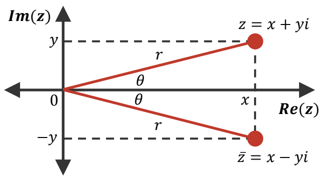
COMPLEX NUMBERS

IMAGINARY NUMBERS

$i^{-4}=1$	$i^0=1$	$i^4=1$
$i^{-3}=\sqrt{-1}$	$i^1=\sqrt{-1}$	$i^5=\sqrt{-1}$
$i^{-2}=-1$	$i^2=-1$	$i^6=-1$
$i^{-1}=-i$	$i^3=-i$	$i^7=-i$

COMPLEX NUMBER NOTATION



- $\boldsymbol{Im}$: imaginary axis (vertical axis)
- $\boldsymbol{Re}$: real axis (horizontal axis)
- $\boldsymbol{z}$: complex number ($z=x+yi$)
- $\bar{\boldsymbol{z}}$: conjugate of a complex number ($\bar{z}=x-yi$) and is reflected in the real axis
- $\boldsymbol{x}$: real components (horizontal axis)
- $\boldsymbol{y}$: imaginary component (vertical axis)
- $\boldsymbol{r}$: modulus (length) of a complex number and can also be represented by $|z|$
- $\boldsymbol{\theta}$: argument (angle that the complex number makes with the real axis) of a complex number and can also be represented by $\arg(z)$

RECTANGULAR (CARTESIAN) FORM

- $z=x+yi$ where:
  - $\boldsymbol{x}$: is the real component
  - $\boldsymbol{y}$: is the imaginary component

Convert Polar to Rectangular (Cartesian):
- $x=r\times\cos(\theta)$ and $y=r\times\sin(\theta)$

Distance between two points A and B:
- $\overline{AB}=\sqrt{(x_B^2-x_A^2)^2+(y_B^2-y_A^2)^2}$

POLAR FORM

- $z=r\times cis(\theta)$ where:
  - $\boldsymbol{r}$: is the modulus
  - $\boldsymbol{\theta}$: is the argument
  - $\boldsymbol{cis(\theta)}$: is short for $\cos(\theta)+i\sin(\theta)$

Convert Rectangular (Cartesian) to Polar:
- $r=|z|=\sqrt{x^2+y^2}$ and $\theta=\tan^{-1}\left(\frac{y}{x}\right)$

Distance between two points A and B:
- $\overline{AB}=\sqrt{r_A^2+r_B^2-2r_Ar_B\cos(\theta_A-\theta_B)}$

COMPLEX NUMBER RULES

Rules for Complex Conjugates

$\overline{z_1\pm z_2}=\bar{z_1}\pm\bar{z_2}$	$\overline{z_1\times z_2}=\bar{z_1}\times\bar{z_2}$
$\bar{z}=x-yi=r cis(-\theta)$	
$z+\bar{z}=2Re(z)=2x=2r\cos\theta$	
$z-\bar{z}=2iIm(z)=2yi=2r(i\sin\theta)$	
$z\times\bar{z}=x^2+y^2=\|z\|^2=r^2$	
$\frac{z}{\bar{z}}=\left(\frac{x^2-y^2}{x^2+y^2}\right)+i\left(\frac{2xy}{x^2+y^2}\right)=cis(2\theta)$	

Rules for Arguments

$\arg(z\times w)=\arg(z)+\arg(w)$
$\arg(z\div w)=\arg(z)-\arg(w)$

Rules for Moduli

$\|z\times w\|=\|z\|\times\|w\|$	$\left\|\frac{z}{w}\right\|=\frac{\|z\|}{\|w\|}$

More Complex Number Rules

$z^{-1}=\frac{1}{z}=\frac{1}{x+yi}\times\frac{x-yi}{x-yi}=\frac{x-yi}{x^2+y^2}=\frac{\bar{z}}{\|z\|}$
$\frac{z}{w}=\frac{a+bi}{c+di}=\frac{a+bi}{c+di}\times\frac{c-di}{c-di}=\frac{z\times\bar{w}}{\|w\|^2}$

DE MOIVRE'S THEOREM

- $(r cis\,\theta)^n=r^n\cos(n\theta)+r^n i\sin(n\theta)$
- $z^n=|z|^n cis(n\theta)$
- $z^{\frac{1}{n}}=|z|^{1/n}\left[cis\left(\frac{\theta+2\pi k}{n}\right)\right]$ for an integer k
- Find the complex n^{th} roots of a non-zero complex number z:
  - Step 1: Write z in polar form: $z=r(cis\theta)$
  - Step 2: z will have n different n^{th} roots (i.e. 3 cube roots, 4 fourth roots etc.)
  - Step 3: All these roots will have the same modulus $|z|^{1/n}=r^{1/n}$
  - Step 4: Roots have different arguments: $\frac{\theta}{n},\frac{\theta+(1\times2\pi)}{n},\frac{\theta+(2\times2\pi)}{n},\ldots,\frac{\theta+((n-1)\times2\pi)}{n}$
  - Step 5: The complex n^{th} roots of z are given in polar form by:
    - $z_1=r^{1/n}cis\left(\frac{\theta}{n}\right)$
    - $z_2=r^{1/n}cis\left(\frac{\theta+(1\times2\pi)}{n}\right)$
    - $z_3=r^{1/n}cis\left(\frac{\theta+(2\times2\pi)}{n}\right)$ and so on…
    - $z_n=r^{1/n}cis\left(\frac{\theta+((n-1)\times2\pi)}{n}\right)$

COMPLEX NUMBER EXAMPLES

1 **Express $\frac{4+3i}{2-i}$ in cartesian form.**

$\frac{4+3i}{2-i}=\frac{4+3i}{2-i}\times\frac{2+i}{2+i}=\frac{(4+3i)(2+i)}{(2-i)(2+i)}$
$=\frac{8+4i+6i+3i^2}{4-i^2}=\frac{5+10i}{5}=1+2i$

2 **Express $(-\sqrt{3}+i)(4+4i)$ in polar form.**

Converting $(-\sqrt{3}+i)$ to polar form:
$r=|z|=\sqrt{(-\sqrt{3})^2+1^2}=\sqrt{4}=2$
$\theta=\arg(z)=tan^{-1}\left(\frac{1}{-\sqrt{3}}\right)=-\frac{\pi}{6}$ but as z is in the second quadrant, $\arg(z)=-\frac{\pi}{6}+\pi=\frac{5\pi}{6}$
Converting $(4+4i)$ to polar form:
$r=|z|=\sqrt{4^2+4^2}=\sqrt{32}=\sqrt{16}\sqrt{2}=4\sqrt{2}$
$\theta=\arg(z)=tan^{-1}\left(\frac{4}{4}\right)=\frac{\pi}{4}$
Multiplying two complex numbers together:
$\left[2cis\left(\frac{5\pi}{6}\right)\right]\times\left[4\sqrt{2}cis\left(\frac{\pi}{4}\right)\right]=8\sqrt{2}cis\left(\frac{5\pi}{6}+\frac{\pi}{4}\right)$
$=8\sqrt{2}cis\left(\frac{26\pi}{24}\right)=8\sqrt{2}cis\left(\frac{13\pi}{12}\right)$

