.

ISBN 978-1-316-63608-4

b ■ Let M be the statement 'It is March'.

■ Let S be the statement 'There are strong winds'.

The compound statement to consider is $[(M \Rightarrow S) \wedge (\neg S)] \Rightarrow \neg M$.

M	S	$M \Rightarrow S$	$\neg S$	$(M \Rightarrow S) \wedge (\neg S)$	$\neg M$	$[(M \Rightarrow S) \wedge (\neg S)] \Rightarrow \neg M$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

Since $[(M \Rightarrow S) \wedge (\neg S)] \Rightarrow \neg M$ is a tautology, the deduction is correct.

c ■ Let M be the statement 'It is Monday'.

■ Let S be the statement 'I swim today'.

The compound statement to consider is $[(M \Rightarrow S) \wedge (\neg M)] \Rightarrow \neg S$.

M	S	$M \Rightarrow S$	$\neg M$	$(M \Rightarrow S) \wedge (\neg M)$	$\neg S$	$[(M \Rightarrow S) \wedge (\neg M)] \Rightarrow \neg S$
T	T	T	F	F	F	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T

The deduction is not valid. It fails to be true when M is false and S is true.

Here is a list of some useful tautologies that can be used to form valid deductions:

1. $[(A \Rightarrow B) \wedge (B \Rightarrow C)] \Rightarrow (A \Rightarrow C)$
2. $[(A \Rightarrow B) \wedge A] \Rightarrow B$
3. $[(A \Rightarrow B) \wedge (\neg B)] \Rightarrow \neg A$
4. $[(A \vee B) \wedge (\neg A)] \Rightarrow B$
5. $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$ (contrapositive)
6. $\neg(A \vee B) \Leftrightarrow (\neg A \wedge \neg B)$ (De Morgan's law)
7. $\neg(A \wedge B) \Leftrightarrow (\neg A \vee \neg B)$ (De Morgan's law)

We have seen the first and third of these tautologies used in Example 16. Others will be considered in Exercise 23C. Here is an example for the second tautology:

> If it is Tuesday, then the flight arrives at 10 a.m. It is Tuesday. Therefore the flight arrives at 10 a.m.

Exercise 23C

1 For each statement (P), write down its negation ($\neg P$):

a Your eyes are blue. **b** The sky is grey. **c** This integer is odd.

d I live in Switzerland. **e** $x > 2$ **f** This number is less than 100.

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2 Consider the following four statements.

- A: It is dark.
- B: It is cold.
- C: It is good.
- D: It is soft.

Using the given table of opposites, write each of the following statements in English:

dark	light
cold	hot
good	bad
soft	hard

a $A \vee B$ **b** $A \wedge B$ **c** $\neg A \wedge B$

d $\neg(A \wedge B)$ **e** $C \vee \neg A$ **f** $\neg(A \vee D)$

g $A \vee \neg D$

3 Consider the following four statements.

- A: It is wet.
- B: It is rough.
- C: It is difficult.
- D: It is expensive.

Write each of the following in symbols:

a It is rough and wet. **b** It is expensive or difficult.

c It is not difficult but is expensive. **d** It is not wet and not rough.

e It is not expensive and not difficult. **f** It is rough or wet.

4 Consider the following four statements.

- A: It is wet.
- B: It is rough.
- C: It is difficult.
- D: It is expensive.

Using the given table of opposites, write each of the following statements in English:

wet	dry
rough	smooth
difficult	easy
expensive	inexpensive

a $A \vee B$ **b** $A \wedge B$

c $\neg A \wedge B$ **d** $\neg(A \wedge B)$

e $C \vee \neg A$ **f** $\neg(A \vee D)$

g $A \vee \neg D$

5 Consider the following three statements for x a natural number.

- A: x is a prime number.
- B: x is an even number.
- C: x is divisible by 6.

Write each of the following in English as concisely as possible:

a $A \vee B$ **b** $B \wedge C$ **c** $A \wedge B$ **d** $\neg A \wedge B$

e $\neg(A \wedge B)$ **f** $C \vee \neg A$ **g** $\neg(A \vee C)$ **h** $A \vee \neg C$

Example 10

6 Show that each of the following pairs of statements are equivalent:

a $\neg(A \vee B)$ $\quad \neg A \wedge \neg B$ **b** $\neg(\neg A)$ $\quad A$

c $A \vee A$ $\quad A$ **d** $A \vee B$ $\quad \neg(\neg A \wedge \neg B)$

e $A \wedge B$ $\quad \neg(\neg A \vee \neg B)$ **f** $A \wedge \neg B$ $\quad \neg(\neg A \vee B)$

Example 12

7 Show that $(A \wedge B) \wedge (\neg B)$ is a contradiction.

8 Show that $(\neg A \wedge B) \wedge A$ is a contradiction.

Example 11

9 Show that $(\neg A \wedge \neg B) \vee B \vee A$ is a tautology.

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Example 13, 14 **10** Give a truth table for each of the following statements:

a $(A \wedge B) \Rightarrow A$ **b** $(A \vee B) \Rightarrow A$ **c** $(\neg B \vee \neg A) \Rightarrow A$
d $(\neg B \wedge A) \Rightarrow A$ **e** $(B \vee \neg A) \Rightarrow \neg A$ **f** $(\neg B \vee \neg A) \Rightarrow (\neg B \wedge A)$
g $(\neg B \vee A) \Rightarrow \neg(B \wedge A)$ **h** $\neg B \wedge (\neg B \Rightarrow A)$

11 Show that each of the following pairs of statements are equivalent:

a $A \wedge B$ $\quad \neg(A \Rightarrow \neg B)$ **b** $A \vee B$ $\quad \neg A \Rightarrow B$
c $A \Leftrightarrow B$ $\quad \neg[(A \Rightarrow B) \Rightarrow \neg(B \Rightarrow A)]$

12 Show that each of the following is a tautology:

a $(A \wedge B) \Rightarrow (A \vee B)$ **b** $[A \wedge (A \Rightarrow B)] \Rightarrow B$ **c** $[(A \vee B) \wedge (\neg A)] \Rightarrow B$

Example 15 **13** Show that $\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$ is a tautology. Give the corresponding result for sets A and B, and illustrate with a Venn diagram.

14 The logical connective **nor** is written symbolically as $\downarrow$. The statement $A \downarrow B$ is true if and only if neither A nor B is true. Thus $A \downarrow B$ is equivalent to $\neg(A \vee B)$.

a Give the truth table for $A \downarrow B$ and $B \downarrow A$.
b Show that $A \downarrow A$ is equivalent to $\neg A$.
c Show that $[(A \downarrow A) \downarrow (B \downarrow B)] \Leftrightarrow (A \wedge B)$ is a tautology.
d Show that $\neg(A \downarrow B) \Leftrightarrow (A \vee B)$ is a tautology.

15 For each of the following statements:

i write the converse **ii** write the contrapositive.

a If $x = 6$, then x is an even integer.
b If I am elected, then public transport will improve.
c If I pass this exam, then I will be qualified as an actuary.

23D Logic circuits

In Section 23A, we introduced circuits composed of switches. In this section, we consider designing circuits composed of logic gates.

We use the Boolean operations $\vee$, $\wedge$ and $\neg$ on the set $\{0, 1\}$.

A	B	$A \vee B$
0	0	0
0	1	1
1	0	1
1	1	1

A	B	$A \wedge B$
0	0	0
0	1	0
1	0	0
1	1	1

A	$\neg A$
0	1
1	0

These correspond to the logical operations ‘or’, ‘and’ and ‘not’ if we interpret 0 as ‘false’ and 1 as ‘true’. In a circuit, we interpret 0 as ‘low voltage’ and 1 as ‘high voltage’.

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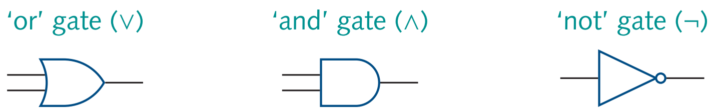
▶ Logic gates

We will create circuits using the following three **logic gates**, which carry out the operations of 'or' (∨), 'and' (∧) and 'not' (¬).



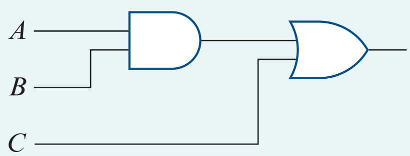
Each gate is shown with the **inputs** on the left and the **output** on the right. For example, if an 'or' gate has inputs 0 and 1, then the output will be 1.

We can build a logic circuit to represent any Boolean expression.

Example 17

Give the gate representation of the Boolean expression $(A \wedge B) \vee C$.

Solution



Note: The inputs to this circuit are labelled A, B and C. The output of the circuit will reflect the state of these inputs according to the logic statement $(A \wedge B) \vee C$.

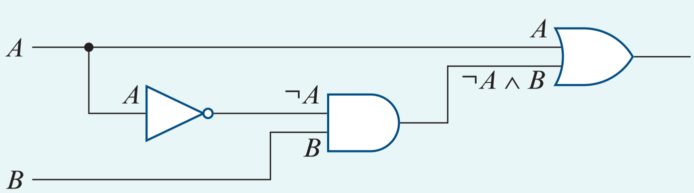
Example 18

a Give the gate representation of the Boolean expression $A \vee (\neg A \wedge B)$.

b Describe the operation of this circuit through a truth table.

Solution

a



b

A	B	$\neg A$	$\neg A \wedge B$	$A \vee (\neg A \wedge B)$
0	0	1	0	0
0	1	1	1	1
1	0	0	0	1
1	1	0	0	1

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In Example 6, we saw a technique for constructing a Boolean expression to match a given truth table of 0s and 1s. Given any truth table, we can construct a matching Boolean expression and therefore build a logic circuit that will operate according to the given table. This is what makes Boolean algebra so central to electronics.

> Given any truth table that specifies the required operation of a circuit, it is possible to build an appropriate circuit using 'or', 'and' and 'not' gates.

Exercise 23D

Example 17, 18

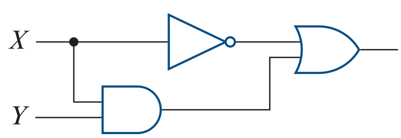
1 Draw a circuit using logic gates for each of the following:

a $A \wedge (\neg B)$ **b** $\neg(A \vee B)$ **c** $\neg(\neg P \wedge Q) \vee R$

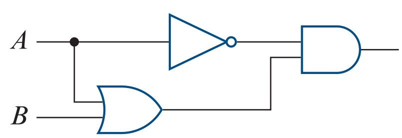
d $A \wedge (\neg B \vee A)$ **e** $(\neg A \vee B) \wedge (\neg B \vee A)$ **f** $(\neg A \wedge B) \vee (B \wedge \neg C) \vee A$

2 For each of the following, give a Boolean expression that represents the circuit and give a truth table describing the operation of the circuit:

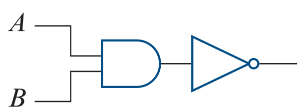
a



b



c



23E Karnaugh maps

We want to be able to simplify Boolean expressions so that the circuits we build from them are simpler and therefore cheaper. In Section 23B, we showed how to simplify Boolean expressions using the axioms. In this section, we introduce a more pictorial approach.

▶ Minimal representation

We start by formalising what it means for a Boolean expression to be 'simplified'.

A **minimal representation** of a Boolean function f is a Boolean expression E which represents f and satisfies the following:

- The expression E has the form $E_1 \vee E_2 \vee \cdots \vee E_n$, where each E_i is an expression such as $x \wedge y$ or $x' \wedge y' \wedge z$ or $y \wedge z'$.
- If F is any other expression of this form which also represents f, then the number of terms F_i is greater than or equal to the number of terms E_i.
- If F and E have the same number of terms, then the number of variables in F is greater than or equal to the number of variables in E.

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Karnaugh maps involving two variables

We demonstrate how a different representation of a truth table can be used to find a minimal expression. The truth table for $f(x, y) = (x' \wedge y') \vee (x' \wedge y) \vee (x \wedge y)$ is shown below.

x	y	$f(x, y)$	
0	0	1	$x' \wedge y'$
0	1	1	$x' \wedge y$
1	0	0	$x \wedge y'$
1	1	1	$x \wedge y$

As in Example 6, each row of the truth table corresponds to some combination using $\wedge$ of either x or x', and either y or y'. This is shown to the right of the truth table above. Using this correspondence, we fill the values of $f(x, y)$ into the following 2×2 table.

	y	y'
x	1	0
x'	1	1

The next step is to shade the 1s which occur in pairs as 1×2 or 2×1 blocks.

	y	y'
x	1	0
x'	1	1

The table above is called a **Karnaugh map**.

To find a minimal expression, we read off the label of each coloured block:

- **Red** The two 1s in the red block have labels $x'y$ and $x'y'$. The common label for the 1s in the red block is x'. So the red block has label x'.
- **Green** The two 1s in the green block have labels xy and $x'y$. The common label for the 1s in the green block is y. So the green block has label y.
- **Together** Combine the block labels using $\vee$. This gives $x' \vee y$.

The minimal expression is $f(x, y) = x' \vee y$.

The following calculation illustrates why this process works:

$$\begin{aligned}(x' \wedge y') \vee (x' \wedge y) \vee (x \wedge y) &= \overbrace{[(x' \wedge y') \vee (x' \wedge y)]}^{\text{red block}} \vee \overbrace{[(x' \wedge y) \vee (x \wedge y)]}^{\text{green block}} \\ &= [x' \wedge (y' \vee y)] \vee [(x' \vee x) \wedge y] \\ &= [x' \wedge 1] \vee [1 \wedge y] \\ &= x' \vee y\end{aligned}$$

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We could have used the original expression $(x' \wedge y') \vee (x' \wedge y) \vee (x \wedge y)$ to fill in the Karnaugh map directly, by putting a 1 for each of the terms $x' \wedge y'$, $x' \wedge y$ and $x \wedge y$ in the expression. A Boolean expression must be of a form like this to enter directly into a Karnaugh map.

Example 19

Simplify $(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y)$.

Solution

Step 1

	y	y'
x	1	1
x'	1	

Explanation

We can fill the 1s into the Karnaugh map directly from the expression. It is not necessary to add the 0s.

Step 2

	y	y'
x	1	1
x'	1	

Complete the shading of the 1s.

Step 3 Block labels:

- Red x
- Green y
- Together $x \vee y$

The simplified expression is $x \vee y$.

Now read off the labels of the coloured blocks:

- The common label for the 1s in the red block is x.
- The common label for the 1s in the green block is y.
- Combine the block labels using $\vee$.

Karnaugh maps involving three variables

A Karnaugh map for three variables x, y and z can be labelled as shown.

	yz	$y'z$	$y'z'$	yz'
x				
x'				

Notes:

- The order of the labels yz, $y'z$, $y'z'$, yz' along the top is important. There is only one change from one label to the next.
- You need to imagine that this Karnaugh map is wrapped around into a *cylinder* so that the xyz and xyz' cells are adjacent and the $x'yz$ and $x'yz'$ cells are adjacent.

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The technique for three variables is similar to that for two variables. We use the truth table or the expression to fill the 1s into the Karnaugh map. Then we cover the 1s using blocks.

Blocks in Karnaugh maps

- You may use an $m \times n$ block in a Karnaugh map if both m and n are powers of 2. (So you may use 1×1, 1×2, 1×4, 2×1, 2×2 and 2×4 blocks.)
- You always try to form the *biggest* blocks that you can, and to use the *least number* of blocks that you can.

Examples of correct shading

	yz	$y'z$	$y'z'$	yz'
x	1			1
x'	1			1

	yz	$y'z$	$y'z'$	yz'
x		1	1	1
x'		1	1	1

	yz	$y'z$	$y'z'$	yz'
x		1	1	
x'			1	1

Examples of incorrect shading

Blocks too small

	yz	$y'z$	$y'z'$	yz'
x	1			1
x'	1			1

Blocks too small

	yz	$y'z$	$y'z'$	yz'
x		1	1	1
x'		1	1	1

Too many blocks

	yz	$y'z$	$y'z'$	yz'
x		1	1	
x'			1	1

Example 20

Simplify $(x \wedge y \wedge z) \vee (x \wedge y \wedge z') \vee (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z')$.

Solution

	yz	$y'z$	$y'z'$	yz'
x	1		1	1
x'				1

Labels of the coloured blocks:

- Red $x \wedge z'$
- Green $y \wedge z'$
- Blue $x \wedge y$

The simplified expression is

$$(x \wedge y) \vee (x \wedge z') \vee (y \wedge z')$$

Explanation

In this example, the blue shading shows a 1×2 block that wraps around the back of the 'cylinder'.

The common labels for the 1s in the red block are x and z'. So the label of the red block is $x \wedge z'$.

We find the other labels similarly, and combine the block labels using $\vee$.

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Example 21

Write a minimal Boolean expression for the following truth table.

x	y	z	$f(x, y, z)$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

Solution

	yz	$y'z$	$y'z'$	yz'
x		1	1	
x'		1	1	1

Labels of the coloured blocks:

- Red y'
- Green $x' \wedge z'$

A minimal expression is
$f(x, y, z) = (x' \wedge z') \vee y'$.

Explanation

We first fill the 1s from the rightmost column of the truth table into the Karnaugh map.

For example, row 1 of the truth table corresponds to $x' \wedge y' \wedge z'$ and thus to the $x'y'z'$ cell.

Note: A Boolean function can have more than one minimal expression. There can be different correct ways to choose the blocks on a Karnaugh map.

Exercise 23E

Example 19

1 Simplify each of the following using a Karnaugh map:

a $(x \wedge y) \vee (x' \wedge y)$

b $(x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y')$

c $(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y')$

Example 20, 21

2 Simplify each of the following using a Karnaugh map:

a $(x \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z')$

b $(x \wedge y \wedge z) \vee (x \wedge y \wedge z') \vee (x' \wedge y \wedge z') \vee (x' \wedge y' \wedge z')$

c $(x \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z) \vee (x' \wedge y \wedge z') \vee (x' \wedge y' \wedge z')$

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Review

Chapter summary

Boolean algebra

- Basic examples of Boolean algebras:
  - the set of all subsets of a set with the operations ∪, ∩ and ′
  - the set {0, 1} with the operations ∨, ∧ and ′.

x	y	$x \vee y$
0	0	0
0	1	1
1	0	1
1	1	1

x	y	$x \wedge y$
0	0	0
0	1	0
1	0	0
1	1	1

x	x'
0	1
1	0

- A **Boolean expression** is an expression formed using ∨, ∧ and ′, such as $(x \vee y) \wedge (y \wedge z)'$.
- Two Boolean expressions are **equivalent** if they give the same Boolean function on {0, 1}.

Logical connectives

- Or

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

- And

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

- Not

A	$\neg A$
T	F
F	T

- Implies

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

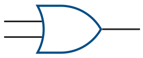
- Equivalence

A	B	$A \Leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

- Two statements are **logically equivalent** if they have the same truth values.
- A **tautology** is a statement which is true under all circumstances.
- A **contradiction** is a statement which is false under all circumstances.
- The **converse** of $A \Rightarrow B$ is the statement $B \Rightarrow A$.
- The **contrapositive** of $A \Rightarrow B$ is the statement $\neg B \Rightarrow \neg A$.

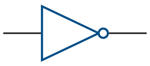
Logic circuits

- 'Or' gate (∨)



- 'And' gate (∧)

- 'Not' gate (¬)



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Short-answer questions

1 Which of the following statements are true?

a $\sqrt{2}$ is rational.

b π is not irrational.

c $\sqrt{2}$ is rational and π is irrational.

d $\sqrt{2}$ is rational or π is irrational.

e If $\sqrt{2}$ is rational, then π is irrational.

f If π is irrational, then $\sqrt{2}$ is rational.

2 For each statement (P), write down the negation ($\neg P$):

a It is raining.

b It is not raining.

c $2 + 3 > 4$

d $x = 5$ and $y = 5$

e $x = 3$ or $x = 5$

f It is not raining or it is not windy.

3 The logical connective 'exclusive or' is denoted by $\oplus$. The statement $A \oplus B$ is true when either A or B is true, but not both. The truth table for $\oplus$ is shown on the right.

Construct a truth table for each of the following compound statements:

a $A \oplus (A \oplus B)$

b $A \oplus (A \vee B)$

Truth table for 'exclusive or'

A	B	$A \oplus B$
T	T	F
T	F	T
F	T	T
F	F	F

4 Construct a truth table to show that the statement $\neg A \Rightarrow (A \Rightarrow B)$ is a tautology.

5 For each pair of Boolean expressions, prove that the two expressions are equivalent by:

i showing that they represent the same Boolean function on $\{0, 1\}$

ii using the axioms and properties of Boolean algebras.

a $x \vee (x' \wedge y)$ and $x \vee y$

b $(x \vee y) \wedge (x' \vee y)$ and y

6 Draw a circuit using logic gates for each of the following:

a $\neg A \vee B$

b $A \wedge (B \vee C)$

c $(A \wedge \neg B) \vee (B \wedge \neg A)$

7 Let P be the statement: If $a < b$, then $a^3 < b^3$.

a Write down the converse of P.

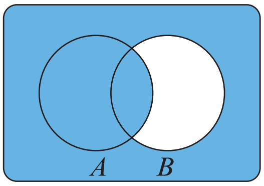
b Write down the contrapositive of P.

Multiple-choice questions

1 The blue region of the Venn diagram is

A B' **B** $A \cup B'$ **C** $A \cap B'$

D $A' \cap B'$ **E** $A' \cup B'$



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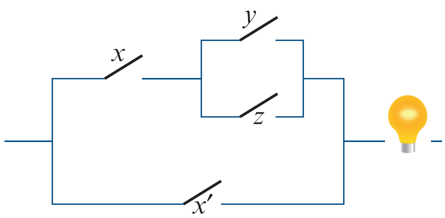
2 The dual of $A \cap (A \cup B)' = \varnothing$ is

A $B \cap (B \cup A)' = \varnothing$ **B** $A \cup (A \cap B)' = \varnothing$ **C** $A \cup (A \cap B)' = \xi$
D $A' \cap (A' \cup B')' = \xi$ **E** $A \cap (A \cup B)' = \xi$

3 Which of the following Boolean expressions represents the switching circuit shown?



A $(x \wedge y \wedge z) \vee x'$ **B** $x \wedge (y \vee z \vee x')$
C $(x \vee x') \wedge (y \vee z)$ **D** $(x \wedge (y \vee z)) \vee x'$
E $x \wedge (y \vee z) \wedge x'$

4 Which of the following is *not* an identity of Boolean algebra?

A $x \wedge x = x$ **B** $x \wedge y = y \wedge x$ **C** $(x \wedge y)' = x' \wedge y'$
D $x \wedge (x \wedge y) = x \wedge y$ **E** $0 \wedge x = 0$

5 Which of the following Boolean expressions corresponds to the truth table shown on the right?

A $(x' \wedge y' \wedge z) \vee (x \wedge y' \wedge z)$
B $(x \wedge y \wedge z') \vee (x' \wedge y \wedge z')$
C $(x \wedge y \wedge z) \vee (x' \wedge y' \wedge z')$
D $(x \wedge y' \wedge z') \vee (x' \wedge y \wedge z)$
E $(x' \wedge y \wedge z') \vee (x \wedge y' \wedge z)$

x	y	z	$f(x, y, z)$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

For Questions 6 and 7, let P be the statement 'I will pass Specialist Mathematics' and let S be the statement 'I study hard'.

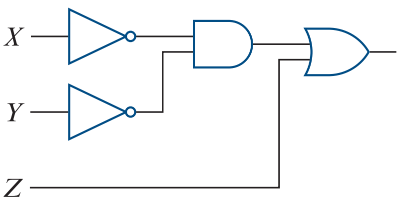
6 Which of the following corresponds to the statement 'If I study hard, then I will pass Specialist Mathematics'?

A $S \Leftrightarrow P$ **B** $S \vee P$ **C** $P \Rightarrow S$ **D** $S \Rightarrow P$ **E** $S \wedge P$

7 Which of the following is *not equivalent* to the statement 'I will not pass Specialist Mathematics unless I study hard'?

A $P \Rightarrow S$ **B** $\neg P \vee S$ **C** $\neg S \Rightarrow \neg P$ **D** $S \vee \neg P$ **E** $P \wedge S$

8 Which of the following Boolean expressions represents the logic circuit shown?



A $(X \wedge Y) \vee Z$ **B** $(\neg X \wedge \neg Y) \vee Z$
C $\neg(X \wedge Y) \vee Z$ **D** $(\neg X \vee \neg Y) \wedge Z$
E $\neg X \wedge (\neg Y \vee Z)$

Review

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9 The minimal Boolean expression represented by the Karnaugh map shown is

	yz	$y'z$	$y'z'$	yz'
x	1	1	1	1
x'			1	

A $x \vee (y' \wedge z')$ **B** $x \wedge (y' \vee z')$

C $x' \vee (y \wedge z)$ **D** $x' \wedge (y \vee z)$

E $x \vee (x' \wedge y' \wedge z')$

Extended-response questions

1 A light in a stairwell is controlled by two switches: one at the bottom of the stairs and one at the top. If both switches are off, then the light should be off. If either of the switches changes state, then the light should change state. In this question, we will create a switching circuit to represent such a two-way switch.

a Use x and y to denote the two switches. Use 0 for 'off' and 1 for 'on'. Complete the table on the right so that it describes the operation of the circuit.

b Based on your table for part **a**, write down a Boolean expression that represents the circuit.

c Draw the switching circuit that is represented by your Boolean expression from part **b**.

x	y	Light
0	0	0
0	1	
1	0	
1	1	

2 Let B be the set of all factors of 30. Thus

$$B = \{1, 2, 3, 5, 6, 10, 15, 30\}$$

It can be shown that axioms 1, 2 and 3 of Boolean algebras hold for B, with the operation $\vee$ as LCM (lowest common multiple) and the operation $\wedge$ as HCF (highest common factor). In this question, we will show that axioms 4 and 5 hold, in order to complete the proof that B is a Boolean algebra.

a **i** The identity element for LCM must be a number ℓ in B such that $\text{LCM}(x, \ell) = x$ for all $x \in B$. What is ℓ?

ii The identity element for HCF must be a number h in B such that $\text{HCF}(x, h) = x$ for all $x \in B$. What is h?

b For each $x \in B$, define $x' = 30 \div x$. By completing the following table, show that this operation $'$ is complementation in B.

x	1	2	3	5	6	10	15	30
x'								
$\text{LCM}(x, x')$								
$\text{HCF}(x, x')$								

Note: Can you see what is special about the number 30 here? Consider its prime factorisation. What goes wrong if you try using the number 12 instead?

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3 A committee with three members reaches its decisions by using a voting machine. The machine has three switches (x, y, z); one for each member of the committee. If at least two of the three members vote 'yes' (1), then the machine's light goes on (1). Otherwise, the light is off (0).

a Construct a table with entries 0s and 1s that describes the operation of the voting machine.

b Give a Boolean expression for the voting machine, based on your table for part **a**.

c Use a Karnaugh map to simplify the Boolean expression obtained in part **b**.

d Draw a circuit for the voting machine using logic gates, based on your simplified Boolean expression from part **c**.

4 Ternary logic In Boolean logic, there are two truth values: true (1) and false (0). In ternary logic, there are three truth values: true (1), false (0) and *don't know* (d). The basic example of a ternary algebra is the set $\{0, d, 1\}$ with the operations $\vee$, $\wedge$ and $'$ given by the following tables.

- Or ($\vee$)

$\vee$	0	d	1
0	0	d	1
d	d	d	1
1	1	1	1

- And ($\wedge$)

$\wedge$	0	d	1
0	0	0	0
d	0	d	d
1	0	d	1

- Not ($'$)

x	x'
0	1
d	d
1	0

Note that $d' = d$, since if we don't know whether a statement is true, then we don't know whether its negation is true. Ternary logic has applications in electronic engineering and database query languages.

a Evaluate each of the following:

i $d \vee 0$ **ii** $(d \wedge 0)'$ **iii** $(d \vee 0) \wedge (d' \vee 1)'$

b Give counterexamples to show that the laws $x \vee x' = 1$ and $x \wedge x' = 0$ for Boolean algebras do not hold for the ternary algebra $\{0, d, 1\}$.

c Prove that the De Morgan law $(x \vee y)' = x' \wedge y'$ holds for the ternary algebra $\{0, d, 1\}$.

Hint: Construct a truth table to show that $(x \vee y)'$ and $x' \wedge y'$ represent the same function on $\{0, d, 1\}$. Since there are two variables, each with three possible values, the truth table will have $3 \times 3 = 9$ rows.

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Glossary

A

Absolute value function [p. 48] The absolute value of a real number x is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Also called the *modulus function*

Acceleration [p. 534] The acceleration of a particle is defined as the rate of change of its velocity with respect to time.

Acceleration, average [p. 534] The average acceleration of a particle for the time interval $[t_1, t_2]$ is given by $\dfrac{v_2 - v_1}{t_2 - t_1}$, where v_2 is the velocity at time t_2 and v_1 is the velocity at time t_1.

Acceleration, instantaneous [p. 534] $a = \dfrac{dv}{dt}$

Addition formulas [p. 318]

- $\cos(u + v) = \cos u \cos v - \sin u \sin v$
- $\cos(u - v) = \cos u \cos v + \sin u \sin v$
- $\sin(u + v) = \sin u \cos v + \cos u \sin v$
- $\sin(u - v) = \sin u \cos v - \cos u \sin v$
- $\tan(u + v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v}$
- $\tan(u - v) = \dfrac{\tan u - \tan v}{1 + \tan u \tan v}$

Addition of complex numbers [p. 388] If $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 + z_2 = (a + c) + (b + d)i$.

Addition of vectors [p. 488] If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$, then $\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1)\boldsymbol{i} + (a_2 + b_2)\boldsymbol{j}$.

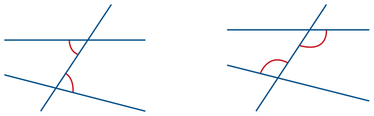
Addition principle [p. 124] Suppose there are m ways of performing one task and n ways of performing another task. If we cannot perform both tasks, then there are $m + n$ ways to perform one of the tasks.

Algebraic number [p. 45] a real number that is a solution to a polynomial equation of the form

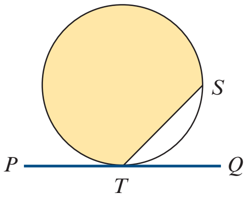
$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where $a_0, a_1, \ldots, a_n$ are integers

Alternate angles [p. 202] In each diagram, the two marked angles are alternate angles. (They are on alternate sides of the transversal.)



Alternate segment [p. 257] The alternate segment to $\angle STQ$ is shaded, and the alternate segment to $\angle STP$ is unshaded.



Altitude of a triangle [p. 207] a line segment from a vertex to the opposite side (possibly extended) which forms a right angle where it meets the opposite side

Amplitude of circular functions [p. 283] The distance between the mean position and the maximum position is called the amplitude. The graph of $y = a \sin x$ has an amplitude of $|a|$.

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Angle between two vectors [p. 501] can be found using the scalar product:

$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$

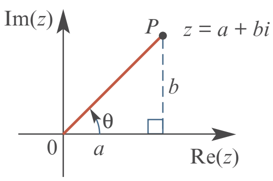
Angle sum and difference identities [p. 318] *see* addition formulas

Angle sum of a polygon [p. 208] The sum of the interior angles of a simple n-sided polygon is $(n-2)180°$.

Arc [p. 251] Two points on a circle divide the circle into arcs; the shorter is the *minor arc*, and the longer is the *major arc*.

Area of image [p. 473] If a linear transformation (with matrix **B**) is applied to a region of the plane, then Area of image = $|\det(\mathbf{B})| \times$ Area of region.

Argand diagram [p. 397] a geometric representation of the set of complex numbers



Argument of a complex number [p. 404]

- The argument of z is an angle θ from the positive direction of the x-axis to the line joining the origin to z.
- The *principal value* of the argument, denoted by $\operatorname{Arg} z$, is the angle in the interval $(-\pi, \pi]$.

Arrangement [p. 127] *see* permutation

Asymptote [p. 358] A straight line is an asymptote of the graph of a function $y = f(x)$ if the graph of $y = f(x)$ gets arbitrarily close to the straight line. An asymptote can be horizontal, vertical or oblique.

Asymptotes of hyperbolas [p. 358] The hyperbola with equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

has asymptotes given by

$$y - k = \pm\frac{b}{a}(x-h)$$

B

Binomial distribution [p. 587] The probability of achieving x successes in a sequence of n trials for a binomial random variable X is

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

where p is the probability of success on each trial.

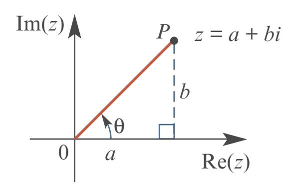
Boolean algebra [p. 622] a set B equipped with operations $\vee, \wedge, '$ that are analogous to the set-theoretic operations $\cup, \cap, '$ and also to the logical connectives 'or', 'and', 'not'

C

$\mathbb{C}$ [p. 387] the set of complex numbers:
$\mathbb{C} = \{\, a + bi : a, b \in \mathbb{R} \,\}$

Cartesian equation [p. 347] an equation that describes a curve in the plane by giving the relationship between the x- and y-coordinates of the points on the curve; e.g. $y = x^2 + 1$

Cartesian form of a complex number [p. 387] A complex number is expressed in Cartesian form as $z = a + bi$, where a is the real part of z and b is the imaginary part of z.



Chord [p. 251] a line segment with endpoints on a circle

Circle, general Cartesian equation [p. 347] The circle with radius r and centre (h, k) has equation $(x-h)^2 + (y-k)^2 = r^2$.

Circular functions [pp. 283, 284] the sine, cosine and tangent functions

Circular functions, exact values [p. 285]

θ^c	$\theta°$	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0°	0	1	0
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	90°	1	0	undefined

Circular functions, solving equations [p. 298]

- For $a \in [-1, 1]$, the general solution of the equation $\cos(x) = a$ is $x = 2n\pi \pm \cos^{-1}(a)$, where $n \in \mathbb{Z}$.
- For $a \in [-1, 1]$, the general solution of the equation $\sin(x) = a$ is $x = 2n\pi + \sin^{-1}(a)$ or $x = (2n+1)\pi - \sin^{-1}(a)$, where $n \in \mathbb{Z}$.
- For $a \in \mathbb{R}$, the general solution of the equation $\tan(x) = a$ is $x = n\pi + \tan^{-1}(a)$, where $n \in \mathbb{Z}$.

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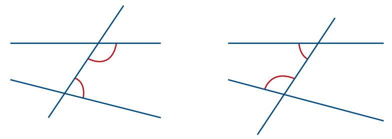
cis θ [p. 404] $\cos\theta + i\sin\theta$

Co-interior angles [p. 202] In each diagram, the two marked angles are co-interior angles. (They are on the same side of the transversal.)



Collinear points [p. 507] Three or more points are collinear if they all lie on a single line.

Column vector [pp. 449, 487] an $n \times 1$ matrix. A column vector $\begin{bmatrix} a \\ b \end{bmatrix}$ can be used to represent an ordered pair, a point in the Cartesian plane, a vector in the plane or a translation of the plane.

Combination [p. 139] a selection where order is not important. The number of combinations of n objects taken r at a time is given by

$${}^nC_r = \frac{n!}{r!\,(n-r)!}$$

An alternative notation for ${}^nC_r$ is $\binom{n}{r}$.

Complement of a set [p. 41] The complement of a set A, written A', is the set of all elements of ξ that are not elements of A.

Complementary angles [p. 200] two angles whose sum is 90°

Complementary relationships [p. 287]

- $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$
- $\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$
- $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$
- $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$

Complex conjugate, $\bar{z}$ [p. 391]

- If $z = a + bi$, then $\bar{z} = a - bi$.
- If $z = r\operatorname{cis}\theta$, then $\bar{z} = r\operatorname{cis}(-\theta)$.

Complex conjugate, properties [p. 392]

- $z + \bar{z} = 2\operatorname{Re}(z)$
- $z\bar{z} = |z|^2$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

Complex number [p. 387] an expression of the form $a + bi$, where a and b are real numbers

Complex plane [p. 397] *see* Argand diagram

Composite [p. 57] A natural number m is a composite number if it can be written as a product $m = a \times b$, where a and b are natural numbers greater than 1 and less than m.

Compound angle formulas [p. 318] *see* addition formulas

Concurrent lines [p. 507] Three or more lines are concurrent if they all pass through a single point.

Conditional statement [pp. 165, 631] a statement of the form 'If P is true, then Q is true', which can be abbreviated to $P \Rightarrow Q$

Congruence tests [p. 211] Two triangles are congruent if one of the following conditions holds:

- **SSS** the three sides of one triangle are equal to the three sides of the other triangle
- **SAS** two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle
- **AAS** two angles and one side of one triangle are equal to two angles and the matching side of the other triangle
- **RHS** the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and one side of another right-angled triangle.

Congruent figures [p. 211] have exactly the same shape and size

Constant acceleration formulas [p. 540]

- $v = u + at$
- $s = ut + \frac{1}{2}at^2$
- $v^2 = u^2 + 2as$
- $s = \frac{1}{2}(u + v)t$

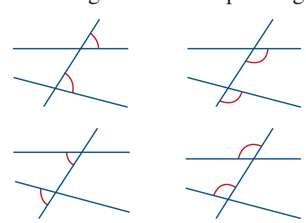
Contradiction [p. 630] a statement which is false under all circumstances; *see also* proof by contradiction

Contrapositive [pp. 171, 633] The contrapositive of $P \Rightarrow Q$ is the statement (not Q) $\Rightarrow$ (not P). The contrapositive is equivalent to the original statement.

Converse [pp. 178, 633] The converse of $P \Rightarrow Q$ is the statement $Q \Rightarrow P$.

Conversion between Cartesian and polar forms [pp. 371, 404] $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$

Corresponding angles [p. 202] In each diagram, the two marked angles are corresponding angles.



Cosecant function [pp. 314, 343]

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \text{ for } \sin\theta \neq 0$$

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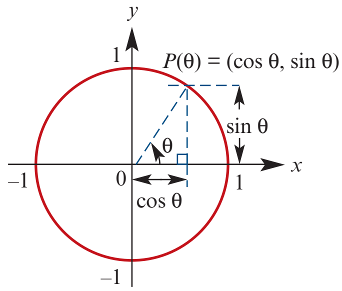
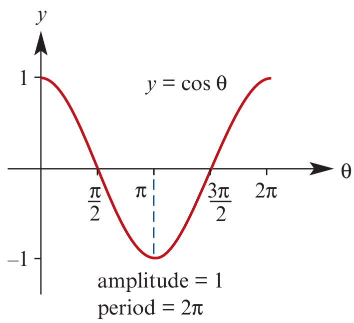
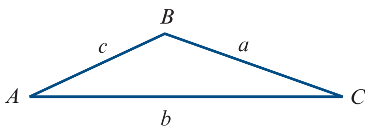
Cosine function [p. 283] cosine θ is defined as the x-coordinate of the point P on the unit circle where OP forms an angle of θ radians with the positive direction of the x-axis.



Cosine function, graph [p. 283]



Cosine rule [p. 499] For triangle ABC:
$a^2 = b^2 + c^2 - 2bc\cos A$



Cotangent function [pp. 314, 344] $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ for $\sin\theta \neq 0$

Counterexample [p. 182] an example that shows that a universal statement is false. For example, the number 2 is a counterexample to the claim 'Every prime number is odd.'

Cyclic quadrilateral [p. 253] a quadrilateral such that all its vertices lie on a circle. Opposite angles of a cyclic quadrilateral are supplementary.

D

De Morgan's laws [pp. 170, 624]

- 'not (P and Q)' is '(not P) or (not Q)'
- 'not (P or Q)' is '(not P) and (not Q)'

Degree of a polynomial [p. 83] given by the highest power of x with a non-zero coefficient; e.g. the polynomial $2x^5 - 7x^2 + 4$ has degree 5

Determinant of a 2 × 2 matrix [p. 437]
If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(\mathbf{A}) = ad - bc$.

Diameter [p. 251] a chord of a circle that passes through the centre

Dilation [p. 454] A dilation scales the x- or y-coordinate of each point in the plane.

- Dilation from the x-axis: $(x, y) \to (x, cy)$
- Dilation from the y-axis: $(x, y) \to (cx, y)$

Direct proof [p. 166] To give a direct proof of a conditional statement $P \Rightarrow Q$, we assume that P is true and show that Q follows.

Discrete random variable [p. 582] a random variable X which can take only a countable number of values, usually whole numbers

Discriminant, Δ, of a quadratic [p. 88] the expression $b^2 - 4ac$, which is part of the quadratic formula. For the quadratic equation $ax^2 + bx + c = 0$:

- If $b^2 - 4ac > 0$, there are two real solutions.
- If $b^2 - 4ac = 0$, there is one real solution.
- If $b^2 - 4ac < 0$, there are no real solutions.

Disjoint sets [p. 40] Sets A and B are said to be disjoint if they have no elements in common, i.e. if $A \cap B = \emptyset$.

Displacement [p. 531] The displacement of a particle moving in a straight line is defined as the change in position of the particle.

Division of complex numbers [pp. 392, 406]

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \times \frac{\overline{z_2}}{\overline{z_2}} = \frac{z_1\overline{z_2}}{|z_2|^2}$$

If $z_1 = r_1 \operatorname{cis}\theta_1$ and $z_2 = r_2 \operatorname{cis}\theta_2$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)$$

Double angle formulas [p. 321]

- $\cos(2u) = \cos^2 u - \sin^2 u$
  $= 2\cos^2 u - 1$
  $= 1 - 2\sin^2 u$
- $\sin(2u) = 2\sin u \cos u$
- $\tan(2u) = \dfrac{2\tan u}{1 - \tan^2 u}$

E

Ellipse [p. 353] The graph of the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

is an ellipse centred at the point (h, k).

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Equality of complex numbers [p. 388]
$a + bi = c + di$ if and only if $a = c$ and $b = d$

Equilibrium [p. 557] A particle is said to be in equilibrium if the resultant force acting on it is zero; the particle will remain at rest or continue moving with constant velocity.

Equivalence of vectors [p. 495]
Let $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$. If $\boldsymbol{a} = \boldsymbol{b}$, then $a_1 = b_1$ and $a_2 = b_2$.

Equivalent statements [pp. 179, 632]
Statements P and Q are equivalent if $P \Rightarrow Q$ and $Q \Rightarrow P$; this is abbreviated to $P \Leftrightarrow Q$.