3 **Determine all roots, real and complex, of the equation $f(z)=z^3-4z^2+z+26$**

Substitute different values of z until $f(z)=0$:
$f(0)=26\neq0$, $f(1)=24\neq0$, $f(-1)=20\neq0$, $f(2)=20\neq0\rightarrow$ these are not factors
$f(-2)=0$ hence $(z+2)$ is a factor
$\therefore z^3-4z^2+z+26=(z+2)(z^2+bz+c)$
Using polynomial long division (on page 2):
$propFrac\left(\frac{z^3-4z^2+z+26}{z+2}\right)=z^2-6z+13$
Find roots of $z^2-6z+13$ by quadratic formula:
$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{6\pm\sqrt{36-4(1)(13)}}{2(1)}$
$=\frac{6\pm\sqrt{-16}}{2}=\frac{6\pm\sqrt{16}\sqrt{-1}}{2}=\frac{6\pm4i}{2}=3\pm2i$
Hence roots are $z=-2,3+2i,3-2i$

4 **Find all the complex numbers that satisfy the equation $|z|^2-iz=36+4i$**

Let $z=x+yi$ and hence:
$|(x+yi)|^2-i(x+yi)=36+4i$
$\left(\sqrt{x^2+y^2}\right)^2-xi-yi^2=36+4i$
$x^2+y^2-xi+y-36-4i=0$
Equating real and imaginary parts:
$x^2+y^2+y-36=0$ and $-x-4=0$
Hence, $x=-4$ and $(-4)^2+y^2+y-36=0$
$16+y^2+y-36=0$
$y^2+y-20=0$ and $(y+5)(y-4)=0$
Giving $y=-5,4$ hence $z=-4-5i,-4+4i$

5 **Let a and b be real numbers with $a\neq b$. If $z=x+yi$ such that $|z-a|^2-|z-b|^2=1$, prove that $x=\frac{a+b}{2}+\frac{1}{2(b-a)}$**

$|(x+yi)-a|^2-|(x+yi)-b|^2=1$
$|(x-a)+yi|^2-|(x-b)+yi|^2=1$
$(x-a)^2+y^2-[(x-b)^2+y^2]=1$
$(x-a)^2-(x-b)^2=1$
$x^2-2ax+a^2-x^2+2bx-b^2=1$
$(2b-2a)x+a^2-b^2=1$
$x=\frac{1-a^2+b^2}{2b-2a}=\frac{a+b}{2}+\frac{1}{2(b-a)}$

DE MOIVRE'S THEOREM EXAMPLES

1 **Find z^{10} given that $z=1-i$**

$r=|z|=\sqrt{1^2+(-1)^2}=\sqrt{2}$ and $\arg(z)=-\frac{\pi}{4}$
Hence, z in polar form is $z=\sqrt{2}cis\left(-\frac{\pi}{4}\right)$
Applying De Moivre's Theorem gives:
$z^{10}=\left(\sqrt{2}\right)^{10}cis\left(10\times-\frac{\pi}{4}\right)=2^5cis\left(-\frac{10\pi}{4}\right)$
$=32cis\left(-\frac{5\pi}{2}\right)=32cis\left(-\frac{5\pi}{2}+2\pi\right)$
$=32cis\left(-\frac{\pi}{2}\right)=32\left[\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right]$
$=32[0+i(-1)]=-32i$

2 **Use De Moivre's Theorem to find the smallest positive angle θ for which: $(cos\theta+i\,sin\theta)^{15}=-i$**

$\cos(15\theta)+i\,sin(15\theta)=0-i$
Equating real and imaginary parts:
$0=\cos(15\theta)$ and $-1=sin(15\theta)$
Considering both conditions, $15\theta=\frac{3\pi}{2}$
Hence, $\theta=\frac{3\pi}{30}=\frac{\pi}{10}$ is the smallest positive angle

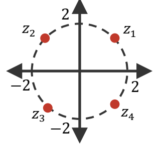
3 **By expanding $(cos\theta+i\,sin\theta)^3$ show that $cos^3\theta=\frac{1}{4}cos3\theta+\frac{3}{4}cos\theta$**

Step 1: expand the brackets of $(cos\theta+i\,sin\theta)^3$:
$(cos\theta+i\,sin\theta)^3=\cos^3\theta+3\cos^2\theta(i\,sin\theta)+3\cos(i\,sin\theta)^2+(i\,sin\theta)^3$
$=\cos^3\theta+3icos^2\theta sin\theta-3cos\theta\sin^2\theta-i\sin^3\theta$
Step 2: simplify $(cos\theta+i\,sin\theta)^3$ using De Moivre's Theorem:
$(cos\theta+i\,sin\theta)^3=cos3\theta+i\,sin3\theta$
Step 3: equating the real parts:
$\cos^3\theta-3cos\theta\sin^2\theta=cos3\theta$
$\cos^3\theta=cos3\theta+3cos\theta\,(1-\cos^2\theta)$
$\cos^3\theta=cos3\theta+3cos\theta-3\cos^3\theta$
$4\cos^3\theta=cos3\theta+3cos\theta$
$\cos^3\theta=\frac{1}{4}cos3\theta+\frac{3}{4}cos\theta$

DE MOIVRE'S THEOREM EXAMPLES

4 **Find and graph all the complex fourth roots of -16 on an argand plane.**

$r=|-16|=\sqrt{(-16)^2}=16$ and $\arg(-16)=\pi$
Hence, -16 in polar form is $z=16cis(\pi)$
We need 4 roots hence $n=4$ and the roots are:
$z_1=16^{\frac{1}{4}}cis\left(\frac{\pi}{4}\right)=2cis\left(\frac{\pi}{4}\right)$
$z_2=16^{\frac{1}{4}}cis\left(\frac{\pi+(1\times2\pi)}{4}\right)=2cis\left(\frac{3\pi}{4}\right)$
$z_3=16^{\frac{1}{4}}cis\left(\frac{\pi+(2\times2\pi)}{4}\right)=2cis\left(\frac{5\pi}{4}\right)$
$z_4=16^{\frac{1}{4}}cis\left(\frac{\pi+(3\times2\pi)}{4}\right)=2cis\left(\frac{7\pi}{4}\right)$



Note that there are $n=4$ roots and that all roots are equally spaced out by an angle of $\frac{2\pi}{n}=\frac{2\pi}{4}=\frac{\pi}{2}$

5 **One of the solutions of $z^3=a$, for some constant a, is $z=4\sqrt{3}-4i$. Determine all other solutions in Cartesian form.**

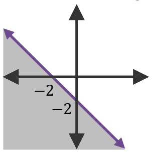
$r^{1/3}=|4\sqrt{3}-4i|=\sqrt{(4\sqrt{3})^2+(-4)^2}=8$ and
$\arg(4\sqrt{3}-4i)=\tan^{-1}\left(\frac{4}{-4\sqrt{3}}\right)=-\frac{\pi}{6}$
Hence, $4\sqrt{3}-4i$ in polar form is $z=8cis\left(-\frac{\pi}{6}\right)$
We need 3 roots hence $n=3$ and the roots are:
$z_1=8cis\left(-\frac{\pi}{6}\right)=4\sqrt{3}-4i$
$z_2=8cis\left(-\frac{\pi}{6}+\frac{2\pi}{3}\right)=8cis\left(\frac{3\pi}{6}\right)=8i$
$z_3=8cis\left(-\frac{\pi}{6}+\frac{4\pi}{3}\right)=8cis\left(\frac{7\pi}{6}\right)=-4\sqrt{3}-4i$

TRANSFORMATIONS

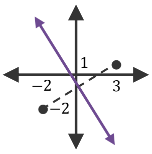
- Multiplying z by i rotates a complex number by 90º anti-clockwise.
- Multiplying z by i^n rotates a complex number by $\left(\frac{n\pi}{2}\right)$ anti-clockwise.
- Multiplying z by n increases the modulus of a complex number by scale factor n.
- Multiplying $Re(z)$ by -1 reflects a complex number in the y-axis.
- Multiplying $Im(z)$ by -1 reflects a complex number in the x-axis.

ARGAND (COMPLEX) PLANE

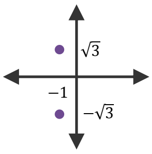
Draw the following on the complex plane:



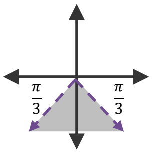
1 $\{z:|z+2+2i|\leq|z|\}$
$|(x+2)+i(y+2)|\leq|x^2+y^2|$
$(x+2)^2+(y+2)^2\leq x^2+y^2$
Simplifying this equation and making y the subject gives $y\leq-x-2$.



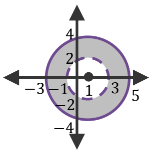
2 $\{z:|z+2+2i|=|z-3-i|\}$
$|z-(-2-2i)|=|z-(3+i)|$
Place a point at (3,1) and $(-2,-2)$, find the halfway point between them and draw a perpendicular.



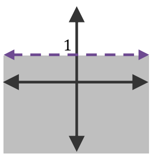
3 $\{z:z^2=-2z-4\}$
$z^2+2z+4=0$
Use quadratic equation to solve for z:
$z=-1\pm\sqrt{3}i$
$\therefore z=-1+\sqrt{3}i$ and $z=-1-\sqrt{3}i$



4 $\left\{z:-\frac{\pi}{3}<arg(iz)<\frac{\pi}{3}\right\}$
$iz=i\times(x+yi)$
$xi+yi^2=xi-y$
$\therefore$ iz rotates a complex number by 90º anti-clockwise. Needs to be reversed in the answer.



5 $\{z:2<|z-1|\leq4\}$
Draw a point at (1,0) and draw a doughnut with outer radius of 4 and inner radius of 2. Always take note of the inequality symbols used in the equation.

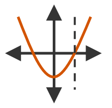


6 $\{z:z-\bar{z}<2i\}$
$(x+yi)-(x-yi)<2i$
$x+yi-x+yi<2i$
$2yi<2i$
$yi<i$
$y<1$
Take note of the inequality symbol used.

FUNCTIONS

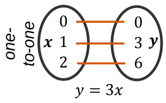
DEFINITION OF A FUNCTION

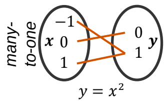
A function is one that:
- Passes the vertical line test



Vertical line cuts the curve once, so it passes the vertical line test. Therefore, this is a function.

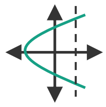
- Is *one-to-one* or *many-to-one*





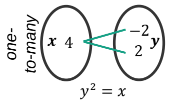
DEFINITION OF A NON - FUNCTION

A non-function (a.k.a. relation) is one that:
- Fails the vertical line test



Vertical line cuts the curve twice, so it fails the vertical line test. Therefore, this is not a function.

- Is *one-to-many*



$y^2=x$
$y=\pm\sqrt{x}$
$y=\sqrt{x}$ and $y=-\sqrt{x}$

COMPOSITE FUNCTIONS

Let $f(x)=ln(x^2+1)$ and $g(x)=2\sqrt{x}$, find:

1 $\boldsymbol{f\circ g(x)}$
$=f(2\sqrt{x})=ln\left[\left(2\sqrt{x}\right)^2+1\right]=\ln(4x+1)$

2 **Find $g(x)$ given $f\circ g(x)$ and $f(x)$**
$f\circ g(x)=f(g(x))$
$\ln(4x+1)=ln(g(x)^2+1)$
Hence $g(x)^2=4x$ and $g(x)=\sqrt{4x}=2\sqrt{x}$

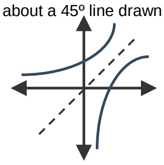
3 **Find $f(x)$ given $f\circ g(x)$ and $g(x)$**
Let $g(x)=2\sqrt{x}=u$
Solve $2\sqrt{x}=u$ for x: $x=\left(\frac{u}{2}\right)^2$
$f(g(x))=\ln(4x+1)=\ln\left[4\left(\frac{u}{2}\right)^2+1\right]$
$=\ln(u^2+1)\therefore f(u)=\ln(u^2+1)$
Change u to x: $f(x)=\ln(x^2+1)$

4 **Let $f(x)=1+\sqrt{x-2}$ and $g(x)=\frac{1}{x-5}$, find the domain and range of $g\circ f(x)$**
$g\circ f(x)=\frac{1}{1+\sqrt{x-2}-5}=\frac{1}{\sqrt{x-2}-4}$
Step 1: Find domain of inside function $f(x)$
Domain of $f(x)=\{x\in\mathbb{R}:x\geq2\}$
Step 2: Find domain of $g\circ f(x)$
Solve $\sqrt{x-2}-4\neq0$, $x-2\neq16$, $x\neq18$
Natural domain of $g\circ f(x)=\{x\in\mathbb{R}:x\neq18\}$
Step 3: The domain of $g\circ f(x)$ is the intersection of the previous two domains
Domain of $g\circ f(x)=\{x\in\mathbb{R}:x\geq2,x\neq18\}$
Step 4: To find the range of $g\circ f(x)$, analyse the critical points from the domain:
- For critical points that are $\leq,\geq$ substitute them directly into $g\circ f(x)$
- For critical points that are $\neq,<,>$ substitute a number that's ever so slightly lower and higher into $g\circ f(x)$
- Also substitute $\infty,-\infty$ into $g\circ f(x)$

$g\circ f(2)=-0.25$
$g\circ f(18.001)\rightarrow\infty$ and $g\circ f(17.999)\rightarrow-\infty$
$g\circ f(\infty)\rightarrow0$ and $g\circ f(-\infty)=N/A$
Range of $g\circ f(x)=\{g\circ f(x)\in\mathbb{R}: g\circ f(x)\leq-0.25, g\circ f(x)>0\}$

INVERSE FUNCTIONS

Inverse functions are diagonally symmetrical about a 45º line drawn through a set of axes.