Euclidean algorithm [p. 66] a method for finding the highest common factor of two numbers and for solving linear Diophantine equations

Existence statement [pp. 181, 183] a statement claiming that a property holds for some member of a given set. Such a statement can be written using the quantifier 'there exists'.

F

Factor [p. 57] A natural number a is a factor of a natural number b if there exists a natural number k such that $b = ak$.

Factorial notation [p. 127] The notation $n!$ (read as 'n factorial') is an abbreviation for the product of all the integers from n down to 1:
$n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 2 \times 1$

Force [p. 556] causes a change in motion; e.g. gravitational force, tension force, normal reaction force. Force is a vector quantity.

Formula [p. 19] an equation containing symbols that states a relationship between two or more quantities; e.g. $A = \ell w$ (area = length × width). The value of A, the subject of the formula, can be found by substituting given values of ℓ and w.

G

g [p. 538] the acceleration of a particle due to gravity. Close to the Earth's surface, the value of g is approximately 9.8 m/s^2.

Geometric mean [p. 239] For $a, b, c \in \mathbb{R}^+$, if $\frac{c}{a} = \frac{b}{c}$, then c is the geometric mean of a and b.

Golden ratio, φ [p. 238] $\varphi = \dfrac{1+\sqrt{5}}{2}$

Golden rectangle [p. 238] a rectangle with ratio of length to width $\varphi : 1$

H

Highest common factor [p. 58] The highest common factor of two natural numbers a and b, denoted by HCF(a, b), is the largest natural number that is a factor of both a and b.

Hyperbola [p. 357] The graph of the equation
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$
is a hyperbola centred at the point (h, k); the asymptotes are given by
$$y - k = \pm\frac{b}{a}(x-h)$$

Hypergeometric distribution [p. 585] The probability of obtaining x defectives in a sample of size n for a hypergeometric random variable X is
$$\Pr(X = x) = \frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}}$$
where N is the size of the population and D is the number of defectives in the population.

I

Imaginary number i [p. 387] $i^2 = -1$

Imaginary part of a complex number [p. 388]
If $z = a + bi$, then $\text{Im}(z) = b$.

Implication [pp. 165, 631] *see* conditional statement

Inclusion–exclusion principle [p. 155] allows us to count the number of elements in a union of sets. In the case of two sets: $|A \cup B| = |A| + |B| - |A \cap B|$

Index laws [p. 2]
- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $(ab)^n = a^n b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
- $a^{-n} = \frac{1}{a^n}$
- $\frac{1}{a^{-n}} = a^n$
- $a^0 = 1$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$
- $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$

Integers [p. 43] the elements of
$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

Intersection of sets [p. 41] The intersection of two sets A and B, written $A \cap B$, is the set of all elements common to A and B.

Interval [p. 46] a subset of the real numbers of the form $[a, b]$, $[a, b)$, (a, ∞), etc.

Irrational number [p. 43] a real number that is not rational; e.g. π and $\sqrt{2}$

ISBN 978-1-316-63608-4

Cambridge University Press

K

Kilogram weight, kg wt [p. 557] a unit of force. If an object on the surface of the Earth has a mass of 1 kg, then the gravitational force acting on this object is 1 kg wt.

L

Like surds [p. 52] surds with the same irrational factor; e.g. $2\sqrt{7}$ and $9\sqrt{7}$

Linear Diophantine equation [p. 62] an equation $ax + by = c$, where the coefficients a, b, c are integers and the aim is to find integer solutions for x, y. If one solution (x_0, y_0) is found, then the general solution is given by

$$x = x_0 + \frac{b}{d}t, \quad y = y_0 - \frac{a}{d}t \qquad \text{for } t \in \mathbb{Z}$$

where d is the highest common factor of a and b.

Linear equation [p. 8] a polynomial equation of degree 1; e.g. $2x + 1 = 0$

Linear transformation [p. 449] a transformation of the plane with a rule of the form

$(x, y) \to (ax + by, cx + dy)$

Each linear transformation can be represented by a 2×2 matrix:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Linear transformation, inverse [p. 465] If $\mathbf{A}$ is the matrix of a linear transformation and $\mathbf{A}$ is invertible, then $\mathbf{A}^{-1}$ is the matrix of the inverse transformation.

Linear transformations, composition [p. 462] If $\mathbf{A}$ and $\mathbf{B}$ are the matrices of two different linear transformations, then the product $\mathbf{BA}$ is the matrix of the transformation $\mathbf{A}$ followed by $\mathbf{B}$.

Literal equation [p. 25] an equation for the variable x in which the coefficients of x, including the constants, are pronumerals; e.g. $ax + b = c$

Locus [p. 347] a set of points described by a geometric condition; e.g. the locus of points P that satisfy $PO = 3$, where O is the origin, is the circle of radius 3 centred at the origin

Logical connectives [p. 628] used to combine statements together to form new statements; e.g. 'and', 'or', 'not', 'implies'

Lowest common multiple [p. 60] The lowest common multiple of two natural numbers a and b, denoted by $\text{LCM}(a, b)$, is the smallest natural number that is a multiple of both a and b.

M

Magnitude of a vector [p. 495] the length of a directed line segment corresponding to the vector.

- If $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$, then $|\boldsymbol{u}| = \sqrt{x^2 + y^2}$.
- If $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, then $|\boldsymbol{u}| = \sqrt{x^2 + y^2 + z^2}$.

Mass [p. 557] The mass of an object is the amount of matter it contains, and can be measured in kilograms. Mass is not the same as weight.

Mathematical induction [p. 185] a proof technique for showing that a statement is true for all natural numbers; uses the *principle of mathematical induction*

Matrices, addition [p. 428] Addition is defined for two matrices of the same size (same number of rows and same number of columns). The sum is found by adding corresponding entries. For example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 4 & 3 \end{bmatrix}$$

Matrices, equal [p. 425] Two matrices $\mathbf{A}$ and $\mathbf{B}$ are equal, and we write $\mathbf{A} = \mathbf{B}$, when:

- they have the same number of rows and the same number of columns, and
- they have the same entry at corresponding positions.

Matrices, multiplication [p. 432] The product of two matrices $\mathbf{A}$ and $\mathbf{B}$ is only defined if the number of columns of $\mathbf{A}$ is the same as the number of rows of $\mathbf{B}$. If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times r$ matrix, then the product $\mathbf{AB}$ is the $m \times r$ matrix whose entries are determined as follows:

> To find the entry in row i and column j of $\mathbf{AB}$, single out row i in matrix $\mathbf{A}$ and column j in matrix $\mathbf{B}$. Multiply the corresponding entries from the row and column and then add up the resulting products.

Matrix [p. 424] a rectangular array of numbers

Matrix, identity [p. 435] For square matrices of a given size (e.g. 2×2), a multiplicative identity $\mathbf{I}$ exists.

For 2×2 matrices, the identity is $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

and $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for each 2×2 matrix $\mathbf{A}$.

Matrix, inverse [p. 436] If $\mathbf{A}$ is a square matrix and there exists a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, then $\mathbf{B}$ is called the inverse of $\mathbf{A}$. When it exists, the inverse of a square matrix $\mathbf{A}$ is unique and is denoted by $\mathbf{A}^{-1}$.

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\mathbf{A}^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

provided $ad - bc \neq 0$.

ISBN 978-1-316-63608-4

Cambridge University Press

Matrix, invertible [p. 436] A square matrix is said to be invertible if its inverse exists.

Matrix, multiplication by a scalar [p. 428] If **A** is an $m \times n$ matrix and k is a real number, then k**A** is an $m \times n$ matrix whose entries are k times the corresponding entries of **A**. For example:

$$3\begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 0 & 3 \end{bmatrix}$$

Matrix, non-invertible [p. 436] A square matrix is said to be non-invertible if it does not have an inverse.

Matrix, size [p. 424] A matrix with m rows and n columns is said to be an $m \times n$ matrix.

Matrix, square [p. 435] A matrix with the same number of rows and columns is called a square matrix; e.g. a 2×2 matrix.

Matrix, zero [p. 429] The $m \times n$ matrix with all entries equal to zero is called the zero matrix.

Mean [p. 579] the most commonly used measure of centre for numerical data; denoted by $\bar{x}$ and calculated by summing all the data values and dividing by the number of values in the data set

Median of a triangle [p. 207] a line segment from a vertex to the midpoint of the opposite side

Modulus–argument form of a complex number [p. 404] *see* polar form of a complex number

Modulus function [p. 48] The modulus of a real number x is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Also called the *absolute value function*

Modulus of a complex number, |z| [p. 404] the distance of the complex number from the origin. If $z = a + bi$, then $|z| = \sqrt{a^2 + b^2}$.

Modulus, properties [p. 406] For complex numbers z_1 and z_2:

- $|z_1 z_2| = |z_1|\,|z_2|$ (the modulus of a product is the product of the moduli)
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ (the modulus of a quotient is the quotient of the moduli)

Multiple [p. 60] A natural number a is a multiple of a natural number b if there exists a natural number k such that $a = kb$.

Multiplication of a complex number by *i* [pp. 399, 406] corresponds to a rotation about the origin by 90° anticlockwise. If $z = a + bi$, then $iz = i(a + bi) = -b + ai$.

Multiplication of a vector by a scalar [p. 489] If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $m \in \mathbb{R}$, then $m\boldsymbol{a} = ma_1\boldsymbol{i} + ma_2\boldsymbol{j}$.

Multiplication of complex numbers [pp. 391, 406] If $z_1 = a + bi$ and $z_2 = c + di$, then

$z_1 z_2 = (ac - bd) + (ad + bc)i$

If $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$, then

$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$

Multiplication principle [p. 123] If there are m ways of performing one task and then there are n ways of performing another task, then there are $m \times n$ ways of performing *both* tasks.

N

***n*!** [p. 127] *see* factorial notation

Natural numbers [p. 43] the elements of $\mathbb{N} = \{1, 2, 3, 4, \dots\}$

Negation [pp. 170, 628] The negation of a statement P is the opposite statement, called 'not P'. For example, the negation of $x \geq 1$ is $x < 1$.

Normal distribution [p. 601] a symmetric, bell-shaped distribution that often occurs for a measure in a population (e.g. height, weight, IQ); its centre is determined by the mean, μ, and its width by the standard deviation, σ.

Normal reaction force [p. 557] A mass placed on a surface (horizontal or inclined) experiences a force perpendicular to the surface, called the normal force.

O

Ordered pair [p. 44] An ordered pair, denoted (x, y), is a pair of elements x and y in which x is considered to be the first coordinate and y the second coordinate.

P

Parallelogram [p. 212] a quadrilateral with both pairs of opposite sides parallel

Parametric equations [p. 362] a pair of equations $x = f(t)$ and $y = g(t)$ describing a curve in the plane, where t is called the *parameter* of the curve

Partial fractions [p. 98] Some rational functions may be expressed as a sum of partial fractions; e.g.

$$\frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2} + \frac{Dx+E}{ex^2+fx+g}$$

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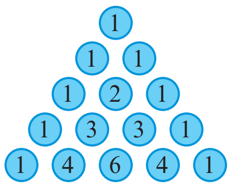
Particle model [p. 557] an object is considered as a point. This can be done when the size of the object can be neglected in comparison with other lengths in the problem being considered, or when rotational motion effects can be ignored.

Pascal's triangle [p. 148] a triangular pattern of numbers formed by the binomial coefficients ${}^nC_r$. Each entry of Pascal's triangle is the sum of the two entries immediately above.



Period of a function [p. 283] A function f with domain $\mathbb{R}$ is periodic if there is a positive constant a such that $f(x + a) = f(x)$ for all x. The smallest such a is called the period of f. For example, the period of the sine function is 2π, as $\sin(x + 2\pi) = \sin x$.

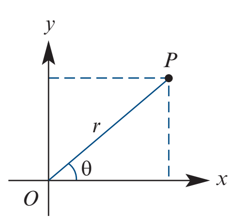
Permutation [p. 127] an ordered arrangement of objects. The number of permutations of n objects taken r at a time is given by

$${}^nP_r = \frac{n!}{(n-r)!}$$

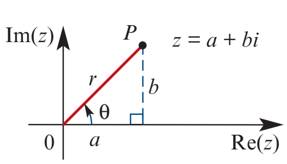
Pigeonhole principle [p. 151] If $n + 1$ or more objects are placed into n holes, then some hole contains at least two objects.

Polar coordinates [p. 371] A point P in the plane has polar coordinates (r, θ), where

- r is the distance from the origin O to P
- θ is the angle between the positive direction of the x-axis and the ray OP.



Polar form of a complex number [p. 404] A complex number is expressed in polar form as $z = r \operatorname{cis} \theta$, where r is the modulus of z and θ is an argument of z. This is also called *modulus–argument form*.



Polynomial function [p. 83] A polynomial has a rule of the type

$$y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad n \in \mathbb{N} \cup \{0\}$$

where $a_0, a_1, \ldots, a_n$ are numbers called coefficients.

Population [p. 576] the set of all eligible members of a group which we intend to study

Population mean, μ [p. 579] the mean of all values of a measure in the entire population

Population parameter [p. 580] a statistical measure that is based on the whole population; the value is constant for a given population

Population proportion, p [p. 579] the proportion of individuals in the entire population possessing a particular attribute

Position [p. 531] For a particle moving in a straight line, the position of the particle relative to a point O on the line is determined by its distance from O and whether it is to the right or left of O. The direction to the right of O is positive.

Position vector [p. 491] A position vector, $\overrightarrow{OP}$, indicates the position in space of the point P relative to the origin O.

Prime [p. 57] A natural number greater than 1 is a prime number if its only factors are itself and 1.

Prime decomposition [p. 57] expressing a composite number as a product of powers of prime numbers; e.g. $500 = 2^2 \times 5^3$

Principle of mathematical induction [p. 185] used to prove that a statement is true for all natural numbers

Probability distribution [p. 583] a function, denoted $p(x)$ or $\Pr(X = x)$, which assigns a probability to each value of a discrete random variable X. It can be represented by a rule, a table or a graph, and must give a probability $p(x)$ for every value x that X can take.

Product-to-sum identities [p. 328]

- $2 \cos A \cos B = \cos(A - B) + \cos(A + B)$
- $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$
- $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

Projection [p. 456] A projection maps each point in the plane onto an axis.

- Projection onto the x-axis: $(x, y) \to (x, 0)$
- Projection onto the y-axis: $(x, y) \to (0, y)$

Proof by contradiction [p. 174] a proof that begins by assuming the negation of what is to be proved

ISBN 978-1-316-63608-4

Pythagoras' theorem [p. 216] For a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides: $(\text{hyp})^2 = (\text{opp})^2 + (\text{adj})^2$

Pythagorean identity [pp. 288, 316]
- $\cos^2\theta + \sin^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $\cot^2\theta + 1 = \operatorname{cosec}^2\theta$

Q

Quadratic formula [p. 87] An equation of the form $az^2 + bz + c = 0$, with $a \neq 0$, may be solved quickly by using the quadratic formula:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic function [p. 87] A quadratic has a rule of the form $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$.

Quadratic surd [p. 51] a number of the form $\sqrt{a}$, where a is a rational number which is not the square of another rational number

Quantifier [p. 181] *see* existence statement, universal statement

R

$\mathbb{R}^+$ [p. 47] $\{x : x > 0\}$, positive real numbers

$\mathbb{R}^-$ [p. 47] $\{x : x < 0\}$, negative real numbers

$\mathbb{R} \setminus \{0\}$ [p. 47] the set of real numbers excluding 0

$\mathbb{R}^2$ [p. 44] $\{(x, y) : x, y \in \mathbb{R}\}$; i.e. $\mathbb{R}^2$ is the set of all ordered pairs of real numbers

Radian [p. 282] One radian (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit:

$$1^c = \frac{180^\circ}{\pi} \quad \text{and} \quad 1^\circ = \frac{\pi^c}{180}$$

Random sample [p. 576] a sample chosen using a random process so that each member of the population has an equal chance of being included

Random variable [p. 581] a variable that takes its value from the outcome of a random experiment; e.g. the number of heads observed when a coin is tossed three times

Rate [p. 93] describes how a certain quantity changes with respect to the change in another quantity (often time)

Rational function [p. 98] a function of the form $f(x) = \dfrac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials

Rational number [p. 43] a number that can be written as $\dfrac{p}{q}$, for some integers p and q with $q \neq 0$

Real part of a complex number [p. 388] If $z = a + bi$, then $\operatorname{Re}(z) = a$.

Reciprocal circular functions [pp. 314, 343] the secant, cosecant and cotangent functions

Reciprocal function [p. 338] The reciprocal of the function $y = f(x)$ is defined by $y = \dfrac{1}{f(x)}$.

Recurrence relation [p. 189] a rule which enables each subsequent term of a sequence to be found from previous terms; e.g. $t_1 = 1$, $t_n = t_{n-1} + 2$

Reflection [p. 453] A reflection in a line ℓ maps each point in the plane to its mirror image on the other side of the line.
- Reflection in the x-axis: $(x, y) \to (x, -y)$
- Reflection in the y-axis: $(x, y) \to (-x, y)$
- Reflection in the line $y = x$: $(x, y) \to (y, x)$
- Reflection in the line $y = -x$: $(x, y) \to (-y, -x)$

Reflection matrix [p. 460] A reflection in the line $y = mx = x\tan\theta$ is expressed using matrix multiplication as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Regular polygon [p. 206] a polygon in which all the angles are equal and all the sides are equal

Resultant force [p. 557] the vector sum of the forces acting at a point

Rhombus [p. 212] a parallelogram with all sides of equal length

Rotation matrix [p. 459] A rotation about the origin by θ degrees anticlockwise is expressed using matrix multiplication as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

S

Sample [p. 576] a subset of the population which we select in order to make inferences about the whole population

Sample mean, $\bar{x}$ [p. 579] the mean of all values of a measure in a particular sample. The values $\bar{x}$ are the values of a random variable $\bar{X}$.

Sample proportion, $\hat{p}$ [p. 579] the proportion of individuals in a particular sample possessing a particular attribute. The values $\hat{p}$ are the values of a random variable $\hat{P}$.

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Sample statistic [p. 580] a statistical measure that is based on a sample from the population; the value varies from sample to sample

Sampling distribution [p. 584] the distribution of a statistic which is calculated from a sample

Sampling with replacement [p. 589] selecting individual objects sequentially from a group of objects, and replacing the selected object, so the probability of obtaining a particular object does not change with each successive selection

Sampling without replacement [p. 589] selecting individual objects sequentially from a group of objects, and not replacing the selected object, so the probability of obtaining a particular object changes with each successive selection

Scalar [p. 489] a real number; name used when working with vectors or matrices

Scalar product [p. 499] The scalar product of two vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$ is given by $\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2$

Scalar product, properties [p. 500]

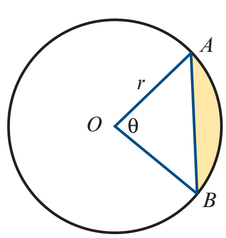
- $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
- $k(\boldsymbol{a} \cdot \boldsymbol{b}) = (k\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (k\boldsymbol{b})$
- $\boldsymbol{a} \cdot \boldsymbol{0} = 0$
- $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$

Scientific notation [p. 5] A number is in standard form when written as a product of a number between 1 and 10 and an integer power of 10; e.g. 6.626×10^{-34}.

Secant function [pp. 314, 343] $\sec\theta = \dfrac{1}{\cos\theta}$ for $\cos\theta \neq 0$

Secant of a circle [p. 251] a line that cuts a circle at two distinct points

Segment [p. 251] Every chord divides the interior of a circle into two regions called segments; the smaller is the *minor segment* (shaded), and the larger is the *major segment*.



Selection [p. 139] *see* combination

Sequence [p. 189] a list of numbers, with the order being important; e.g. 1, 1, 2, 3, 5, 8, 13, . . . The numbers of a sequence are called its *terms*, and the *n*th term is often denoted by t_n.

Set notation [p. 40]

$\in$	means 'is an element of'
$\notin$	means 'is not an element of'
$\subseteq$	means 'is a subset of'
$\cap$	means 'intersection'
$\cup$	means 'union'
$\varnothing$	is the empty set, containing no elements
ξ	is the universal set, containing all elements being considered
A'	is the complement of a set A
$\lvert A \rvert$	is the number of elements in a finite set A

Sets of numbers [pp. 43, 387]

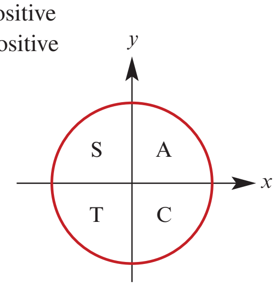
$\mathbb{N}$	is the set of natural numbers
$\mathbb{Z}$	is the set of integers
$\mathbb{Q}$	is the set of rational numbers
$\mathbb{R}$	is the set of real numbers
$\mathbb{C}$	is the set of complex numbers

Shear [p. 455] A shear moves each point in the plane by an amount proportional to its distance from an axis.

- Shear parallel to the x-axis: $(x, y) \to (x + cy, y)$
- Shear parallel to the y-axis: $(x, y) \to (x, cx + y)$

Signs of circular functions [p. 286]

1st quadrant	all are positive
2nd quadrant	sin is positive
3rd quadrant	tan is positive
4th quadrant	cos is positive



Similar figures [p. 222] Two figures are similar if we can enlarge one figure so that its enlargement is congruent to the other figure.

Similarity tests [p. 223] Two triangles are similar if one of the following conditions holds:

- **AAA** two angles of one triangle are equal to two angles of the other triangle
- **SAS** the ratios of two pairs of matching sides are equal and the included angles are equal
- **SSS** the ratios of matching sides are equal
- **RHS** the ratio of the hypotenuses of two right-angled triangles equals the ratio of another pair of sides.

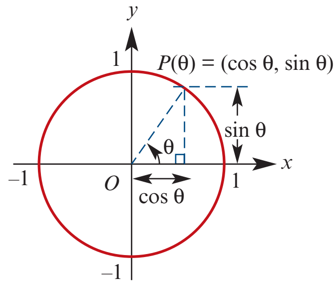
Simplest form [p. 52] a surd $\sqrt{a}$ is in simplest form if the number under the square root has no factors which are squares of a rational number

ISBN 978-1-316-63608-4

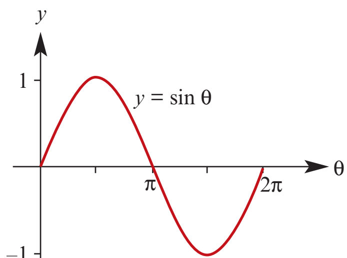
Simulation [p. 594] using technology (calculators or computers) to repeat a random process many times; e.g. random sampling

Simultaneous equations [pp. 8, 105, 440] equations of two or more lines or curves in the Cartesian plane, the solutions of which are the points of intersection of the lines or curves

Sine function [p. 283] sine θ is defined as the *y*-coordinate of the point *P* on the unit circle where *OP* forms an angle of θ radians with the positive direction of the *x*-axis.



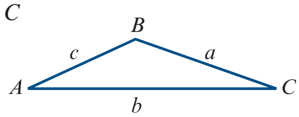
Sine function, graph [p. 283]



amplitude = 1
period = 2π

Sine rule [p. 499] For triangle *ABC*:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Speed [p. 532] the magnitude of velocity

Speed, average [p. 532]

$$\text{average speed} = \frac{\text{total distance travelled}}{\text{total time taken}}$$

Standard deviation a measure of the spread or variability of the distribution of numerical data about the mean, denoted *s* and defined by

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

Standard form [p. 5] *see* scientific notation

Subtraction of complex numbers [p. 389] If $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 - z_2 = (a - c) + (b - d)i$.

Subtraction of vectors [p. 489] If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$, then $\boldsymbol{a} - \boldsymbol{b} = (a_1 - b_1)\boldsymbol{i} + (a_2 - b_2)\boldsymbol{j}$.

Sum-to-product identities [p. 329]

- $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
- $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$
- $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
- $\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$

Supplementary angles [p. 200] two angles whose sum is 180°

Surd of order *n* [p. 51] a number of the form $\sqrt[n]{a}$, where *a* is a rational number which is not a perfect *n*th power

Surd, quadratic [p. 51] a number of the form $\sqrt{a}$, where *a* is a rational number which is not the square of another rational number

T

Tangent function [pp. 284, 294] $\tan\theta = \frac{\sin\theta}{\cos\theta}$ for $\cos\theta \neq 0$

Tangent to a circle [p. 256] a line that touches the circle at exactly one point, called the *point of contact*

Tautology [p. 630] a statement which is true under all circumstances

Transformation [p. 449] A transformation of the plane maps each point (x, y) in the plane to a new point (x', y'). We say that (x', y') is the *image* of (x, y).

Translation [p. 457] a transformation that moves each point in the plane in the same direction and over the same distance: $(x, y) \to (x + a, y + b)$

Trapezium [p. 212] a quadrilateral with at least one pair of opposite sides parallel

Triangle inequality [p. 207] If *a*, *b* and *c* are the side lengths of a triangle, then $a < b + c$, $b < c + a$ and $c < a + b$.

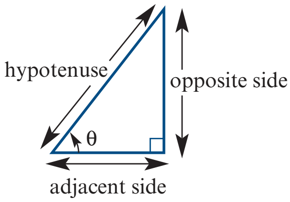
ISBN 978-1-316-63608-4

Trigonometric ratios [p. 284]

$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$

$\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

$\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$



Truth table [p. 628] gives the truth value of a compound statement for each combination of truth values of the constituent statements

U

Union of sets [p. 40] The union of two sets A and B, written $A \cup B$, is the set of all elements which are in A or B or both.

Unit vector [p. 496] a vector of magnitude 1. The unit vectors in the positive directions of the x-, y- and z-axes are $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ respectively. The unit vector in the direction of $\boldsymbol{a}$ is given by

$$\hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$$

Universal statement [pp. 181, 182] a statement claiming that a property holds for all members of a given set. Such a statement can be written using the quantifier 'for all'.

V

Vector [p. 487] a set of equivalent directed line segments

Vector quantity [p. 487] a quantity determined by its magnitude and direction; e.g. position, displacement, velocity, acceleration, force

Vectors, parallel [p. 490] Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if and only if $\boldsymbol{a} = k\boldsymbol{b}$ for some $k \in \mathbb{R} \setminus \{0\}$.

Vectors, perpendicular [p. 500] Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

Vectors, properties [pp. 487–489]

- $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$ commutative law
- $(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$ associative law
- $\boldsymbol{a} + \boldsymbol{0} = \boldsymbol{a}$ zero vector
- $\boldsymbol{a} + (-\boldsymbol{a}) = \boldsymbol{0}$ additive inverse
- $m(\boldsymbol{a} + \boldsymbol{b}) = m\boldsymbol{a} + m\boldsymbol{b}$ distributive law

Vectors, resolution [p. 503] A vector $\boldsymbol{a}$ is resolved into rectangular components by writing it as a sum of two vectors, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.

The *vector resolute* of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is given by

$$\boldsymbol{u} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$$

Velocity [p. 532] The velocity of a particle is defined as the rate of change of its position with respect to time.

Velocity, average [p. 532]

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}}$$

Velocity, instantaneous [p. 532] $v = \dfrac{dx}{dt}$

Velocity–time graph [p. 543]

- Acceleration is given by the gradient.
- Distance travelled is given by the sum of the areas of the regions between the graph and the t-axis.
- Displacement is given by the sum of the signed areas of the regions between the graph and the t-axis.

W

Weight [p. 557] On the Earth's surface, a mass of m kg has a force of m kg wt (or mg newtons) acting on it; this force is known as the weight.

Z

Zero vector, 0 [p. 489] a line segment of zero length with no direction

ISBN 978-1-316-63608-4

Cambridge University Press

Answers

Chapter 1

Exercise 1A

1 a x^7 **b** a^2 **c** x^3 **d** y^{-4}
e x^{12} **f** p^{-7} **g** $a^{-\frac{1}{6}}$ **h** a^{-8}
i y^{14} **j** x^{15} **k** a^{-12} **l** x^2
m n^2 **n** $8x^{\frac{7}{2}}$ **o** a **p** x^4
q $\frac{1}{2n^6}$ **r** $-8x^2$ **s** $a^{-2}b^5$ **t** 1

2 a 5 **b** 4 **c** $\frac{4}{3}$ **d** $\frac{1}{4}$
e $\frac{6}{7}$ **f** 3 **g** 12 **h** 16
i 27 **j** $\frac{3}{2}$ **k** 1 **l** 8

3 a 18.92 **b** 79.63 **c** 5.89
d 125 000 **e** 0.9 **f** 1.23
g 0.14 **h** 1.84 **i** 0.4

4 a a^4b^7 **b** $64a^4b^7$ **c** b
d a^6b^9 **e** $2a^4b^7$ **f** $\frac{a^2b^5}{128}$

5 2^{2n-4}

6 6^{3x}

7 a $\left(\frac{1}{2}\right)^{\frac{1}{6}}$ **b** $a^{\frac{11}{20}}$ **c** $2^{\frac{5}{6}}$
d $2^{\frac{19}{6}}$ **e** $2^{\frac{3}{5}}$

8 a $a^{\frac{1}{3}}b$ **b** $a^{\frac{5}{2}}b^{\frac{1}{2}}$ **c** $ab^{\frac{1}{5}}$
d $\left(\frac{b}{a}\right)^{\frac{1}{2}}$ **e** $a^{\frac{5}{2}}b^{\frac{1}{2}}c^{-4}$ **f** $a^{\frac{1}{5}}b^{\frac{3}{5}}$
g $a^{-4}b^{\frac{7}{2}}c^5$

Exercise 1B

1 a 4.78×10 **b** 6.728×10^3
c 7.923×10 **d** 4.358×10^4
e 2.3×10^{-3} **f** 5.6×10^{-7}
g $1.200\,034 \times 10$ **h** 5.0×10^7
i 2.3×10^{10} **j** 1.3×10^{-9}
k 1.65×10^5 **l** 1.4567×10^{-5}

2 a 2.99×10^{-23} **b** 1×10^{-8}
c 3.432×10^2 **d** 3.1536×10^7
e 6.09×10^9 **f** 3.057×10^{21}

3 a 1 390 000 000 **b** 0.000 0075
c 0.000 000 000 000 0056

4 a 4.569×10^2 **b** 3.5×10^4
c 5.6791×10^3 **d** 4.5×10^{-2}
e 9.0×10^{-2} **f** 4.5682×10^3

5 a 0.000 0567 **b** $\frac{262}{2625}$

6 a 11.8 **b** 4.76×10^7

Exercise 1C

1 a $x = \frac{8}{3}$ **b** $x = 48$ **c** $x = -\frac{20}{3}$
d $x = 63$ **e** $x = -0.7$ **f** $x = 2.4$
g $x = 0.3$ **h** $x = -6$ **i** $x = -\frac{15}{92}$
j $x = -\frac{21}{17}$

2 a $x = \frac{160}{9}$ **b** $x = 19.2$ **c** $x = -4$
d $x = \frac{80}{51}$ **e** $x = 6.75$ **f** $x = -\frac{85}{38}$
g $x = \frac{487}{13}$ **h** $x = \frac{191}{91}$

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3 **a** $x = \frac{18}{13}, y = -\frac{14}{13}$ **b** $x = \frac{16}{11}, y = -\frac{18}{11}$
c $x = 12, y = 17$ **d** $x = 8, y = 2$
e $x = 0, y = 2$ **f** $x = 1, y = 6$

Exercise 1D

1 **a** $4(x - 2) = 60;\ x = 17$
b $\left(\frac{2x + 7}{4}\right)^2 = 49;\ x = 10.5$
c $x - 5 = 2(12 - x);\ x = \frac{29}{3}$ **d** $y = 6x - 4$
e $Q = np$ **f** $R = 1.1pS$
g $\frac{60n}{5} = 2400$ **h** $a = \frac{\pi}{3}(x + 3)$

2 \$2500
3 Eight dresses and three handbags
4 8.375 m by 25.125 m
5 \$56.50
6 Nine
7 20, 34 and 17
8 Annie 165, Belinda 150, Cassie 189
9 15 km/h
10 2.04×10^{-23} g
11 30 pearls
12 Oldest \$48, Middle \$36, Youngest \$12
13 98%
14 25 students
15 After 20 minutes
16 **a** 40 minutes **b** 90 minutes **c** 20 minutes
17 200 km
18 39 km/h

Exercise 1E

1 140.625 km **2** 50 **3** 10 000 adults
4 Men \$220; boys \$160 **5** 127 and 85
6 252 litres 40% and 448 litres 15%
7 120 and 100; 60 **8** \$370 588
9 500 adults, 1100 students

Exercise 1F

1 **a** 25 **b** 330 **c** 340.47 **d** 1653.48
e 612.01 **f** 77.95 **g** 2.42 **h** 2.1
i 9.43 **j** 9.54

2 **a** $a = \frac{v - u}{t}$ **b** $\ell = \frac{2S}{n} - a$ **c** $b = \frac{2A}{h}$
d $I = \pm\sqrt{\frac{P}{R}}$ **e** $a = \frac{2(s - ut)}{t^2}$
f $v = \pm\sqrt{\frac{2E}{m}}$ **g** $h = \frac{Q^2}{2g}$ **h** $x = \frac{-z}{y}$
i $x = \frac{-b(c + y)}{a - c}$ **j** $x = \frac{-b(c + 1)}{m - c}$

3 **a** 82.4°F **b** $C = \frac{5(F - 32)}{9}$; 57.22°C
4 **a** 1080° **b** $n = \frac{S}{180} + 2$; 9 sides
5 **a** 115.45 cm^3 **b** 12.53 cm **c** 5.00 cm
6 **a** 66.5 **b** 4 **c** 11

Exercise 1G

1 **a** $\frac{13x}{6}$ **b** $\frac{5a}{4}$ **c** $\frac{-h}{8}$ **d** $\frac{5x - 2y}{12}$
e $\frac{3y + 2x}{xy}$ **f** $\frac{7x - 2}{x(x - 1)}$
g $\frac{5x - 1}{(x - 2)(x + 1)}$ **h** $\frac{-7x^2 - 36x + 27}{2(x + 3)(x - 3)}$
i $\frac{4x + 7}{(x + 1)^2}$ **j** $\frac{5a^2 + 8a - 16}{8a}$
k $\frac{4(x^2 + 1)}{5x}$ **l** $\frac{2x + 5}{(x + 4)^2}$
m $\frac{3x + 14}{(x - 1)(x + 4)}$ **n** $\frac{x + 14}{(x - 2)(x + 2)}$
o $\frac{7x^2 + 28x + 16}{(x - 2)(x + 2)(x + 3)}$ **p** $\frac{(x - y)^2 - 1}{x - y}$
q $\frac{4x + 3}{x - 1}$ **r** $\frac{3 - 2x}{x - 2}$

2 **a** $2xy^2$ **b** $\frac{xy}{8}$ **c** $\frac{2}{x}$ **d** $\frac{x}{y^2}$
e $\frac{a}{3}$ **f** $\frac{1}{2x}$ **g** $\frac{x - 1}{x + 4}$ **h** $x + 2$
i $\frac{x - 1}{x}$ **j** $\frac{a}{4b}$ **k** $\frac{2x}{x + 2}$ **l** $\frac{x - 1}{4x}$
m $\frac{x + 1}{2x}$ **n** $\frac{1}{3}x(x + 3)$
o $\frac{x - 2}{3x(3x - 2)(x + 5)}$

3 **a** $\frac{3}{x - 3}$ **b** $\frac{4x - 14}{x^2 - 7x + 12}$
c $\frac{5x - 1}{x^2 + x - 12}$ **d** $\frac{2x^2 + 10x - 6}{x^2 + x - 12}$
e $\frac{2x - 9}{x^2 - 10x + 25}$ **f** $\frac{5x - 8}{(x - 4)^2}$
g $\frac{1}{3 - x}$ **h** $\frac{23 - 3x}{x^2 + x - 12}$
i $\frac{5x^2 - 3x}{x^2 - 9}$ **j** $\frac{11 - 2x}{x^2 - 10x + 25}$
k $\frac{12}{(x - 6)^3}$ **l** $\frac{9x - 25}{x^2 - 7x + 12}$

4 **a** $\frac{3 - x}{\sqrt{1 - x}}$ **b** $\frac{2\sqrt{x - 4} + 6}{3\sqrt{x - 4}}$ **c** $\frac{5}{\sqrt{x + 4}}$
d $\frac{x + 7}{\sqrt{x + 4}}$ **e** $-\frac{12x^2}{\sqrt{x + 4}}$ **f** $\frac{9x^2(x + 2)}{2\sqrt{x + 3}}$

5 **a** $\frac{6x - 4}{(6x - 3)^{\frac{2}{3}}}$ **b** $\frac{3}{(2x + 3)^{\frac{2}{3}}}$ **c** $\frac{3 - 3x}{(x - 3)^{\frac{2}{3}}}$

ISBN 978-1-316-63608-4

Cambridge University Press

Exercise 1H

1 a $x = \frac{m-n}{a}$ **b** $x = \frac{b}{b-a}$ **c** $x = -\frac{bc}{a}$

d $x = \frac{5}{p-q}$ **e** $x = \frac{m+n}{n-m}$ **f** $x = \frac{ab}{1-b}$

g $x = 3a$ **h** $x = -mn$ **i** $x = \frac{a^2-b^2}{2ab}$

j $x = \frac{p-q}{p+q}$ **k** $x = \frac{3ab}{b-a}$ **l** $x = \frac{1}{3a-b}$

m $x = \frac{p^2+p^2t+t^2}{q(p+t)}$ **n** $x = -\frac{5a}{3}$

4 a $x = \frac{d-bc}{1-ab}, y = \frac{c-ad}{1-ab}$

b $x = \frac{a^2+ab+b^2}{a+b}, y = \frac{ab}{a+b}$

c $x = \frac{t+s}{2a}, y = \frac{t-s}{2b}$ **d** $x = a+b, y = a-b$

e $x = c, y = -a$ **f** $x = a+1, y = a-1$

5 a $s = a(2a+1)$ **b** $s = \frac{2a^2}{1-a}$

c $s = \frac{a^2+a+1}{a(a+1)}$ **d** $s = \frac{a}{(a-1)^2}$

e $s = 3a^3(3a+1)$ **f** $s = \frac{3a}{a+2}$

g $s = 2a^2 - 1 + \frac{1}{a^2}$ **h** $s = \frac{5a^2}{a^2+6}$

Exercise 1I

1 a $x = a-b$ **b** $x = 7$

c $x = -\frac{a \pm \sqrt{a^2+4ab-4b^2}}{2}$ **d** $x = \frac{a+c}{2}$

2 a $(x-1)(x+1)(y-1)(y+1)$

b $(x-1)(x+1)(x+2)$

c $(a^2-12b)(a^2+4b)$

d $(a-c)(a-2b+c)$

3 a $x = \frac{a+b+c}{a+b}, y = \frac{a+b}{c}$

b $x = \frac{-(a-b-c)}{a+b-c}, y = \frac{a-b+c}{a+b-c}$

Chapter 1 review

Short-answer questions

1 a x^{12} **b** y^{-9} **c** $-15x^{\frac{11}{2}}$ **d** x^{-1}

2 3.22×10^{11}

3 a $\frac{2x+y}{10}$ **b** $\frac{4y-7x}{xy}$

c $\frac{7x-1}{(x+2)(x-1)}$ **d** $\frac{7x+20}{(x+2)(x+4)}$

e $\frac{13x^2+2x+40}{2(x+4)(x-2)}$ **f** $\frac{3(x-4)}{(x-2)^2}$

4 a $\frac{2}{x}$ **b** $\frac{x-4}{4x}$ **c** $\frac{x^2-4}{3}$ **d** $4x^2$

5 a 2×10^6 photos, i.e. 2 million photos

b 2×10^5 seconds ($\approx$ 55.6 hours)

6 12

7 88 classical, 80 blues, 252 heavy metal

8 a 300π cm^3 **b** $h = \frac{V}{\pi r^2}$; $\frac{117}{5\pi}$ cm

c $r = \sqrt{\frac{V}{\pi h}}$; $\sqrt{\frac{128}{\pi}}$ cm $= \frac{8\sqrt{2}}{\sqrt{\pi}}$ cm

9 a $x = \frac{b}{a+y}$ **b** $x = \frac{a+b}{c}$

c $x = \frac{2ab}{b-a}$ **d** $x = \frac{ab+b^2d-d^2}{d(a+b)}$

10 a $\frac{p^2+q^2}{p^2-q^2}$ **b** $\frac{x+y}{x(y-x)}$

c $(x-2)(2x-1)$ **d** $\frac{2}{a}$

11 A 36; B 12; C 2

12 a $a = 8, b = 18$ **b** $x = p+q, y = 2q$

13 $x = 3.5$

14 a $4n^2k^2$ **b** $\frac{40cx^2}{ab^2}$

15 $x = -1$

Multiple-choice questions

1 A **2** A **3** C **4** A **5** B
6 E **7** B **8** B **9** B **10** B

Extended-response questions

1 a $\frac{5x}{4}$ hours **b** $\frac{4x}{7}$ hours **c** $\frac{19x}{28}$ hours

d i $x = \frac{14}{19} \approx 0.737$

ii Jack $\frac{140}{19} \approx 7$ km; Benny $\frac{560}{19} \approx 29$ km

2 a 18 000 cm^3 per minute

b $V = 18\,000t$ **c** $h = \frac{45t}{4\pi}$ **d** After

3 a Thomas a; George $\frac{3a}{2}$; Sally $a - 18$;

Zeb $\frac{a}{3}$; Henry $\frac{5a}{6}$

b $\frac{3a}{2} + a - 18 + \frac{a}{3} = a + \frac{5a}{6} + 6$

c $a = 24$; Thomas 24; Henry 20; George 36; Sally 6; Zeb 8

4 a 1.9×10^{-8} N **b** $m_1 = \frac{Fr^2 10^{11}}{6.67 m_2}$

c 9.8×10^{24} kg

5 a $V = (1.8 \times 10^7)d$ **b** 5.4×10^8 m^3

c $k = 9.81 \times 10^3$ **d** 1.325×10^{15} J

e 1202 days (to the nearest day)

6 $\frac{10\sqrt{3}}{3}$ cm

7 $-40°$

8 $\frac{240}{11}$ km/h

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9 a $h = 20 - r$

b i $V = \left(20r^2 - \dfrac{r^3}{3}\right)\pi$

ii $r = 5.94$ cm; $h = 14.06$ cm

10 a $\dfrac{2}{3}$ litre from A; $\dfrac{1}{3}$ litre from B

b 600 mL from A; 400 mL from B

c $\dfrac{(p-q)(n+m)}{2(np-qm)}$ litres from A,

$\dfrac{(n-m)(p+q)}{2(np-qm)}$ litres from B,

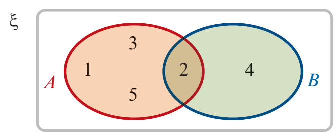
where $\dfrac{n}{m} \neq \dfrac{q}{p}$ and one of $\dfrac{n}{m}$ or $\dfrac{q}{p}$ is ≥ 1 and the other is ≤ 1