Step 1: solve the function for x.
Step 2: swap all x's with y's, this new equation, is the inverse.

Inverse Function Rules

$f\circ f^{-1}(x)=f(f^{-1}(x))=x$
$f^{-1}\circ f(x)=f^{-1}(f(x))=x$

1 **Determine $f^{-1}(x)$ of $f(x)=ln(x+3)+1$**
$f^{-1}(x)$ is the inverse of $f(x)$:
$f(x)=y=\ln(x+3)+1\rightarrow y-1=\ln(x+3)$
$e^{y-1}=x+3\rightarrow e^{y-1}-3=x\rightarrow y=e^{x-1}-3$

2 **Prove that $f(x)=2x-3$ and $g(x)=0.5x+1.5$ are inverse functions.**
$f(g(x))=2(0.5x+1.5)-3=x+3-3=x$

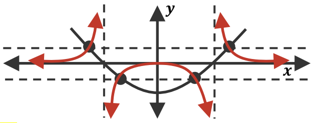
RECIPROCAL FUNCTIONS

Sketch $1/f(x)$ given $f(x)$

- Any x - intercepts on $f(x)$ are vertical asymptotes on $1/f(x)$
- Any intersections that $f(x)$ has with $y=1$ or $y=-1$ are points on $1/f(x)$
- As $f(x)$ approaches ∞ or $-\infty$ it moves toward the x - axis on $1/f(x)$

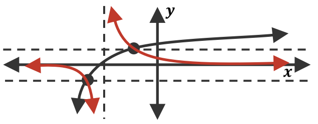
1 Sketch the function $y=\frac{1}{x^2-2}$

Let $f(x)=x^2-2$ and hence, $\frac{1}{f(x)}=\frac{1}{x^2-2}$



2 Sketch the function $y=\frac{1}{\ln(x+4)}$

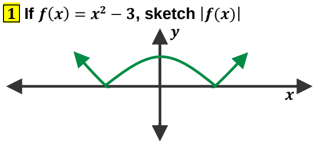
Let $f(x)=\ln(x+4)$ and hence, $\frac{1}{f(x)}=\frac{1}{\ln(x+4)}$



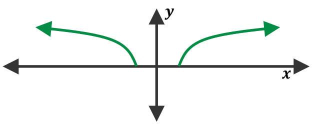
ABSOLUTE VALUE FUNCTIONS

- **Sketch $|f(x)|$:** Any points below the x - axis are reflected in the x - axis and any points above the x - axis aren't changed.
- **Sketch $f(|x|)$:** Reflects functions that cannot have negative x values (e.g. square root and logarithm functions) in the y – axis.

1 If $f(x)=x^2-3$, sketch $|f(x)|$



2 If $f(x)=\sqrt{x-2}$, sketch $f(|x|)$

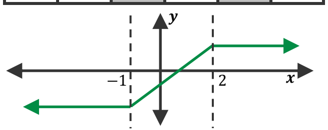


3 Sketch $y=|x+1|-|x-2|$

Solve each individual absolute value brackets for when it equals each individual absolute value brackets for when it equals 0:
$|x+1|=0$, $x=-1$ and $|x-2|=0$, $x=2$
Hence, $x=1,2$ are the critical values.

Create a x/y table with each critical value above. Insert columns between each critical value and choose a random number between them. Solve the entire table for y:

x	-2	-1	0	2	3
y	-3	-3	-1	3	3



POLYNOMIAL FRACTION FUNCTIONS

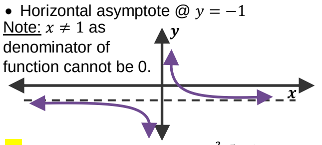
1 Sketch the function $y=\frac{-3+4x-x^2}{x^2-x}$

$=\frac{-(x^2-4x+3)}{x(x-1)}=\frac{-(x-3)(x-1)}{x(x-1)}$

$=\frac{-(x-3)}{x}=\frac{3-x}{x}=\frac{3}{x}-1$

- Vertical asymptote @ $x=0$
- Horizontal asymptote @ $y=-1$

Note: $x\neq1$ as denominator of function cannot be 0.

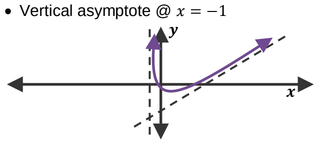


2 Sketch the function $y=\frac{x^2-5x+6}{x+1}$

Using polynomial long division (on the right):

$propFrac\left(\frac{x^2-5x+6}{x+1}\right)=x-6+\frac{12}{x+1}$

- Oblique asymptote @ $y=x-6$
- Vertical asymptote @ $x=-1$



ABSOLUTE VALUE

Absolute Value Piecewise Function

$$|x|=\begin{cases}x & x\geq0\\-x & x<0\end{cases}$$

1 If $f(x)=x+2$ and $g(x)=(x+1)^2-5$, solve $|f(x)|=|g(x)|$

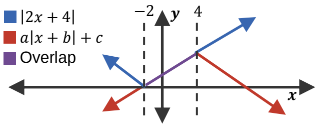
$|g(x)|=|x^2+2x-4|=|x+2|=|f(x)|$

Solving for when absolute value is positive:
$x^2+2x-4=x+2\to x^2+x-6=0$
$(x+3)(x-2)=0\to x=-3,2$

Solving for when absolute value is negative:
$x^2+2x-4=-x-2\to x^2+3x-2=0$
$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-3\pm\sqrt{9+8}}{2}$
$=\frac{-3\pm\sqrt{17}}{2}=0.5616,-3.5616$
$x=-3,2,0.5616,-3.5616$

2 If $|2x+4|=a|x+b|+c$, determine the values of the real constants a, b and c that over the domain $\{x\in\mathbb{R}:-2\leq x\leq4\}$

In order for two absolute functions to be equal over a given domain, the two functions cannot have the same concavity:



From the graph, we can find the signs for the values for a, b and c: a is negative (concave), b is negative (positive x - intercept) and c is positive (positive y – intercept). Hence, when $x=4, y=12\to c=12$. Also, $b=-4$ as there is a cusp at $x=4$ and substituting $(-2,0)$ into $y=a|x-4|+12$ gives $a=-2$.