11 a $h = 2(10 - r)$ **b** $V = 2\pi r^2(10 - r)$

c $r = 3.4986$, $h = 13.0029$ or $r = 9.0224$, $h = 1.9551$

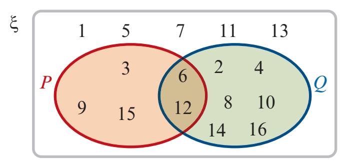
Chapter 2

Exercise 2A

1 

a {4} **b** {1, 3, 5} **c** {1, 2, 3, 4, 5} = ξ

d ∅ **e** ∅

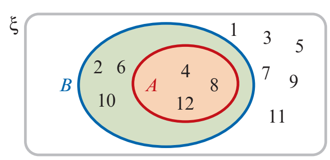
2 

a {1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16}

b {1, 3, 5, 7, 9, 11, 13, 15}

c {2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16}

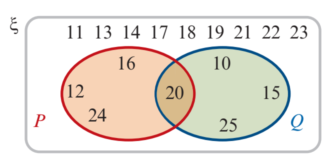
d {1, 5, 7, 11, 13} **e** {1, 5, 7, 11, 13}

3 

a {1, 2, 3, 5, 6, 7, 9, 10, 11}

b {1, 3, 5, 7, 9, 11} **c** {2, 4, 6, 8, 10, 12}

d {1, 3, 5, 7, 9, 11} **e** {1, 3, 5, 7, 9, 11}

4 

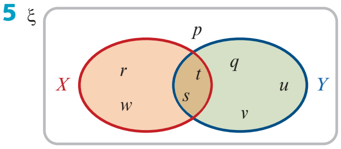
a {10, 11, 13, 14, 15, 17, 18, 19, 21, 22, 23, 25}

b {11, 12, 13, 14, 16, 17, 18, 19, 21, 22, 23, 24}

c {10, 12, 15, 16, 20, 24, 25}

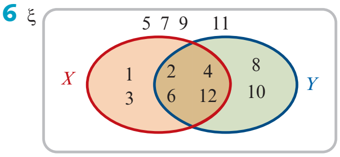
d {11, 13, 14, 17, 18, 19, 21, 22, 23}

e {11, 13, 14, 17, 18, 19, 21, 22, 23}

5 

a $\{p, q, u, v\}$ **b** $\{p, r, w\}$ **c** $\{p\}$

d $\{p, q, r, u, v, w\}$ **e** $\{q, r, s, t, u, v, w\}$ **f** $\{p\}$

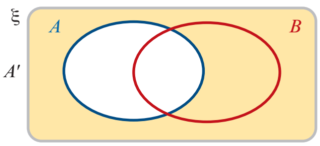
6 

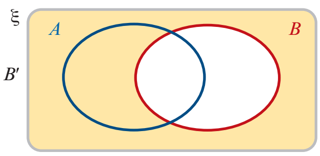
a {5, 7, 8, 9, 10, 11} **b** {1, 3, 5, 7, 9, 11}

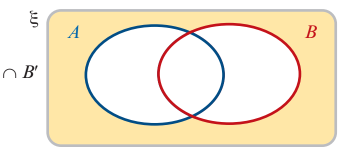
c {1, 3, 5, 7, 8, 9, 10, 11}

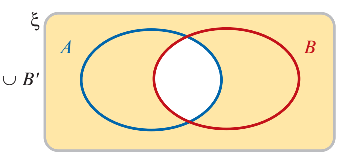
d {1, 3, 5, 7, 8, 9, 10, 11}

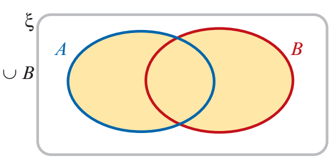
e {1, 2, 3, 4, 6, 8, 10, 12} **f** {5, 7, 9, 11}

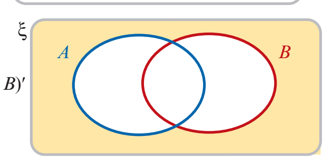
7 a A' 

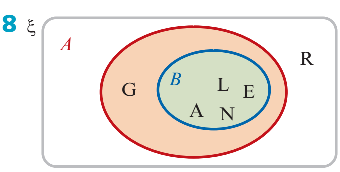
b B' 

c $A' \cap B'$ 

d $A' \cup B'$ 

e $A \cup B$ 

f $(A \cup B)'$ 

8 

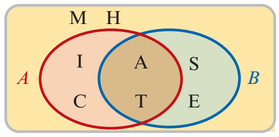
a {R} **b** {G, R} **c** {L, E, A, N}

d {A, N, G, E, L} **e** {R} **f** {G, R}

Answers 2A

ISBN 978-1-316-63608-4

9 ξ



a {E, H, M, S} **b** {C, H, I, M}
c {A, T} **d** {H, M} **e** {C, E, H, I, M, S}
f {H, M}

Exercise 2B

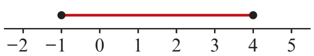
1 a Yes **b** Yes **c** Yes

2 a No **b** No **c** No

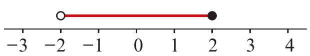
3 a $\frac{9}{20}$ **b** $\frac{3}{11}$ **c** $\frac{3}{25}$ **d** $\frac{2}{7}$ **e** $\frac{4}{11}$ **f** $\frac{2}{9}$

4 a $0.\dot{2}8571\dot{4}$ **b** $0.\dot{4}\dot{5}$ **c** 0.35
d $0.\dot{3}0769\dot{2}$ **e** $0.\dot{0}58823529411764\dot{7}$

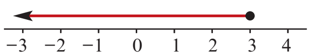
5 a



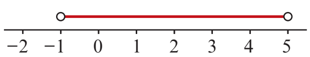
b



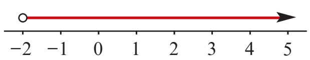
c



d



e



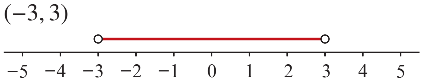
6 a $(-\infty, 3)$ **b** $[-3, \infty)$ **c** $(-\infty, -3]$
d $(5, \infty)$ **e** $[-2, 3)$ **f** $[-2, 3]$
g $(-2, 3]$ **h** $(-5, 3)$

Exercise 2C

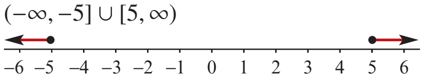
1 a 8 **b** 8 **c** 2 **d** −2 **e** −2 **f** 4

2 a 3, −1 **b** $\frac{7}{2}, -\frac{1}{2}$ **c** $\frac{12}{5}, -\frac{6}{5}$ **d** 12, −6
e −1, 7 **f** $\frac{4}{3}, -4$ **g** $-\frac{2}{5}, -4$

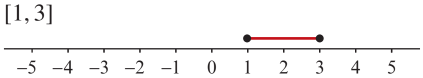
3 a $(-3, 3)$



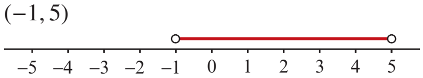
b $(-\infty, -5] \cup [5, \infty)$



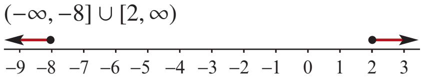
c $[1, 3]$



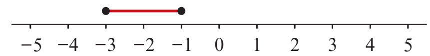
d $(-1, 5)$



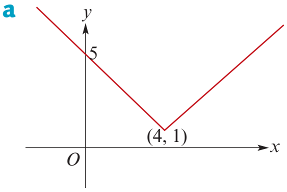
e $(-\infty, -8] \cup [2, \infty)$



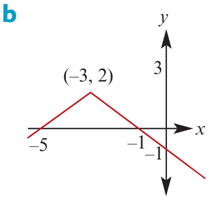
f $[-3, -1]$



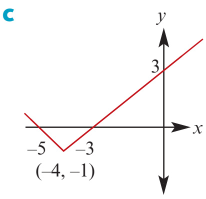
4 a Range $[1, \infty)$



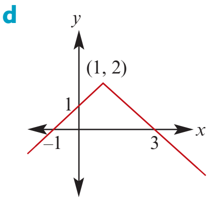
b Range $(-\infty, 2]$



c Range $[-1, \infty)$



d Range $(-\infty, 2]$



5 a $\{x : -5 \le x \le 5\}$
b $\{x : x \le -2\} \cup \{x : x \ge 2\}$
c $\{x : 1 \le x \le 2\}$ **d** $\{x : -\frac{1}{5} < x < 1\}$
e $\{x : x \le -4\} \cup \{x : x \ge 10\}$
f $\{x : 1 \le x \le 3\}$

6 a $x \le -2$ **b** $x = -9$ or $x = 11$
c $x = -\frac{5}{4}$ or $x = \frac{15}{4}$

7 $a = 1,\ b = 1$

Exercise 2D

1 a $2\sqrt{2}$ **b** $2\sqrt{3}$ **c** $3\sqrt{3}$ **d** $5\sqrt{2}$
e $3\sqrt{5}$ **f** $11\sqrt{10}$ **g** $7\sqrt{2}$ **h** $6\sqrt{3}$
i 5 **j** $5\sqrt{3}$ **k** $16\sqrt{2}$

2 a $3\sqrt{2}$ **b** $6\sqrt{3}$ **c** $4\sqrt{7}$
d $5\sqrt{10}$ **e** $28\sqrt{2}$ **f** 0

3 a $11\sqrt{3} + \sqrt{14}$ **b** $5\sqrt{7}$
c 0 **d** $\sqrt{2} + \sqrt{3}$
e $5\sqrt{2} + 15\sqrt{3}$ **f** $\sqrt{2} + \sqrt{5}$

4 a $\frac{\sqrt{5}}{5}$ **b** $\frac{\sqrt{7}}{7}$ **c** $\frac{-\sqrt{2}}{2}$
d $\frac{2\sqrt{3}}{3}$ **e** $\frac{\sqrt{6}}{2}$ **f** $\frac{\sqrt{2}}{4}$

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g $\sqrt{2} - 1$ **h** $2 + \sqrt{3}$ **i** $\dfrac{4 + \sqrt{10}}{6}$

j $\sqrt{6} - 2$ **k** $\dfrac{\sqrt{5} + \sqrt{3}}{2}$ **l** $3(\sqrt{6} + \sqrt{5})$

m $3 + 2\sqrt{2}$

5 a $6 + 4\sqrt{2}$ **b** $9 + 4\sqrt{5}$ **c** $-1 + \sqrt{2}$

d $4 - 2\sqrt{3}$ **e** $\dfrac{2\sqrt{3}}{9}$ **f** $\dfrac{8 + 5\sqrt{3}}{11}$

g $\dfrac{3 + \sqrt{5}}{2}$ **h** $\dfrac{6 + 5\sqrt{2}}{14}$

6 a $4a - 4\sqrt{a} + 1$

b $3 + 2x + 2\sqrt{(x+1)(x+2)}$

7 a $5 - 3\sqrt{2}$ **b** $7 - 2\sqrt{6}$

8 a $\dfrac{3}{\sqrt{2}}$ **b** $\dfrac{\sqrt{5}}{2}$ **c** $\dfrac{\sqrt{5}}{5}$ **d** $\dfrac{8}{\sqrt{3}}$

9 a $b = 0, c = -3$ **b** $b = 0, c = -12$

c $b = -2, c = -1$ **d** $b = -4, c = 1$

e $b = -6, c = 1$

f $b = -7 + 5\sqrt{5}, c = -58 - 13\sqrt{5}$

10 $\dfrac{3\sqrt{2} + 2\sqrt{3} - \sqrt{30}}{12}$

11 b $-1 - 2^{\frac{1}{3}} - 2^{\frac{2}{3}}$

Exercise 2E

1 a $2^2 \times 3 \times 5$ **b** $2^2 \times 13^2$

c $2^2 \times 3 \times 19$ **d** $2^2 \times 3^2 \times 5^2$

e $2^2 \times 3^2 \times 7$ **f** $2^2 \times 3^2 \times 5^2 \times 7$

g $2^5 \times 3 \times 5 \times 11 \times 13$

h $2^5 \times 3 \times 7 \times 11 \times 13$

i $2^5 \times 7 \times 11 \times 13$

j $2^5 \times 7 \times 11 \times 13 \times 17$

2 a 1 **b** 27 **c** 5 **d** 31 **e** 6

3 a 18: 1, 2, 3, 6, 9, 18; 36: 1, 2, 3, 4, 6, 9, 18, 36

b 36 is a square number (36 = 6 × 6)

c 121 has factors 1, 11 and 121

4 5, 14 and 15 **5** $n = 121$

6 105 **7** 8

8 4 **9** 1:12 p.m.

10 600 and 108 000;
2400 and 27 000;
3000 and 21 600;
5400 and 12 000

Exercise 2F

1 a $x = 2 + 3t,\ y = -7 - 11t,\ t \in \mathbb{Z}$

b $x = 1 + 7t,\ y = -2t,\ t \in \mathbb{Z}$

c $x = 12 + 21t,\ y = -3 - 8t,\ t \in \mathbb{Z}$

d $x = 2 + 3t,\ y = -7 - 11t,\ t \in \mathbb{Z}$

e $x = 11 + 7t,\ y = -2t,\ t \in \mathbb{Z}$

f $x = 11 + 7t,\ y = -2t,\ t \in \mathbb{Z}$

2 $x = 4, y = 2$

4 a $8s + 6b = 54$

b $s = 6, b = 1$ or $s = 3, b = 5$

5

50c	0	2	4	6	8	10
20c	25	20	15	10	5	0

6 $x = 17, y = 20$;
Solutions of $19x + 98y = 1998$ with $x, y \in \mathbb{N}$ are $x = 100, y = 1$ and $x = 2, y = 20$

7 (10, 0), (9, 5), (8, 10), (7, 15), (6, 20), (5, 25), (4, 30), (3, 35), (2, 40), (1, 45), (0, 50)

8 $63x - 23y = -7$;
$x = 5 + 23t,\ y = 14 + 63t$ for $t \in \mathbb{N} \cup \{0\}$

9 5 and 15

10 20; $20 + 77t$ for $t \in \mathbb{N} \cup \{0\}$

11 Pour two full 5 litre jugs into a container and remove one 3 litre jugful

12 All amounts greater than or equal to 3c, except 4c and 7c

13 8 type A, 16 type B

14 The highest common factor of 6 and −9 is 3, which does not divide 10

15 221 **16** 52 and 97

Exercise 2G

1 a $43 = 8 \times 5 + 3$;
$\text{HCF}(43, 5) = \text{HCF}(5, 3) = 1$

b $39 = 3 \times 13 + 0$;
$\text{HCF}(39, 13) = \text{HCF}(13, 0) = 13$

c $37 = 2 \times 17 + 3$;
$\text{HCF}(37, 17) = \text{HCF}(17, 3) = 1$

d $128 = 16 \times 8 + 0$;
$\text{HCF}(128, 16) = \text{HCF}(16, 0) = 16$

3 a 1 **b** 27 **c** 6 **d** 5

4 a $x = 44 + 393t,\ y = -15 - 134t,\ t \in \mathbb{Z}$

b $x = -1 + 4t,\ y = 1 - 3t,\ t \in \mathbb{Z}$

c $x = 2 + 4t,\ y = 118 - 3t,\ t \in \mathbb{Z}$

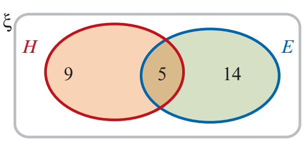
d $x = 1 + 5t,\ y = -7 + 3t,\ t \in \mathbb{Z}$

e $x = 107 + 224t,\ y = -32 - 67t,\ t \in \mathbb{Z}$

f $x = -37 + 336t,\ y = 25 - 227t,\ t \in \mathbb{Z}$

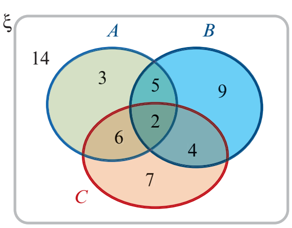
Exercise 2H

1 a



b i 19 **ii** 9 **iii** 23

2 a



b i 23 **ii** 37 **iii** 9

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3 20%

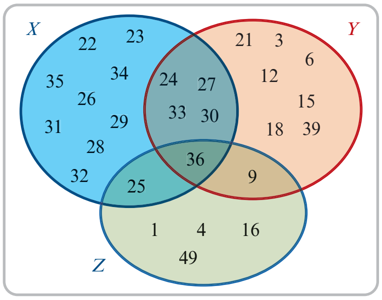
4 7

5 a 5 **b** 10

6 45

7 a $x = 5$ **b** 16 **c** 0

8 a ξ



b i $X \cap Y \cap Z = \{36\}$ **ii** $|X \cap Y| = 5$

9 31 students; 15 black, 12 green, 20 red

10 $|M \cap F| = 11$ **11** 1

12 $x = 6$; 16 students **13** 102 students

Chapter 2 review

Short-answer questions

1 a $\frac{7}{90}$ **b** $\frac{5}{11}$ **c** $\frac{1}{200}$

d $\frac{81}{200}$ **e** $\frac{4}{15}$ **f** $\frac{6}{35}$

2 $2^3 \times 3^2 \times 7$

3 a $n = \pm 2$ or $n = \pm 4$

b i $x = \pm 1$ **ii** $x \le 0$

c $x < -1$ or $x > 1$

4 a $\frac{2\sqrt{6} - \sqrt{2}}{2}$ **b** $4\sqrt{5} + 9$ **c** $2\sqrt{6} + 5$

5 $-23 - 12\sqrt{3}$

6 a $2\sqrt{6} + 6$ **b** $\frac{a - \sqrt{a^2 - b^2}}{b}$

7 a 15 **b** 15

8 a 1 **b** 22 **c** 22

9 5 **10** 2 cm^2

11 $-15\sqrt{7}$ **12** $x = \pm 2$

13 $\sqrt{5} - \sqrt{6}$ **14** $\frac{51\sqrt{3}}{5}$

15 a 57 **b** 3 **c** 32

16 $2\sqrt{2} + 3$

17 HCF(1885, 365) = 5

18 a $x = -4 + 43t,\ y = 1 - 9t,\ t \in \mathbb{Z}$

b No solutions for $x, y \in \mathbb{N}$

20 HCF(10 659, 12 121) = 17

21 a $x = 3 + 7t,\ y = -2 - 5t,\ t \in \mathbb{Z}$

b $x = 6 + 7t,\ y = 10 - 5t,\ t \in \mathbb{Z}$

c $x = 3 + 7t,\ y = -2 - 5t,\ t \in \mathbb{Z}^-$

22 Tom is 36 and Fred is 27

Multiple-choice questions

1 A **2** D **3** D **4** D **5** C
6 D **7** B **8** B **9** C **10** A
11 D **12** D **13** B

Extended-response questions

1 c i $\sqrt{11} + \sqrt{3}$

ii $2\sqrt{2} - \sqrt{7}$ or $\sqrt{7} - 2\sqrt{2}$

iii $3\sqrt{3} - 2\sqrt{6}$ or $2\sqrt{6} - 3\sqrt{3}$

2 a $a = 6,\ b = 5$ **b** $p = 26,\ q = 16$

c $a = -1,\ b = \frac{2}{3}$

d i $\frac{12\sqrt{3} - 19}{71}$ **ii** $3 \pm \sqrt{3}$ **iii** $\frac{1 \pm \sqrt{3}}{2}$

e $\mathbb{Q} = \{a + 0\sqrt{3} : a \in \mathbb{Q}\}$

3 d $x = \pm 2$

5 a $b = -4,\ c = 1$ **b** $2 + \sqrt{3}$

6 a (20, 21, 29)

7 a i 4 factors **ii** 8 factors

b $n + 1$ factors

c i 32 factors **ii** $(n + 1)(m + 1)$ factors

d $(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_n + 1)$ factors

e 24

8 a $1080 = 2^3 \times 3^3 \times 5$;
$25\,200 = 2^4 \times 3^2 \times 5^2 \times 7$

b 75 600

d i 3470, 3472, 3474, 3476

ii 1735, 1736, 1737, 1738

9 a i Region 8

ii Male, red hair, blue eyes

iii Male, not red hair, blue eyes

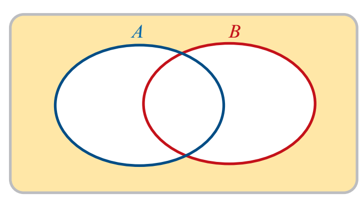
b i 5 **ii** 182

10 a i Students shorter than or equal to 180 cm

ii Students who are female or taller than 180 cm

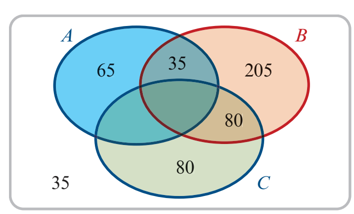
iii Students who are male and shorter than or equal to 180 cm

b



$(A \cup B)' = A' \cap B'$ is shaded

11 a



$|A \cap C| = 0$

b 160 **c** 65 **d** 0

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12 **a** $6 \times 5c + 1 \times 8c$
b $15 \times 8c$, $8 \times 5c + 10 \times 8c$, $16 \times 5c + 5 \times 8c$, $24 \times 5c$

Chapter 3

Exercise 3A

1 $a = 10,\ b = 0,\ c = -7$
2 $a = 1,\ b = -2$
3 $a = 2,\ b = -1,\ c = 7$
4 $a = 2,\ b = 1,\ c = 3$
5 $(x+2)^2 - 4(x+2) + 4$
6 $(x+1)^3 - 3(x+1)^2 + 3(x+1) - 1$
7 $a = 1,\ b = -2,\ c = -1$
8 **a** It is impossible to find a, b and c such that $a = 3$, $3ab = -9$, $3ab^2 = 8$ and $ab^3 + c = 2$
b $a = 3,\ b = -1,\ c = 5$
9 $a = 1,\ b = -6,\ c = 7,\ d = -1$
10 **a** If $a = -\frac{5}{3}b$ and $a = -3b$, then both a and b are zero, but then $a + b = 1$ is not satisfied
b $(n+1)(n+2) - 3(n+1) + 1$
11 **a** $ax^2 + 2abx + ab^2 + c$
b $a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$
13 $a = -3,\ b = -\frac{1}{3},\ c = 3$ or $a = -\frac{1}{3},\ b = -3,\ c = 3$
14 $a = 3,\ b = -3,\ c = 1$
15 If $c = 5$, then $a = 1$ and $b = -5$; if $c = -27$, then $a = -3$ and $b = 3$

Exercise 3B

1 **a** $x = -1$ **b** $x = 3$
c $x = 1 \pm \frac{\sqrt{30}}{5}$ **d** $x = 1 \pm \frac{\sqrt{2}}{2}$
e $x = -1 \pm \frac{3\sqrt{2}}{2}$ **f** $x = \frac{-13 \pm \sqrt{145}}{12}$
2 **a** $m > \frac{9}{4}$ **b** $m < \frac{25}{4}$ **c** $m = -\frac{25}{32}$
d $m \le -6$ or $m \ge 6$ **e** $-4 < m < 4$
f $m = 0$ or $m = -16$
3 **a** $x = \frac{1 \pm \sqrt{32t+1}}{4},\ t \ge \frac{-1}{32}$
b $x = \frac{-1 \pm \sqrt{t+3}}{2},\ t \ge -3$
c $x = \frac{-2 \pm \sqrt{5t-46}}{5},\ t \ge \frac{46}{5}$
d $x = -2 \pm \frac{\sqrt{5t(t-2)}}{t},\ t < 0$ or $t \ge 2$
4 **a** $x = \frac{-p \pm \sqrt{p^2+64}}{2}$
b $p = 0$ and $p = 6$
5 **a** $\Delta = (3p-4)^2$
b $p = \frac{4}{3}$
c **i** $x = 1$ or $x = \frac{1}{2}$ **ii** $x = 1$ or $x = 2$
iii $x = 1$ or $x = -\frac{5}{2}$
6 **a** $\Delta = 16(2p-3)^2$
b $p = \frac{3}{2}$
c **i** $x = \frac{3}{2}$ or $x = \frac{1}{2}$ **ii** $x = \frac{1}{2}$ or $x = \frac{3}{10}$
iii $x = \frac{1}{2}$ or $x = -\frac{3}{14}$
7 $x = 2$
8 side length 37.5 cm
9 **a** $x = 4$ or $x = 36$ **b** $x = 16$ **c** $x = 49$
d $x = 1$ or $x = 512$ **e** $x = 27$ or $x = -8$
f $x = 16$ or $x = 625$
10 $a = 3,\ b = -\frac{5}{6},\ c = -\frac{13}{12}$; Minimum $-\frac{13}{12}$
12 $x = 1$ or $x = \frac{a-b}{b-c}$
13 $m = 8$
14 **a** $16[(a-c)^2 + 2b^2] \ge 0$
b $a = c$ and $b = 0$
15 $-8 < k < 0$
16 $p = 10$

Exercise 3C

1 **a** $\frac{18}{x(x+3)}$ **b** $x = -6$ or $x = 3$
2 $x = -30$ or $x = 25$
3 17 and 19
4 **a** $\frac{40}{x}$ hours **b** $\frac{40}{x-2}$ hours **c** 10 km/h
5 **a** Car $\frac{600}{x}$ km/h; Plane $\left(\frac{600}{x} + 220\right)$ km/h
b Car 80 km/h; Plane 300 km/h
6 $x = 20$
7 6 km/h
8 **a** $x = 50$ **b** 72 minutes
9 Slow train 30 km/h; Express train 50 km/h
10 60 km/h
11 Small pipe 25 minutes; Large pipe 20 minutes
12 Each pipe running alone takes 14 minutes
13 Rail 43 km/h; Sea 18 km/h
14 22 km
15 10 litres
16 32.23 km/h, 37.23 km/h

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17 a $a + \sqrt{a^2 - 24a}$ minutes, $a - 24 + \sqrt{a^2 - 24a}$ minutes

b **i** 84 minutes, 60 minutes
ii 48 minutes, 24 minutes
iii 36 minutes, 12 minutes
iv 30 minutes, 6 minutes

18 a 120 km **b** 20 km/h, 30 km/h

Exercise 3D

1 a $\frac{2}{x-1} + \frac{3}{x+2}$ **b** $\frac{1}{x+1} - \frac{2}{2x+1}$

c $\frac{2}{x+2} + \frac{1}{x-2}$ **d** $\frac{1}{x+3} + \frac{3}{x-2}$

e $\frac{3}{5(x-4)} - \frac{8}{5(x+1)}$

2 a $\frac{2}{x-3} + \frac{9}{(x-3)^2}$

b $\frac{4}{1+2x} + \frac{2}{1-x} + \frac{3}{(1-x)^2}$

c $\frac{-4}{9(x+1)} + \frac{4}{9(x-2)} + \frac{2}{3(x-2)^2}$

3 a $\frac{-2}{x+1} + \frac{2x+3}{x^2+x+1}$ **b** $\frac{x+1}{x^2+2} + \frac{2}{x+1}$

c $\frac{x-2}{x^2+1} - \frac{1}{2(x+3)}$

4 $3 + \frac{3}{x-1} + \frac{2}{x-2}$

5 It is impossible to find A and C such that $A = 0$, $C - 2A = 2$ and $A + C = 10$

6 a $\frac{1}{2(x-1)} - \frac{1}{2(x+1)}$ **b** $\frac{2}{5(x-2)} + \frac{3}{5(x+3)}$

c $\frac{1}{x-2} + \frac{2}{x+5}$

d $\frac{2}{5(2x-1)} - \frac{1}{5(x+2)}$

e $\frac{3}{3x-2} - \frac{1}{2x+1}$ **f** $\frac{2}{x-1} - \frac{2}{x}$

g $\frac{1}{x} + \frac{3-x}{x^2+1}$ **h** $\frac{2}{x} + \frac{x}{x^2+4}$

i $\frac{1}{4(x-4)} - \frac{1}{4x}$ **j** $\frac{7}{4(x-4)} - \frac{3}{4x}$

k $x + \frac{1}{x} - \frac{1}{x-1}$ **l** $-x - 1 - \frac{3}{x} - \frac{1}{2-x}$

m $\frac{2}{3(x+1)} + \frac{x-4}{3(x^2+2)}$

n $\frac{2}{3(x-2)} + \frac{1}{3(x+1)} - \frac{1}{(x+1)^2}$

o $\frac{2}{x} + \frac{1}{x^2+4}$ **p** $\frac{8}{2x+3} - \frac{5}{x+2}$

q $\frac{26}{9(x+2)} + \frac{1}{9(x-1)} - \frac{1}{3(x-1)^2}$

r $\frac{16}{9(2x+1)} - \frac{8}{9(x-1)} + \frac{4}{3(x-1)^2}$

s $x - 2 + \frac{1}{4(x+2)} + \frac{3}{4(x-2)}$

t $x - \frac{1}{x+1} + \frac{2}{x-1}$ **u** $\frac{3}{x+1} - \frac{7}{3x+2}$

Exercise 3E

1 a $(1,1), (0,0)$ **b** $(0,0), \left(\frac{1}{2}, \frac{1}{2}\right)$

c $\left(\frac{3+\sqrt{13}}{2}, 4+\sqrt{13}\right), \left(\frac{3-\sqrt{13}}{2}, 4-\sqrt{13}\right)$

2 a $(13,3), (3,13)$ **b** $(10,5), (5,10)$

c $(-8,-11), (11,8)$ **d** $(9,4), (4,9)$

e $(9,5), (-5,-9)$

3 a $(11,17), (17,11)$ **b** $(37,14), (14,37)$

c $(14,9), (-9,-14)$

4 $(0,0), (2,4)$

5 $\left(\frac{5+\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}\right), \left(\frac{5-\sqrt{5}}{2}, \frac{5-\sqrt{5}}{2}\right)$

6 $\left(\frac{15}{2}, \frac{5}{2}\right), (3,1)$

7 $\left(\frac{-130+80\sqrt{2}}{41}, \frac{60+64\sqrt{2}}{41}\right)$, $\left(\frac{-130-80\sqrt{2}}{41}, \frac{60-64\sqrt{2}}{41}\right)$

8 $\left(\frac{1+\sqrt{21}}{2}, \frac{-1-\sqrt{21}}{2}\right), \left(\frac{1-\sqrt{21}}{2}, \frac{-1+\sqrt{21}}{2}\right)$

9 $\left(\frac{4}{9}, 2\right)$

10 $\left(\frac{-6\sqrt{5}}{5}, \frac{3\sqrt{5}}{5}\right)$

11 $\left(-2, \frac{1}{2}\right)$

12 $(0,-1), (3,2)$

13 a $\left(\frac{2}{3}, -\frac{7}{9}\right)$ **b** $\left(-\frac{1}{2}, 0\right), (1,0)$

c $\left(-\frac{3}{2}, \frac{7}{4}\right)$ **d** $(-1,4), (0,2)$

14 a $k = -2,\ k = 1$ **b** $-10 < c < 10$

c $p = 5$

Chapter 3 review

Short-answer questions

1 $a = 3,\ b = 2,\ c = 1$

2 $(x-1)^3 + 3(x-1)^2 + 3(x-1) + 1$

5 a $x = -4$ or $x = 3$ **b** $x = -1$ or $x = 2$

c $x = -2$ or $x = 5$ **d** $x = \frac{2 \pm \sqrt{2}}{2}$

e $x = \frac{1 \pm \sqrt{3t - 14}}{3}$ **f** $x = \frac{t \pm \sqrt{t^2 - 16t}}{2t}$

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6 $x = \dfrac{-3 \pm \sqrt{73}}{2}$

7 **a** $\dfrac{-1}{x-3} - \dfrac{2}{x+2}$ **b** $\dfrac{3}{x+2} + \dfrac{4}{x-2}$

c $\dfrac{1}{2(x-3)} - \dfrac{3}{2(x+5)}$ **d** $\dfrac{1}{x-5} + \dfrac{2}{x+1}$

e $\dfrac{13}{x+2} - \dfrac{13}{x+3} - \dfrac{10}{(x+2)^2}$

f $\dfrac{4}{x+4} + \dfrac{2}{x-1} - \dfrac{3}{(x-1)^2}$

g $\dfrac{1}{x+1} - \dfrac{6}{x^2+2}$ **h** $\dfrac{1}{x-1} - \dfrac{x+3}{x^2+x+1}$

i $\dfrac{1}{3-x} - \dfrac{3}{x+4}$ **j** $\dfrac{2}{7(x-3)} - \dfrac{16}{7(x+4)}$

8 **a** $\dfrac{1}{x-3} - \dfrac{x-10}{x^2+x+2}$

b $\dfrac{1}{4(x+1)} - \dfrac{x-2}{4(x^2-x+2)}$

c $3x + 15 + \dfrac{64}{x-4} - \dfrac{1}{x-1}$

9 **a** $(0, 0), (-1, 1)$ **b** $(0, 4), (4, 0)$

c $(1, 4), (4, 1)$

10 $(-4, -1), (2, 1)$

11 **a** $t = \dfrac{135}{x}$ **b** $t = \dfrac{135}{x-15}$ **c** $x = 60$

d 60 km/h, 45 km/h

Multiple-choice questions

1 C **2** D **3** D **4** C **5** E
6 E **7** C **8** D **9** B **10** B

Extended-response questions

1 **a** 24 km/h

b Speed $= \dfrac{a + \sqrt{a(a+480)}}{2}$, $a > 0$;

When $a = 60$, speed $= 120$ km/h, which is a very fast constant speed for a train. If we choose this as the upper limit for the speed, then $0 < a < 60$ and $0 < \text{speed} < 120$

c

a	1	8	14	22	34	43	56	77	118
speed	16	20	24	30	40	48	60	80	120

2 **a** $\dfrac{a + \sqrt{a^2 + 4abc}}{2ac}$

b e.g. $a = 3, b = 1, c = \dfrac{4}{3}$

3 **a** Smaller pipe $(b + \sqrt{b^2 - ab})$ minutes;
Larger pipe $(b - a + \sqrt{b^2 - ab})$ minutes

b Smaller pipe 48 minutes;
Larger pipe 24 minutes

c

a	3	8	15	24	35
b	4	9	16	25	36

Chapter 4

Short-answer questions

1 **a** $-\dfrac{\sqrt{2}+3}{7}$ **b** $\dfrac{3(\sqrt{5}+1)}{4}$ **c** $\dfrac{4\sqrt{2}+2}{7}$

d $\dfrac{3(\sqrt{5}+\sqrt{3})}{2}$ **e** $\dfrac{\sqrt{7}+\sqrt{2}}{5}$ **f** $\dfrac{2\sqrt{5}+\sqrt{3}}{17}$

2 **b** **i** $x = \dfrac{-1-\sqrt{5}}{2}$ **ii** $x = \dfrac{-1+\sqrt{5}}{2}$

3 $a = -7,\ b = -5,\ c = 1$

5 **a** $576 = 2^6 \times 3^2,\ \sqrt{576} = 24$

b $1225 = 5^2 \times 7^2,\ \sqrt{1225} = 35$

c $1936 = 4^2 \times 11^2,\ \sqrt{1936} = 44$

d $1296 = 6^4,\ \sqrt{1296} = 36$

6 $x = -b - c$

7 $x = \dfrac{2ab}{a+b}$

8 $(5, 14), (17, 9), (29, 4)$

9 Two at \$25 and four at \$35

10 $a = -\dfrac{1}{3},\ b = -2,\ \lambda = -\dfrac{4}{3}$;

$a = -\dfrac{1}{2},\ b = -1,\ \lambda = -\dfrac{3}{2}$

11 **a** 5, 1 **b** $\dfrac{8}{3}, 0$ **c** $3, -\dfrac{3}{5}$ **d** 14, −6

e 1, 9 **f** $4, -\dfrac{4}{3}$ **g** $\dfrac{5}{2}, -\dfrac{15}{2}$

12 **a** $\{x : -2 \le x \le 2\}$

b $\{x : x \le -1\} \cup \{x : x \ge 1\}$

c $\left\{x : \dfrac{1}{2} \le x \le \dfrac{9}{2}\right\}$ **d** $\{x : -1 < x < 2\}$

e $\left\{x : x \le -\dfrac{1}{2}\right\} \cup \left\{x : x \ge \dfrac{7}{2}\right\}$

f $\left\{x : -\dfrac{1}{3} \le x \le \dfrac{5}{3}\right\}$

13 **a** $x = \dfrac{(y-3)^2+1}{2}$ **b** $x = \dfrac{1}{3}\left(\dfrac{4}{(y+2)^2} - 1\right)$

14 150 minutes

15 **a** $x = \dfrac{51}{25},\ y = \dfrac{32}{25}$

b $x = \dfrac{a(b^2+1)}{a^2+b^2},\ y = \dfrac{b(a^2-1)}{a^2+b^2}$

16 **a** 3 **b** 12 **c** 8

17 **a** $\Delta = 4a(a-1)$

b **i** $a = 1$ **ii** $a > 1$ or $a < 0$

iii $0 < a < 1$

Multiple-choice questions

1 E **2** B **3** C **4** C **5** A
6 C **7** C **8** A **9** B **10** B
11 B **12** A **13** D **14** B **15** B
16 A **17** A **18** C **19** A **20** E
21 C **22** C **23** E **24** D **25** A
26 A **27** D **28** D **29** C **30** B

ISBN 978-1-316-63608-4

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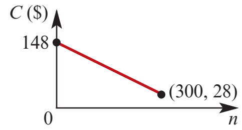
Extended-response questions

1 **a** 8 **b** 7.7 **c** 6 cm **d** 15 cm

2 **a** $a = -0.4,\ b = 148$

b



c \$68 **d** 248

3 **a** **i** 178 **ii** 179 **iii** 179.5 **iv** 179.95

b **i** 180 **ii** Circle

c 20 **d** $n = \dfrac{360}{180 - A}$ **e** Square

4 **a** Volume of hemisphere $= \dfrac{2}{3}\pi t^3$,

Volume of cylinder $= \pi t^2 s$,

Volume of cone $= \dfrac{1}{3}\pi t^2 w$

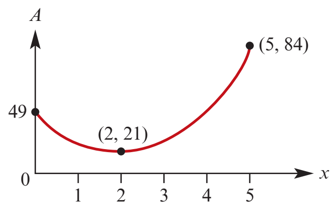
b **i** 6 : 2 : 3 **ii** 54π cubic units

5 **a** $a = 6000,\ b = -15\ 000$ **b** \$57 000

c 2016

6 **a** $8x$ cm **b** $28 - 8x$ cm **c** $7 - 2x$ cm

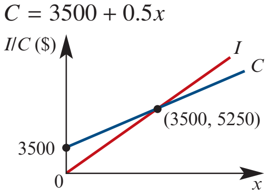
e



f $A = 21$ when $x = 2$

7 **a** $C = 3500 + 0.5x$ **b** $I = 1.5x$

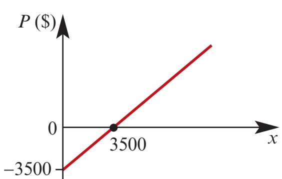
c



d 3500

e 5500

f

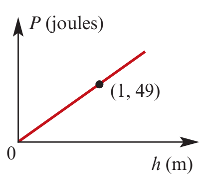


P represents profit

8 **b** **i** $x = \dfrac{1}{24}$ **ii** $x = \dfrac{25}{24}$

9 **c** 11, 24 and 39

10 **a** **i** $g = 9.8$ **ii**



iii 7 kg

b **i** Doubled **ii** Halved

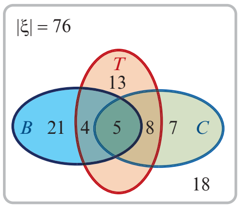
c **i** 14 m/s **ii** 42 m/s

d 4

11 **a** 1 hour 35 minutes **b** 2.5 km

12 **a** $|B' \cap C' \cap T| = |C \cap T|$,
$|B \cap C' \cap T'| = 3|B' \cap C \cap T'|$,
$|B \cap C' \cap T| = 4$

b



c **i** 5 **ii** 0

Chapter 5

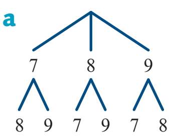
Exercise 5A

1 45

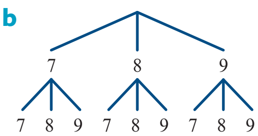
2 8

3 120

4 **a**



b



5 **a** 27 **b** 6

6 30

7 **a** 6 **b** 18 **c** 20 **d** 15

8 BB, BR, BG, RB, RG, GB, GR, GG

9 12

10 9

11 **a** 6 **b** 13

12 16

Exercise 5B

1 1, 1, 2, 6, 24, 120, 720, 5040, 40 320, 362 880, 3 628 800

2 **a** 5 **b** 90 **c** 66 **d** 161 700

3 **a** $n + 1$ **b** $n + 2$ **c** $n(n - 1)$ **d** $\dfrac{n + 2}{(n + 1)!}$

4 1, 4, 12, 24, 24

5 DOG, DGO, ODG, OGD, GOD, GDO

6 120

7 362 880

8 FR, FO, FG, RF, RO, RG, OF, OR, OG, GF, GR, GO

9 **a** 720 **b** 720 **c** 360

10 **a** 120 **b** 120 **c** 60

11 20 160

12 **a** 125 **b** 60

13 **a** 120 **b** 360 **c** 720

14 60

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15 a 17 576 000 **b** 11 232 000

16 a 384 **b** 3072

17 $(m, n) = (6, 0), (6, 1), (5, 3)$

18 $(n^2 - n) \cdot (n-2)! = n \cdot (n-1) \cdot (n-2)! = n!$

19 30

Exercise 5C

1 a 120 **b** 72 **c** 24 **d** 96

2 a 120 **b** 48 **c** 72 **d** 12

3 a 360 **b** 144 **c** 144 **d** 72

4 a 1152 **b** 1152

5 a 600 **b** 108 **c** 431 **d** 52

6 a 720 **b** 48 **c** 144 **d** 96 **e** 48

7 a 900 **b** 900

8 84

9 32

10 a 480 **b** 192

11 144

Exercise 5D

1 35 **2** 34 650

3 4 989 600 **4** 56

5 27 720

6 a 420 **b** 105 **c** 90 **d** 12 **e** 105

7 35

8 a 15 **b** $\dfrac{(m+n)!}{m! \cdot n!}$

9 a 52! **b** $\dfrac{104!}{(2!)^{52}}$ **c** $\dfrac{(52n)!}{(n!)^{52}}$

10 4900

11 89

Exercise 5E

1 1, 5, 10, 10, 5, 1

2 a 7 **b** 6 **c** 66 **d** 56 **e** 100 **f** 499 500

3 a n **b** $\dfrac{n(n-1)}{2}$ **c** n **d** $n+1$ **e** $\dfrac{(n+2)(n+1)}{2}$ **f** $\dfrac{n(n+1)}{2}$

4 a 720 **b** 120

5 2 598 960

6 a 10 **b** 45 **c** 45 **d** 10

7 45 379 620

8 56

9 a 45 **b** 16

10 15

11 $^nC_{n-r} = \dfrac{n!}{(n-r)!\,(n-(n-r))!}$

$= \dfrac{n!}{(n-r)!\,r!} = {}^nC_r$

12 Each diagonal is obtained by choosing 2 vertices from n vertices. This can be done in $^nC_2$ ways. But n of these choices define a side of the polygon, not a diagonal. Therefore there are $^nC_2 - n$ diagonals.

13 There are $^{10}C_5$ ways to choose 5 students for team A. The remaining 5 students will belong to team B. However, the labelling of the teams does not matter, so we must divide by 2.

14 462

15 $^{n-1}C_{r-1} + {}^{n-1}C_r$

$$= \frac{(n-1)!}{(r-1)!\,(n-1-(r-1))!} + \frac{(n-1)!}{r!\,(n-1-r)!}$$

$$= \frac{(n-1)!}{(r-1)!\,(n-r)!} + \frac{(n-1)!}{r!\,(n-r-1)!}$$

$$= \frac{(n-1)!}{(r-1)!\,(n-r-1)!}\left(\frac{1}{n-r} + \frac{1}{r}\right)$$

$$= \frac{(n-1)!}{(r-1)!\,(n-r-1)!} \cdot \frac{n}{r(n-r)}$$

$$= \frac{n!}{r!\,(n-r)!} = {}^nC_r$$

16 a 2300 **b** 152 **c** 2148

Exercise 5F

1 153 **2** 126

3 1176 **4** 140

5 a 1716 **b** 700 **c** 980 **d** 1568

6 a 25 200 **b** 4200

7 a 1 392 554 592 **b** 5 250 960

8 a 15 504 **b** 10 800 **c** 15 252

9 a 21 **b** 10 **c** 11

10 2100

11 a 204 490 **b** 7 250 100

12 a 48 **b** 210

13 1440 **14** 3600

15 14 400 **16** 150

17 3744

Exercise 5G

1 $^7C_2 = 21, {}^6C_2 = 15, {}^6C_1 = 6$

2 1, 7, 21, 35, 35, 21, 7, 1; $^7C_2 = 21, {}^7C_4 = 35$

3 1, 8, 28, 56, 70, 56, 28, 8, 1; $^8C_4 = 70, {}^8C_6 = 28$

4 $2^6 = 64$

5 $2^5 = 32$

6 $2^{10} = 1024$

7 $2^6 - 1 = 63$

8 $2^8 - {}^8C_1 - {}^8C_0 = 247$

9 $2^8 = 256$

10 $2^4 - 1 = 15$

11 a 128 **b** 44

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Exercise 5H

1 4

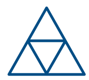
2 Label 26 holes from A to Z. Put each of the 27 words into the hole labelled by its first letter. Some hole contains at least two words.

3 Label 4 holes by 0, 1, 2, 3. Put each of the 5 numbers into the hole labelled by its remainder when divided by 4. Some hole contains at least two numbers.

4 **a** 3 **b** 5 **c** 14

5 Divide $[0, 1]$ into 10 subintervals: $[0, 0.1)$, $[0.1, 0.2), \dots, [0.9, 1]$. Some interval contains at least two of the 11 numbers.

6 Divide into 4 equilateral triangles of side length 1 unit as shown. Some triangle contains at least two of the 5 points.



7 Divide the rectangle into squares of size 2×2. There are 12 squares and 13 points, so some square contains at least two points.
The distance between two points in the same square cannot exceed the length of the square's diagonal, $\sqrt{2^2 + 2^2} = 2\sqrt{2}$.

8 **a** For two-digit numbers, the possible digital sums are $1, 2, \dots, 18$. Since $19 > 18$, some digital sum occurs at least twice.
b For three-digit numbers, the possible digital sums are $1, 2, \dots, 27$. Since $82 = 3 \times 27 + 1$, some digital sum occurs at least 4 times.

9 Label 4 holes by 0, 1, 2, 3. Place each number into the hole labelled by its remainder when divided by 4. Since $13 = 3 \times 4 + 1$, some hole contains at least 4 numbers.

10 Two teams can be chosen in ${}^{8}C_2 = 28$ ways. Since there are 29 games, some pair of teams play each other at least twice.

11 At least 26 students. To show that 26 numbers suffice, label 25 holes by (1 or 49), (2 or 48), …, (24 or 26), (25). To show that 25 numbers do not, consider $1, 2, 3, \dots, 25$.

12 Label the chairs $1, 2, \dots, 14$. There are 14 groups of three consecutive chairs:
$\{1, 2, 3\}, \{2, 3, 4\}, \dots, \{13, 14, 1\}, \{14, 1, 2\}$
Each of the 10 people belongs to 3 groups, so there are 30 people to be allocated to 14 groups. Since $30 \geq 2 \times 14 + 1$, some group contain at least 3 people.

13 Draw a diameter through one of the 4 points. This creates 2 half circles. One half circle contains at least two of the 3 remaining points (and the chosen point).

14 There are 195 possible sums: $3, 4, \dots, 197$. There are ${}^{35}C_2 = 595$ ways to choose a pair of players. Since $595 \geq 3 \times 195 + 1$, at least 4 pairs have the same sum.

15 Label the chairs $1, 2, \dots, 12$. There are 6 pairs of opposite seats:
$\{1, 7\}, \{2, 8\}, \{3, 9\}, \{4, 10\}, \{5, 11\}, \{6, 12\}$
Some pair contains two of the 7 boys.

16 Label n holes by $0, 1, 2, \dots, n - 1$. Place each guest in the hole labelled by the number of hands they shake. The first or last hole must be empty. (If a guest shakes 0 hands, then no guest shakes n hands. If a guest shakes n hands, then no guest shakes 0 hands.) This leaves $n - 1$ holes, so some hole contains at least two guests.

Exercise 5I

1 **a** $\{1, 3, 4\}$ **b** $\{1, 3, 4, 5, 6\}$ **c** $\{4\}$
d $\{1, 2, 3, 4, 5, 6\}$ **e** 3
f $\varnothing, \{4\}, \{5\}, \{6\}, \{4, 5\}, \{4, 6\}, \{5, 6\}, \{4, 5, 6\}$

2 36 **3** 4

4 150

5 **a** 64 **b** 32

6 **a** 72 **b** 72 **c** 36 **d** 108

7 **a** 12 **b** 38

8 88 **9** 80

10 4

11 **a** 756 **b** 700 **c** 360 **d** 1096

12 1 452 555 **13** 3417

14 5

Chapter 5 review

Short-answer questions

1 **a** 20 **b** 190 **c** 300 **d** 4950

2 11

3 **a** 27 **b** 6

4 120 **5** 60

6 18 **7** 31

8 10 **9** 3

10 12 **11** 192

Multiple-choice questions

1 C **2** B **3** A **4** D **5** B **6** B
7 C **8** D **9** C **10** C **11** A

Extended-response questions

1 **a** 120 **b** 360 **c** 72 **d** 144

2 **a** 20 **b** 80 **c** 60

3 **a** 210 **b** 84 **c** 90 **d** 195

4 **a** 420 **b** 15 **c** 105 **d** 12

5 **a** **i** 20 **ii** 10 **iii** 64
b 8

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6 a 210 **b** 100 **c** 10 **d** 80
7 a 676 **b** 235 **c** 74
8 a 24 **b** 4 **c** 24 **d** $\frac{3}{4}$
9 a 924
b There are at least $365 \times 3 = 1095$ days in three years and there are 924 different paths, so some path is taken at least twice.
c i 6 **ii** 70 **iii** 420
d 624
10 196

Chapter 6

See solutions supplement

Chapter 7

Exercise 7A

1 a i Obtuse **ii** Straight
iii Acute **iv** Right
b i $\angle HFB$ **ii** $\angle BFE$
iii $\angle HFG$ **iv** $\angle BFE$
c i $\angle CBD, \angle BFE, \angle ABF, \angle HFG$
ii $\angle CBA, \angle BFH, \angle DBF, \angle EFG$
2 a $a = 65°$, $b = 65°$
b $x = 40°$, $y = 130°$
c $a = 60°$, $b = 70°$, $c = 50°$, $d = 60°$, $e = 50°$, $f = 130°$
d $\alpha = 60°$, $\beta = 120°$
e $\alpha = 90°$, $\beta = 93°$
f $\alpha = 108°$, $\beta = 90°$, $\theta = 108°$
4 a $\angle B = \angle D = 180° - \alpha$ **b** $\angle C = \alpha$
9 a $\theta = 107°$ **b** $\theta = 55°$

Exercise 7B

1 a Yes **b** Yes **c** Yes **d** No
2 a Scalene **b** Isosceles **c** Equilateral
3 Must be greater than 10 cm
4 a 6, 6.5, 7 **b** No
6 a $\theta = 46°$, straight angle;
$\beta = 70°$, supplementary to $\angle EBC$;
$\gamma = 64°$, alternate angles ($\angle CBD$);
$\alpha = 46°$, corresponding angles ($\angle EBD$)
b $\gamma = 80°$, angle sum of triangle;
$\beta = 80°$, vertically opposite (γ);
$\theta = 100°$, supplementary to β;
$\alpha = 40°$, alternate angles ($\angle BAD$)
c $\alpha = 130°$, supplementary to $\angle ADC$;
$\beta = 65°$, co-interior angles $\angle CDA$;
$\gamma = 65°$, co-interior angles $\angle CDA$
d $\alpha = 60°$, equilateral triangle
e $\alpha = 60°$, straight angle;
$\beta = 60°$, angle sum of triangle
f $a = 55°$, straight angle;
$b = 55°$, corresponding angles (a);
$g = 45°$, vertically opposite;
$c = 80°$, angle sum of triangle;
$e = 100°$, straight angle;
$f = 80°$, corresponding angles (c)
g $m = 68°$, corresponding angles;
$n = 60°$, angle sum of triangle;
$p = 52°$, straight angle;
$q = 60°$, alternate angles (n);
$r = 68°$, alternate angles (m)
7 a Sum = 720°; Angles = 120°
b Sum = 1800°; Angles = 150°
c Sum = 3240°; Angles = 162°
8 a Together they form 10 straight angles
b 360°
10 10

Exercise 7C

1 a A and C (SAS)
b All of them (AAS)
c A and B (SSS)
2 a $\triangle ABC \equiv \triangle CDA$ (SSS)
b $\triangle CBA \equiv \triangle CDE$ (SAS)
c $\triangle CAD \equiv \triangle CAB$ (SAS)
d $\triangle ADC \equiv \triangle CBA$ (RHS)
e $\triangle DAB \equiv \triangle DCB$ (SSS)
f $\triangle DAB \equiv \triangle DBC$ (SAS)
6 a $a = b = c = d = 60°$
7 a $a = 108°$, $b = 36°$, $c = 72°$, $d = 36°$, $e = 36°$, $f = 36°$

Exercise 7D

1 16.58 m **2** 41 m
3 13.9 cm **4** 18.38 cm
5 a 50 cm^2 **b** 32 cm^2
6 $2\sqrt{5}$ cm **7** 2 cm^2
8 $2\sqrt{6} \approx 4.9$ cm **9 b, d**
12 13.86 cm **14** $XY = 2.8$ cm
15 $x = 1.375$, $y = 2.67$ **16** $3\sqrt{2}$ cm

Exercise 7E

1 2 parts = 2000, 7 parts = 7000
2 1 part = 3000, 2 parts = 6000
3 3.6 **4** 264
5 22.5 **6** 60°, 50°, 70°
7 \$14 **8** 30 g zinc, 40 g tin
9 16 white beads, 8 green beads
10 5.625 km **11** \$1200
12 $\frac{3}{5}$ **13** $\pi : 1$
14 1 : 1 **15** 6 : 7
16 8.75 km

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Exercise 7F

1 a AAA, 11.25 cm **b** AAA, $11\frac{2}{3}$ cm **c** AAA, 3 cm **d** AAA, 7.5 cm

2 a AAA, 6 cm **b** AAA, $1\frac{1}{3}$ cm **c** AAA, $2\frac{2}{3}$ cm **d** AAA, 7.5 cm

3 $AC = 17.5,\ AE = 16,\ AB = 20$

4 4.42 m **5** 7.5 m

6 15 m **7** 22.5 m

8 $10\frac{10}{31}$ m **9** $x = 6\frac{2}{3}$

10 83.6 cm **11** $x = \dfrac{39}{46}$

12 $40\frac{1}{7}$ m **13** 7.2 m

14 $1\frac{14}{15}$ m

15 b $x = 10$ **c** $y = 2\sqrt{5},\ z = 5\sqrt{5}$

16 $a = \dfrac{36}{7}$ **17** 7.11 m

18 1.6 cm **19** $2\frac{1}{7}$ m

20 $a = 3\sqrt{5},\ x = 5,\ y = 2\sqrt{5}$

Exercise 7G

See solutions supplement

Exercise 7H

1 a 1 : 2 : 3 : 4 **b** 1 : 4 : 9 : 16 **c** Yes, second ratio is the square of the first

2 a 1 : 2 : 3 : 4 **b** 1 : 4 : 9 : 16 **c** Yes, second ratio is the square of the first

3 $19\frac{4}{9}$ cm^2

4 4.54 cm^2

5 a $\sqrt{3}$ cm **b** $\dfrac{4\sqrt{3}}{3}$ cm **c** $\dfrac{4}{3}$

6 4 : 5 **7** 22.5

8 a 1 : 2 : 3 **b** 1 : 2 : 3 **c** 1 : 8 : 27 **d** Yes, the third ratio is the cube of the first

9 a i 2 : 3 **ii** 2 : 3 **iii** 2 : 3
b 8 : 27
c Yes, the ratios in **a** are cubed to form the ratios in **b**

10 a 3 : 2 : 5
b Volumes (in cm^3) are 36π, $\dfrac{32}{3}\pi$ and $\dfrac{500}{3}\pi$;
Ratio of volumes is 27 : 8 : 125
c Yes, the ratios in **a** are cubed to form the ratios in **b**

11 8 : 1 **12** 27 : 64

13 2 : 3

14 a 4 : 3 **b** 4 : 3

15 a 4 : 1 **b** 8 : 1

16 a 1 : 100 **b** 1 : 1000 **c** 1 : 10 **d** 1 : 1

17 $\dfrac{27}{16}$ litres, 4 litres

18 125 mL, 216 mL

19 a 1 : 50 **b** 1 : 125 000 **c** 3 cm **d** 7500 cm^2

20 a 12 : 13 **b** 1728 : 2197

21 a 4 **b** 3.75

22 3 : 4 **23** 4.5 cm

Exercise 7I

2 b i 4 **ii** $\sqrt{10}$

4 a i 36° **ii** 72° **c** 0.62

5 $\varphi^0 = 1,\ \varphi^1 = \dfrac{1+\sqrt{5}}{2},\ \varphi^2 = \dfrac{3+\sqrt{5}}{2},$
$\varphi^3 = 2+\sqrt{5},\ \varphi^4 = \dfrac{7+3\sqrt{5}}{2},\ \varphi^{-1} = \dfrac{\sqrt{5}-1}{2},$
$\varphi^{-2} = \dfrac{3-\sqrt{5}}{2},\ \varphi^{-3} = \sqrt{5}-2,\ \varphi^{-4} = \dfrac{7-3\sqrt{5}}{2}$

Chapter 7 review

Short-answer questions

1 a Rectangle **b** 16 cm

3 $\sqrt{34}$ cm

4 a $x = 7$ cm, $y = 7$ cm, $\alpha = 45°$, $\beta = 40°$
b $\alpha = 125°$, $\beta = 27.5°$
c $\theta = 52°$, $\alpha = 52°$, $\beta = 65°$, $\gamma = 63°$

5 8 m

7 b i 20 cm **ii** 10 cm
c $XP : PY = 2 : 1,\ PQ : YZ = 2 : 3$

8 a 3 cm **b** 5 : 3 **c** 3 : 5

9 $\dfrac{210}{23}$ m **10** $\dfrac{15}{8}$

11 12.25 **12** 12

13 a 96 g **b** 2 : 1 **c** 1000 cm^3 **d** 100 mm

14 b 25 : 36 **c** 48 cm

15 a 20 : 3 **b** 1.6 m^2 **c** $\dfrac{8}{27}$ m^3

16 a 2% **b** 3%

17 a $\dfrac{1}{3}$ **b** $\dfrac{1}{3}$ **c** $\dfrac{2}{3}$ **d** $\dfrac{2}{3}$ **e** $\dfrac{1}{9}$ **f** $\dfrac{4}{9}$

Multiple-choice questions

1 C **2** B **3** B **4** B **5** A
6 D **7** B **8** D **9** C **10** B
11 D **12** C **13** E **14** E **15** E

Extended-response questions

1 a $\triangle EBC$ **c** $\dfrac{h}{q} = \dfrac{x}{x+y}$ **e** $\dfrac{20}{9}$

2 a Rhombus; $CF = 1$
c $\triangle ACF$, $\triangle ABC$ and $\triangle AED$ **e** $\dfrac{1+\sqrt{5}}{2}$

3 $x = 8$ or $x = 11$

4 a $\triangle BDR$ and $\triangle CDS$; $\triangle BDT$ and $\triangle BCS$; $\triangle RSB$ and $\triangle DST$
b $\dfrac{z}{y} = \dfrac{p}{p+q}$ **c** $\dfrac{z}{x} = \dfrac{q}{p+q}$

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5 **a** **i** 9 cm **ii** 12 cm **iii** $\frac{1}{16}$ **iv** $\frac{9}{16}$
b **i** $16a$ cm^2 **ii** $3a$ cm^2
7 $15\sqrt{26}$ m

Chapter 8

Exercise 8A

1 **a** $x = 100,\ y = 50$
b $x = 126,\ y = 252,\ z = 54$
c $y = 145,\ z = 290$
d $x = 180,\ y = 90$
e $x = 45,\ y = 90,\ z = 270$
2 **a** $x = 68,\ y = 121$ **b** $x = 112,\ y = 87$
c $x = 50,\ y = 110$
3 $110°,\ 110°,\ 140°$
4 $\angle ABC = 98°,\ \angle BCD = 132°,\ \angle CDE = 117°,\ \angle DEA = 110°,\ \angle EAB = 83°$
7 $60°$ or $120°$
8 $\angle P = 78°,\ \angle Q = 74°,\ \angle R = 102°,\ \angle S = 106°$

Exercise 8B

1 **a** $x = 73,\ y = 81$ **b** $x = 57,\ q = 57$
c $x = 53,\ y = 74,\ z = 53$
d $x = 60,\ y = 60,\ z = 20,\ w = 100$
e $w = 54,\ x = 54,\ y = 72,\ z = 54$
2 **a** $40°$ **b** $40°$ **c** $80°$
3 $32°$ and $148°$
4 $\angle ACB = 40°,\ \angle ABC = 70°,\ \angle BAT = 40°$

Exercise 8C

1 **a** 10 cm **b** 6 cm
2 7 cm
3 $5\sqrt{6}$ cm

Chapter 8 review

Short-answer questions

1 $\angle MCN = 18°$
2 **a** $x = 110,\ y = 70$ **b** $x = 35,\ y = 35$
c $x = 47,\ y = 53,\ z = 100$
d $x = 40,\ y = 40,\ z = 70$
6 **a** $x = 66$ **b** $x = 116$ **c** $x = 66,\ y = 114$
8 3 cm

Multiple-choice questions

1 B **2** A **3** E **4** A **5** C
6 A **7** C **8** B **9** A **10** A

Extended-response questions

5 **b** 24 cm^2

Chapter 9

Short-answer questions

1 24
2 360
3 **a** 125 **b** 60
4 **a** 9 **b** 25
5 **a** 24 **b** 30 **c** 28 **d** 45
6 **a** 120 **b** 120
7 **a** 120 **b** 36
8 **a** 96 **b** 24 **c** 72 **d** 60
9 10
10 **a** 20 **b** 325 **c** 210 **d** 56
11 **a** 28 **b** 21 **c** $2^8 = 256$
12 60 **13** 120
14 7 **15** 51
16 80
19 **a** If n is odd, then $5n + 3$ is even.
c If n is even, then $5n + 3$ is odd.
26 **a** $90°$ **b** $54°$ **c** $80°$ **d** $220°$
e $x = 96°,\ y = 70°$ **f** $46°$
27 **a** $40°$ **b** $140°$ **c** $50°$
28 **a** $38°$ **b** $52°$ **c** $68°$
29 **a** 4 **b** $\frac{3\sqrt{10}}{2}$ **c** 12
31 **a** 156 **b** 144 **c** 25
32 $30°$
35 Yes

Multiple-choice questions

1 A **2** C **3** C **4** A **5** B
6 D **7** D **8** D **9** A **10** E
11 B **12** E **13** E **14** D **15** C
16 E **17** E **18** B **19** B **20** D
21 D **22** A **23** E **24** C **25** B
26 E **27** A **28** C **29** C **30** E
31 D **32** B **33** C **34** B **35** D
36 C **37** C **38** B **39** D

Extended-response questions

1 **a** 2160 **b** 360 **c** 900 **d** 1260
2 **a** 70 **b** 30 **c** 15 **d** 55
3 **a** 20 **b** 4 **c** 68
4 **a** 420 **b** 60 **c** 120 **d** 24
5 **a** 300 **b** 10 and 15
6 **a** 495 **b** 60
c The two points diametrically opposite
d 15 **e** $\frac{1}{33}$
7 **a** No
b Yes; both a and b are odd, and c is even
8 **a** $(a, b, c) = (2, 3, 6)$
b $(a, b, c, d) = (1, 2, 3, 4)$ or $(a, b, c, d) = (1, 2, 3, 5)$

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10 a 10

11 a 49, 50, 51 and 52 **b** 93 and 94 **d** 44

12 b 21 coins **c** 10

14 a No **b** $n = 4k$ or $n = 4k - 1$

16 a $a = 1,\ b = 3,\ c = 1$ **c** 41^2

18 b $\angle BCA = x°,\ \angle BOA = 2x°,\ \angle TAB = x°,$ $\angle TBA = x°$

Chapter 10

Exercise 10A

1 a 4π **b** 3π **c** $-\frac{5\pi}{2}$ **d** $\frac{\pi}{12}$

e $-\frac{\pi}{18}$ **f** $-\frac{7\pi}{4}$

2 a 225° **b** −120° **c** 105° **d** −330°

e 260° **f** −165°

3 a 0 **b** −1 **c** 1 **d** −1

4 a −1 **b** −1 **c** 1 **d** 1

Exercise 10B

1 a $-\frac{1}{\sqrt{2}}$ **b** $-\frac{1}{\sqrt{2}}$ **c** 1 **d** 1 **e** $\frac{1}{\sqrt{2}}$

f $\frac{1}{\sqrt{2}}$ **g** 0 **h** $\frac{\sqrt{3}}{2}$ **i** 0 **j** 0

k 1 **l** 0 **m** $-\frac{1}{2}$ **n** −1 **o** −1

2 a $\frac{1}{\sqrt{2}}$ **b** $\frac{1}{2}$ **c** $\frac{\sqrt{3}}{2}$ **d** $-\frac{1}{2}$

e $\frac{1}{\sqrt{2}}$ **f** $\frac{\sqrt{3}}{2}$

3 a 0.6 **b** 0.6 **c** 0.3 **d** −0.3

e −0.3 **f** 0.6 **g** −0.6 **h** −0.3

4 $\cos x = -\frac{\sqrt{3}}{2},\quad \tan x = -\frac{1}{\sqrt{3}}$

5 $\sin x = -\frac{\sqrt{51}}{10},\quad \tan x = \frac{\sqrt{51}}{7}$

6 $\cos x = -\frac{\sqrt{3}}{2},\quad \tan x = \frac{1}{\sqrt{3}}$

7 $\cos x = \frac{\sqrt{91}}{10},\quad \tan x = -\frac{3\sqrt{91}}{91}$

Exercise 10C

1 $2\pi - a,\ 2\pi - b,\ 2\pi - c,\ 2\pi - d$

2 a $\frac{4\pi}{3}, \frac{5\pi}{3}$ **b** $\frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}$

c $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ **d** $\frac{5\pi}{6}, \frac{3\pi}{2}$

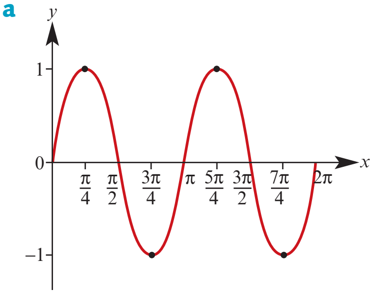
e $0, \frac{\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$ **f** $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$

3 a $-\frac{5\pi}{6}, -\frac{\pi}{6}$ **b** $0, -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, -\pi, \pi$

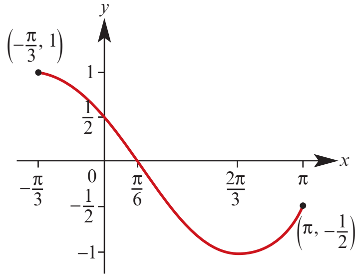
c 0 **d** $0, -\frac{2\pi}{3}$ **e** $-\frac{5\pi}{6}, -\frac{\pi}{2}, \frac{\pi}{6}, \frac{\pi}{2}$

Exercise 10D

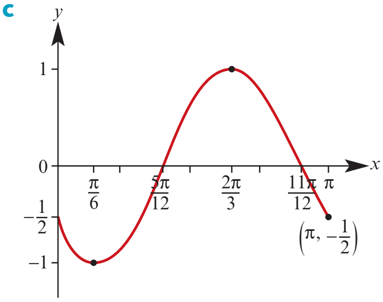
1 a



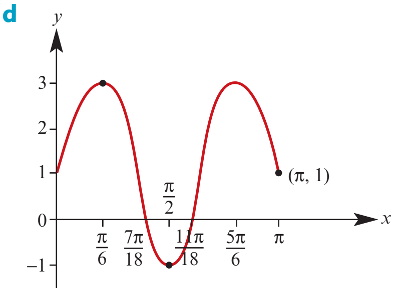
b



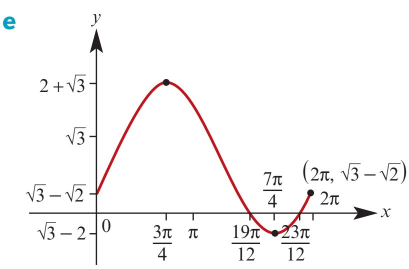
c



d



e



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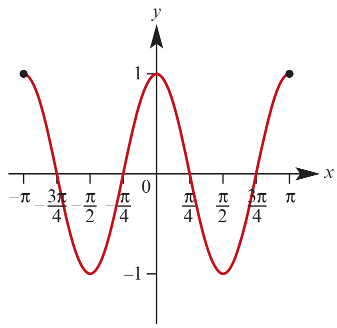
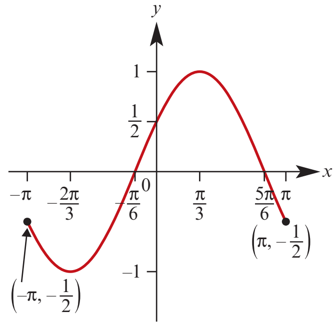
f



2 a



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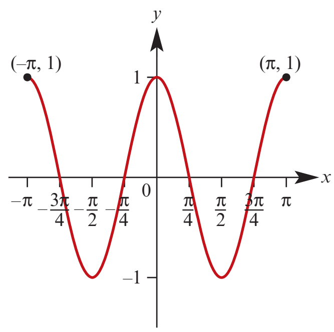

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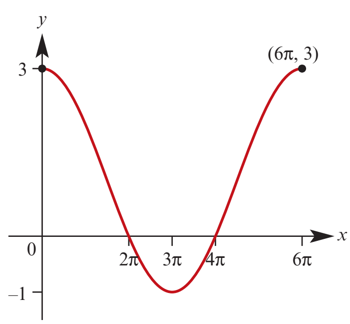

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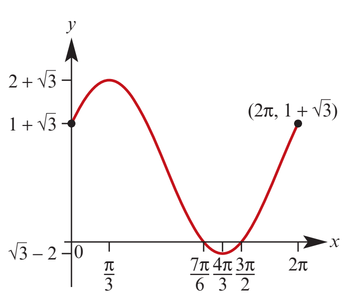

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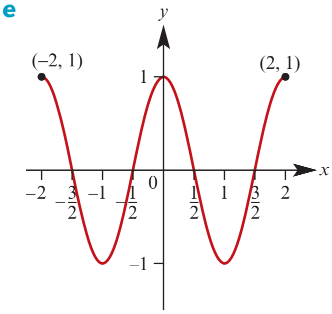

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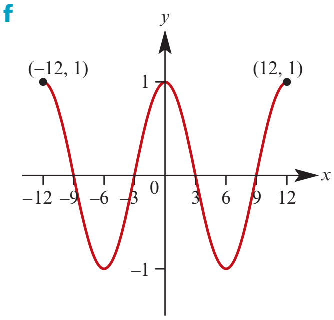


Exercise 10E

1 **a** 1 **b** $\sqrt{3}$ **c** $\frac{1}{\sqrt{3}}$

2 **a** $\sqrt{3}$ **b** $\frac{1}{\sqrt{3}}$ **c** -1

3 **a** $\frac{-\sqrt{17}}{17}$ **b** $\frac{-4\sqrt{17}}{17}$ **c** $\frac{-1}{4}$ **d** $\frac{-1}{4}$

4 **a** $\frac{\sqrt{21}}{7}$ **b** $\frac{-2\sqrt{7}}{7}$ **c** $\frac{\sqrt{3}}{2}$ **d** $\frac{-\sqrt{3}}{2}$

5 **a** $\frac{3\pi}{4}, \frac{7\pi}{4}$ **b** $\frac{\pi}{3}, \frac{4\pi}{3}$ **c** $\frac{\pi}{6}, \frac{7\pi}{6}$

d $-\frac{7\pi}{8}, -\frac{3\pi}{8}, \frac{\pi}{8}, \frac{5\pi}{8}$ **e** $-\frac{5\pi}{6}, -\frac{\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}$

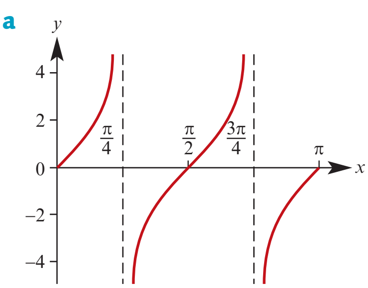
f $-\frac{7\pi}{12}, -\frac{\pi}{12}, \frac{5\pi}{12}, \frac{11\pi}{12}$

6 **a** $\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

b $-\frac{7\pi}{8}, -\frac{3\pi}{8}, \frac{\pi}{8}, \frac{5\pi}{8}$

c $-\frac{13\pi}{18}, -\frac{7\pi}{18}, -\frac{\pi}{18}, \frac{5\pi}{18}, \frac{11\pi}{18}, \frac{17\pi}{18}$ **d** $-\frac{\pi}{6}$

7 **a**



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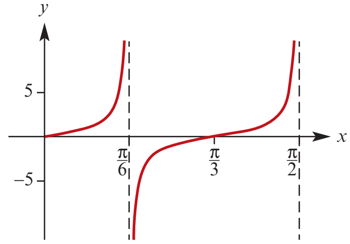
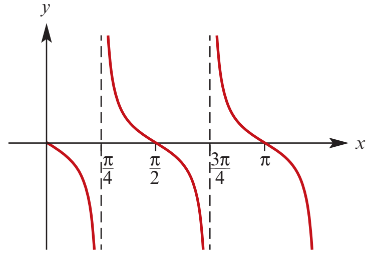
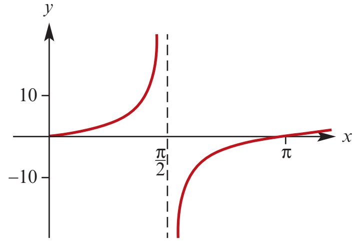
b



c



d



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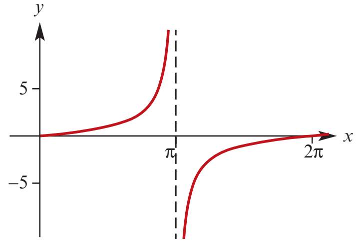

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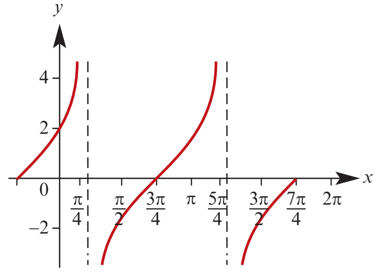
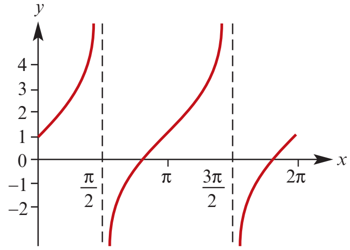
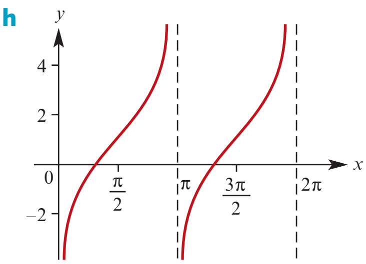


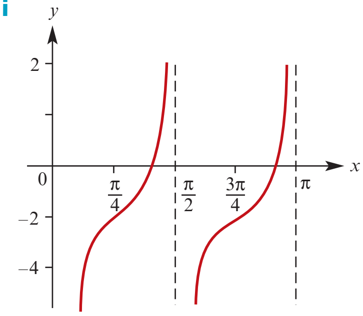
g



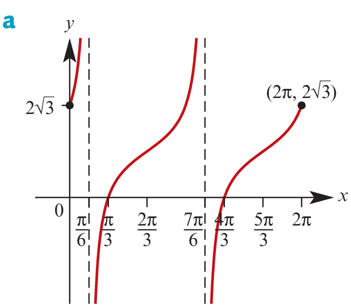
h



i



8 a



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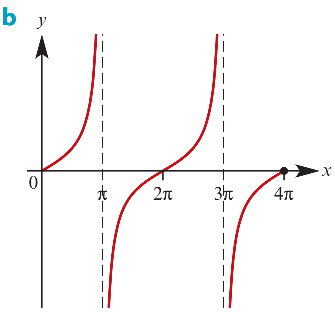
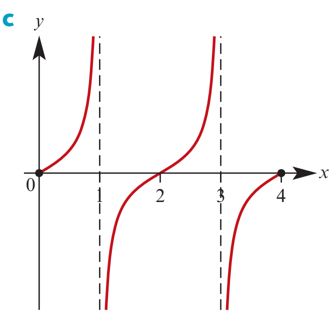


c



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Exercise 10F

1 a i 2π **ii** 4π **iii** -4π

b i $\frac{4\pi}{3}, \frac{8\pi}{3}$ **ii** $\frac{14\pi}{3}, \frac{10\pi}{3}$ **iii** $\frac{-14\pi}{3}, \frac{-10\pi}{3}$

2 a $2n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}$

b $\frac{2n\pi}{3} + \frac{\pi}{9}$ or $\frac{2n\pi}{3} + \frac{2\pi}{9}, n \in \mathbb{Z}$

c $n\pi + \frac{\pi}{3}, n \in \mathbb{Z}$

3 a $\frac{\pi}{6}, \frac{5\pi}{6}$ **b** $\frac{\pi}{12}, \frac{11\pi}{12}$ **c** $\frac{\pi}{3}, \frac{5\pi}{6}$

4 $\frac{-11\pi}{6}, \frac{-7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$

5 $\frac{-\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}$

6 a $x = n\pi - \frac{\pi}{6}$ or $x = n\pi - \frac{\pi}{2}, n \in \mathbb{Z}$

b $x = \frac{n\pi}{2} - \frac{\pi}{12}, n \in \mathbb{Z}$

c $x = 2n\pi + \frac{5\pi}{6}$ or $x = 2n\pi - \frac{\pi}{2}, n \in \mathbb{Z}$

7 $x = \frac{(4n-1)\pi}{4}$ or $x = n\pi, n \in \mathbb{Z}$;
$\left\{\frac{-5\pi}{4}, -\pi, \frac{\pi}{4}, 0, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}\right\}$

8 $x = \frac{n\pi}{3}, n \in \mathbb{Z}; \left\{-\pi, \frac{-2\pi}{3}, \frac{-\pi}{3}, 0\right\}$

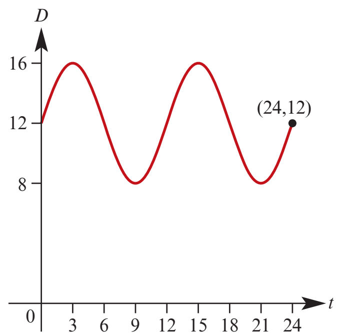
9 $x = \frac{6n-1}{12}$ or $x = \frac{3n+2}{6}, n \in \mathbb{Z}$;
$\left\{\frac{-2}{3}, \frac{-7}{12}, \frac{-1}{6}, \frac{-1}{12}, \frac{1}{3}, \frac{5}{12}, \frac{5}{6}, \frac{11}{12}\right\}$

Exercise 10G

1 a i 0.00 hours **ii** 24.00 hours

b 13 February ($t = 1.48$), 24 October ($t = 9.86$)

2 a



b $t \in [0, 6] \cup [12, 18]$ **c** 15.9 m

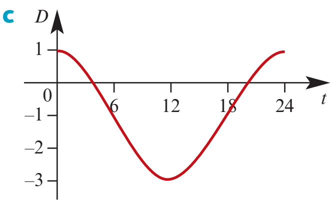
3 a 7 metres **b** 1 metre

c $t = \frac{1}{4}$ or $t = \frac{5}{4}$ **d** $t = \frac{3}{4}$ or $t = \frac{7}{4}$

e Particle oscillates between $x = 1$ and $x = 7$

4 a 19.5°C **b** $D = -1 + 2\cos\left(\frac{\pi t}{12}\right)$

c



d $\{t : 4 < t < 20\}$

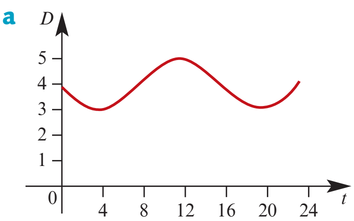
5 a 2 a.m. **b** 8 a.m. and 8 p.m.

6 a i $\frac{3}{2}$ **ii** 12 **iii** $d(t) = \frac{7}{2} - \frac{3}{2}\cos\left(\frac{\pi}{6}t\right)$

iv 1.5 m

b Between 9 p.m. and 3 a.m. and between 9 a.m. and 3 p.m. each day

7 a



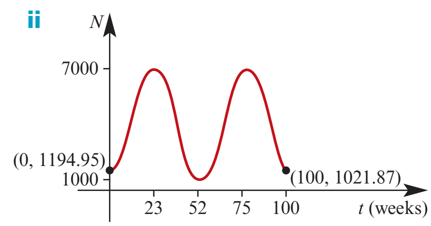
b The boat can enter at 8 a.m. and must leave by 4 p.m.

c The boat can enter at 6:40 a.m. and must leave by 5:20 p.m.