PARTIAL FRACTIONS

- Partial Fractions: ClassPad → Main → Action → Transformation → Expand

$expand(equation, x)$

- ClassPad output:

$expand\left(\frac{3x+11}{x^2-x-6},x\right)=\frac{4}{x-3}-\frac{1}{x+2}$

1 Simplify $\frac{3x+11}{x^2-x-6}$

$\frac{3x+11}{x^2-x-6}=\frac{3x+11}{(x-3)(x+2)}=\frac{A}{x-3}+\frac{B}{x+2}$

$\frac{3x+11}{x^2-x-6}=\frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$

$3x+11=A(x+2)+B(x-3)$
$3x+11=Ax+2A+Bx-3B$
Hence, $3=A+B$ and $11=2A-3B$
Simultaneously solving on the ClassPad:
$A=4, B=-1$

2 Simplify $\frac{x^2-29x+5}{(x-4)^2(x^2+3)}$

$\frac{x^2-29x+5}{(x-4)^2(x^2+3)}=\frac{A}{x-4}+\frac{B}{(x-4)^2}+\frac{Cx+D}{x^2+3}$

$x^2-29x+5=A(x-4)(x^2+3)+B(x^2+3)+(Cx+D)(x-4)^2$
$=(A+C)x^3+(-4A+B-8C+D)x^2+(3A+16C-8D)x-12A+3B+16D$

Equating co-efficients and solving:

$$\left.\begin{array}{ll}x^3: & A+C=0\\x^2: & -4A+B-8C+D=1\\x^1: & 3A+16C-8D=-29\\x^0: & -12A+3B+16D=5\end{array}\right\}\Rightarrow\begin{array}{l}A=1\\B=-5\\C=-1\\D=2\end{array}$$

POLYNOMIAL LONG DIVISION

- Polynomial Long Division: ClassPad → Main → Action → Transformation → Fraction → propFrac

$propFrac(equation\ 1/equation\ 2)$

- ClassPad output:

$propFrac\left(\frac{x^2-9x-10}{x+1}\right)=x-10$

1 Determine $\frac{3x^3-5x^2+10x-3}{3x+1}$

$$\begin{array}{rl} & x^2-2x+4\\ 3x+1\,) & 3x^3-5x^2+10x-3\\ - & 3x^3+1x^2\\ & -6x^2+10x\\ - & -6x^2-2x\\ & +12x-3\\ - & +12x+4\\ & -7\end{array}$$

$=x^2-2x+4-\frac{7}{3x+1}$

Step 1: divide the highest order polynomials and multiply this answer by the divisor.

Step 2: subtract the two equations

Step 3: repeat steps 1 and 2 until a single number remains.

SYSTEMS OF LINEAR EQUATIONS

- Echelon Form: ClassPad → Main → Action → Matrix → Calculation → ref

$ref\left(\begin{bmatrix}a&b&c&d\\e&f&g&h\\i&j&k&l\end{bmatrix}\right)$ This returns the matrix in echelon form.

- ClassPad output:

$ref\left(\begin{bmatrix}2&6&4&14\\6&12&3&18\\4&10&6&22\end{bmatrix}\right)=\begin{bmatrix}1&3&2&7\\0&1&1.5&4\\0&0&1&2\end{bmatrix}$

- Reduced Echelon Form: ClassPad → Main → Action → Matrix → Calculation → rref

$rref\left(\begin{bmatrix}a&b&c&d\\e&f&g&h\\i&j&k&l\end{bmatrix}\right)$ This returns the matrix in reduced echelon form (i.e. will give the answers for x, y and z).

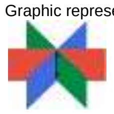
- ClassPad output:

$rref\left(\begin{bmatrix}2&6&4&14\\6&12&3&18\\4&10&6&22\end{bmatrix}\right)=\begin{bmatrix}1&0&0&0\\0&1&0&1\\0&0&1&2\end{bmatrix}$

SOLUTIONS OF LINEAR EQUATIONS

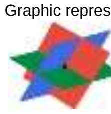
There are three types of solutions for a system of linear equations. To solve for these different solutions, the last row of matrix in echelon form must have the following forms:

- **Infinite Solutions:** more than 1 solution
  - Graphic representation:



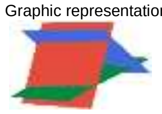
The three planes produce an intersection that is a line.

  - Last row of matrix in echelon form has the form: $[0\ \ 0\ \ 0\ \ |\ \ 0]$
- **Unique Solution:** only 1 solution
  - Graphic representation:



The three planes have a single point of intersection.

  - Last row of matrix in echelon form has the form: $[0\ \ 0\ \ A\ \ |\ \ B]\ \ A,B\neq0$
- **No Solutions:** 0 solutions
  - Graphic representation:



None of the three planes have a common intersection.

  - Last row of matrix in echelon form has the form: $[0\ \ 0\ \ 0\ \ |\ \ B]\ \ B\neq0$

LINEAR EQUATIONS EXAMPLES

1 Reduce this matrix to echelon form

$\left[\begin{array}{ccc|c}1&1&1&3\\1&2&7+a&5\\2&3&a^2+2&a+10\end{array}\right]$

$\left[\begin{array}{ccc|c}1&1&1&3\\0&1&6+a&2\\0&1&a^2&a+4\end{array}\right]\begin{array}{l}r_1\\r_2-r_1\\r_3-2r_1\end{array}$

$\left[\begin{array}{ccc|c}1&1&1&3\\0&1&6+a&2\\0&0&a^2-a-6&a+2\end{array}\right]\begin{array}{l}r_1\\r_2\\r_3-r_2\end{array}$

Note: ensure that row operations are written aside the matrix.