8 a i 52 weeks **ii** 3000 **iii** [1000, 7000]

b i $N(0) = 1194.95$, $N(100) = 1021.87$

ii



c i $t = 23, 75$ **ii** 49

d $(14\frac{1}{3}, 31\frac{2}{3}) \cup (66\frac{1}{3}, 83\frac{2}{3})$

e $d = 25\,000$, $a = 15\,000$, $b = 10$, $c = 5$

Chapter 10 review

Short-answer questions

1 a $\frac{13\pi}{6}$ **b** $\frac{14\pi}{3}$ **c** $\frac{37\pi}{6}$ **d** $\frac{71\pi}{12}$ **e** $\frac{11\pi}{12}$

f $\frac{5\pi}{2}$ **g** $\frac{7\pi}{3}$ **h** $\frac{13\pi}{6}$ **i** $\frac{2\pi}{9}$

2 a 330° **b** 765° **c** 405° **d** 105°

e 1530° **f** −495° **g** −225° **h** −585°

i 1035°

3 a $\frac{1}{\sqrt{2}}$ **b** $-\frac{1}{\sqrt{2}}$ **c** -1 **d** 0 **e** $\frac{\sqrt{3}}{2}$

f -1 **g** $-\sqrt{3}$ **h** 1

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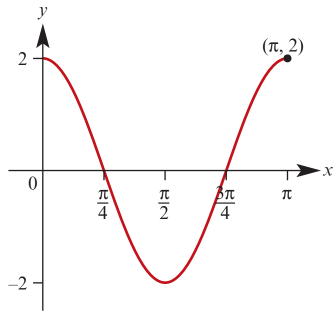
4 a $4, 4\pi$ **b** $5, \frac{\pi}{3}$ **c** $\frac{1}{3}, \frac{\pi}{2}$ **d** $2, \frac{2\pi}{5}$

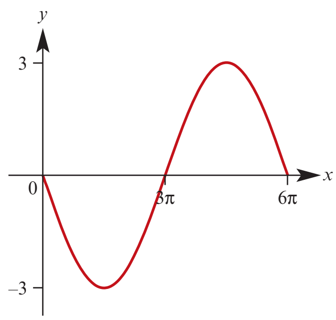
e $7, 8$ **f** $\frac{2}{3}, 3$

5 a $5, 1$ **b** $9, -1$

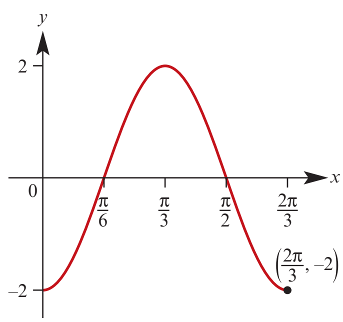
6 a $y = 2\cos(2x)$



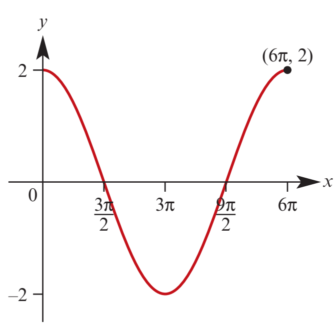
b $y = -3\sin\left(\frac{x}{3}\right)$



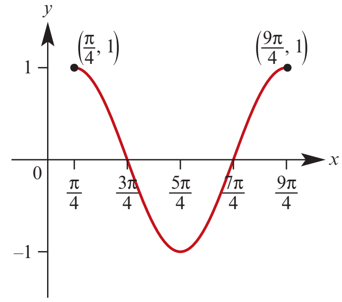
c $y = -2\cos(3x)$



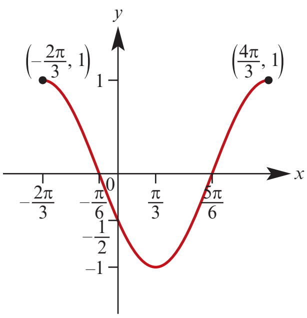
d $y = 2\cos\left(\frac{x}{3}\right)$



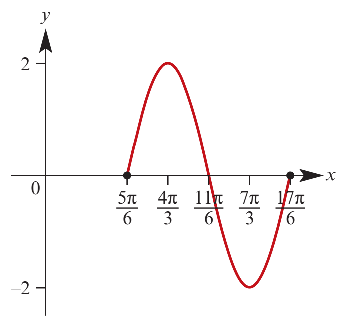
e $y = \cos\left(x - \frac{\pi}{4}\right)$



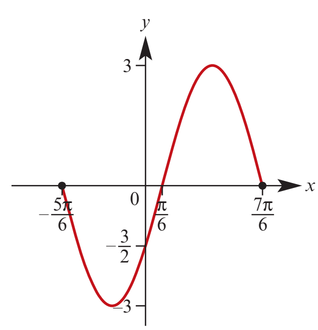
f $y = \cos\left(x + \frac{2\pi}{3}\right)$



g $y = 2\sin\left(x - \frac{5\pi}{6}\right)$



h $h = -3\sin\left(x + \frac{5\pi}{6}\right)$



7 a $-\frac{5\pi}{6}, \frac{5\pi}{6}$ **b** $-\frac{7\pi}{12}, -\frac{5\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}$

c $\pi, \frac{5\pi}{3}$ **d** $\frac{2\pi}{3}$ **e** $\pi, \frac{5\pi}{3}$

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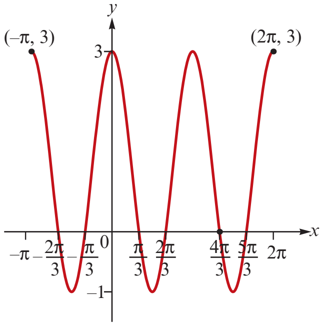
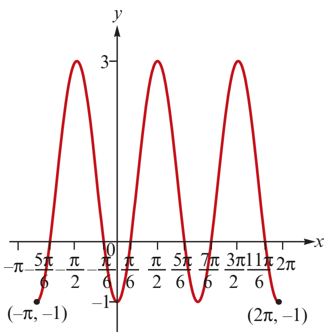
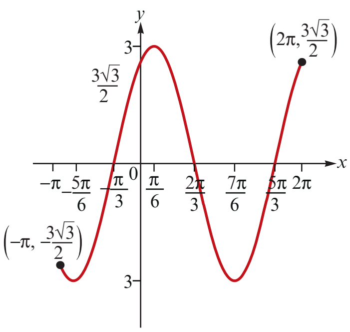
8 a $f(x) = 2\cos(2x) + 1$



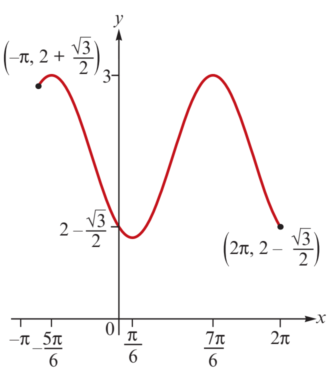
b $f(x) = 1 - 2\cos(2x)$



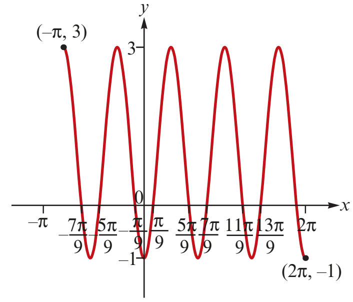
c $f(x) = 3\sin\left(x + \frac{\pi}{3}\right)$



d $f(x) = 2 - \sin\left(x + \frac{\pi}{3}\right)$



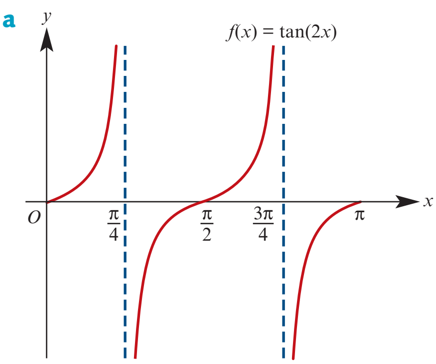
e $f(x) = 1 - 2\cos(3x)$



9 a $\frac{2\pi}{3}, \frac{5\pi}{3}$ **b** $\frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, \frac{10\pi}{9}, \frac{13\pi}{9}, \frac{16\pi}{9}$

c $\frac{3\pi}{2}$ **d** $\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

10 a



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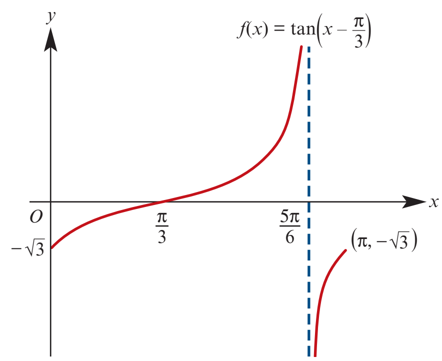
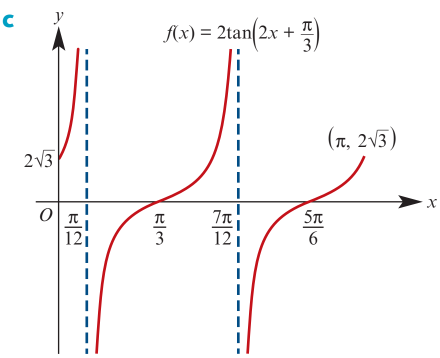


c



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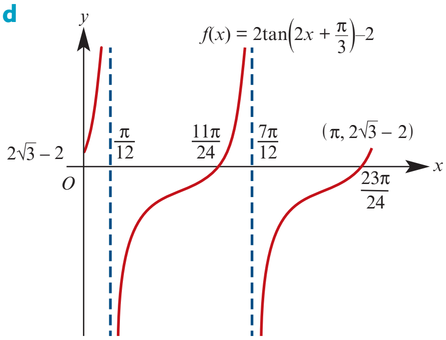
d



11 a $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **b** $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

c $\frac{\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{23\pi}{18}, \frac{25\pi}{18}, \frac{35\pi}{18}$

d $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

12 0°, 60°, 180°, 300°, 360°

13 a $n\pi - \frac{\pi}{4}, n \in \mathbb{Z}$ **b** $\frac{2n\pi}{3}, n \in \mathbb{Z}$

c $-\frac{\pi}{4} + n\pi, n \in \mathbb{Z}$

Multiple-choice questions

1 A **2** D **3** D **4** C **5** D
6 E **7** E **8** C **9** D **10** B

Extended-response questions

1 a i $a = 13.4$ **ii** $b = 2$ **iii** $k = 12$

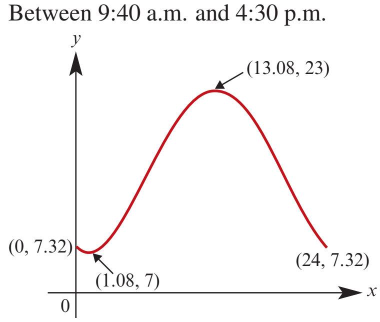
b 3 a.m., 9 a.m., 3 p.m., 9 p.m.

c $2 < t < 10, \ 14 < t < 22$

2 a 7.3°C **b** Min = 7°C; Max = 23°C

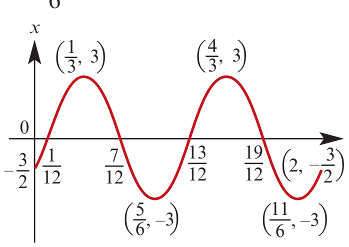
c Between 9:40 a.m. and 4:30 p.m.

d



3 a $a = \frac{\pi}{6}$

b



c 3 m **d** $\frac{5}{6}$ s **e** 1 s **f** $\frac{1}{4}$ s

g i 24 m **ii** 30 m

4 a $p = 6, \ q = 4.2$ **b** 3 a.m., 3 p.m.

c 6 m **d** 7 a.m., 11 a.m., 7 p.m., 11 p.m.

e 8 hours

5 a i $-1 < k < 1$

ii $k = -1$ or $k = 3$

iii $k < -1$ or $k > 3$

b A translation of 1 unit in the negative direction of the y-axis, followed by a dilation of factor $\frac{1}{2}$ from the x-axis and a dilation of factor 3 from the y-axis

c i $h = \frac{\pi}{2}$ **ii** $h = \frac{\pi}{6}$

6 a A translation of $\frac{\pi}{2}$ units in the positive direction of the x-axis

b A dilation of factor $\frac{1}{2}$ from the y-axis, followed by a translation of $\frac{\pi}{4}$ units in the negative direction of the x-axis, and then a dilation of factor $\frac{1}{4}$ from the x-axis

c i $y = -\sin\left(\frac{\pi x}{2}\right) + 4$

ii Range = [3, 5]; Period = 4

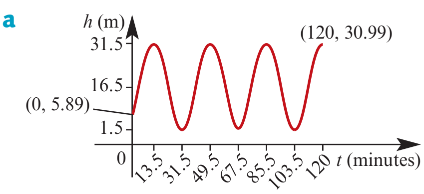
7 a For N: Max = 7000, occurs in April ($t = 4$); Min = 1000, occurs in October ($t = 10$).
For M: Max = 8500, occurs in June ($t = 6$); Min = 2500, occurs at end of January and November ($t = 1$ and $t = 11$)

b $t = 4.31$ (April), population is 6961; $t = 0.24$ (January), population is 2836

c 145 556 in May ($t = 5.19$)

d $t = 7.49$ (July)

8 a



b 5.89 m **c** 27.51 s **d** 6 times **e** 20 times

f 4.21 m **g** 13.9 m

Chapter 11

Exercise 11A

1 a -1 **b** $-\sqrt{2}$ **c** $\frac{-2}{\sqrt{3}} = \frac{-2\sqrt{3}}{3}$ **d** 1

e -2 **f** 2 **g** $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ **h** 2

2 a -1 **b** $\frac{-2}{\sqrt{3}} = \frac{-2\sqrt{3}}{3}$ **c** 1

d $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ **e** $-\sqrt{2}$ **f** $\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

g -1 **h** $\frac{-2}{\sqrt{3}} = \frac{-2\sqrt{3}}{3}$ **i** $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

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3 a $\frac{\pi}{6}, \frac{5\pi}{6}$ **b** $\frac{\pi}{6}, \frac{7\pi}{6}$ **c** $\frac{3\pi}{4}, \frac{5\pi}{4}$ **d** $\frac{\pi}{4}, \frac{5\pi}{4}$

4 a $-\frac{8}{17}$ **b** $\frac{15}{17}$ **c** $-\frac{15}{8}$

5 $\cos\theta = \frac{24}{25},\ \sin\theta = -\frac{7}{25}$

6 $-\frac{\sqrt{29}}{5}$

7 $\frac{8}{31}$

8 $\frac{15}{4(9+\sqrt{5})} = \frac{15(6-\sqrt{5})}{124}$

Exercise 11B

1 a $\frac{\sqrt{2}+\sqrt{6}}{4}$ **b** $\frac{1-\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}-\sqrt{6}}{4}$

2 a $\frac{\sqrt{6}-\sqrt{2}}{4}$ **b** $\frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3}$

3 a $\frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$

b $\frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$

c $\frac{1-\sqrt{3}}{1+\sqrt{3}} = -2+\sqrt{3}$

4 For $u, v \in \left(0, \frac{\pi}{2}\right)$, $\sin(u+v) = \frac{63}{65}$;

For $u, v \in \left(\frac{\pi}{2}, \pi\right)$, $\sin(u+v) = -\frac{63}{65}$;

For $u \in \left(0, \frac{\pi}{2}\right)$, $v \in \left(\frac{\pi}{2}, \pi\right)$, $\sin(u+v) = -\frac{33}{65}$;

For $u \in \left(\frac{\pi}{2}, \pi\right)$, $v \in \left(0, \frac{\pi}{2}\right)$, $\sin(u+v) = \frac{33}{65}$

5 a $\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta$ **b** $\frac{1}{\sqrt{2}}(\cos\varphi + \sin\varphi)$

c $\frac{\tan\theta + \sqrt{3}}{1-\sqrt{3}\tan\theta}$ **d** $\frac{1}{\sqrt{2}}(\sin\theta - \cos\theta)$

6 a $\sin u$ **b** $\cos u$

7 a $-\frac{119}{169}$ **b** $\frac{24}{25}$ **c** $\frac{24}{7}$ **d** $-\frac{169}{119}$

e $-\frac{33}{65}$ **f** $-\frac{16}{65}$ **g** $-\frac{65}{33}$ **h** $\frac{7}{24}$

8 a $\frac{63}{16}$ **b** $-\frac{24}{7}$ **c** $\frac{56}{65}$ **d** $\frac{24}{25}$

9 a $\frac{7}{25}$ **b** $\frac{3}{5}$ **c** $\frac{117}{44}$ **d** $\frac{-336}{625}$

10 a $-\frac{\sqrt{3}}{2}$ for $\theta = \frac{5\pi}{3}$ **b** $-\frac{1}{2}$

11 a $1-\sin(2\theta)$ **b** $\cos(2\theta)$

Exercise 11C

1 a $5, -5$ **b** $2, -2$ **c** $\sqrt{2}, -\sqrt{2}$

d $\sqrt{2}, -\sqrt{2}$ **e** $2\sqrt{3}, -2\sqrt{3}$

f $2, -2$ **g** $4, 0$ **h** $5+\sqrt{13}, 5-\sqrt{13}$

2 a $\frac{\pi}{2}, \pi$ **b** $0, \frac{2\pi}{3}, 2\pi$

c $\frac{\pi}{6}, \frac{3\pi}{2}$ **d** $0, \frac{5\pi}{3}, 2\pi$

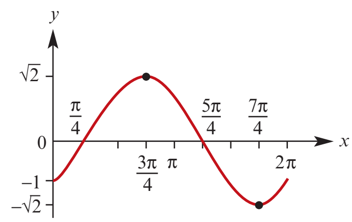
e $53.13°$ **f** $95.26°, 155.26°$

3 $2\cos\left(2x+\frac{\pi}{6}\right)$

4 $\sqrt{2}\sin\left(3x-\frac{5\pi}{4}\right)$

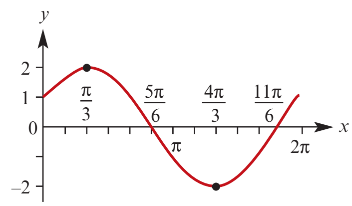
5 a $f(x) = \sin x - \cos x = \sqrt{2}\cos\left(x-\frac{3\pi}{4}\right)$

$= \sqrt{2}\sin\left(x+\frac{7\pi}{4}\right) = \sqrt{2}\sin\left(x-\frac{\pi}{4}\right)$



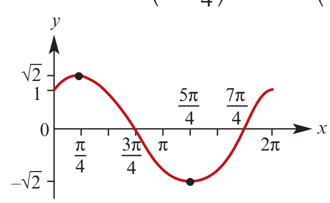
b $f(x) = \sqrt{3}\sin x + \cos x$

$= 2\cos\left(x-\frac{\pi}{3}\right) = 2\sin\left(x+\frac{\pi}{6}\right)$



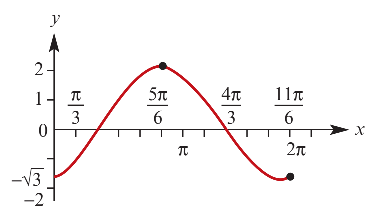
c $f(x) = \sin x + \cos x$

$= \sqrt{2}\cos\left(x-\frac{\pi}{4}\right) = \sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$



d $f(x) = \sin x - \sqrt{3}\cos x = 2\cos\left(x-\frac{5\pi}{6}\right)$

$= 2\sin\left(x+\frac{5\pi}{3}\right) = 2\sin\left(x-\frac{\pi}{3}\right)$



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Exercise 11D

1 a $\sin(5\pi t) + \sin(\pi t)$ **b** $\frac{1}{2}(\sin 50° - \sin 10°)$
c $\sin(\pi x) + \sin\left(\frac{\pi x}{2}\right)$ **d** $\sin(A) + \sin(B + C)$

2 $\cos(\theta) - \cos(5\theta)$

3 $\sin A - \sin B$

5 a $2 \sin 39° \cos 17°$ **b** $2 \cos 39° \cos 17°$
c $2 \cos 39° \sin 17°$ **d** $-2 \sin 39° \sin 17°$

6 a $2 \sin(4A) \cos(2A)$ **b** $2 \cos\left(\frac{3x}{2}\right) \cos\left(\frac{x}{2}\right)$
c $2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{7x}{2}\right)$ **d** $-2 \sin(2A) \sin(A)$

11 a $-\frac{5\pi}{6}, -\frac{3\pi}{4}, -\frac{\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{6}$
b $-\pi, -\frac{2\pi}{3}, -\frac{\pi}{2}, -\frac{\pi}{3}, 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$
c $-\pi, -\frac{3\pi}{4}, -\frac{2\pi}{3}, -\frac{\pi}{3}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}, \pi$
d $-\pi, -\frac{5\pi}{6}, -\frac{\pi}{2}, -\frac{\pi}{6}, 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$

12 a $\frac{\pi}{6}, \frac{5\pi}{6}$ **b** $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$
c $0, \frac{\pi}{12}, \frac{\pi}{3}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{2\pi}{3}, \frac{11\pi}{12}, \pi$
d $\frac{\pi}{10}, \frac{\pi}{6}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{5\pi}{6}, \frac{9\pi}{10}$

17 $\dfrac{1 - \cos(100x)}{2 \sin(x)}$

Chapter 11 review

Short-answer questions

1 a 5, 1 **b** 4, −2 **c** 4, −4 **d** 2, 0 **e** $1, \frac{1}{3}$

2 a $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **b** $\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$
c $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **d** $\frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$

4 a $\frac{140}{221}$ **b** $\frac{-21}{221}$ **c** $\frac{171}{140}$

5 a $\frac{1}{2}$ **b** 1

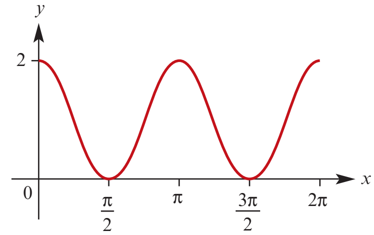
6 a 1 **b** 0

8 a $\frac{-1}{9}$ **b** $\frac{-4\sqrt{5}}{9}$ **c** $\frac{8\sqrt{5}}{81}$

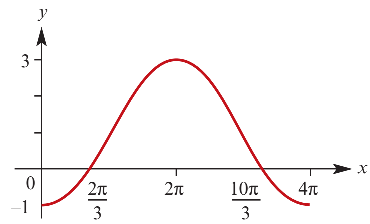
10 $2 - \sqrt{3}$

11 a $0, \frac{\pi}{2}, 2\pi$ **b** $\frac{7\pi}{6}, \frac{11\pi}{6}$ **c** $0, \pi, 2\pi$
d $\frac{\pi}{2}, \frac{3\pi}{2}$ **e** $\frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$
f $\frac{7\pi}{12}, \frac{3\pi}{4}, \frac{19\pi}{12}, \frac{7\pi}{4}$

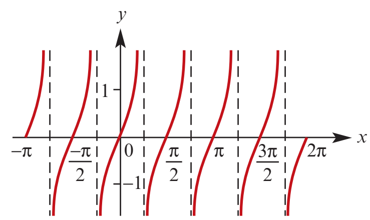
12 a $y = 2\cos^2 x$



b $y = 1 - 2\sin\left(\frac{\pi}{2} - \frac{x}{2}\right)$



c $f(x) = \tan(2x)$



13 $\frac{2}{9}$

14 a $\sqrt{85}\cos(\theta - \alpha)$ where $\alpha = \cos^{-1}\left(\frac{2}{\sqrt{85}}\right)$
b i $\sqrt{85}$ **ii** $\frac{2}{\sqrt{85}}$
iii $\theta = \cos^{-1}\left(\frac{2}{\sqrt{85}}\right) + \cos^{-1}\left(\frac{1}{\sqrt{85}}\right)$

15 a $0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$ **b** $0, \frac{\pi}{3}, \pi$

Multiple-choice questions

1 A **2** A **3** B **4** A **5** C
6 E **7** E **8** A **9** D **10** E

Extended-response questions

1 b $P = 10\sqrt{5}\cos(\theta - \alpha)$ where $\alpha = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$;
$\theta = 70.88°$
c $k = 25$ **d** $\theta = 45°$

2 a $AD = \cos\theta + 2\sin\theta$
b $AD = \sqrt{5}\cos(\theta - \alpha)$ where
$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63°$
c Max length of AD is $\sqrt{5}$ m when $\theta = 63°$
d $\theta = 79.38°$

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3 b ii $a = 1,\ b = 1$

c $\dfrac{1+\sqrt{2}-\sqrt{3}}{1+\sqrt{3}+\sqrt{6}} = \dfrac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1}$

$= \sqrt{6}+\sqrt{2}-\sqrt{3}-2$

4 a ii $2\cos\left(\dfrac{\pi}{5}\right)$

b iii $4\cos^2\left(\dfrac{\pi}{5}\right) - 2\cos\left(\dfrac{\pi}{5}\right) - 1 = 0$

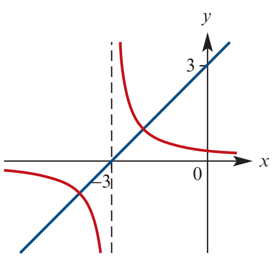
iv $\dfrac{1+\sqrt{5}}{4}$

5 b $-\dfrac{2}{3}$ or $\dfrac{1}{2}$

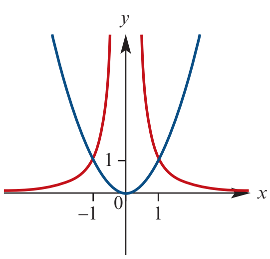
Chapter 12

Exercise 12A

1 a



b



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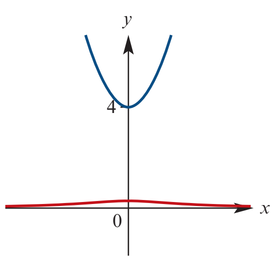
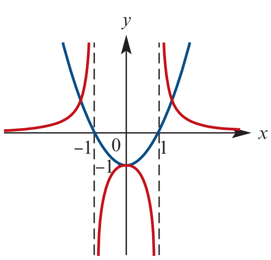


d



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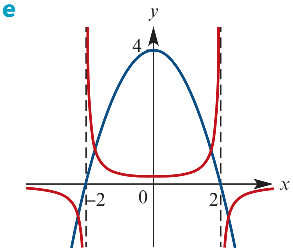

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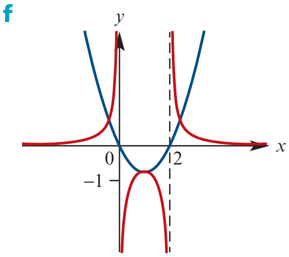

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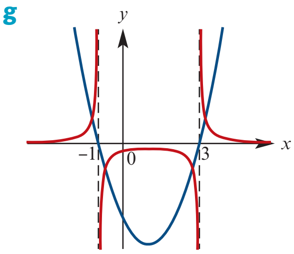

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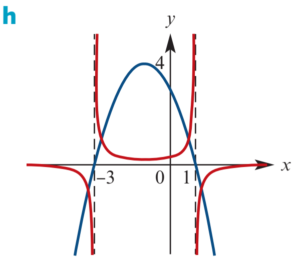

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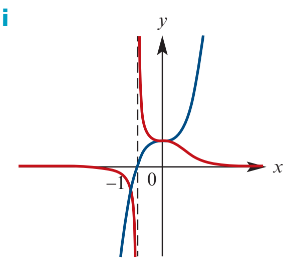
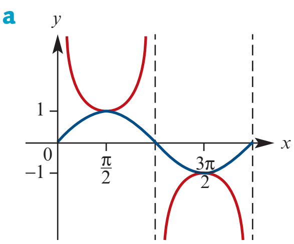


2 a



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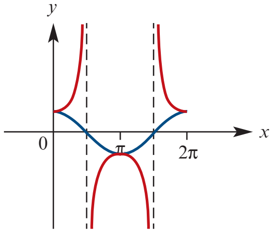
b



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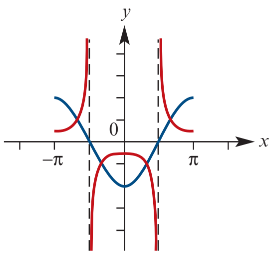

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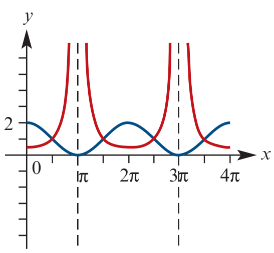

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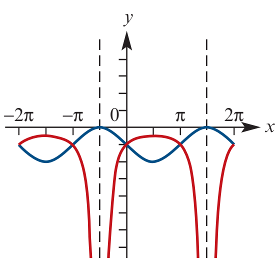

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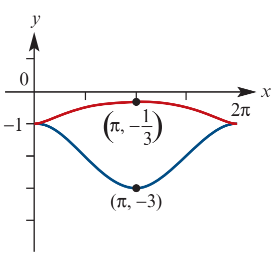

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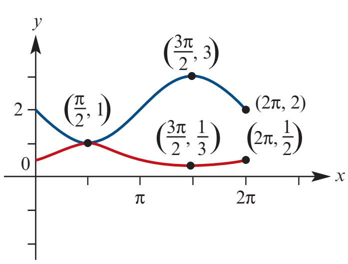

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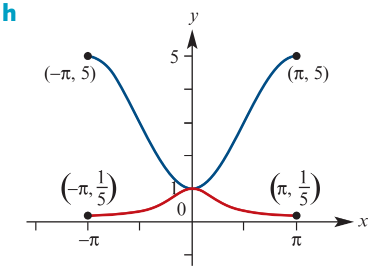
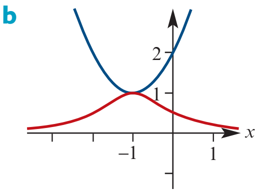
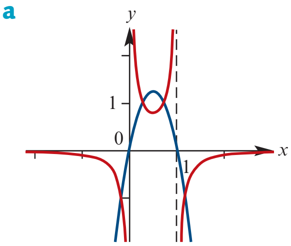


3 a $(-1, 1)$

b

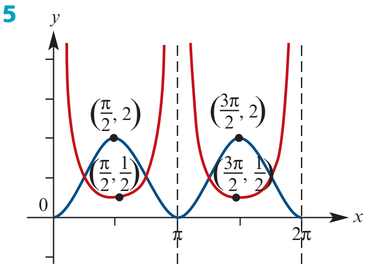


4 a



b $\left(\dfrac{5 \pm 3\sqrt{5}}{10}, -1\right), \left(\dfrac{5 \pm \sqrt{5}}{10}, 1\right)$

5



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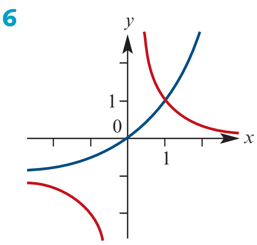

7 a $f(x) = (x + k)^2 + 1 - k^2$

b **i** $-1 < k < 1$

ii $k = \pm 1$

iii $k > 1$ or $k < -1$

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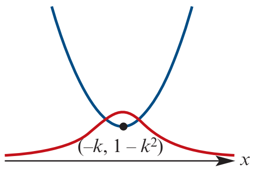
c i



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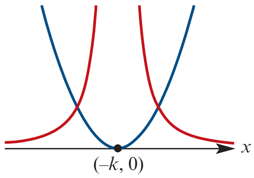
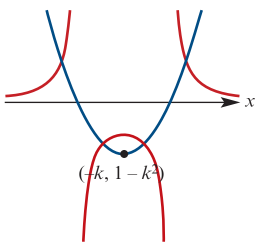
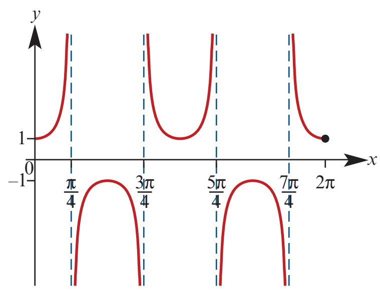


iii

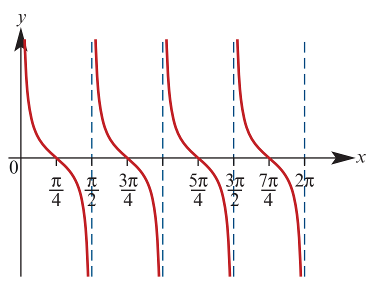


Exercise 12B

1 a



b



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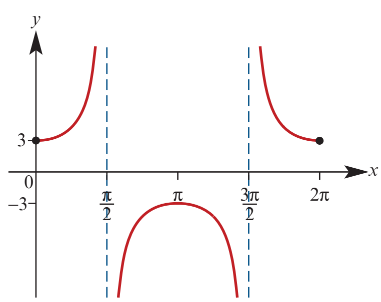
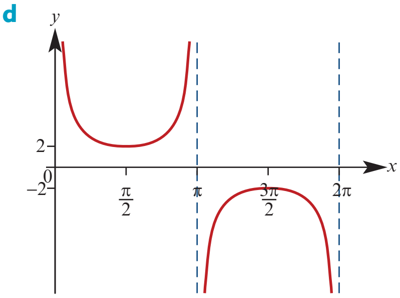
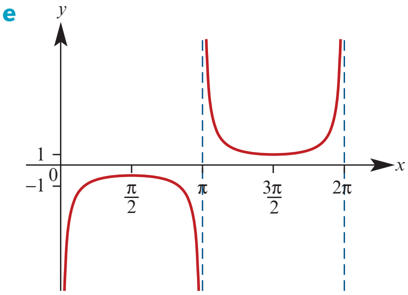


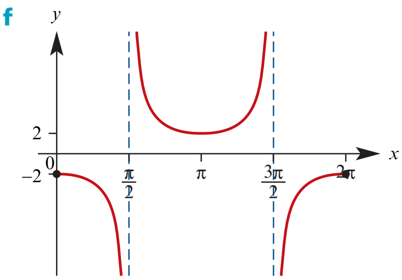
d



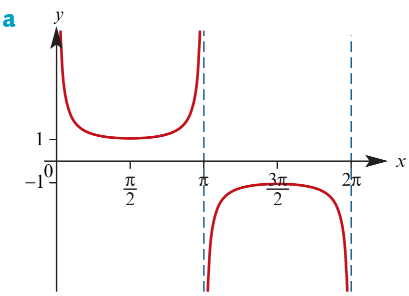
e



f



2 a



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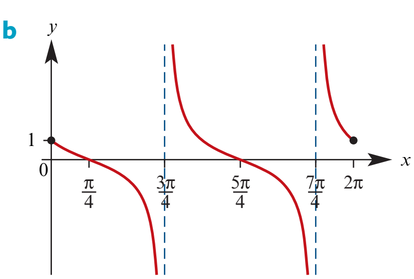

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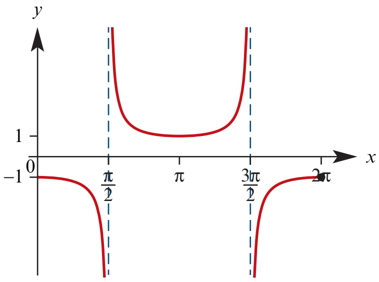
c



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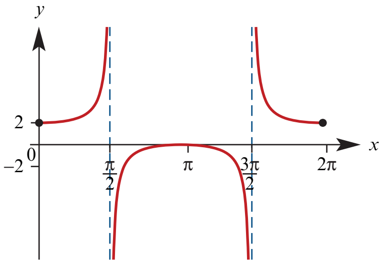

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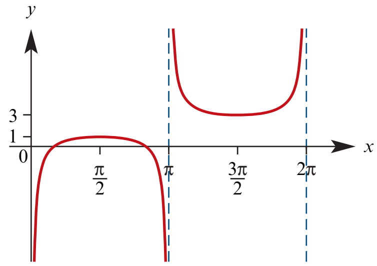

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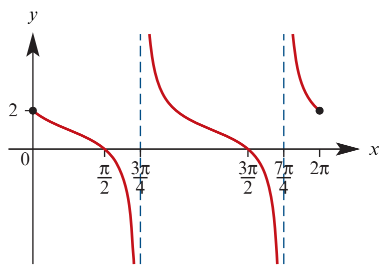
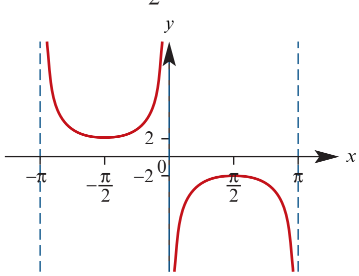
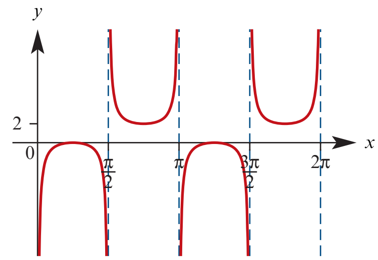


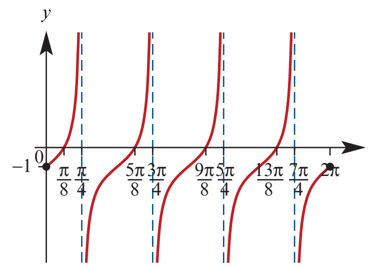
3
- Reflection in the x-axis
- Dilation of factor 2 from the x-axis
- Translation $\frac{\pi}{2}$ units to the right



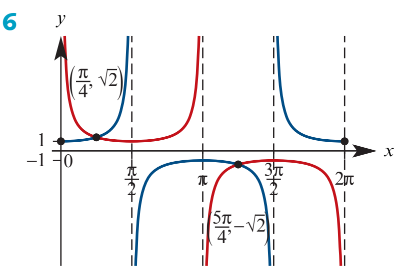
4
- Reflection in the y-axis
- Dilation of factor $\frac{1}{2}$ from the y-axis
- Translation 1 unit up



5
- Reflection in the x-axis
- Dilation of factor $\frac{1}{2}$ from the y-axis
- Translation $\frac{\pi}{4}$ units to the right and 1 unit down



6



Exercise 12C

1 $(x-1)^2 + (y+2)^2 = 4^2$

2 $(x+4)^2 + (y-3)^2 = 5^2$

3 **a** $y = -x$

4 **a** $y = \frac{x}{2} + \frac{3}{4}$

5 $(0, 3)$ or $(3, 0)$

6 $\left(\frac{9}{10}, \frac{3}{10}\right)$

7 $(6, 8)$ or $\left(\frac{72}{17}, \frac{154}{17}\right)$

8 **a** $2y - x = 1$ **b** $x + y = 2$ **c** $P(1, 1)$
d $(x-1)^2 + (y-1)^2 = 5^2$

9 $y = 2x + 1$

10 $y = 6$

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11 a The lines $x = 0$ and $y = 0$

b $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$

12 $(x - 4)^2 + y^2 = 4$

13 The lines $y = 1$ and $y = 5$

14 3

Exercise 12D

1 $y = \frac{x^2}{12}$

2 $y = -\frac{x^2}{12} - 1$

3 $x = \frac{y^2}{12} - 1$

4 a $x = \frac{y^2}{4c}$ **b** $\left(\frac{1}{12}, 0\right)$

5 a $y = \frac{1}{2b - 2c}(x^2 - 2ax + a^2 + b^2 - c^2)$

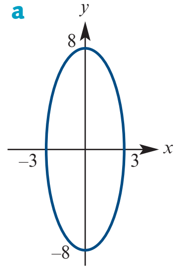
b $y = -\frac{1}{2}(x^2 - 2x - 4)$

6 $y = -1$ or $y = 19$

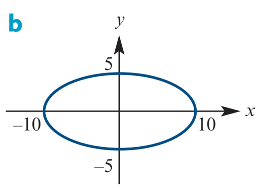
7 $(2, 1 + \sqrt{3})$ or $(2, 1 - \sqrt{3})$

Exercise 12E

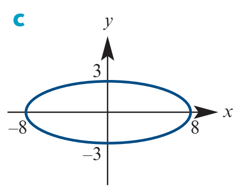
1 a



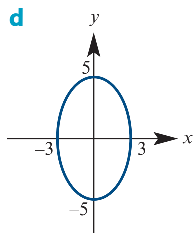
b



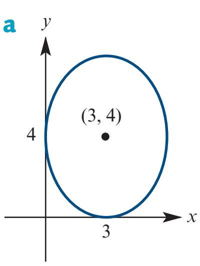
c



d



2 a



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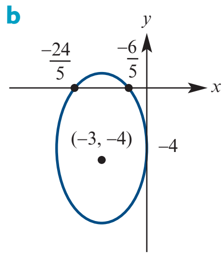
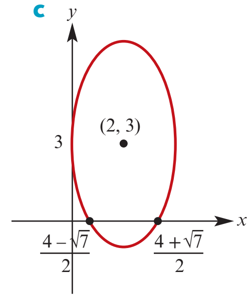


c



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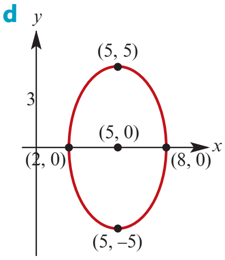


3 a $\frac{x^2}{25} + \frac{y^2}{16} = 1$ **b** $\frac{(x-2)^2}{9} + \frac{y^2}{4} = 1$

c $\frac{(x+1)^2}{4} + (y - 1)^2 = 1$

4 $\frac{x^2}{4} + \frac{y^2}{3} = 1$

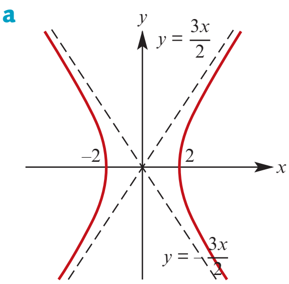
5 $\frac{x^2}{5} + \frac{y^2}{9} = 1$

6 $\frac{(x-4)^2}{16} + \frac{y^2}{12} = 1$

7 $\frac{x^2}{25} + \frac{y^2}{9} = 1$

Exercise 12F

1 a



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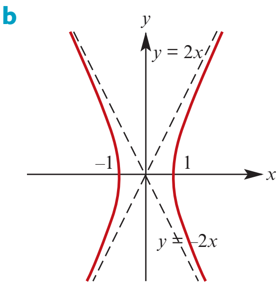

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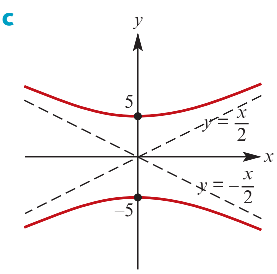

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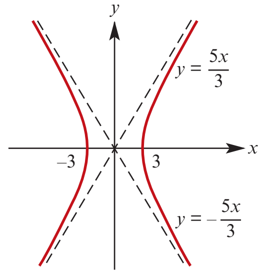
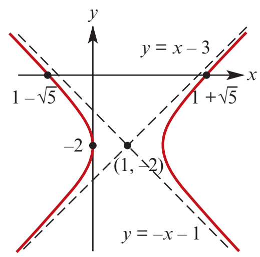
d



2 a



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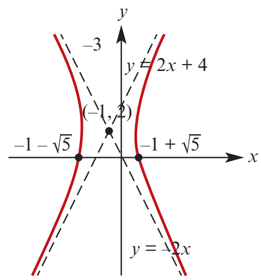

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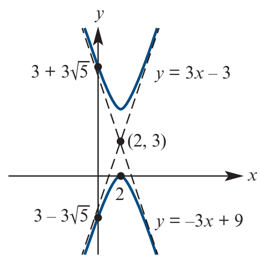

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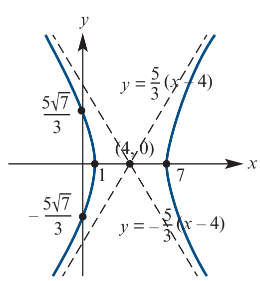

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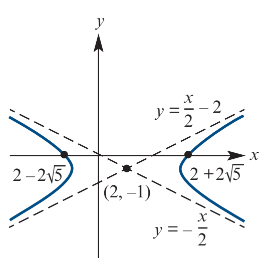

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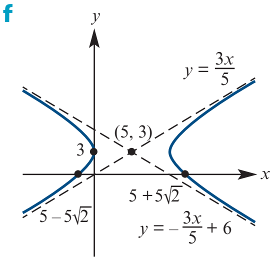


3 $\dfrac{x^2}{9} - \dfrac{y^2}{7} = 1$

4 $5x^2 - 4y^2 = 20$

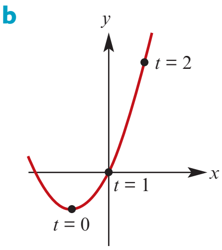
5 $\dfrac{(x+3)^2}{16} - \dfrac{y^2}{48} = 1$

6 $\dfrac{(y+5)^2}{4} - \dfrac{x^2}{12} = 1$

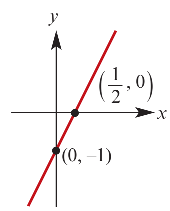
Exercise 12G

1 a $y = x^2 + 2x$

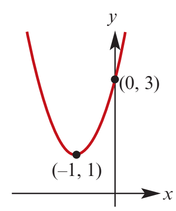
b



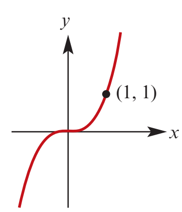
2 a $y = 2x - 1$



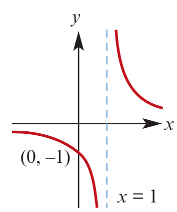
b $y = 2(x+1)^2 + 1$



c $y = x^3$



d $y = \dfrac{1}{x-1}$



3 a $x^2 + y^2 = 2^2$

b $\dfrac{(x+1)^2}{3^2} + \dfrac{(y-2)^2}{2^2} = 1$

c $x = 3\cos t - 3$ and $y = 3\sin t + 2$ (other answers are possible)

d $x = 3\cos t - 2$ and $y = 2\sin t + 1$ (other answers are possible)

4 $x = t$ and $y = 3t + 1$ (other answers are possible)

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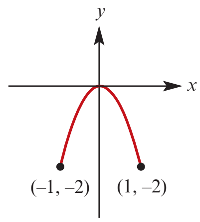
5 a $y = -2x^2$ where $-1 \le x \le 1$

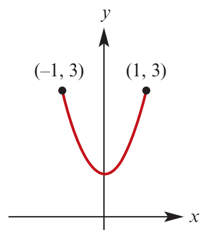
b



6 $\left(-\frac{3}{5}, -\frac{4}{5}\right), \left(\frac{3}{5}, \frac{4}{5}\right)$

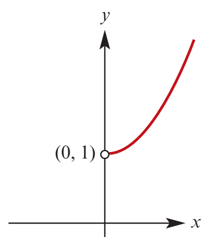
7 a $y = 2x^2 + 1$ **b** $-1 \le x \le 1$ **c** $1 \le y \le 3$

d

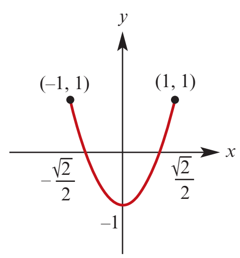


8 a $y = x^2 + 1$ **b** $x > 0$ **c** $y > 1$

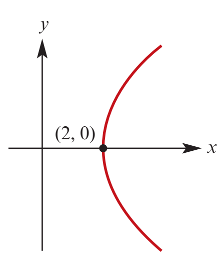
d



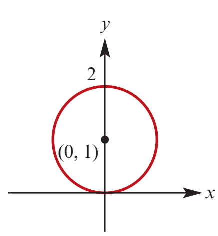
9 $y = -1 + 2x^2$ where $-1 \le x \le 1$



10 b



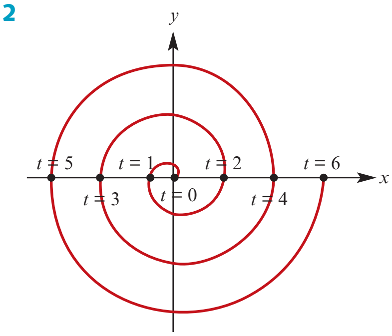
11 a



c $x = \dfrac{2t}{t^2 + 1}$

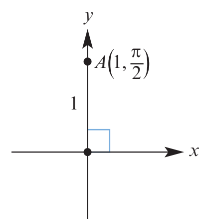
$y = \dfrac{2}{t^2 + 1}$

12

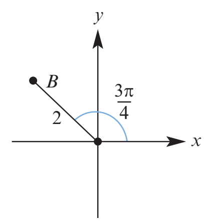


Exercise 12H

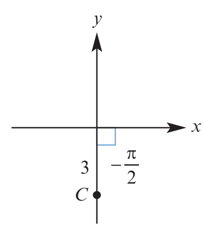
1 a $(0, 1)$



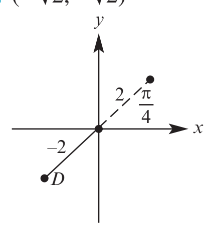
b $(-\sqrt{2}, \sqrt{2})$



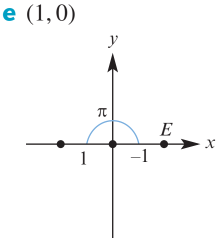
c $(0, -3)$



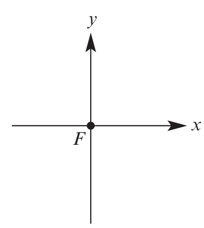
d $(-\sqrt{2}, -\sqrt{2})$



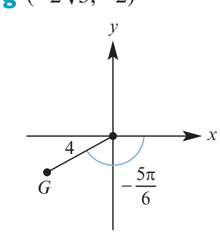
e $(1, 0)$



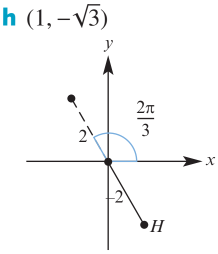
f $(0, 0)$



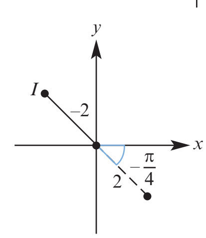
g $(-2\sqrt{3}, -2)$



h $(1, -\sqrt{3})$



i $(-\sqrt{2}, \sqrt{2})$



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2 a $\left(\sqrt{2}, -\frac{\pi}{4}\right), \left(-\sqrt{2}, \frac{3\pi}{4}\right)$

b $\left(2, \frac{\pi}{3}\right), \left(-2, \frac{4\pi}{3}\right)$

c $\left(2\sqrt{2}, -\frac{\pi}{4}\right), \left(-2\sqrt{2}, \frac{3\pi}{4}\right)$

d $\left(2, -\frac{3\pi}{4}\right), \left(-2, \frac{\pi}{4}\right)$

e $(3, 0), (-3, \pi)$

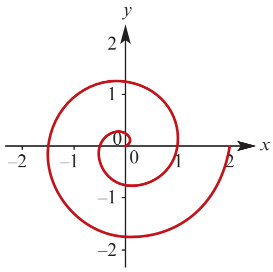
f $\left(2, -\frac{\pi}{2}\right), \left(-2, \frac{\pi}{2}\right)$

3 $\sqrt{7}$

4 $PQ = \sqrt{(r_1)^2 + (r_2)^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)}$

Exercise 12I

1



2 a $r = \dfrac{4}{\cos\theta}$ **b** $r^2 = \dfrac{1}{\cos\theta \sin\theta}$

c $r = \tan\theta \sec\theta$ **d** $r = 3$ or $r = -3$

e $r^2 = \dfrac{1}{\cos(2\theta)}$ **f** $r = \dfrac{5}{2\cos\theta - 3\sin\theta}$

3 a $x = 2$ **b** $x^2 + y^2 = 2^2$

c $y = x$ **d** $3x - 2y = 4$

4 a $(x-3)^2 + y^2 = 9$ **b** $x^2 + (y-2)^2 = 4$

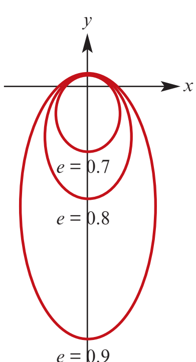
c $(x+3)^2 + y^2 = 9$ **d** $x^2 + (y+4)^2 = 16$

5 $(x-a)^2 + y^2 = a^2$

6 a Equation $x = a$ **b** $r = \dfrac{a}{\sin\theta}$

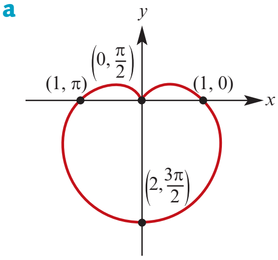
Exercise 12J

1 a

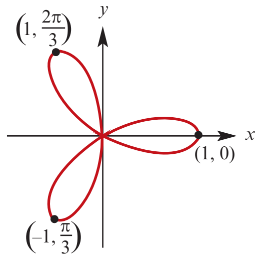


b Ellipse becomes larger and narrower

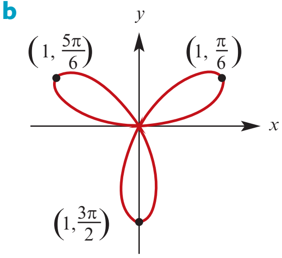
2 a



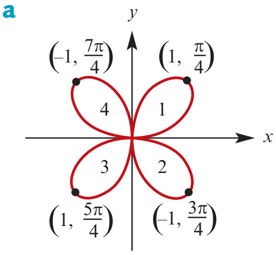
3 a



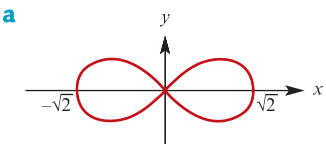
b



4 a



5 a

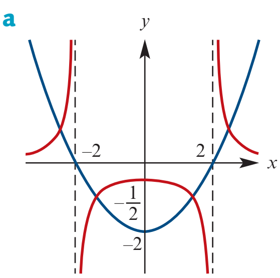


b $(x^2 + y^2)^2 = 2x^2 - 2y^2$

Chapter 12 review

Short-answer questions

1 a



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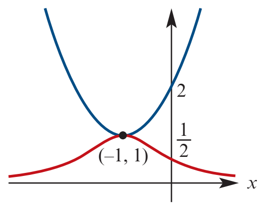
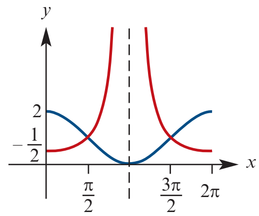
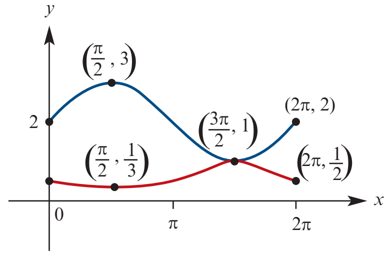
b



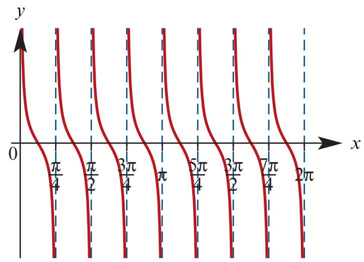
c



d



2 a



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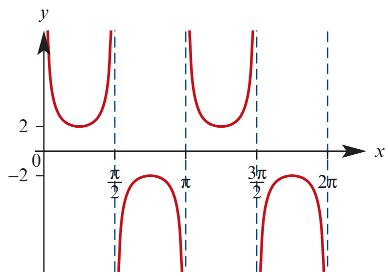

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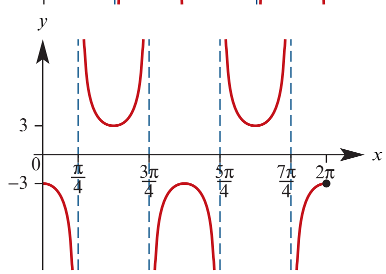
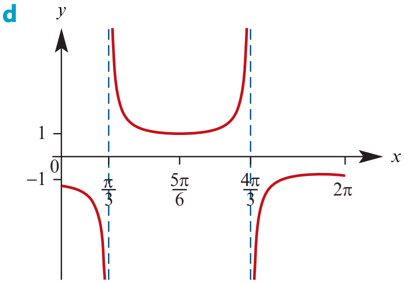
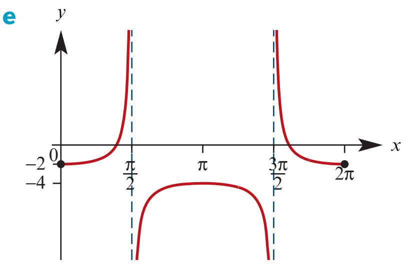


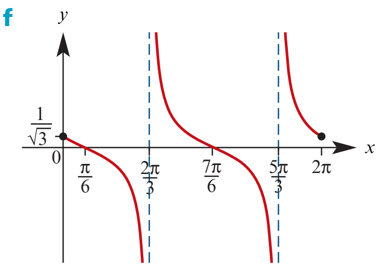
d



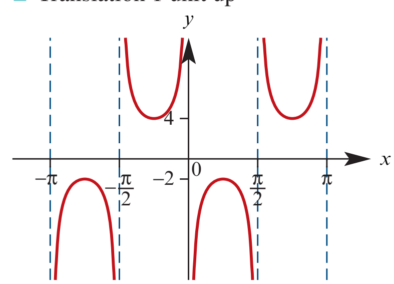
e



f



3
- Reflection in the x-axis
- Dilation of factor 3 from the x-axis
- Dilation of factor $\frac{1}{2}$ from the y-axis
- Translation 1 unit up

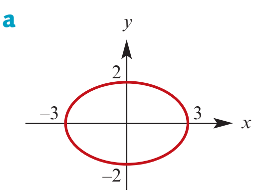


4 $y = \dfrac{x}{3}$

5 $(x - 3)^2 + (y - 2)^2 = 6^2$

6 $C(-2, 4)$, $r = \sqrt{20}$

7 a



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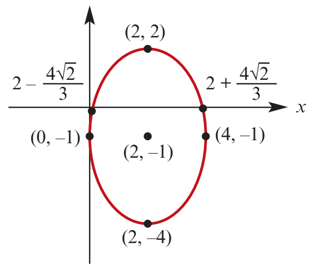
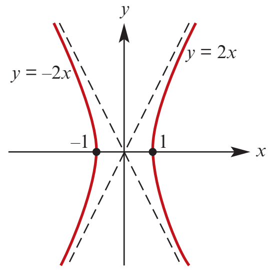
b

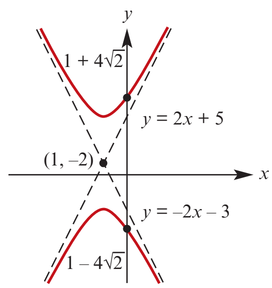


8 $C(-2, 0)$; Intercepts $(0, 0)$, $(-4, 0)$

9 a



b



10 $\dfrac{(x-2)^2}{4} - \dfrac{(y-5)^2}{12} = 1$

11 a $y = 4 - 2x$ **b** $x^2 + y^2 = 2^2$

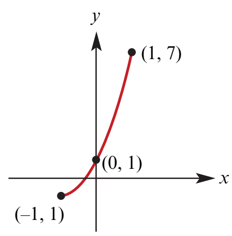
c $\dfrac{(x-1)^2}{3^2} + \dfrac{(y+1)^2}{5^2} = 1$

d $y = 1 - 3x^2$ where $-1 \le x \le 1$

12 a $y = 2(x+1)^2 - 1$ **b** $-1 \le x \le 1$

c $-1 \le y \le 7$

d



13 $(-\sqrt{2}, \sqrt{2})$

14 $\left(4, -\dfrac{\pi}{3}\right), \left(-4, \dfrac{2\pi}{3}\right)$

15 $r = \dfrac{5}{2\cos\theta + 3\sin\theta}$

16 $x^2 + (y-3)^2 = 9$

Multiple-choice questions

1 B **2** B **3** A **4** D **5** C
6 D **7** C **8** C **9** E **10** B

Extended-response questions

1 a $y = 2x - \dfrac{9}{2}$

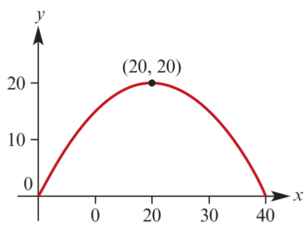
b $(x-8)^2 + (y+1)^2 = 20$

2 a $y = \dfrac{x^2}{12} + 1$ **b** $\dfrac{x^2}{12} + \dfrac{(y-6)^2}{16} = 1$

c $\dfrac{(y+4)^2}{16} - \dfrac{x^2}{48} = 1$

3 a $y = \dfrac{1}{20}x(40 - x)$

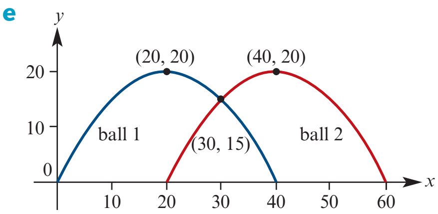
b



c 20 metres

d $y = -\dfrac{1}{20}(x-20)(x-60)$

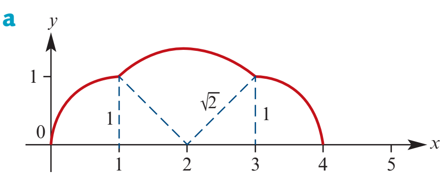
e



f (30, 15)

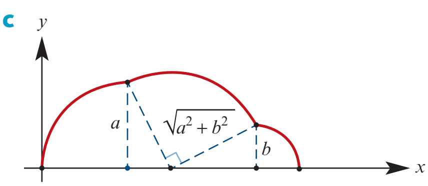
g Yes (same position at same time)

5 a



b $\dfrac{1}{2}(2\pi + \pi\sqrt{2})$

c



Distance $= \dfrac{\pi}{2}(a + \sqrt{a^2 + b^2} + b)$

d Area $= \dfrac{\pi}{2}(a^2 + b^2) + ab$

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Chapter 13

Exercise 13A

1

	Re(z)	Im(z)
a	2	3
c	$\frac{1}{2}$	$-\frac{3}{2}$
e	0	3

	Re(z)	Im(z)
b	4	5
d	−4	0
f	$\sqrt{2}$	$-2\sqrt{2}$

2 **a** $a = 2,\ b = -2$
b $a = 3,\ b = 2$ or $a = 2,\ b = 3$
c $a = 5,\ b = 0$ **d** $a = \frac{2}{3},\ b = -\frac{1}{3}$

3 **a** $6 - 8i$ **b** $6 - i$ **c** $-6 - 2i$
d $7 - 3\sqrt{2}i$ **e** $-2 - 3i$ **f** $4 + 2i$
g $6 - 4i$ **h** $-4 + 6i$ **i** $-1 + 11i$
j -1

4 **a** $4i$ **b** $6i$ **c** $\sqrt{2}i$
d $-i$ **e** -1 **f** 1
g -2 **h** -12 **i** -4

5 **a** $1 + 2i$ **b** $-3 + 4i$
c $-\sqrt{2} - 2i$ **d** $-\sqrt{6} - 3i$

Exercise 13B

1 **a** $15 + 8i$ **b** $-8i$ **c** $-2 + 16i$
d $2i$ **e** 5 **f** $-4 + 19i$

2 **a** $2 + 5i$ **b** $-1 - 3i$
c $\sqrt{5} + 2i$ **d** $5i$

3 **a** $2 + i$ **b** $-3 - 2i$ **c** $-4 + 7i$
d $-4 - 7i$ **e** $-4 - 7i$ **f** $-1 + i$
g $-1 - i$ **h** $-1 - i$

4 **a** $2 + 4i$ **b** 20 **c** 4
d $8 - 16i$ **e** $-8i$ **f** 8
g $\frac{1}{10}(1 + 2i)$ **h** $-4 - 2i$

5 $a = \frac{1}{29},\ b = -\frac{17}{29}$

6 **a** $\frac{7}{17} - \frac{6}{17}i$ **b** i **c** $\frac{7}{2} - \frac{1}{2}i$
d $-\frac{1}{2} - \frac{1}{2}i$ **e** $\frac{2}{13} + \frac{3}{13}i$ **f** $\frac{3}{20} + \frac{1}{20}i$

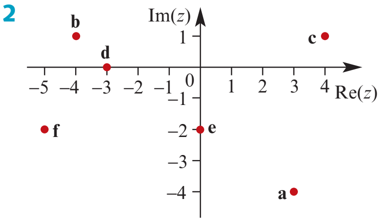
7 $a = \frac{5}{2},\ b = -\frac{3}{2}$

8 **a** $-\frac{42}{5}(1 - 2i)$ **b** $-\frac{1}{2}(1 - i)$
c $\frac{1}{17}(4 + i)$ **d** $\frac{1}{130}(6 + 43i)$
e $2 - 2i$

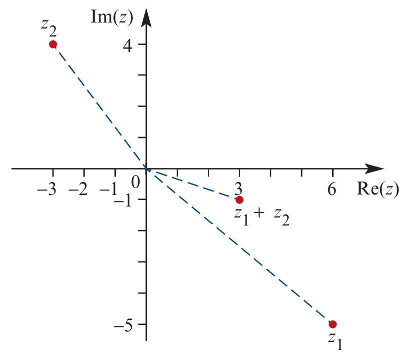
Exercise 13C

1 $A = 3 + i,\quad B = 2i,\quad C = -3 - 4i$
$D = 2 - 2i,\quad E = -3,\quad F = -1 - i$

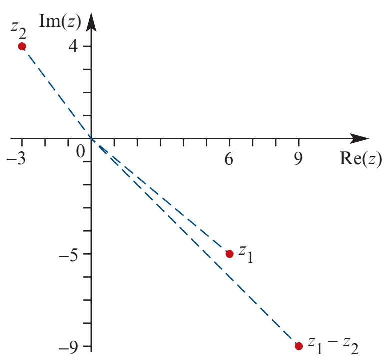
2



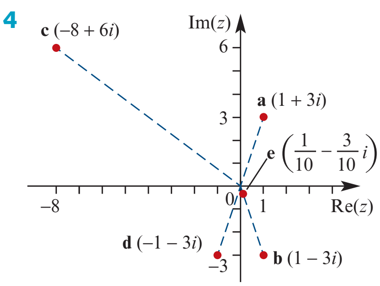
3 **a** $z_1 + z_2 = 3 - i$



b $z_1 - z_2 = 9 - 9i$



4



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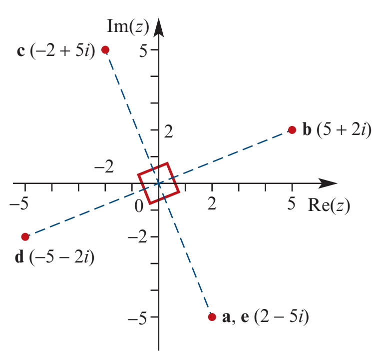

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Exercise 13D

1 **a** $\pm 2i$ **b** $\pm 3i$ **c** $\pm\sqrt{5}i$ **d** $2 \pm 4i$
e $-1 \pm 7i$ **f** $1 \pm \sqrt{2}i$ **g** $\frac{1}{2}(-3 \pm \sqrt{3}i)$
h $\frac{1}{4}(-5 \pm \sqrt{7}i)$ **i** $\frac{1}{6}(1 \pm \sqrt{23}i)$
j $1 \pm 2i$ **k** $\frac{1}{2}(3 \pm \sqrt{11}i)$ **l** $3 \pm \sqrt{5}i$

2 **a** $(z + 3i)(z - 3i)$ **b** $(z + \sqrt{3}i)(z - \sqrt{3}i)$
c $3(z + 2i)(z - 2i)$
d $(z + 1 + 2i)(z + 1 - 2i)$
e $\left(z - \frac{3}{2} + \frac{\sqrt{15}}{2}i\right)\left(z - \frac{3}{2} - \frac{\sqrt{15}}{2}i\right)$
f $2\left(z + \frac{1}{2} + \frac{1}{2}i\right)\left(z + \frac{1}{2} - \frac{1}{2}i\right)$

Exercise 13E

1 **a** $2\operatorname{cis}\left(\frac{\pi}{3}\right)$ **b** $\sqrt{2}\operatorname{cis}\left(\frac{-\pi}{4}\right)$
c $4\operatorname{cis}\left(\frac{5\pi}{6}\right)$ **d** $4\sqrt{2}\operatorname{cis}\left(\frac{-3\pi}{4}\right)$
e $24\operatorname{cis}\left(\frac{-\pi}{3}\right)$ **f** $\frac{1}{\sqrt{2}}\operatorname{cis}\left(\frac{3\pi}{4}\right)$

2 **a** $3i$ **b** $\frac{1}{\sqrt{2}}(1 + \sqrt{3}i) = \frac{\sqrt{2}}{2}(1 + \sqrt{3}i)$
c $\sqrt{3} + i$ **d** $\frac{-5}{\sqrt{2}}(1 - i) = -\frac{5\sqrt{2}}{2}(1 - i)$
e $-6(\sqrt{3} - i)$ **f** $3(1 - i)$
g $-\frac{5}{2}(1 + \sqrt{3}i)$ **h** $-\frac{5}{2}(1 + \sqrt{3}i)$

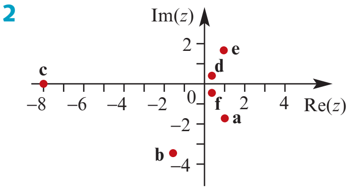
3 **a** $3\sqrt{2}(1 + i)$ **b** $6(1 + \sqrt{3}i)$
c $-\frac{5}{2}(1 - \sqrt{3}i)$ **d** $18(1 + \sqrt{3}i)$
e $-18(1 + \sqrt{3}i)$ **f** $\sqrt{3}(1 + i)$
g $\sqrt{3} + i$ **h** -4
i $-4(1 - \sqrt{3}i)$ **j** $-\frac{5}{2}$

Chapter 13 review

Short-answer questions

1 **a** $(2m + 3p) + (2n + 3q)i$ **b** $p - qi$
c $(mp + nq) + (np - mq)i$
d $\frac{(mp + nq) + (np - mq)i}{p^2 + q^2}$ **e** $2m$
f $(m^2 - n^2 - p^2 + q^2) + (2mn - 2pq)i$
g $\frac{m - ni}{m^2 + n^2}$
h $\frac{(mp + nq) + (mq - np)i}{m^2 + n^2}$
i $\frac{3((mp + nq) + (np - mq)i)}{p^2 + q^2}$

2

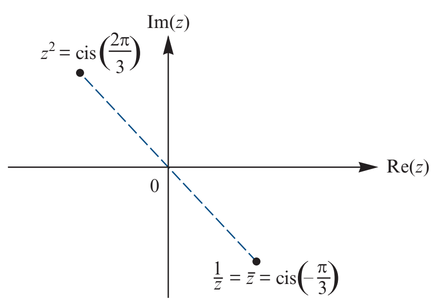


a $1 - \sqrt{3}i$ **b** $-2 - 2\sqrt{3}i$ **c** -8
d $\frac{1}{4}(1 + \sqrt{3}i)$ **e** $1 + \sqrt{3}i$ **f** $\frac{1}{4}(1 - \sqrt{3}i)$

3 **a** $\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$ **b** $2\operatorname{cis}\left(\frac{-\pi}{3}\right)$
c $\sqrt{13}\operatorname{cis}\left(\tan^{-1}\left(\frac{\sqrt{3}}{6}\right)\right)$ **d** $6\operatorname{cis}\left(\frac{\pi}{4}\right)$
e $6\operatorname{cis}\left(\frac{-3\pi}{4}\right)$ **f** $2\operatorname{cis}\left(\frac{-\pi}{6}\right)$

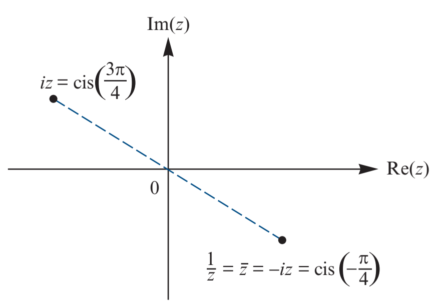
4 **a** $-1 - \sqrt{3}i$ **b** $\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$
c $\frac{-3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$ **d** $\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$
e $\frac{-3\sqrt{3}}{2} - \frac{3}{2}i$ **f** $1 - i$

5



a $z^2 = \operatorname{cis}\left(\frac{2\pi}{3}\right)$ **b** $\bar{z} = \operatorname{cis}\left(-\frac{\pi}{3}\right)$
c $\frac{1}{z} = \operatorname{cis}\left(-\frac{\pi}{3}\right)$ **d** $\operatorname{cis}\left(\frac{2\pi}{3}\right)$

6



a $iz = \operatorname{cis}\left(\frac{3\pi}{4}\right)$ **b** $\bar{z} = \operatorname{cis}\left(-\frac{\pi}{4}\right)$
c $\frac{1}{z} = \operatorname{cis}\left(-\frac{\pi}{4}\right)$ **d** $-iz = \operatorname{cis}\left(-\frac{\pi}{4}\right)$

Multiple-choice questions

1 C **2** D **3** C **4** D **5** D
6 E **7** D **8** D **9** B **10** D

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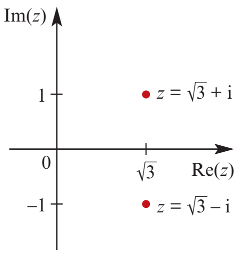
Extended-response questions

1 a $z = \sqrt{3} + i$ or $z = \sqrt{3} - i$

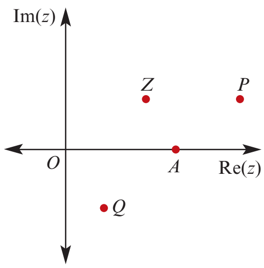
b i



ii $x^2 + y^2 = 4$ **iii** $a = 2$

2 a i $6\sqrt{2}$ **ii** 6

3 a



b $\sqrt{2} + 1$

6 a $|z+1| = \sqrt{2 + 2\cos\theta} = 2\cos\left(\frac{\theta}{2}\right)$,

$\text{Arg}(z+1) = \frac{\theta}{2}$

b $|z-1| = \sqrt{2 - 2\cos\theta} = 2\sin\left(\frac{\theta}{2}\right)$,

$\text{Arg}(z-1) = \frac{\pi+\theta}{2}$

c $\left|\frac{z-1}{z+1}\right| = \tan\left(\frac{\theta}{2}\right)$, $\text{Arg}\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$

7 a $\Delta = b^2 - 4ac$

b $b^2 < 4ac$

c i $-\frac{b}{a}, \frac{\sqrt{4ac}}{2a}$ **ii** $\frac{b^2}{2ac} - 1$

8 a $z_1 = \frac{1}{2}(-1 + \sqrt{3}i)$, $z_2 = \frac{1}{2}(-1 - \sqrt{3}i)$

c $|z_1| = 1$, $\text{Arg}(z_1) = \frac{2\pi}{3}$;

$|z_2| = 1$, $\text{Arg}(z_2) = -\frac{2\pi}{3}$

d $\frac{\sqrt{3}}{4}$

Chapter 14

Short-answer questions

1 a $\frac{\pi}{2}, \frac{5\pi}{2}, \frac{13\pi}{2}, \frac{17\pi}{2}$

b $\frac{-17\pi}{24}, \frac{-11\pi}{24}, \frac{7\pi}{24}, \frac{13\pi}{24}$

2 $x = 2n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}$

3 a $\frac{5}{4}$ **b** $\frac{4}{3}$ **c** $-\frac{\sqrt{3}}{3}$ **d** $\frac{2\sqrt{3}}{3}$

4 $\pm\frac{\sqrt{6}}{3}$

6 a $\frac{1}{2}\sin(4x) - \frac{1}{2}\sin(2x)$

b $\theta = \frac{(2n+1)\pi}{2}$ or $\theta = \frac{(2n+1)\pi}{4}, n \in \mathbb{Z}$

8 a 6 **b** $4i$ **c** 13 **d** 10
e 36 **f** −16 **g** $24i$ **h** $24i$

9 a $3 - 5i$ **b** $-1 + i$ **c** $-4 - 7i$ **d** $\frac{8-i}{13}$
e $2 + i$ **f** $\frac{-2+i}{5}$ **g** $-2 - i$ **h** $\frac{8+i}{5}$
i $\frac{13-i}{34}$ **j** $3 - i$ **k** $\frac{-1-3i}{2}$ **l** $-3 - 4i$

10 a $(z - 7i)(z + 7i)$

b $(z - 1 - 3i)(z - 1 + 3i)$

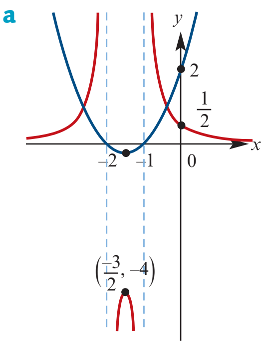
c $9\left(z - \frac{1}{3} - \frac{2}{3}i\right)\left(z - \frac{1}{3} + \frac{2}{3}i\right)$

d $4\left(z + \frac{3}{2} - i\right)\left(z + \frac{3}{2} + i\right)$

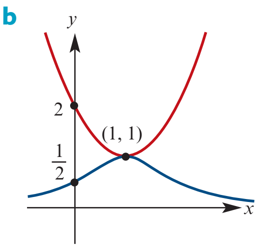
11 a $2 + i$, $-2 - i$

b $z = -1 - i$ or $z = -i$

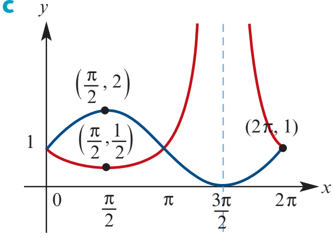
12 a



b



c



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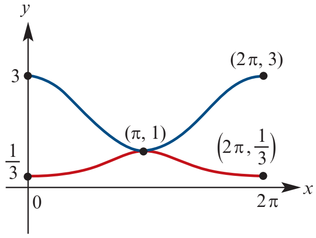
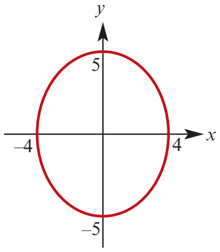
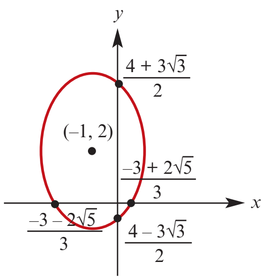
d



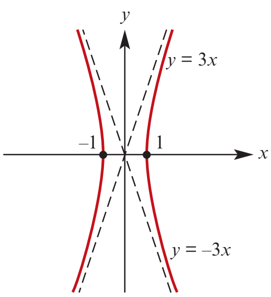
13 a



b



14 a



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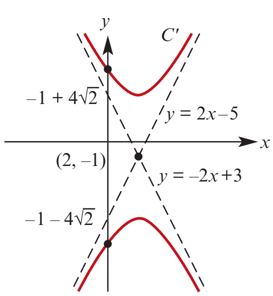


15 $2x + 4y = 17$

16 $\dfrac{(x-1)^2}{4} + \dfrac{(y-1)^2}{3} = 1$

17 $y = \dfrac{x^2}{8} - 1$

18 a $3x + 2y = 7$

b $x^2 + y^2 = 1$

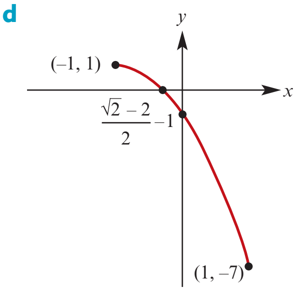
c $\dfrac{(x-2)^2}{4} + \dfrac{(y-3)^2}{9} = 1$

d $\dfrac{y^2}{9} - \dfrac{x^2}{4} = 1$

19 a $y = 1 - 2(x+1)^2$ **b** $-1 \le x \le 1$

c $-7 \le y \le 1$

d



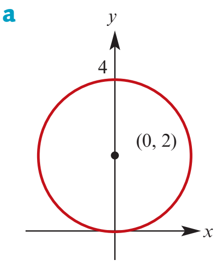
20 $(-\sqrt{3}, -1)$

21 $\left(2\sqrt{2}, -\dfrac{\pi}{4}\right), \left(-2\sqrt{2}, \dfrac{3\pi}{4}\right)$

22 a $x^2 + y^2 = 5^2$ **b** $y = \sqrt{3}x$ **c** $y = 3$

d $3y + 4x = 2$ **e** $y = \dfrac{1}{2x}$

23 a



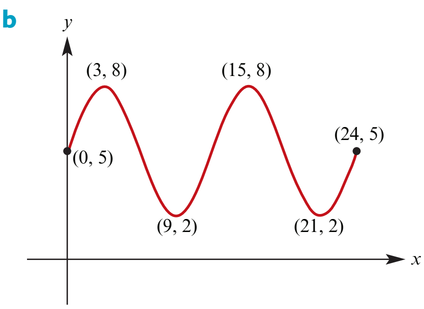
Multiple-choice questions

1 B	**2** B	**3** E	**4** E	**5** D	**6** D
7 C	**8** A	**9** D	**10** E	**11** A	**12** A
13 E	**14** C	**15** C	**16** C	**17** B	**18** D
19 E	**20** A	**21** C	**22** C	**23** A	**24** C
25 E	**26** C	**27** A	**28** B	**29** C	**30** B
31 B	**32** C	**33** B	**34** D	**35** A	**36** E
37 D	**38** A				

Extended-response questions

1 a i 5 m **ii** 8 m

b



c i 8 m **ii** 2 m

d i 0, 6, 12, 18, 24

ii 0.65, 5.35, 12.65, 17.35

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2 **a** $x = 4, \ y = 9$

b **i** 4 **ii** 2

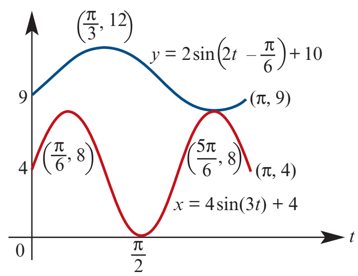
c **i** 8, 0 **ii** 12, 8

d **i** $\frac{2\pi}{3}$ **ii** π

e $\frac{\pi}{6}$ s **f** $\frac{5\pi}{6}, \frac{3\pi}{2}, \frac{13\pi}{6}, \frac{17\pi}{6}$

g $\frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \frac{23\pi}{6}$

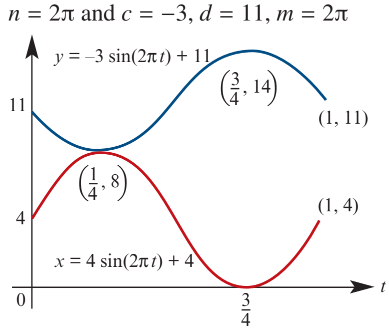
h



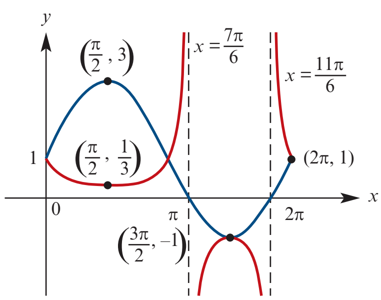
i $\frac{5\pi}{6}$ s **j** 2π s

3 **a** One possible set of values is $a = 4$, $b = 4$, $n = 2\pi$ and $c = -3$, $d = 11$, $m = 2\pi$

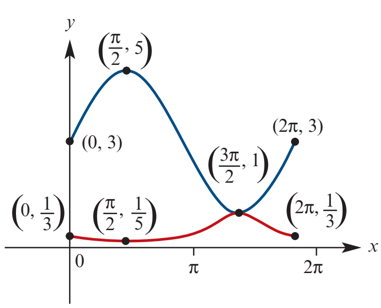
b



4 **a** **i**

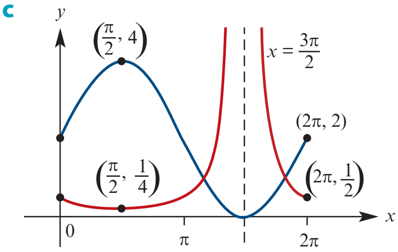


ii



b $k = 2$

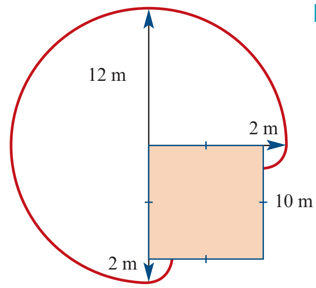
c



5 **a** $y = \frac{x}{4} - \frac{3}{8}$ **c** $\frac{\sqrt{17}}{2}$ km

6 **c** $x = t$ and $y = -t + 3$ **e** $k > \frac{8}{\sqrt{5}}$ or $k < -\frac{8}{\sqrt{5}}$

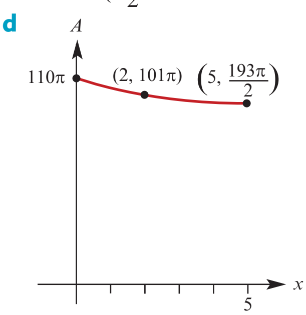
7 **a**



b 110π m^2

c $A(x) = \begin{cases} \frac{3\pi x^2}{4} - 6\pi x + 110\pi, & 0 \le x \le 2 \\ \frac{\pi x^2}{2} - 5\pi x + 109\pi, & 2 < x \le 5 \end{cases}$

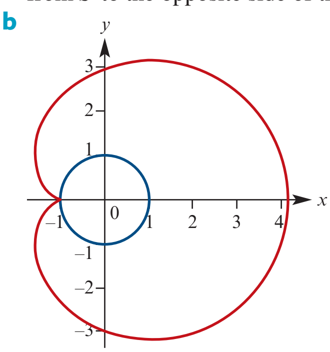
d



e **i** $x = 0$ **ii** $x = 5$

8 **a** Length of rope, π, is equal to the arc length from S to the opposite side of the circle

b



c **i** θ **ii** $\pi - \theta$ **iii** θ

iv $(\pi - \theta)\sin\theta$ **v** $(\pi - \theta)\cos\theta$

d $x = \cos\theta - (\pi - \theta)\sin\theta$ and $y = \sin\theta + (\pi - \theta)\cos\theta$

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9 d $z = \dfrac{1}{\sqrt[3]{16}}\left(\sqrt[3]{1+\sqrt{3}i} + \sqrt[3]{1-\sqrt{3}i}\right)$

Chapter 15

Exercise 15A

1 a 2×2 **b** 2×3 **c** 1×4 **d** 4×1

2 a $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

3 a $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$

c $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

4 $\begin{bmatrix} 200 & 180 & 135 & 110 & 56 & 28 \\ 110 & 117 & 98 & 89 & 53 & 33 \end{bmatrix}$

5 a $\begin{bmatrix} 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 4 \end{bmatrix}$ if $x = 4$

b $\begin{bmatrix} 4 & 7 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} x & 7 \\ 1 & -2 \end{bmatrix}$ if $x = 4$

c $\begin{bmatrix} 2 & x & 4 \\ -1 & 10 & 3 \end{bmatrix} = \begin{bmatrix} y & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix} =$
$\begin{bmatrix} 2 & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix}$ if $x = 0, y = 2$

6 a $x = 2, y = 3$ **b** $x = 3, y = 2$
c $x = 4, y = -3$ **d** $x = 3, y = -2$

7 $\begin{bmatrix} 21 & 5 & 5 \\ 8 & 2 & 3 \\ 4 & 1 & 1 \\ 14 & 8 & 60 \\ 0 & 1 & 2 \end{bmatrix}$

Exercise 15B

1 $\mathbf{X} + \mathbf{Y} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ $2\mathbf{X} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ $4\mathbf{Y} + \mathbf{X} = \begin{bmatrix} 13 \\ -2 \end{bmatrix}$

$\mathbf{X} - \mathbf{Y} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ $-3\mathbf{A} = \begin{bmatrix} -3 & 3 \\ -6 & -9 \end{bmatrix}$