Using the matrix above, find a that gives:

2 No solutions

Last row in form of: $[0\ \ 0\ \ 0\ \ |\ \ B]\ B\neq0$
$\therefore a^2-a-6=0$ and $a+2\neq0$
Solving to get $a=3,-2$ and $a\neq-2$
$\therefore a=3$ gives no solutions

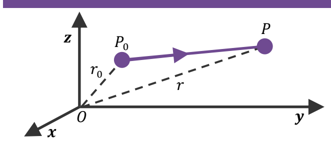
3 Infinite solutions

Last row in form of: $[0\ \ 0\ \ 0\ \ |\ \ 0]$
$\therefore a^2-a-6=0$ and $a+2=0$
Solving to get $a=3,-2$ and $a=-2$
$\therefore a=-2$ gives no solutions

4 A unique solution

Last row in form of: $[0\ \ 0\ \ A\ \ |\ \ B]\ A,B\neq0$
$\therefore a^2-a-6\neq0$ and $a+2\neq0$
Solving to get $a\neq3,-2$ and $a\neq-2$
$\therefore a\neq-2$ gives unique solution ($a\in\mathbb{R}: a\neq-2$)

DRAWING LINES



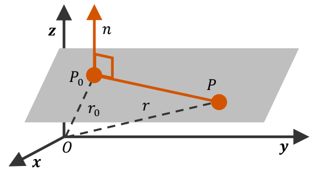
Parametric form of vector equation of a line

- $x=a+\lambda d$, $y=b+\lambda e$, $z=c+\lambda f$ where:
  - (a,b,c) is r_0 and (d,e,f) is $r-r_0$
  - λ determines the magnitude and direction

Cartesian equation of a line

- $\frac{x-a}{d}=\frac{y-b}{e}=\frac{z-c}{f}$ where:
  - (a,b,c) is r_0 and (d,e,f) is $r-r_0$

DRAWING PLANES



Vector Equation of a Plane

- $(r-r_0).n=0$ where:
  - P and P_0 are points on the plane
  - n is normal (perpendicular) to the plane
  - This equation can be simplified to: $r.n-r_0.n=0\to r.n=r_0.n\to r.n=c$

Cartesian Equation of a Plane

- $Ax+By+Cz+D=0$ where:
  - A,B,C and D are real-valued parameters
  - Vector (A,B,C) is normal (perpendicular) to the plane

VECTOR RULES

- Given $\tilde{x}=(a,b,c)$ and $\tilde{y}=(d,e,f)$:

General Vector Rules

$\overrightarrow{XY}=\tilde{y}-\tilde{x}$	$\lvert x\rvert=\sqrt{a^2+b^2+c^2}$
$\lvert\overrightarrow{XY}\rvert=\sqrt{(d-a)^2+(e-b)^2+(f-c)^2}$	

Unit Vector ($\hat{x}$)

- Returns vector with the same direction but with a magnitude of 1.

$\hat{x}=\frac{x}{\lvert x\rvert}$	$\lvert\hat{x}\rvert=1$

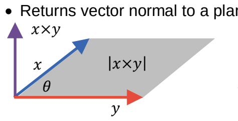
Dot Product ($x\,.\,y$)

- Dot product gives scalar result (a number).

$x\,.\,y=(a\times d)+(b\times e)+(c\times f)$	
$x\,.\,y=\lvert x\rvert\lvert y\rvert cos\theta$	$x.x=\lvert x\rvert^2$
x and y are perpendicular if $x\,.\,y=0$	
$dotP([a,b,c],[d,e,f])$ ClassPad → Main → Action → Vector → dotP	

Cross Product ($x\times y$)

- Cross product gives vector result (a vector).
- Returns vector normal to a plane.



$|x\times y|$ is area of the parallelogram with sides x and y.

$x\times y=(bf-ce,cd-af,ae-bd)$
$x\times y=\hat{n}\lvert x\rvert\lvert y\rvert sin\theta$ Where $\hat{n}$ is the unit vector perpendicular to vectors x and y
$crossP([a,b,c],[d,e,f])$ ClassPad → Main → Action → Vector → crossP

VECTOR EXAMPLES

1 Vector equation of a line passing through two given points

Points A and B have co-ordinates $(2,1,-3)$ and $(4,5,-1)$ respectively.

$\overrightarrow{AB}=\tilde{b}-\tilde{a}=2i+4j+2k$ and hence,
$r=(2i+j-3k)+\lambda(2i+4j+2k)$

2 Test if a point is perpendicular to a line

Point to test is $A(1,2,1)$ and the equation of the line is $r=(i+2j+3k)+\lambda(4i+2j-8k)$

$(i+2j+k)\,.\,(4i+2j-8k)=4+4-8=0$
Hence, the point is perpendicular to the line.

3 Intersection of two moving vectors

Find point of intersection between the lines $A=(-7i+9j-5k)+\lambda(5i-4j+2k)$ and $B=(-6i-5j+2k)+\mu(9i+6j-3k)$

Solve the i, j and k parts for λ and μ:
$-7+5\lambda=-6+9\mu$, $9-4\lambda=-5+6\mu$ and $-5+2\lambda=2-3\mu$ and hence, $\lambda=6$, $\mu=1$ therefore point of intersection is $(3,1,-1)$

4 Collision of two moving vectors

Find collision between moving vectors $A=(2i+1j-3k)+\lambda(7i+10j-3k)$ and $B=(5i+28j-6k)+\mu(6i+j-2k)$ where velocity is measured in km/h.

Equating i – coefficients: $2+7\lambda=5+6\mu$
Equating j – coefficients: $1+10\lambda=28+1\mu$
Equating k – coefficients: $-3-3\lambda=-6-2\mu$
Solving the first two equations (i and j coefficients) for λ and μ: $\lambda=3$ and $\mu=3$
Substitute into third equation (k coefficient): $-3-3(3)=-6-2(3)\to6=6$ which is consistent so a collision occurs as times λ and μ are the same (@ $t=3$). Finding collision point, substitute $t=3$ back into A or B:
$A=(2i+1j-3k)+3(7i+10j-3k)$
$=(23i+31j-12k)$

3-D VECTORS

VECTOR EXAMPLES

5 Intersection of two moving vectors
Find intersection between moving vectors $A = (-7i + 9j - 5k) + \lambda(5i - 4j + 2k)$ and $B = (-6i - 5j + 2k) + \mu(9i + 6j - 3k)$

Equating i – coefficients: $-7 + 5\lambda = -6 + 9\mu$
Equating j – coefficients: $9 - 4\lambda = -5 + 6\mu$
Equating k – coefficients: $-5 + 2\lambda = 2 - 3\mu$
Solving the first two equations (i and j coefficients) for λ and μ: $\lambda = 2$ and $\mu = 1$
λ and μ are different hence intersection at $A = (-7i + 9j - 5k) + 2(5i - 4j + 2k)$
$= (3i + j - k)$

6 Shortest distance between two moving vectors Find shortest distance between the two moving vectors where velocity is measured in km/h.
$A = (2i + j - 3k) + \lambda(7i + 10j - 3k)$ and $B = (-5i + 20j + k) + \mu(-3i - j + 7k)$

$\vec{d} = \overrightarrow{BA} + ({}_AV_B)t$ and $\vec{d}\,.\,{}_AV_B = 0$ where:

- $\vec{d}$: shortest displacement between A and B
- $\overrightarrow{BA} = \tilde{a} - \tilde{b}$: vector between A and B
- ${}_AV_B = V_A - V_B$: relative velocity of B to A

$\overrightarrow{BA} = \begin{bmatrix} 7 \\ -19 \\ -4 \end{bmatrix}$ and ${}_AV_B = \begin{bmatrix} 7 \\ 10 \\ -3 \end{bmatrix} - \begin{bmatrix} -3 \\ -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \\ -10 \end{bmatrix}$

$\vec{d} = \overrightarrow{BA} + ({}_AV_B)t = \begin{bmatrix} 7 \\ -19 \\ -4 \end{bmatrix} + t\begin{bmatrix} 10 \\ 11 \\ -10 \end{bmatrix}$

Using ClassPad to find time, $\vec{d}\,.\,{}_AV_B$

$= dotP\left(\begin{bmatrix} 7 \\ -19 \\ -4 \end{bmatrix} + t\begin{bmatrix} 10 \\ 11 \\ -10 \end{bmatrix}, \begin{bmatrix} 10 \\ 11 \\ -10 \end{bmatrix}\right) = 0.308\ hr$

Using ClassPad to find distance,

$= \left|\begin{bmatrix} 7 \\ -19 \\ -4 \end{bmatrix} + 0.308\begin{bmatrix} 10 \\ 11 \\ -10 \end{bmatrix}\right| = 19.89km$

7 Vector equation of a plane
A plane contains the point $(5, -7, 2)$ and has a normal parallel to $(3, 0, -1)$

$\begin{bmatrix} x - 5 \\ y + 7 \\ z - 2 \end{bmatrix} . \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = 0$ hence, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} . \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = 13$

8 Locating where a line intersects with a plane A plane contains the point $(5, -7, 2)$ and has a normal parallel to $(3, 0, -1)$, where does it intersect with the line $A = (-10i + 4j - 9k) + \lambda(2i + j - 6k)$

$\begin{bmatrix} -10 + 2\lambda \\ 4 + \lambda \\ -9 - 6\lambda \end{bmatrix} . \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = 13$ Solving on ClassPad

$\lambda = 17/6$ and substituting into $A = \begin{bmatrix} -26/6 \\ 41/6 \\ -26 \end{bmatrix}$

9 Equation of a plane using three non-collinear points Find the equation of a plane that passes through the points $A(1, 1, 1)$, $B(-1, 1, 0)$ and $C(2, 0, 3)$

$\overrightarrow{AB} = (-2, 0, -1)$ and $\overrightarrow{AC} = (1, -1, 2)$
$\overrightarrow{AB} \times \overrightarrow{AC} = (-1, 3, 2)$ and hence equation of the plane is $-x + 3y + 2z + D = 0$. Sub any point to find D: $-(2) + 3(0) + 2(3) + D = 0$
$D = -4$ hence $-x + 3y + 2z - 4 = 0$

10 Cartesian equation of a sphere
Find the radius and co-ordinates of the centre of the sphere with the equation $x^2 + y^2 + z^2 + 2x + 4y - 6z - 50 = 0$

$x^2 + y^2 + z^2 + 2x + 4y - 6z = 50$
$LHS = (x + 1)^2 + (y + 2)^2 + (z - 3)^2$
$RHS = 50 + 1 + 4 + 9 = 64 = 8^2$
Hence, centre at $(-1, -2, 3)$ and radius of 8.

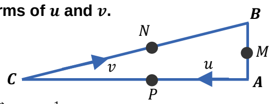
11 Cartesian equation of a hyperbola
Find the cartesian equation of the hyperbola with the vector equation $A = [3\tan(t)]i + [4\sec(t)]j$

$\frac{x}{3} = \tan(t)$ and $\frac{y}{4} = \sec(t)$

$1 + \tan^2\theta = \sec^2\theta \rightarrow 1 + \left(\frac{x}{3}\right)^2 = \left(\frac{y}{4}\right)^2$

$1 + \frac{x^2}{9} = \frac{y^2}{16} \rightarrow \frac{y^2}{16} - \frac{x^2}{9} = 1$

12 Vectors in practice
Triangle ABC is below with the midpoints of each side M, N and P shown. Let $\overrightarrow{AC} = u$ and $\overrightarrow{CB} = v$. Express $\overrightarrow{AN} + \overrightarrow{CM} + \overrightarrow{BP}$ in terms of u and v.



$\overrightarrow{AN} = u + \frac{1}{2}v$

$\overrightarrow{CM} = -u + \frac{1}{2}(u + v) = \frac{1}{2}v - \frac{1}{2}u$

$\overrightarrow{BP} = -\frac{1}{2}u - v$

$\overrightarrow{AN} + \overrightarrow{CM} + \overrightarrow{BP} = u + \frac{1}{2}v + \frac{1}{2}v - \frac{1}{2}u - \frac{1}{2}u - v = u - u + v - v = 0$

CALCULUS

TRIGONOMETRY IDENTITIES

Reciprocal Identities

$\sin(x) = \frac{1}{cosec(x)}$	$\cos(x) = \frac{1}{sec(x)}$	$\tan(x) = \frac{1}{cot(x)}$
$cosec(x) = \frac{1}{\sin(x)}$	$\sec(x) = \frac{1}{\cos(x)}$	$\cot(x) = \frac{1}{\tan(x)}$

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$	$1 + \tan^2\theta = \sec^2\theta$

Quotient Identities

$\tan(x) = \frac{\sin(x)}{\cos(x)}$	$\cot(x) = \frac{\cos(x)}{\sin(x)}$

Co-Function Identities

$\sin\left(\frac{\pi}{2} - x\right) = \cos(x)$	$\cos\left(\frac{\pi}{2} - x\right) = \sin(x)$

Parity Identities (Even and Odd)

$\sin(-x) = -\sin(x)$	$\cos(-x) = \cos(x)$
$\tan(-x) = -\tan(x)$	$\sec(-x) = \sec(x)$

Sum and Difference

$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$

$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$

$\tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)}$

Double Angle

$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$

$\sin(2x) = 2\sin(x)\cos(x)$

$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$

Power Reducing

$\sin^2(x) = \frac{1 - \cos(2x)}{2}$	$\cos^2(x) = \frac{1 + \cos(2x)}{2}$

Limits of Sine and Cosine

$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$	$\lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0$

DIFFERENTIATION RULES

Product, Quotient and Chain Rules

$y = uv \rightarrow \frac{dy}{dx} = u'v + uv'$

$y = \frac{u}{v} \rightarrow \frac{dy}{dx} = \frac{u'v - uv'}{v^2}$

$y = [f(x)]^n \rightarrow \frac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)$