$-3\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 3 \\ -7 & -7 \end{bmatrix}$

2 $2\mathbf{A} = \begin{bmatrix} 2 & -2 \\ 0 & 4 \end{bmatrix}$ $-3\mathbf{A} = \begin{bmatrix} -3 & 3 \\ 0 & -6 \end{bmatrix}$

$-6\mathbf{A} = \begin{bmatrix} -6 & 6 \\ 0 & -12 \end{bmatrix}$

3 a Yes **b** Yes

4 a $\begin{bmatrix} 6 & 4 \\ -4 & -4 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & -9 \\ 12 & 3 \end{bmatrix}$

c $\begin{bmatrix} 6 & -5 \\ 8 & -1 \end{bmatrix}$ **d** $\begin{bmatrix} -6 & -13 \\ 16 & 7 \end{bmatrix}$

5 a $\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} -2 & 3 \\ 6 & 3 \end{bmatrix}$

c $\begin{bmatrix} 3 & 3 \\ -1 & 7 \end{bmatrix}$

6 $\mathbf{X} = \begin{bmatrix} 2 & 4 \\ 0 & -3 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} \frac{-9}{2} & \frac{-23}{2} \\ \frac{-1}{2} & 11 \end{bmatrix}$

7 $\mathbf{X} + \mathbf{Y} = \begin{bmatrix} 310 & 180 & 220 & 90 \\ 200 & 0 & 125 & 0 \end{bmatrix}$
represents the total production at two factories in two successive weeks

Exercise 15C

1 $\mathbf{AX} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$ $\mathbf{BX} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ $\mathbf{AY} = \begin{bmatrix} -5 \\ 8 \end{bmatrix}$

$\mathbf{IX} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\mathbf{AC} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

$\mathbf{CA} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ $(\mathbf{AC})\mathbf{X} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\mathbf{C}(\mathbf{BX}) = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$ $\mathbf{AI} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

$\mathbf{IB} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ $\mathbf{AB} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\mathbf{BA} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\mathbf{A}^2 = \begin{bmatrix} 3 & -8 \\ -4 & 11 \end{bmatrix}$

$\mathbf{B}^2 = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$ $\mathbf{A}(\mathbf{CA}) = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$

$\mathbf{A}^2\mathbf{C} = \begin{bmatrix} -2 & -5 \\ 3 & 7 \end{bmatrix}$

2 Defined: **AY**, **CI**;
Not defined: **YA**, **XY**, $\mathbf{X}^2$, **XI**

3 $\mathbf{AB} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

4 No

5 One possible answer is $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

6 $\mathbf{LX} = \begin{bmatrix} 7 \end{bmatrix}$, $\mathbf{XL} = \begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$

7 **AB** and **BA** are not defined unless $m = n$

8 b $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

9 One possible answer is
$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$

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10 One possible answer is

$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix}$,

$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \begin{bmatrix} -1 & 11 \\ -4 & 24 \end{bmatrix}$, $\mathbf{AB} + \mathbf{AC} = \begin{bmatrix} -1 & 11 \\ -4 & 24 \end{bmatrix}$,

$(\mathbf{B} + \mathbf{C})\mathbf{A} = \begin{bmatrix} 11 & 7 \\ 16 & 12 \end{bmatrix}$

11 For example: $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

12 a $\begin{bmatrix} 29 \\ 8.50 \end{bmatrix}$, John took 29 minutes to eat food costing \$8.50

b $\begin{bmatrix} 29 & 22 & 12 \\ 8.50 & 8.00 & 3.00 \end{bmatrix}$,
John's friends took 22 and 12 minutes to eat food costing \$8.00 and \$3.00 respectively

13 $\mathbf{A}^2 = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix}$, $\mathbf{A}^4 = \begin{bmatrix} -7 & -24 \\ 24 & -7 \end{bmatrix}$,

$\mathbf{A}^8 = \begin{bmatrix} -527 & 336 \\ -336 & -527 \end{bmatrix}$

14 $\mathbf{A}^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $\mathbf{A}^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, $\mathbf{A}^4 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$,

$\mathbf{A}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

Exercise 15D

1 a 1 **b** $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ **c** 2 **d** $\frac{1}{2}\begin{bmatrix} 2 & 2 \\ -3 & -2 \end{bmatrix}$

2 a $\begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} \frac{2}{7} & \frac{-1}{14} \\ \frac{1}{7} & \frac{3}{14} \end{bmatrix}$

c $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$ **d** $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

4 a $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & -1 \end{bmatrix}$, $\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

b $\mathbf{AB} = \begin{bmatrix} 5 & 1 \\ -3 & -1 \end{bmatrix}$, $(\mathbf{AB})^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix}$

c $\mathbf{A}^{-1}\mathbf{B}^{-1} = \begin{bmatrix} -1 & \frac{1}{2} \\ 3 & -1 \end{bmatrix}$,

$\mathbf{B}^{-1}\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix}$, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

5 a $\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ 1 & -2 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 7 \\ 1 & -8 \end{bmatrix}$

c $\begin{bmatrix} \frac{5}{2} & -\frac{7}{2} \\ \frac{11}{2} & -\frac{21}{2} \end{bmatrix}$

6 a $\begin{bmatrix} \frac{-3}{8} & \frac{11}{8} \\ \frac{1}{16} & \frac{7}{16} \end{bmatrix}$ **b** $\begin{bmatrix} \frac{-11}{16} & \frac{17}{16} \\ \frac{-1}{4} & \frac{3}{4} \end{bmatrix}$

7 $\begin{bmatrix} \frac{1}{a_{11}} & 0 \\ 0 & \frac{1}{a_{22}} \end{bmatrix}$

9 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$,

$\begin{bmatrix} 1 & 0 \\ k & -1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} -1 & k \\ 0 & 1 \end{bmatrix}$, $k \in \mathbb{R}$,

$\begin{bmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{bmatrix}$, $b \neq 0$

10 $a = \pm\sqrt{2}$

Exercise 15E

1 a $\begin{bmatrix} 3 \\ 10 \end{bmatrix}$ **b** $\begin{bmatrix} 5 \\ 17 \end{bmatrix}$

2 a $x = -\frac{1}{7}$, $y = \frac{10}{7}$ **b** $x = 4$, $y = 1.5$

3 $(2, -1)$

4 Book \$12, CD \$18

5 a $\begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

b $\begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}$ is non-invertible

c System has solutions (not a unique solution)

d Solution set contains infinitely many pairs

6 a $\mathbf{A}^{-1}\mathbf{C}$ **b** $\mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{C}$ **c** $\mathbf{A}^{-1}\mathbf{C}\mathbf{B}^{-1}$

d $\mathbf{A}^{-1}\mathbf{C} - \mathbf{B}$ **e** $\mathbf{A}^{-1}(\mathbf{C} - \mathbf{B})$

f $(\mathbf{A} - \mathbf{B})\mathbf{A}^{-1} = \mathbf{I} - \mathbf{B}\mathbf{A}^{-1}$

Chapter 15 review

Short-answer questions

1 a $\begin{bmatrix} 0 & 0 \\ 12 & 8 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 0 \\ 8 & 8 \end{bmatrix}$

2 $\begin{bmatrix} a \\ 2 - \frac{3}{4}a \end{bmatrix}$, $a \in \mathbb{R}$

3 a Exist: **AC**, **CD**, **BE**; Does not exist: **AB**

b $\mathbf{DA} = \begin{bmatrix} 14 & 0 \end{bmatrix}$, $\mathbf{A}^{-1} = \frac{1}{7}\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$

4 $\mathbf{AB} = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}$, $\mathbf{C}^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

5 $\begin{bmatrix} -1 & 2 \\ -3 & 5 \end{bmatrix}$

6 $\mathbf{A}^2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$

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7 8

8 a i $\begin{bmatrix} 3 & -5 \\ 5 & 8 \end{bmatrix}$ **ii** $\begin{bmatrix} 1 & -18 \\ 18 & 19 \end{bmatrix}$ **iii** $\frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

b $x = 2, \ y = 1$

Multiple-choice questions

1 B **2** E **3** C **4** E **5** C
6 A **7** E **8** A **9** E **10** D

Extended-response questions

1 a i $\begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

ii $\det(\mathbf{A}) = 14, \ \mathbf{A}^{-1} = \frac{1}{14}\begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}$

iii $\frac{1}{7}\begin{bmatrix} 9 \\ -1 \end{bmatrix}$

iv $\left(\frac{9}{7}, -\frac{1}{7}\right)$ is the point of intersection of the two lines

b i $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$

ii $\det(\mathbf{A}) = 0$, so $\mathbf{A}$ is non-invertible

c Equations of two parallel lines

2 a $\begin{bmatrix} 79 & 78 & 80 \\ 80 & 78 & 82 \end{bmatrix}$ **b** $\begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}$

c Semester 1: 79.2; Semester 2: 80.4
d Semester 1: 83.8; Semester 2: 75.2
e No, total score is 318.6
f 3 marks

3 a $\begin{bmatrix} 10 & 2 \\ 8 & 4 \\ 8 & 8 \\ 6 & 10 \end{bmatrix}$ **b** $\begin{bmatrix} 70 \\ 60 \end{bmatrix}$

c Term 1: \$820; Term 2: \$800;
Term 3: \$1040; Term 4: \$1020

d $\begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & 4 & 2 \\ 3 & 4 & 2 \end{bmatrix}$ **e** $\begin{bmatrix} 60 \\ 55 \\ 40 \end{bmatrix}$

f Term 1: \$270; Term 2: \$270;
Term 3: \$480; Term 4: \$480

g Term 1: \$1090; Term 2: \$1070;
Term 3: \$1520; Term 4: \$1500

Chapter 16

Exercise 16A

1 a $(-2, 6)$ **b** $(-8, 22)$
c $(26, 2)$ **d** $(-4, -2)$

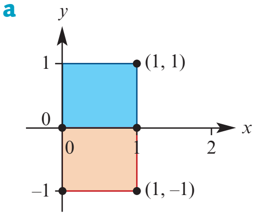
2 a $(3, 2)$ **b** $(-4, 9)$
c $(8, 3)$ **d** $(7, 11)$

3 a $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ **b** $\begin{bmatrix} 11 & -3 \\ 3 & -8 \end{bmatrix}$

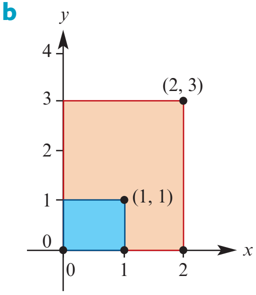
c $\begin{bmatrix} 2 & 0 \\ 1 & -3 \end{bmatrix}$ **d** $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

4 Unit square is blue; image is red

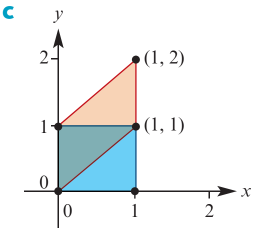
a



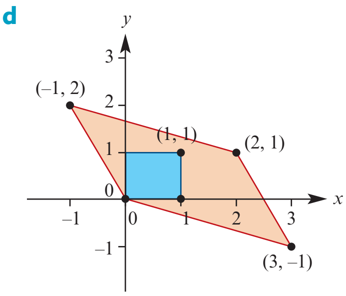
b



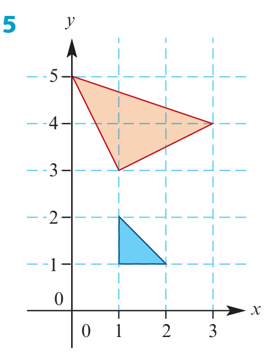
c



d



5



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6 $\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \end{bmatrix}$

7 $\begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

8 a $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

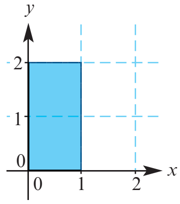
b $\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$ or $\begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$

c $\begin{bmatrix} 1 & -2 \\ -1 & -3 \end{bmatrix}$ or $\begin{bmatrix} -2 & 1 \\ -3 & -1 \end{bmatrix}$

Exercise 16B

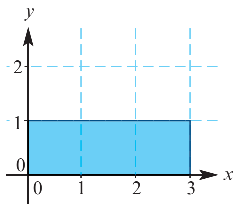
1 a i $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

ii



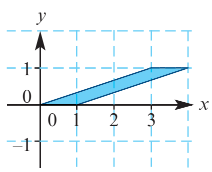
b i $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

ii



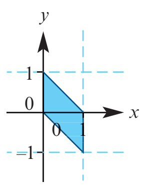
c i $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

ii



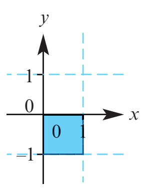
d i $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

ii



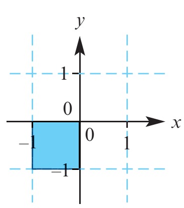
e i $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

ii



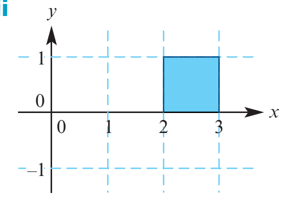
f i $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

ii



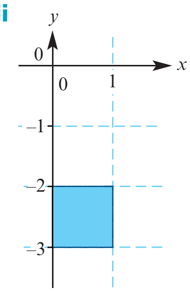
2 a i $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} x+2 \\ y \end{bmatrix}$

ii



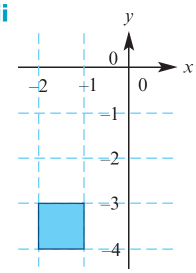
b i $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} x \\ y-3 \end{bmatrix}$

ii



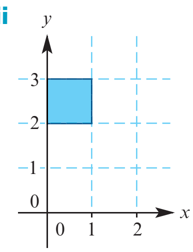
c i $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} x-2 \\ y-4 \end{bmatrix}$

ii



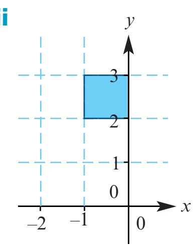
d i $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y+2 \end{bmatrix}$

ii



e i $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x-1 \\ y+2 \end{bmatrix}$

ii



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Exercise 16C

1 a $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

c $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ **d** $\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

2 a $(-3, 2)$ **b** $\left(\frac{5\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

3 a $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

c $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ **d** $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$

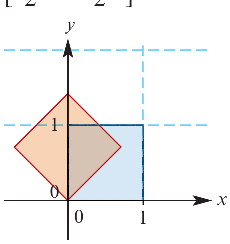
4 a $\begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$ **b** $\begin{bmatrix} -\frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{bmatrix}$

c $\begin{bmatrix} \frac{5}{13} & \frac{12}{13} \\ \frac{12}{13} & -\frac{5}{13} \end{bmatrix}$ **d** $\begin{bmatrix} -\frac{4}{5} & -\frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{bmatrix}$

5 a $\begin{bmatrix} \frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} \end{bmatrix}$ **b** $\left(\frac{-23}{37}, \frac{47}{37}\right)$

6 a $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

b



c $\sqrt{2} - 1$

7 a $C\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), B\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

b Equilateral

c $y = -\sqrt{3}x,\ y = 0,\ y = \sqrt{3}x$

Exercise 16D

1 $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ **2** $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

3 a $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

b $\begin{bmatrix} \cos 180^\circ & -\sin 180^\circ \\ \sin 180^\circ & \cos 180^\circ \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

4 a $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ **c** No

5 a $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **c** Yes

6 a $(x, y) \to (-x-3, y+5)$

b $(x, y) \to (-x+3, y+5)$ **c** Yes

7 a $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

c $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **d** $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

8 a $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

b $\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

9 $\theta = 180^\circ k$, where $k \in \mathbb{Z}$

10 a 2θ

b $\begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$

c $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$\sin(2\theta) = 2\sin\theta\cos\theta$

11 a $x' = y + 1$

$y' = x + 2$

b $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

12 a $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ **b** $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

c $\begin{bmatrix} \frac{\sqrt{2}+\sqrt{6}}{4} & \frac{\sqrt{2}-\sqrt{6}}{4} \\ \frac{\sqrt{6}-\sqrt{2}}{4} & \frac{\sqrt{6}+\sqrt{2}}{2} \end{bmatrix}$

d $\cos 15^\circ = \frac{\sqrt{2}+\sqrt{6}}{4}$, $\sin 15^\circ = \frac{\sqrt{6}-\sqrt{2}}{4}$

13 $\begin{bmatrix} \cos(2\theta - 2\varphi) & -\sin(2\theta - 2\varphi) \\ \sin(2\theta - 2\varphi) & \cos(2\theta - 2\varphi) \end{bmatrix}$,

rotation matrix for angle $2\theta - 2\varphi$

Exercise 16E

1 a $\begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} \frac{2}{7} & \frac{1}{14} \\ \frac{1}{7} & -\frac{3}{14} \end{bmatrix}$

c $\begin{bmatrix} \frac{2}{3} & -\frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix}$ **d** $\begin{bmatrix} \frac{5}{7} & -\frac{3}{7} \\ \frac{4}{7} & -\frac{1}{7} \end{bmatrix}$

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2 a $(x, y) \to (x - 2y, 2x - 5y)$
b $(x, y) \to (y, -x + y)$

3 a $(-1, 1)$ **b** $(-\frac{1}{2}, 1)$

4 $\begin{bmatrix} -4 & 3 \\ -1 & 1 \end{bmatrix}$

5 $(0, 0), (-1, -2), (1, 1), (0, -1)$

6 a $\mathbf{A} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ **b** $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{bmatrix}$

7 a $\mathbf{A} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ **b** $\mathbf{A}^{-1} = \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$

8 a $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
b Reflecting twice in the same axis will return any point (x, y) to its original position

9 a $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$
b Reflecting twice in the same line will return any point (x, y) to its original position

Exercise 16F

1 a $y = -3x - 1$ **b** $y = \frac{x}{2} + 1$ **c** $y = \frac{9x}{2} + 3$
d $y = 3x - 1$ **e** $y = -9x + 3$ **f** $y = \frac{-x - 1}{3}$
g $y = \frac{x - 1}{3}$

2 a $y = 6 - \frac{9x}{2}$ **b** $y = \frac{x + 2}{3}$
c $y = \frac{2 - 3x}{7}$ **d** $y = \frac{5x - 2}{12}$

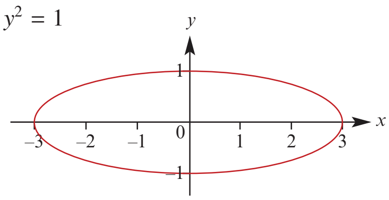
3 $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

4 $\begin{bmatrix} -3 & 0 \\ 0 & 6 \end{bmatrix}$

5 $y = -(x + 1)^2 - 1$

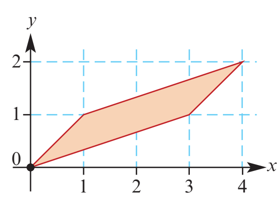
6 $y = (x - 1)^2 - 3$

7 $\frac{x^2}{3^2} + y^2 = 1$

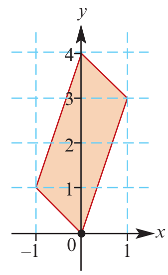


Exercise 16G

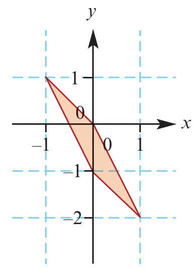
1 a Area = 2



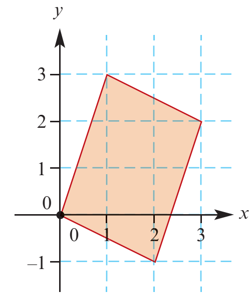
b Area = 4



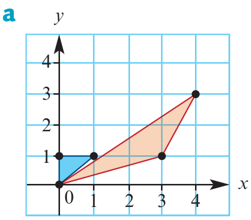
c Area = 1



d Area = 7

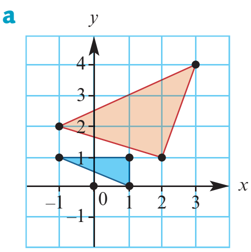


2 a



b Original area = $\frac{1}{2}$; image area = $\frac{5}{2}$

3 a



b Original area = 1; image area = 5

4 $m = \pm 2$

5 $m = -1, 2$

6 a i $\det \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = 1$
ii $\det \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = 1$
iii $\det \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} = -1$
b i Dilation of factor k from the y-axis and dilation of factor $\frac{1}{k}$ from the x-axis
ii Determinant of matrix is 1

7 b $x = -1$

8 $m > 2$ or $m < 1$

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9 $\begin{bmatrix} 1 & \pm\frac{\sqrt{3}}{2} \\ 0 & \pm\frac{1}{2} \end{bmatrix}$

10 a $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Exercise 16H

1 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -x+4 \end{bmatrix}$ **2** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x-2 \\ -y+2 \end{bmatrix}$

3 a $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y+1 \\ x-1 \end{bmatrix}$ **b** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -y-1 \\ -x-1 \end{bmatrix}$

c $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ -y+2 \end{bmatrix}$ **d** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x-4 \\ y \end{bmatrix}$

4 a $\mathbf{A} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ **b** $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

c $\mathbf{C} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

d $\mathbf{CBA} =$

$$\begin{bmatrix} \cos^2\theta + k\sin^2\theta & \cos\theta\sin\theta - k\sin\theta\cos\theta \\ \cos\theta\sin\theta - k\sin\theta\cos\theta & \sin^2\theta + k\cos^2\theta \end{bmatrix}$$

5 $\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$

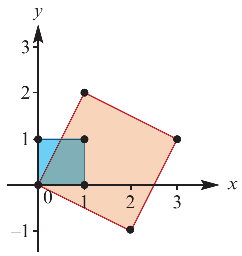
6 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x+1 \\ y-1 \end{bmatrix}$

Chapter 16 review

Short-answer questions

1 a $(7, 4)$ **b** $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

c Area = 5



d $(x, y) \to \left(\frac{2}{5}x - \frac{1}{5}y, \frac{1}{5}x + \frac{2}{5}y\right)$

2 a $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

d $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ **e** $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ **f** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

3 a $\begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$ **b** $\left(\frac{4}{5}, \frac{22}{5}\right)$

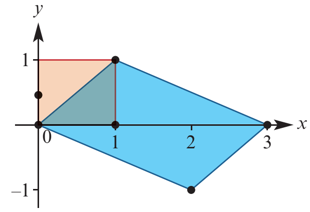
4 a $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$ **c** $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$

5 a $(x, y) \to (x-3, -y+4)$

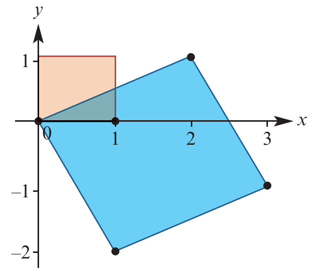
b $(x, y) \to (x-3, -y-4)$

6 a $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ **b** $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}$

7 a Area of image = 3 square units

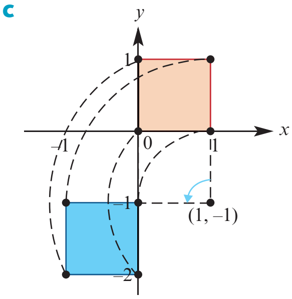


b Area of image = 5 square units



8 a $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -y \\ x-2 \end{bmatrix}$ **b** $(1, 0)$

c



Multiple-choice questions

1 B **2** D **3** A **4** D **5** C
6 A **7** D **8** E **9** D

Extended-response questions

1 a $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ **b** $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

c Product of these two matrices:

$$\begin{bmatrix} \frac{-1+\sqrt{3}}{2\sqrt{2}} & -\frac{1+\sqrt{3}}{2\sqrt{2}} \\ \frac{1+\sqrt{3}}{2\sqrt{2}} & \frac{-1+\sqrt{3}}{2\sqrt{2}} \end{bmatrix}$$

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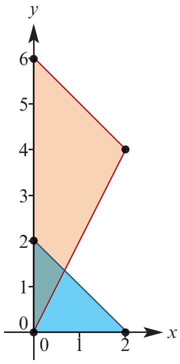
d $\cos 75^\circ = \dfrac{-1+\sqrt{3}}{2\sqrt{2}} = \dfrac{-\sqrt{2}+\sqrt{6}}{4}$

$\sin 75^\circ = \dfrac{1+\sqrt{3}}{2\sqrt{2}} = \dfrac{\sqrt{2}+\sqrt{6}}{4}$

2 a



b Original area = 2 square units; Image area = 6 square units

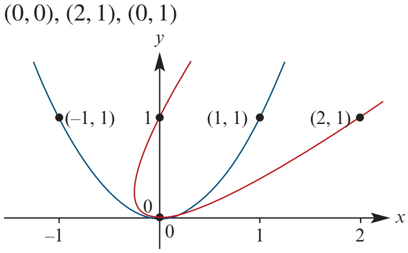
c 8π cubic units

3 a $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

b Shear of factor 1 parallel to the x-axis

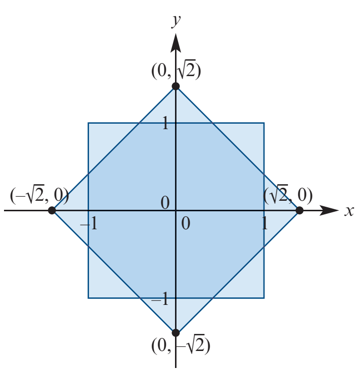
c $(0, 0), (2, 1), (0, 1)$

d



4 a $(0, \sqrt{2}), (\sqrt{2}, 0), (0, -\sqrt{2}), (-\sqrt{2}, 0)$

b



c $13 - 8\sqrt{2}$ square units

5 b i The composition of two rotations is a rotation

ii The composition of two reflections is a rotation

iii The composition of a reflection followed by a rotation is a reflection

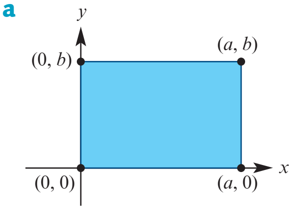
iv The composition of a rotation followed by a reflection is a reflection

c $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

6 a $\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$ **b** $A'(-1, -3)$ **c** $2\sqrt{10}$

d Isosceles **f** $2\sqrt{10}$

7 a



b $O(0, 0)$, $A(a\cos\theta, a\sin\theta)$, $B(-b\sin\theta, b\cos\theta)$, $C(a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta)$

8 a $y = \dfrac{1}{m} - \dfrac{x}{m}$; $(1, 0)$, $\left(\dfrac{1-m^2}{1+m^2}, \dfrac{2m}{1+m^2}\right)$

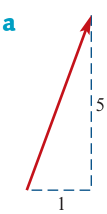
b $y = 1 - \dfrac{x}{m}$; $(0, 1)$, $\left(\dfrac{2m}{1+m^2}, \dfrac{m^2-1}{1+m^2}\right)$

c $\begin{bmatrix} \dfrac{1-m^2}{1+m^2} & \dfrac{2m}{1+m^2} \\ \dfrac{2m}{1+m^2} & \dfrac{m^2-1}{1+m^2} \end{bmatrix}$

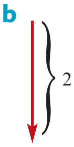
Chapter 17

Exercise 17A

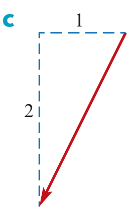
1 a



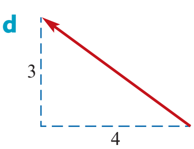
b



c



d



2 $a = 5, \ b = 1$

3 $a = 3, \ b = -15$

4 a $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ **b** $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ **c** $\begin{bmatrix} -1 \\ -3 \end{bmatrix}$ **d** $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ **e** $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

5 a i $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ **ii** $\begin{bmatrix} -5 \\ 0 \end{bmatrix}$ **iii** $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$

b $\boldsymbol{a} + \boldsymbol{b} = -\boldsymbol{c}$

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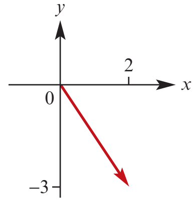
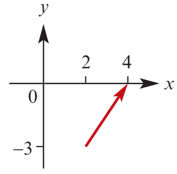
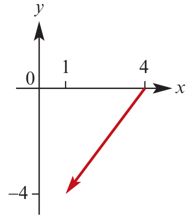
6 a



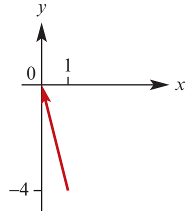
b



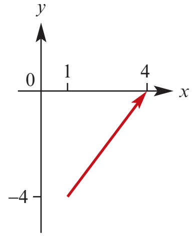
c



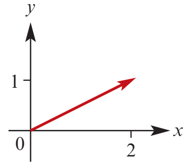
d



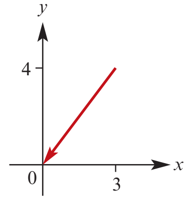
e



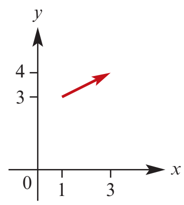
7 a



b



c



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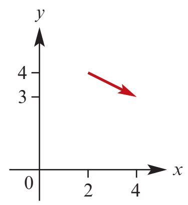
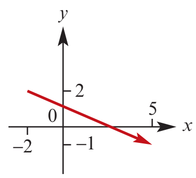
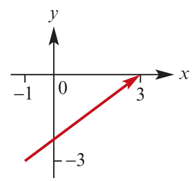


e

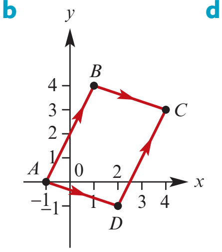


f



8 a and **c**

9 a b



d Parallelogram

10 $m = -11,\ n = 7$

11 a i $\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$ **ii** $\boldsymbol{b}$

b $\overrightarrow{MN} = \overrightarrow{AD}$

12 a $\overrightarrow{CB} = \boldsymbol{a} - \boldsymbol{b},\ \overrightarrow{MN} = \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$

b $\overrightarrow{CB} = -2\overrightarrow{MN}$

13 a $\boldsymbol{a}$ **b** $\boldsymbol{b}$ **c** $2\boldsymbol{a}$ **d** $2\boldsymbol{b}$
e $-\boldsymbol{a}$ **f** $\boldsymbol{b} - \boldsymbol{a}$ **g** $\boldsymbol{a} + \boldsymbol{b}$

14 a $\boldsymbol{a}$ **b** $-\boldsymbol{b}$ **c** $\boldsymbol{a} + \boldsymbol{b}$
d $-\boldsymbol{a} - \boldsymbol{b}$ **e** $\boldsymbol{b} - \boldsymbol{a}$

15 a $\boldsymbol{a} - \boldsymbol{b}$ **b** $\frac{1}{3}(\boldsymbol{b} - \boldsymbol{a})$ **c** $\frac{1}{3}(\boldsymbol{a} + 2\boldsymbol{b})$

d $\frac{1}{9}(\boldsymbol{a} + 2\boldsymbol{b})$ **e** $\frac{1}{9}(4\boldsymbol{a} - \boldsymbol{b})$

16 a $\boldsymbol{u} + \boldsymbol{v}$ **b** $\boldsymbol{v} + \boldsymbol{w}$ **c** $\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}$

17 a $\overrightarrow{OB} = \boldsymbol{u} + \boldsymbol{v},\ \overrightarrow{OM} = \boldsymbol{u} + \frac{1}{2}\boldsymbol{v}$ **b** $\boldsymbol{u} - \frac{1}{2}\boldsymbol{v}$

c $\frac{2}{3}\left(\boldsymbol{u} - \frac{1}{2}\boldsymbol{v}\right)$

d $\overrightarrow{OP} = \frac{2}{3}(\boldsymbol{u} + \boldsymbol{v}) = \frac{2}{3}\overrightarrow{OB}$ **e** 2 : 1

Exercise 17B

1 $2\boldsymbol{i} - 7\boldsymbol{j}$

2 a $5\boldsymbol{i} + 6\boldsymbol{j}$ **b** $-5\boldsymbol{i} + 6\boldsymbol{j}$ **c** $5\boldsymbol{i} - 6\boldsymbol{j}$

3 a 5 **b** 2 **c** 5 **d** 13

4 a 13 **b** $x = 2,\ y = -7$

5 $7\boldsymbol{i} + \frac{5}{2}\boldsymbol{j}$

6 a i $\frac{2}{5}\boldsymbol{i}$ **ii** $\frac{-2}{5}\boldsymbol{i} + \boldsymbol{j}$ **iii** $\frac{1}{6}\left(-\frac{2}{5}\boldsymbol{i} + \boldsymbol{j}\right)$

iv $\frac{1}{3}\boldsymbol{i} + \frac{1}{6}\boldsymbol{j}$ **v** $2\boldsymbol{i} + \boldsymbol{j}$

b i $\overrightarrow{ON} = \frac{1}{6}\overrightarrow{OA}$ **ii** 1 : 5

7 $4\sqrt{2}$ units

8 a $k = \frac{3}{2},\ \ell = \frac{1}{2}$ **b** $x = 6,\ y = 2$

c $x = 3,\ y = 3$ **d** $k = -\frac{1}{3},\ \ell = -\frac{5}{3}$

9 $3\boldsymbol{i} - 2\boldsymbol{j},\ \sqrt{13}$

10 a $-2\boldsymbol{i} + 4\boldsymbol{j}$ **b** $-6\boldsymbol{i} + \boldsymbol{j}$ **c** 5

11 a $D(-6, 3)$ **b** $F(4, -3)$ **c** $G\left(\frac{3}{2}, -\frac{3}{2}\right)$

12 $A(-1, -4),\ B(-2, 2),\ C(0, 10)$

13 a i $2\boldsymbol{i} - \boldsymbol{j}$ **ii** $-5\boldsymbol{i} + 4\boldsymbol{j}$ **iii** $\boldsymbol{i} + 7\boldsymbol{j}$
iv $6\boldsymbol{i} + 3\boldsymbol{j}$ **v** $6\boldsymbol{i} + 3\boldsymbol{j}$

b $D(8, 2)$

14 a $\overrightarrow{OP} = 12\boldsymbol{i} + 5\boldsymbol{j},\ \overrightarrow{PQ} = 6\boldsymbol{i} + 8\boldsymbol{j}$ **b** 13, 10

15 a i $\sqrt{29}$ **ii** $\sqrt{116}$ **iii** $\sqrt{145}$
b $(\sqrt{29})^2 + (\sqrt{116})^2 = (\sqrt{145})^2$

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16 a i $-i - 3j$ **ii** $4i + 2j$ **iii** $-3i + j$
b i $\sqrt{10}$ **ii** $2\sqrt{5}$ **iii** $\sqrt{10}$

17 a i $-3i + 2j$ **ii** $7j$
iii $-3i - 5j$ **iv** $\frac{1}{2}(-3i - 5j)$
b $M\left(\frac{-3}{2}, \frac{9}{2}\right)$

18 a $\frac{1}{5}(3i + 4j)$ **b** $\frac{1}{\sqrt{10}}(3i - j)$
c $\frac{1}{\sqrt{2}}(-i + j)$ **d** $\frac{1}{\sqrt{2}}(i - j)$
e $\frac{6}{\sqrt{13}}\left(\frac{1}{2}i + \frac{1}{3}j\right)$ **f** $\frac{1}{\sqrt{13}}(3i - 2j)$

Exercise 17C

1 a 17 **b** 13 **c** 8 **d** -10
e -4 **f** 3 **g** -58

2 a 5 **b** 13 **c** 8 **d** -5 **e** 13

3 a $15\sqrt{2}$ **b** $-15\sqrt{2}$

4 a $|a|^2 + 4|b|^2 + 4a \cdot b$ **b** $4a \cdot b$
c $|a|^2 - |b|^2$ **d** $|a|$

5 a $-3i + j$ **b** $\sqrt{10}$ **c** 116.57°

6 $\sqrt{66}$

7 a $-\frac{11}{2}$ **b** $\frac{10}{3}$ **c** -1 **d** $\frac{-2 \pm \sqrt{76}}{6}$

8 a $-a + qb$ **b** $\frac{22}{29}$ **c** $\left(\frac{44}{29}, \frac{110}{29}\right)$

9 a 139.40° **b** 71.57° **c** 26.57° **d** 126.87°

11 a $\frac{3}{2}i$ **b** 45° **c** 116.57°

12 a i $\frac{3}{2}i + 2j$ **ii** $\frac{1}{2}i + 3j$ **b** 27.41° **c** 55.30°

Exercise 17D

1 a $\frac{1}{\sqrt{10}}(i + 3j)$ **b** $\frac{1}{\sqrt{2}}(i + j)$ **c** $\frac{1}{\sqrt{2}}(i - j)$

2 a i $\frac{1}{5}(3i + 4j)$ **ii** $\sqrt{2}$
b $\frac{\sqrt{2}}{5}(3i + 4j)$

3 a i $\frac{1}{5}(3i + 4j)$ **ii** $\frac{1}{13}(5i + 12j)$
b $\frac{1}{\sqrt{65}}(4i + 7j)$

4 a $-\frac{11}{17}(i - 4j)$ **b** $\frac{13}{17}(i - 4j)$ **c** $4i$

5 a 2 **b** $\frac{1}{\sqrt{5}}$ **c** $\frac{2\sqrt{3}}{\sqrt{7}}$ **d** $\frac{-1 - 4\sqrt{5}}{\sqrt{17}}$

6 a $a = u + w$ where $u = 2i$ and $w = j$
b $a = u + w$ where $u = 2i + 2j$ and $w = i - j$
c $a = u + w$ where $u = 0$ and $w = -i + j$

7 a $2i + 2j$ **b** $\frac{1}{\sqrt{2}}(-i + j)$

8 a $\frac{3}{2}(i - j)$ **b** $\frac{5}{2}(i + j)$ **c** $\frac{5\sqrt{2}}{2}$

9 a i $i - j$ **ii** $i - 5j$
b $\frac{3}{13}(i - 5j)$ **c** $\frac{2\sqrt{26}}{13}$ **d** 2

Exercise 17E

1 a i $\frac{4}{5}p$ **ii** $\frac{1}{5}p$ **iii** $-p$ **iv** $\frac{1}{5}(q - p)$ **v** $\frac{1}{5}q$
b RS and OQ are parallel
c $ORSQ$ is a trapezium
d 120 cm²

2 a i $\frac{1}{3}a + \frac{2}{3}b$ **ii** $\frac{k}{7}a + \frac{6}{7}b$
b i 3 **ii** $\frac{7}{2}$

3 a i $\overrightarrow{OD} = 2i - 0.5j$, $\overrightarrow{OE} = \frac{15}{4}i + \frac{9}{4}j$
ii $\frac{\sqrt{170}}{4}$
b i $p\left(\frac{15}{4}i + \frac{9}{4}j\right)$
ii $(q + 2)i + (4q - 0.5)j$
c $p = \frac{2}{3}$, $q = \frac{1}{2}$

5 a i $\overrightarrow{AB} = c$ **ii** $\overrightarrow{OB} = a + c$ **iii** $\overrightarrow{AC} = c - a$
b $|c|^2 - |a|^2$

6 a $r + t$ **b** $\frac{1}{2}(s + t)$

Exercise 17F

1 a $-i + 2j - k$ **b** $3i - 5j + 6k$
c $\sqrt{14}$ **d** $3\sqrt{2}$
e $-5i + 6j - k$

2 a $2j + 2k$ **b** $i + 2j$
c $i + 2k$ **d** $i + 2j + 2k$
e $-2j$ **f** $-2j + 2k$
g $i + 2j - 2k$ **h** $i - 2j - 2k$

3 a i $\frac{3}{\sqrt{11}}i + \frac{1}{\sqrt{11}}j - \frac{1}{\sqrt{11}}k$
ii $-\frac{6}{\sqrt{11}}i - \frac{2}{\sqrt{11}}j + \frac{2}{\sqrt{11}}k$
b $\frac{15}{\sqrt{11}}i + \frac{5}{\sqrt{11}}j - \frac{5}{\sqrt{11}}k$

4 $\frac{\sqrt{14}}{3\sqrt{3}}(i - j + 5k)$

5 a $i - 3j$ **b** $\sqrt{10}$ **c** $\frac{3}{2}i + \frac{1}{2}j - k$

6 a $\frac{1}{6}i + 2j + 2k$ **b** $\frac{17}{6}$

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Chapter 17 review

Short-answer questions

1 a $\frac{12}{7}$ **b** ± 9

2 $A(2, -1),\ B(5, 3),\ C(3, 8),\ D(0, 4)$

3 $p = \frac{1}{6},\ q = -\frac{11}{12}$

4 a $3\sqrt{10}$ **b** $\frac{1}{3\sqrt{10}}(\boldsymbol{i} - 5\boldsymbol{j} + 8\boldsymbol{k})$

5 6

6 a $\frac{1}{5}(4\boldsymbol{i} + 3\boldsymbol{j})$ **b** $\frac{16}{25}(4\boldsymbol{i} + 3\boldsymbol{j})$

7 a i $\boldsymbol{a} + \boldsymbol{b}$ **ii** $\frac{1}{3}(\boldsymbol{a} + \boldsymbol{b})$ **iii** $\boldsymbol{b} - \boldsymbol{a}$

iv $\frac{1}{3}(2\boldsymbol{a} - \boldsymbol{b})$ **v** $\frac{2}{3}(2\boldsymbol{a} - \boldsymbol{b})$

b $\overrightarrow{TR} = 2\overrightarrow{PT}$, so P, T and R are collinear

8 a $s = -2,\ t = 5,\ u = 2$
b $\sqrt{33}$

9 $\sqrt{109}$ units

10 a $11\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ **b** $\sqrt{30}$

c $\frac{1}{\sqrt{30}}(5\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$ **d** $2\boldsymbol{i} + 4\boldsymbol{j}$

11 a $(-1, 10)$ **b** $h = 3,\ k = -2$

12 $m = 2,\ n = 1$

13 a $\boldsymbol{b} = \boldsymbol{a} + \boldsymbol{c}$ **b** $\boldsymbol{b} = \frac{2}{5}\boldsymbol{a} + \frac{3}{5}\boldsymbol{c}$

14 a 13 **b** 10 **c** 8 **d** -11
e -9 **f** 0 **g** -27

16 a $\frac{6}{5}$ **b** $\pm\frac{3}{\sqrt{2}}$ **c** $\frac{7}{3}$

17 a i $\overrightarrow{AB} = -\boldsymbol{i}$ **ii** $\overrightarrow{AC} = -5\boldsymbol{j}$
b $\boldsymbol{0}$
c 1

Multiple-choice questions

1 C **2** C **3** E **4** A **5** B
6 B **7** A **8** C **9** D **10** C

Extended-response questions

1 a $\begin{bmatrix} -31 \\ -32 \end{bmatrix}$ **b** $\begin{bmatrix} -15 \\ -20 \end{bmatrix}$ **c** $|\overrightarrow{OR}| = 25$

2 a $\sqrt{34}$ **b** $\sqrt{10} - \sqrt{20}$ **c** $\boldsymbol{r} = \boldsymbol{i} - 9\boldsymbol{j}$

3 a $\frac{1}{2}$ **b** $x = -2,\ y = 2$
c $p = 4,\ q = 2,\ r = 2$

4 a $(25, -7),\ \begin{bmatrix} 7 \\ 24 \end{bmatrix}$ **b** $\begin{bmatrix} -20 \\ 15 \end{bmatrix}$

5 a $(12, 4)$ **b** $\begin{bmatrix} k - 12 \\ -4 \end{bmatrix}$

c $\sqrt{160},\ k,\ \sqrt{(k-12)^2 + 16},\ k = \frac{40}{3}$

d 34.7°

Chapter 18

Short-answer questions

1 a All defined except **AB**

b $\mathbf{DA} = \begin{bmatrix} 6 & -12 \end{bmatrix},\quad \mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{9} & \frac{4}{9} \\ \frac{2}{9} & -\frac{1}{9} \end{bmatrix}$

2 a $\begin{bmatrix} -2 & 4 \\ 18 & -24 \end{bmatrix}$ **b** $\begin{bmatrix} -10 & -19 \\ 7 & -16 \end{bmatrix}$

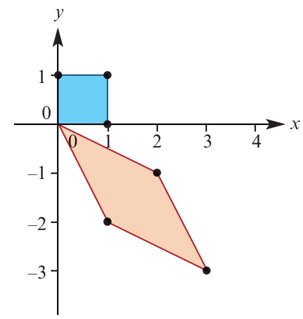
3 8

4 $\mathbf{A} = \begin{bmatrix} t \\ 3t - 5 \end{bmatrix},\ t \in \mathbb{R}$

5 $\mathbf{AB} = \begin{bmatrix} -9 & -8 \\ -15 & 10 \end{bmatrix},\quad \mathbf{C}^{-1} = \begin{bmatrix} 2 & 1 \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}$

6 a $(7, -8)$ **b** $\begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$

c Area = 3



d $(x, y) \to \left(\frac{2}{3}x + \frac{1}{3}y, -\frac{1}{3}x - \frac{2}{3}y\right)$

7 a $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

d $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ **e** $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

f $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ **g** $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ **h** $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

8 a $\begin{bmatrix} -\frac{15}{17} & \frac{8}{17} \\ \frac{8}{17} & \frac{15}{17} \end{bmatrix}$ **b** $\left(\frac{2}{17}, \frac{76}{17}\right)$

9 a $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **c** $\begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}$

10 a $(x, y) \to (-x + 2, y - 1)$
b $(x, y) \to (-x - 2, y - 1)$

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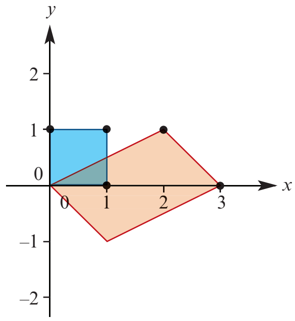
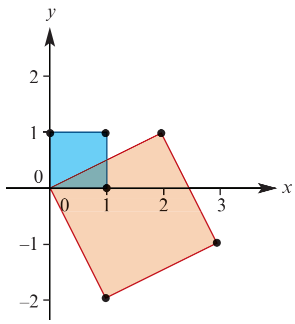
11 a Area = 3

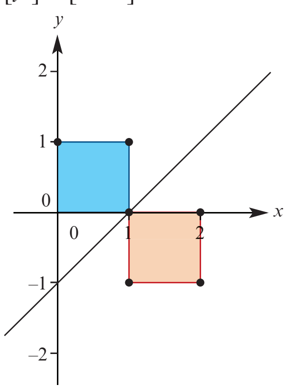


b Area = 5



12 a $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y+1 \\ x-1 \end{bmatrix}$ **b** $(0,0) \to (1,-1)$

c



13 a 13 **b** 13 **c** 13 **d** −13
e 5 **f** 0 **g** −13

14 a $m = \frac{46}{11}$, $n = -\frac{18}{11}$ **b** $p = -48$
c $p = 3, 5$

Multiple-choice questions

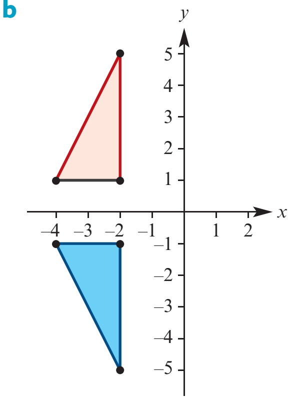
1 A	**2** B	**3** E	**4** A	**5** B
6 C	**7** A	**8** B	**9** D	**10** D
11 C	**12** B	**13** A	**14** E	**15** B
16 C	**17** D	**18** D	**19** E	**20** D
21 B	**22** B	**23** B	**24** D	**25** B
26 A	**27** B	**28** D	**29** A	**30** B
31 C	**32** B	**33** B		

Extended-response questions

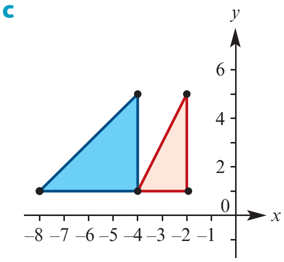
1 a i $\begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & d^2+bc \end{bmatrix}$ **ii** $\begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$

2 a $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x+6 \\ y+3 \end{bmatrix}$

b



c



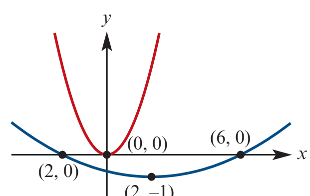
d $y = 2(x+3)^2 + 2$ **e** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x+3 \\ -2y+4 \end{bmatrix}$

3 a $(4, 1)$
b i Rectangle with vertices $A'(0,0)$, $B'(0,1)$, $C'(4,1)$, $E'(4,0)$
ii 1 **iii** 4 **iv** k

c $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4x \\ y \end{bmatrix}$

d i $y = \frac{1}{16}x^2$ **ii** $y = \frac{1}{16}(x-2)^2 - 1$

iii



e $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x+2 \\ \frac{1}{5}(y+3) \end{bmatrix}$

4 b i $x^2 + (y-1)^2 = 1$

ii $\left(x - \frac{4}{5}\right)^2 + \left(y - \frac{3}{5}\right)^2 = 1$

c $(0,0), \left(\frac{4}{5}, \frac{8}{5}\right)$

5 a $(-3, 11)$ **b** $\frac{1}{10}\begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$

c $a = 2,\ b = 3$ **d** $(5a, 5a)$

e $\lambda = 2,\ b = -2a;\ \lambda = 5,\ b = a$

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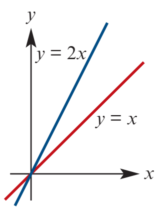
6 a $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ **b** $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

c $a = \sqrt{2},\ b = 0$ **d** $c = \frac{3\sqrt{2}}{2},\ d = \frac{\sqrt{2}}{2}$

e i $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y' \\ -\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y' \end{bmatrix}$

ii $\sqrt{2}(y - x) = (x + y)^2$

7 a

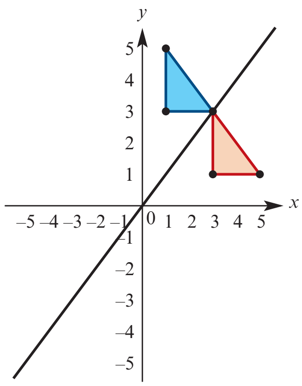


b $a = 2,\ b = \frac{\pi}{4}$

c $\begin{bmatrix} \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$

8 a i $(3, 1)$ **ii** $A'(3, 1),\ B'(5, 1),\ C'(3, 3)$

iii



b ii $(-1, -1), (2, 2)$

iv $(-1, -1), (2, 2)$, $\left(\frac{1}{2}(-1 + \sqrt{5}), \frac{1}{2}(-1 - \sqrt{5})\right)$, $\left(\frac{1}{2}(-1 - \sqrt{5}), \frac{1}{2}(-1 + \sqrt{5})\right)$

9 a $\overrightarrow{AE} = \frac{1}{t+1}(2\boldsymbol{a} + t\boldsymbol{b})$

b $\overrightarrow{AE} = \frac{1}{8}(7\boldsymbol{a} + \overrightarrow{AF})$ **d** $t = \frac{9}{7}$

10 b $(n - 1)\boldsymbol{a} - n\boldsymbol{b} + \boldsymbol{c}$

11 a $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a},\ \overrightarrow{PQ} = \frac{-3}{10}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$

b i $n\left(\frac{-3}{10}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}\right)$ **ii** $\left(k + \frac{1}{2}\right)\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$

c $n = \frac{5}{3},\ k = \frac{1}{3}$

12 a $4\sqrt{2}$ km/h blowing from the south-west

b $\sqrt{5}$ km/h; 200 m downstream

c 43.1 km/h at bearing 80° **d** 222°

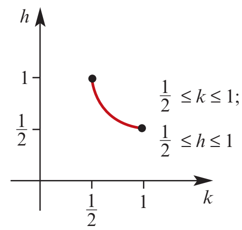
13 b ii $\overrightarrow{ZG} = \frac{1}{3h}\overrightarrow{ZH} + \frac{1}{3k}\overrightarrow{ZK}$

iii $\frac{1}{h} + \frac{1}{k} = 3$ **iv** $h = \frac{2}{3}$; similarity

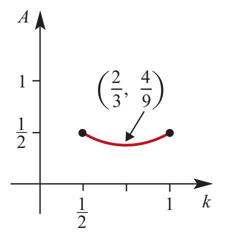
v $\frac{4}{9}$ cm^2

vi $h = \frac{1}{2}$; H is midpoint of ZX, $K = Y$

vii



viii



Chapter 19

Exercise 19A

1 a 12 cm to the right of O

b 2 cm to the right of O

c Moving to the left at 7 cm/s

d When $t = 3.5$ s and the particle is 0.25 cm to the left of O

e -2 cm/s

f 2.9 cm/s

2 a After 3.5 s **b** 2 m/s^2 **c** 14.5 m

d When $t = 2.5$ s and the particle is 1.25 m to the left of O

3 a 3 cm to the left of O moving to the right at 24 cm/s

b $v = 3t^2 - 22t + 24$

c After $\frac{4}{3}$ s and 6 s

d $11\frac{22}{27}$ cm to the right of O and 39 cm to the left of O

e $4\frac{2}{3}$ s

f $a = 6t - 22$

g When $t = \frac{11}{3}$ s and the particle is $13\frac{16}{27}$ cm left of O moving to the left at $16\frac{1}{3}$ cm/s

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4 a When $t = \frac{2}{3}$ s and $a = -2$ cm/s^2; when $t = 1$ and $a = 2$ cm/s^2
b When $t = \frac{5}{6}$ s and the particle is moving to the left at $\frac{1}{6}$ cm/s

5 When $t = 2$ s, $v = 6$ cm/s, $a = -14$ cm/s^2; when $t = 3$ s, $v = -5$ cm/s, $a = -8$ cm/s^2; when $t = 8$ s, $v = 30$ cm/s, $a = 22$ cm/s^2

6 a $t = 4$ s and $t = -1$ s b $t = \frac{3}{2}$ s

Exercise 19B

1 a $x = 2t^2 - 6t$ b At the origin O
c 9 cm d 0 cm/s e 3 cm/s

2 a $x = t^3 - 4t^2 + 5t + 4,\ a = 6t - 8$
b When $t = 1$, $x = 6$; when $t = \frac{5}{3}$, $x = 5\frac{23}{27}$
c When $t = 1$, $a = -2$ m/s^2; when $t = \frac{5}{3}$, $a = 2$ m/s^2

3 20 m to the left of O

4 $x = 215\frac{1}{3}$ m, $v = 73$ m/s

5 a $v = -10t + 25$ b $x = -5t^2 + 25t$
c 2.5 s d $31\frac{1}{4}$ m e 5 s

6 29th floor

Exercise 19C

1 $2\sqrt{10}$ s

2 37.5 m

3 a 3 m/s^2 b $6\frac{2}{3}$ s c 337.5 m d $\frac{500}{27}$ s

4 a 2.5 m/s^2 b 31.25 m

5 a 50 s b 625 m

6 a 20 s b 10 m/s

7 a −19.2 m/s b 1.6 m

8 a −59.2 m/s b −158.4 m

9 a 10 s b After 3 s and 7 s

10 a $4.9(1 - 2t)$ m/s b $4.9t(1 - t) + 3$ m
c 4.225 m d $\frac{10}{7}$ s

11 a 2 s b 44.1 m c 4 s d 5 s

12 $10\sqrt{10}$ m/s

Exercise 19D

1 65 m

2 a 562.5 m b 450 m c 23.75 s

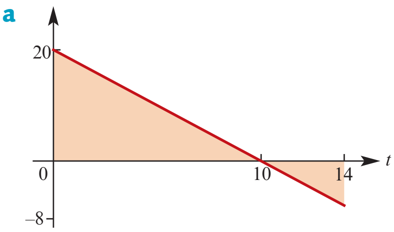
3 $\frac{200}{3}$ m/s

4 210 m

5 a 500 m b 375 m c 17.57 s

6 a 12.5 s b 187.5 m

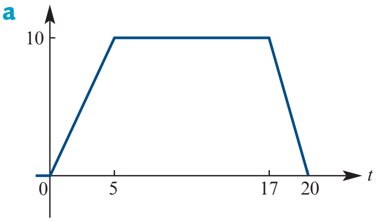
7 a



b From initial position O, the particle moves to the right with initial velocity 20 m/s. It slows until after 10 seconds it is 100 m from O and momentarily stops. It then moves to the left towards O, getting faster.
c 116 m
d 84 m to the right of initial position

8 a 1 m/s^2 b −2.5 m/s^2 c 215 m
d 125 m to the right of initial position

9 a



b $-\frac{10}{3}$ m/s^2

10 No, the first train will stop after 6.25 km and the second train will stop after 6 km.

11 a 57.6 km/h b 1 minute $6\frac{2}{3}$ seconds
c 0.24

Chapter 19 review

Short-answer questions

1 a 5 cm to the left of O b 8 cm to the left of O
c −4 cm/s d $t = 2$ s, 9 cm to the left of O
e −1 cm/s f $1\frac{2}{3}$ cm/s

2 a 8 cm to the right, 0 cm/s, −4 cm/s^2
b At $t = 0$ s, 8 cm to the right, −4 cm/s^2; at $t = \frac{4}{3}$ s, $6\frac{22}{27}$ cm to the right, 4 cm/s^2

3 a 3.5 s, −40.5 cm/s, −36 cm/s^2 b 2 s
c 31 cm

4 a i $\frac{1}{8}$ cm to the left ii 1 cm/s^2 iii 1 cm/s
b i 0 s, 2 s ii $\frac{32}{27}$ cm

5 a 12 m/s b $x = t^3$

6 a 4 s b $18\frac{2}{3}$ m to the right c −5 m/s^2
d 1.5 s e $6\frac{1}{4}$ m/s

7 a $\frac{1}{12}$ m to the left b −1 m/s c −5 m/s^2

8 a $a = -\frac{1}{t^3}$ b $x = \frac{1}{2} - \frac{1}{2t}$

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9 a $a = 3t^2 - 22t + 24$ **b** -15 m/s^2
c $2\frac{1}{12}$ m to the left, $60\frac{7}{12}$ m

10 40 m

11 a 2.5 m/s^2 **b** 8 s **c** 500 m **d** $\frac{100}{9}$ s

12 a $41\frac{2}{3}$ s **b** $347\frac{2}{9}$ m

13 a 7.143 s **b** $2\frac{6}{7}$ s, $4\frac{2}{7}$ s

14 a 2 s **b** 39.6 m **c** 4 s **d** 4.84 s

15 437.5 m

16 a 288 m **b** 16 s

17 16 m/s

18 $\frac{80}{81}$ m/s^2

19 a 0 m/s **b** -3 m/s^2 **c** -4 m/s
d $4\frac{2}{3}$ m **e** $\frac{11}{12}$ m

20 a $2t - t^2 + 8$ **b** $t^2 - \frac{t^3}{3} + 8t$

21 b i 8 m/s **ii** 2 s **iii** 18 m

22 a 27 m/s^2 **b** 50 m/s **c** 4.5 s

23 a -10 m/s **b** 0 m

24 a 4 s, 6 s **b** 36 m **c** $0 \le t < 5$

Multiple-choice questions

1 A **2** E **3** C **4** C **5** E
6 C **7** D **8** E **9** A **10** D

Extended-response questions

1 a $2\frac{1}{3}$ cm to the left of O **b** 4 cm/s
c 2 cm/s^2 **d** At 2 s
e $\frac{1}{3}$ cm to the right of O **f** At 1 s

3 a After 6 s at -36 m/s
b When $t = 0$ or $t = 4$; when $t = 4$, the maximum height is 32 m
c After 2 s

4 $x(1) - x(0) = 15.1$, $x(2) - x(1) = 5.3$,
$x(3) - x(2) = -4.5$, $x(4) - x(3) = -14.3$,
$x(5) - x(4) = -24.1$, $x(6) - x(5) = -33.9$,
$x(7) - x(6) = -43.7$, $x(8) - x(7) = -53.5$,
$x(9) - x(8) = -63.3$, $x(10) - x(9) = -73.1$

The constant difference between successive numbers is -9.8 (acceleration due to gravity)

6 33 m

7 a $v = -5t + 25,\ 0 \le t \le 5$ **b** 62.5 m

8 25 m to the left of O

9 b The second particle is projected upwards at the instant the first particle lands.
c The second particle is projected upwards after the first particle has landed, so there is no collision.

Chapter 20

Exercise 20A

1 $T_1 = 3$ kg wt, $T_2 = 7$ kg wt

2 $T_1 = T_2 = \frac{5\sqrt{2}}{2}$ kg wt

3 90°

4 $T_1 = 14.99$ kg wt, $T_2 = 12.10$ kg wt

5 28.34 kg wt, W48.5°S

6 $T = 40$ kg wt, $N = 96$ kg wt

7 $F = 6.39$ kg wt

8 a No **b** Yes

9 146.88°, 51.32°, 161.8°

10 a 7.5 kg wt **b** 9.64 kg wt **c** 7.62 kg wt

11 32.97 kg wt, 26.88 kg wt, 39.29 kg wt, $W = 39.29$ kg

Exercise 20B

1 13.05 kg wt

2 5.74 kg wt

3 3.73 kg wt, 8.83 kg wt

4 4.13 kg wt

5 6.93 kg wt

6 31.11 kg, 23.84 kg wt

7 44.10 kg, 22.48° to the vertical

8 6.43 kg wt, 7.66 kg wt, 11.92 kg

9 3.24 kg wt

Chapter 20 review

Short-answer questions

1 9 kg wt, 12 kg wt

2 $10\sqrt{3}$ kg wt, 150° to the 10 kg wt

3 $14\sqrt{5}$ kg wt, $28\sqrt{5}$ kg wt

4 $5\sqrt{3}$ kg wt

5 $-\frac{7}{8}$

6 $\frac{40\sqrt{3}}{3}$ kg wt

7 $\frac{15\sqrt{2}}{2}$ kg wt

8 28 kg, $14\sqrt{3}$ kg wt

9 $4\sqrt{3}$ kg wt

Multiple-choice questions

1 E **2** C **3** E **4** A **5** C
6 B **7** B **8** A **9** C **10** B

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Chapter 21

Short-answer questions

1 **a** $\frac{6}{5}$ s **b** $\frac{25}{6}$ m/s^2 **c** $\frac{12}{5}$ s **d** 12 m

2 $F = 7$ kg wt, $\cos\theta = \frac{-31}{49}$

3 $\cos\theta = \frac{-5}{8}$ **4** $10\sqrt{10}$ m/s

5 $Q^2 = 100 - 48\sqrt{2}$ **6** 6 m/s^2

7 5 s

8 **a** $T = 5$ kg wt, $N = 5\sqrt{3}$ kg wt

b $T = \frac{10\sqrt{3}}{3}$ kg wt, $N = \frac{20\sqrt{3}}{3}$ kg wt

9 $T = 10$ kg wt, $\tan\theta = \frac{3}{4}$

10 $\frac{50}{13}$ kg wt, $\frac{120}{13}$ kg wt

11 **a** $\frac{1125}{2}$ m **b** $\frac{225}{22}$ m/s **c** $\frac{3}{4}$ m/s^2

d $t = \frac{40}{3}$ and $t = 45$

12 **a** 70 m/s **b** 245 m **c** 8 s

Multiple-choice questions

1 D	2 E	3 B	4 B	5 C
6 C	7 A	8 D	9 C	10 B
11 D	12 D	13 C	14 E	15 D
16 A	17 D	18 B	19 A	20 C
21 C	22 E	23 B		

Extended-response questions

1 **a** $p = -4$, $q = 3$ **b** $4\frac{2}{9}$ m/s

2 **a** 14.7 m/s **b** 24.66 m/s

c 1.043 s, 1.957 s **d** 3.80 s

3 **b** **i** $\frac{2V(a+r)}{3ar}$ **ii** $\frac{7V(a+r)}{6ar}$

c $\frac{4}{7}V$

4 **a** $\frac{16\,000}{49}$ m **b** 52.568 m/s

5 **a** **i** $4 - 10t - 3t^2$ **ii** $-10 - 6t$

b **i** 4 m/s **ii** -10 m/s^2

c $2\frac{1}{3}$ s

d Y: $-9\frac{2}{3}$ m/s; X: $-35\frac{2}{3}$ m/s;
X and Y are moving in the same direction, X is moving faster and catches up to Y.

Chapter 22

Exercise 22A

1 No, the sample is biased towards students who use the internet, because of the email collection method.

2 No, the sample is biased because she is collecting the data at a particular time of day. Some age groups would be more likely to use the restaurant at that time – probably school children and 'young' families.

3 No, the sample is biased towards viewers of that station. Only people with strong opinions will call, and people may call more than once.

4 Answers will vary

5 **a** 0.48 **b** $\hat{p}$

6 **a** All students at the school **b** 0.42 **c** 0.37

7 **a** All Australian adults **b** 4 **c** 3.5

Exercise 22B

1 **a**

x	0	1	2
$\Pr(X = x)$	0.16	0.48	0.36

b $\Pr(X \geq 1) = 0.84$

2 **a** $\Pr(X = 3) = 0.35$ **b** $\Pr(X < 3) = 0.20$

c $\Pr(X \geq 4) = 0.45$

d $\Pr(1 < X < 5) = 0.75$

e $\Pr(X \neq 5) = 0.85$

f $\Pr(1 < X < 5 \mid X > 1) = \frac{15}{19}$

3 **a** 0.0034 **b** 0.0035 **c** 0.7432

d 0.2533 **e** 0.2567 **f** 0.2212

4 **a** $p = 0.5$ **b** $0, \frac{1}{3}, \frac{2}{3}, 1$

c

$\hat{p}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\Pr(\hat{P} = \hat{p})$	0.1	0.4	0.4	0.1

d 0.9

5 **a** $p = 0.6$

b 0, 0.2, 0.4, 0.6, 0.8, 1

c

$\hat{p}$	0	0.2	0.4
$\Pr(\hat{P} = \hat{p})$	0.0036	0.0542	0.2384

$\hat{p}$	0.6	0.8	1
$\Pr(\hat{P} = \hat{p})$	0.3973	0.2554	0.0511

d $\Pr(\hat{P} > 0.7) = 0.3065$

e $\Pr(0 < \hat{P} < 0.8) = 0.6899$,
$\Pr(\hat{P} < 0.8 \mid \hat{P} > 0) = 0.6924$

6 **a** $p = 0.3$

b 0, 0.25, 0.5, 0.75, 1

c

$\hat{p}$	0	0.25	0.5
$\Pr(\hat{P} = \hat{p})$	0.2274	0.4263	0.2713

$\hat{p}$	0.75	1
$\Pr(\hat{P} = \hat{p})$	0.0691	0.0059

d $\Pr(\hat{P} > 0.5) = 0.075$

e $\Pr(0 < \hat{P} < 0.5) = 0.4263$,
$\Pr(\hat{P} < 0.5 \mid \hat{P} > 0) = 0.5518$

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7 **a** 0.028
b $\Pr(0 < \hat{P} < 0.6) = 0.243$,
$\Pr(\hat{P} < 0.6 \mid \hat{P} > 0) = 0.897$

8 **a** $p = 0.5$
b 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1
c

$\hat{p}$	0	0.1	0.2	0.3
$\Pr(\hat{P} = \hat{p})$	0.00098	0.0098	0.0439	0.1172

$\hat{p}$	0.4	0.5	0.6	0.7
$\Pr(\hat{P} = \hat{p})$	0.2051	0.2461	0.2051	0.1172

$\hat{p}$	0.8	0.9	1
$\Pr(\hat{P} = \hat{p})$	0.0439	0.0098	0.00098

d $\Pr(\hat{P} > 0.5) = 0.3770$

9 **a** $0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1$
b

$\hat{p}$	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$
$\Pr(\hat{P} = \hat{p})$	0.0122	0.0795	0.2153	0.3110

$\hat{p}$	$\frac{2}{3}$	$\frac{5}{6}$	1
$\Pr(\hat{P} = \hat{p})$	0.2527	0.1095	0.0198

c 0.307
d $\Pr(\hat{P} < 0.3 \mid \hat{P} < 0.8) = 0.1053$

10 **a** $0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}, 1$
b

$\hat{p}$	0	$\frac{1}{8}$	$\frac{1}{4}$
$\Pr(\hat{P} = \hat{p})$	0.000003	0.00008	0.00115

$\hat{p}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$
$\Pr(\hat{P} = \hat{p})$	0.0092	0.0459	0.1468

$\hat{p}$	$\frac{3}{4}$	$\frac{7}{8}$	1
$\Pr(\hat{P} = \hat{p})$	0.2936	0.3355	0.1678

c 0.9437
d $\Pr(\hat{P} > 0.6 \mid \hat{P} > 0.25) = 0.9449$

11 **a**

$\hat{p}$	0	0.25	0.5	0.75	1
Hyp	0.0587	0.2499	0.3827	0.2499	0.0587
Bin	0.0625	0.25	0.375	0.25	0.0625

b

$\hat{p}$	0	0.1	0.2	0.3
Hyp	0.0006	0.0072	0.0380	0.1131
Bin	0.00098	0.0098	0.0440	0.1172

$\hat{p}$	0.4	0.5	0.6	0.7
Hyp	0.2114	0.2593	0.2114	0.1131
Bin	0.2051	0.2461	0.2051	0.1172

$\hat{p}$	0.8	0.9	1
Hyp	0.0380	0.0072	0.0006
Bin	0.0439	0.0098	0.00098

c Not much

Exercise 22C

1 **a** $\Pr(\hat{P} \geq 0.8) = 0.08$ **b** $\Pr(\hat{P} \leq 0.5) = 0.01$
2 **a** $\Pr(\hat{P} \geq 0.7) = 0.01$ **b** $\Pr(\hat{P} \leq 0.25) = 0.07$
3 **c** **i** ≈ 0.04 **ii** ≈ 0.01
4 **c** **i** ≈ 0.04 **ii** ≈ 0.04
5 **c** **i** ≈ 0.06 **ii** ≈ 0.13
6 **c** **i** ≈ 0.01 **ii** ≈ 0.06

Exercise 22D

1 **a** $\Pr(\bar{X} \geq 25) = 0.02$ **b** $\Pr(\bar{X} \leq 23) = 0.01$
2 **a** $\Pr(\bar{X} \geq 163) = 0.04$ **b** $\Pr(\bar{X} \leq 158) = 0.05$
3 **c** **i** ≈ 0.04 **ii** ≈ 0.04
4 **c** **i** ≈ 0.02 **ii** ≈ 0.01
5 **c** **i** ≈ 0.07 **ii** ≈ 0.01
6 **c** **i** ≈ 0 **ii** ≈ 0

Chapter 22 review

Short-answer questions

1 **a** Employees of the company
b 0.35 **c** 0.4
2 No, this sample (people already interested in yoga) is not representative of the population
3 No, people who choose to live in houses with gardens may not be representative of the population
4 **a** People with Type II diabetes
b Population is too large and dispersed
c Unknown **d** $\bar{x} = 1.5$
5 **a** Employees of the company
b $p = 0.2$ **c** $\hat{p} = 0.22$
6 **a** $p = \frac{3}{5}$ **b** $\frac{1}{3}, \frac{2}{3}, 1$
c

$\hat{p}$	$\frac{1}{3}$	$\frac{2}{3}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{3}{10}$	$\frac{6}{10}$	$\frac{1}{10}$

d $\Pr(\hat{P} > 0.5) = \frac{7}{10}$
e $\Pr(0 < \hat{P} < 0.5) = \frac{3}{10}$,
$\Pr(\hat{P} < 0.5 \mid \hat{P} > 0) = \frac{3}{10}$
7 **a** 0, 0.25, 0.5, 0.75, 1
b

$\hat{p}$	0	0.25	0.5	0.75	1
$\Pr(\hat{P} = \hat{p})$	0.0625	0.25	0.375	0.25	0.0625

c $\Pr(\hat{P} < 0.5) = 0.3125$
d $\Pr(\hat{P} < 0.5 \mid \hat{P} < 0.8) = \frac{1}{3}$
8 **a** **i** $\Pr(\hat{P} \geq 0.7) = 0.03$
ii $\Pr(\hat{P} \leq 0.38) = 0.04$
b **i** 0.42 **ii** 0.08

Multiple-choice questions

1 B **2** C **3** A **4** B **5** B **6** E
7 A **8** D **9** E **10** B **11** C **12** E

ISBN 978-1-316-63608-4

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Extended-response questions

1 a

p	a	b
0.1	0.03	0.17
0.2	0.11	0.29
0.3	0.19	0.41
0.4	0.29	0.51
0.5	0.38	0.62
0.6	0.49	0.71

b i $\hat{p} = 0.34$ **ii** $p = 0.3$ or $p = 0.4$

2 a iii mean ≈ 50, s.d. ≈ 1.12

b iii mean ≈ 50, s.d. ≈ 0.71

c iii mean ≈ 50, s.d. ≈ 0.50

Chapter 23

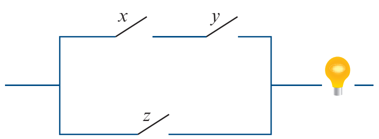
Exercise 23A

3 a $(A \cap \varnothing) \cup (A \cup \xi) = \xi$

b If $A \cup B = \xi$, then $A' \cap B = A'$.

c $A \cup B \supseteq A \cap B$

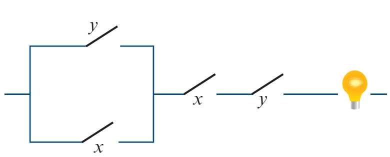
4 a i



ii

x	y	z	$x \wedge y$	$(x \wedge y) \vee z$
0	0	0	0	0
0	0	1	0	1
0	1	0	0	0
0	1	1	0	1
1	0	0	0	0
1	0	1	0	1
1	1	0	1	1
1	1	1	1	1

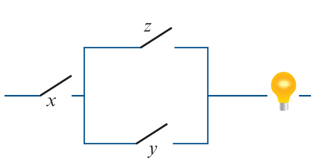
b i



ii

x	y	$x \vee y$	$x \wedge y$	$(x \vee y) \wedge (x \wedge y)$
0	0	0	0	0
0	1	1	0	0
1	0	1	0	0
1	1	1	1	1

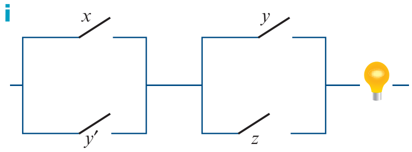
c i



ii

x	y	z	$y \vee z$	$x \wedge (y \vee z)$
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	1	0
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

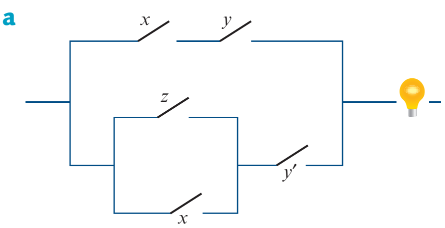
d i



ii

x	y	z	$a = x \vee y'$	$b = y \vee z$	$a \wedge b$
0	0	0	1	0	0
0	0	1	1	1	1
0	1	0	0	1	0
0	1	1	0	1	0
1	0	0	1	0	0
1	0	1	1	1	1
1	1	0	1	1	1
1	1	1	1	1	1

5 a



b

x	y	z	$a = x \wedge y$	$b = (z \vee x) \wedge y'$	$a \vee b$
0	0	0	0	0	0
0	0	1	0	1	1
0	1	0	0	0	0
0	1	1	0	0	0
1	0	0	0	1	1
1	0	1	0	1	1
1	1	0	1	0	1
1	1	1	1	0	1

Exercise 23B

3 a

x	y	y'	$x \wedge y'$	$f(x, y)$
0	0	1	0	0
0	1	0	0	0
1	0	1	1	1
1	1	0	0	0

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b

x	y	z	$x \vee y$	$y \vee z$	$z \vee x$	$f(x, y, z)$
0	0	0	0	0	0	0
0	0	1	0	1	1	0
0	1	0	1	1	0	0
0	1	1	1	1	1	1
1	0	0	1	0	1	0
1	0	1	1	1	1	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

4 a $(x \wedge y') \vee (x' \wedge y') = y' \wedge (x \vee x')$

$= y' \wedge 1$

$= y'$

The circuit can be simplified to a y' switch

b $(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y')$

$= (x \wedge (y \vee y')) \vee (x' \wedge (y \vee y'))$

$= (x \wedge 1) \vee (x' \wedge 1)$

$= x \vee x'$

$= 1$

The globe is always on; the circuit can be a single wire with no switches

5 a $(x' \wedge y') \vee (x' \wedge y) \vee (x \wedge y)$

b $(x' \wedge y' \wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge y \wedge z) \vee (x \wedge y' \wedge z')$

Exercise 23C

1 a Your eyes are not blue.
b The sky is not grey.
c This integer is even.
d I do not live in Switzerland.
e $x \leq 2$
f This number is greater than or equal to 100.

2 a It is dark or it is cold.
b It is dark and cold. **c** It is light and cold.
d It is light or hot. **e** It is good or light.
f It is light and hard. **g** It is dark or hard.

3 a $B \wedge A$ **b** $D \vee C$
c $\neg C \wedge D$ **d** $\neg A \wedge \neg B$
e $\neg D \wedge \neg C$ **f** $B \vee A$

4 a It is wet or rough. **b** It is wet and rough.
c It is dry and rough. **d** It is dry or smooth.
e It is difficult or dry.
f It is dry and inexpensive.
g It is wet or inexpensive.

5 a x is a prime number or an even number.
b x is divisible by 6.
c x is 2.
d x is an even number greater than 2.
e x is not 2.
f x is not prime.
g x is neither prime nor divisible by 6.
h x is not divisible by 6.

6 a

A	B	$\neg(A \vee B)$	$\neg A \wedge \neg B$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

b

A	$\neg(\neg A)$
T	T
F	F

c

A	$A \vee A$
T	T
F	F

d

A	B	$A \vee B$	$\neg(\neg A \wedge \neg B)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

e

A	B	$A \wedge B$	$\neg(\neg A \vee \neg B)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

f

A	B	$A \wedge \neg B$	$\neg(\neg A \vee B)$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	F	F

10 a

A	B	$A \wedge B$	$(A \wedge B) \Rightarrow A$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

b

A	B	$A \vee B$	$(A \vee B) \Rightarrow A$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	T

c

A	B	$\neg A$	$\neg B$	$C: \neg B \vee \neg A$	$C \Rightarrow A$
T	T	F	F	F	T
T	F	F	T	T	T
F	T	T	F	T	F
F	F	T	T	T	F

d

A	B	$\neg B$	$\neg B \wedge A$	$(\neg B \wedge A) \Rightarrow A$
T	T	F	F	T
T	F	T	T	T
F	T	F	F	T
F	F	T	F	T

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e

A	B	$\neg A$	$B \vee \neg A$	$(B \vee \neg A) \Rightarrow \neg A$
T	T	F	T	F
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

f

A	B	C: $\neg B \vee \neg A$	D: $\neg B \wedge A$	$C \Rightarrow D$
T	T	F	F	T
T	F	T	T	T
F	T	T	F	F
F	F	T	F	F

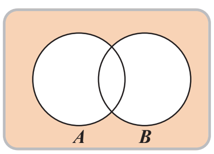
g

A	B	C: $\neg B \vee A$	D: $\neg(B \wedge A)$	$C \Rightarrow D$
T	T	T	F	F
T	F	T	T	T
F	T	F	T	T
F	F	T	T	T

h

A	B	$\neg B$	$\neg B \Rightarrow A$	$\neg B \wedge (\neg B \Rightarrow A)$
T	T	F	T	F
T	F	T	T	T
F	T	F	T	F
F	F	T	F	F

13



$(A \cup B)' = A' \cap B'$

14 a

A	B	$A \downarrow B$	$B \downarrow A$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

b

A	$A \downarrow A$	$\neg A$
T	F	F
F	T	T

c Note: $A \downarrow A$ is equivalent to $\neg A$ by part b

A	B	$\neg A \downarrow \neg B$	$A \wedge B$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

d

A	B	$\neg(A \downarrow B)$	$A \vee B$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

15 a **i** If x is an even integer, then $x = 6$.

ii If x is not an even integer, then $x \neq 6$.

b **i** If public transport improves, then I was elected.

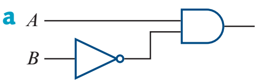
ii If public transport does not improve, then I was not elected.

c **i** If I qualify as an actuary, then I passed the exam.

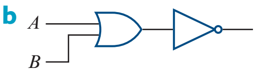
ii If I do not qualify as an actuary, then I failed the exam.

Exercise 23D

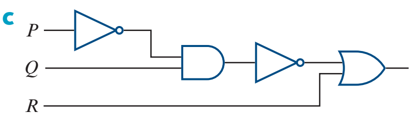
1 a



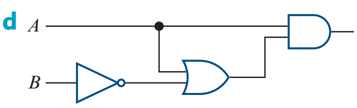
b



c

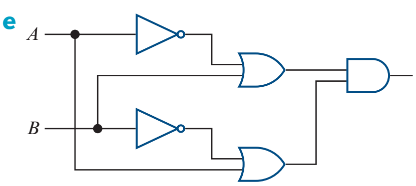


d

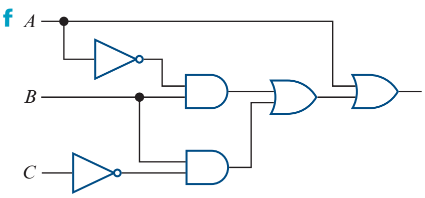


Note: Equivalent to A

e



f



Note: Equivalent to $A \vee B$

2 a $\neg X \vee (X \wedge Y) \equiv \neg X \vee Y$

X	Y	$\neg X$	$X \wedge Y$	$\neg X \vee (X \wedge Y)$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	0
1	1	0	1	1

b $\neg A \wedge (A \vee B) \equiv \neg A \wedge B$

A	B	$\neg A$	$A \vee B$	$\neg A \wedge (A \vee B)$
0	0	1	0	0
0	1	1	1	1
1	0	0	1	0
1	1	0	1	0

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c $\neg(A \wedge B)$

A	B	$A \wedge B$	$\neg(A \wedge B)$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	0

Exercise 23E

1 a y **b** $x' \vee y'$ **c** $x \vee y'$

2 a $(x \wedge y') \vee (x \wedge z) \vee (x' \wedge y \wedge z')$

b $(x \wedge y) \vee (x' \wedge z')$

c $(x \wedge y') \vee (x' \wedge z') \vee (y \wedge z)$
or $(x \wedge z) \vee (x' \wedge y) \vee (y' \wedge z')$

Chapter 23 review

Short-answer questions

1 True: d, e

2 a It is not raining. **b** It is raining.

c $2 + 3 \le 4$ **d** $x \ne 5$ or $y \ne 5$

e $x \ne 3$ and $x \ne 5$ (i.e. $x \notin \{3, 5\}$)

f It is raining and windy.

3 a

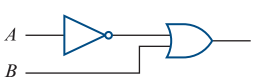
A	B	$A \oplus B$	$A \oplus (A \oplus B)$
T	T	F	T
T	F	T	F
F	T	T	T
F	F	F	F

Note: $A \oplus (A \oplus B) \equiv B$

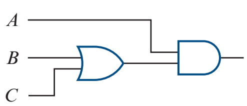
b

A	B	$A \vee B$	$A \oplus (A \vee B)$
T	T	T	F
T	F	T	F
F	T	T	T
F	F	F	F

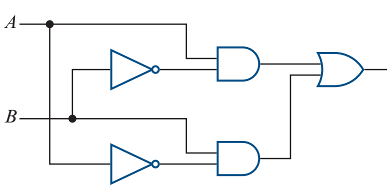
6 a



b



c



7 a If $a^3 < b^3$, then $a < b$.

b If $a^3 \ge b^3$, then $a \ge b$.

Multiple-choice questions

1 B **2** C **3** D **4** C **5** A
6 D **7** E **8** B **9** A

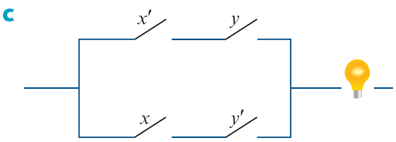
Extended-response questions

1 a

x	y	Light
0	0	0
0	1	1
1	0	1
1	1	0

b $(x' \wedge y) \vee (x \wedge y')$

c



2 a i $\ell = 1$ **ii** $h = 30$

b $\text{LCM}(x, x') = 30 = h$, for all $x \in B$;
$\text{HCF}(x, x') = 1 = \ell$, for all $x \in B$

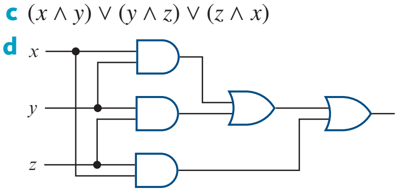
3 a

x	y	z	Light
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

b $(x' \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x \wedge y \wedge z') \vee (x \wedge y \wedge z)$

c $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$

d



4 a i d **ii** 1 **iii** 0

b $d \vee d' = d \ne 1$ and $d \wedge d' = d \ne 0$

ISBN 978-1-316-63608-4