Common Derivatives

$y = ax^n \rightarrow \frac{dy}{dx} = n \times ax^{n-1}$

$y = e^{f(x)} \rightarrow \frac{dy}{dx} = f'(x) \times e^{f(x)}$

$y = \frac{1}{x} = x^{-1} \rightarrow \frac{dy}{dx} = \frac{-1}{x^2} = -x^{-2}$

$y = \pm\sin(x) \rightarrow \frac{dy}{dx} = \pm\cos(x)$

$y = \pm\cos(x) \rightarrow \frac{dy}{dx} = \mp\sin(x)$

$y = \pm\tan(x) \rightarrow \frac{dy}{dx} = \pm\sec^2(x) = \frac{\pm 1}{\cos^2(x)}$

$y = \ln[f(x)] \rightarrow \frac{dy}{dx} = \frac{f'(x)}{f(x)}$

$y = a^x \rightarrow \frac{dy}{dx} = \ln(a) \times a^x$

INTEGRATION RULES

Integral Rules

$\int_a^b f(x) = -\int_b^a f(x)$

$\int ax^n\ dx = a\int x^n\ dx$

Fundamental Theorem of Calculus

$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$

$\int_a^b f'(x)dx = f(b) - f(a)$

Integration by Parts

$\int uv'\ dx = uv - \int u'v\ dx$

Area Between Curves

$\int_a^b \text{upper curve}\ dx - \int_a^b \text{lower curve}\ dx$

INTEGRATION RULES

Common Integrals

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$\int f'(x) \times [f(x)]^n\ dx = \frac{[f(x)]^{n+1}}{n+1} + c$

$\int e^{f(x)}\ dx = \frac{e^{f(x)}}{f'(x)} + c$

$\int \frac{f'(x)}{f(x)}\ dx = \ln|f(x)| + c$

$\int \sin(x)\ dx = -\cos(x) + c$

$\int \cos(x)\ dx = \sin(x) + c$

$\int \sec^2(x)\ dx = \tan(x) + c$

IMPLICIT DIFFERENTIATION

1 The point (a, b) lies on the curves $x^2 - y^2 = 5$ and $xy = 6$. Prove that the tangents to those curves at (a, b) are perpendicular.

Differentiating $x^2 - y^2$ with respect to x:
$x^2 - y^2 = 5 \rightarrow 2x - 2y\frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = \frac{x}{y}$
At point (a, b) the slope is $m_1 = \frac{x}{y}$
Differentiating $xy = 6$ with respect to x:
$xy = 6 \rightarrow y + x\frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = -\frac{y}{x}$
At point (a, b) the slope is $m_2 = -\frac{y}{x}$
Lines are perpendicular if $m_1 \times m_2 = -1$
$m_1 m_2 = \frac{x}{y} \times -\frac{y}{x} = -1$

2 Find the gradient at the point $(2, -1)$ on the curve $x + x^2y^3 = -2$

Differentiating with respect to x:
$1 + 2xy^3 + x^2 3y^2\frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = \frac{-1 - 2xy^3}{x^2 3y^2}$

$\left.\frac{dy}{dx}\right|_{x=2, y=-1} = \frac{-1 - 2 \times 2 \times (-1)^3}{2^2 \times 3 \times (-1)^2} = \frac{1}{4}$

3 Determine the derivative of $\sqrt{x} + \sqrt{y} = 1$

Differentiating with respect to x:
$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = -\sqrt{\frac{y}{x}}$

4 Find the co-ordinates of the points where the tangent to the curve $x^2 + 2xy + 3y^2 = 18$ is horizontal.

Differentiating with respect to x:
$2x + 2y + 2x\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$
$\frac{dy}{dx}(2x + 6y) = -2x - 2y = -\left(\frac{x + y}{x + 3y}\right)$
Solve for when $\frac{dy}{dx} = 0$ hence $x = -y$
Substitute into original: $y^2 - 2y^2 + 3y^2 = 18$
$y^2 = 9$ and hence, $y = \pm 3, x = \pm 3$

$\frac{80}{(20 + x)(20 - x)} = \frac{A(20 - x) + B(20 + x)}{40 - x^2}$
$80 = 20A - Ax + 20B + Bx$
Hence, $80 = 20A + 20B$ and $0 = B - A$
$A = 2, B = -2$ hence, integral is $\int \frac{2}{20+x} - \frac{2}{20-x}dx$
$= 2ln(|20 + x|) - 2ln(|20 - x|) + c$

DIFFERENTIAL EQUATIONS

1 A solution of a differential equation is $y = Ae^{-2t} + Be^{-t}$. When $t = 0$, it given that $y = 0$ and $\frac{dy}{dt} = 1$. Find the values of A and B.

$y = Ae^{-2t} + Be^{-t} \rightarrow \frac{dy}{dx} = -2Ae^{-2t} - Be^{-t}$
Using that $y = 0$ when $t = 0$: $0 = A + B$
Using that $\frac{dy}{dx} = 1$ when $t = 0$: $-1 = -2A + B$
Solving for A and B: $A = -1$ and $B = 1$
Hence, $y = -e^{-2t} + e^{-t}$

2 Determine the equation of the graph from the following conditions:

- **Gradient of the tangent at all points is given by $-\frac{x}{3y}$**
- **The graph passes through $(3, 1)$**

$\frac{dy}{dx} = -\frac{x}{3y} \rightarrow \int 3ydy = \int -xdx$
$\frac{3y^2}{2} = -\frac{x^2}{2} + C \rightarrow 3y^2 = -x^2 + C$
Applying initial condition (3,1):
$3(1)^2 = -(3)^2 + C \rightarrow 3 = -9 + C \rightarrow C = 12$
Hence, $2y^2 = 3x^2 + 12$

3 Determine the general solution for $y' = 6y^2x$ given that $x = 1, y = \frac{1}{25}$

$\frac{dy}{dx} = 6y^2x \rightarrow \int\frac{dy}{y^2} = \int 6xdx \rightarrow -\frac{1}{y} = 3x^2 + c$
Applying initial condition (1/25,1):
$-25 = 3 + c \rightarrow c = -28$ hence, $-\frac{1}{y} = 3x^2 - 28$

LOGISTIC EQUATION

Logistic Equation Differential Equation

- Used in biology, mathematics, economics, chemistry, probability and statistics

Form	$\frac{dP}{dt} = aP - bP^2$
Solution	$P = \frac{a}{b + ke^{-at}}$

1 Show that if $P = \frac{a}{b+ke^{-at}}$, then the derivative is in the form $\frac{dP}{dt} = aP - bP^2$

From these two equations, deduce that:
$ke^{-at} = \frac{a}{P} - b$

$\frac{dP}{dt} = a\left(\frac{a}{b + ke^{-at}}\right) - b\left(\frac{a}{b + ke^{-at}}\right)^2$
$= \frac{a^2}{b + ke^{-at}} - \frac{a^2b}{(b + ke^{-at})^2}$
$= \frac{a^2}{b + ke^{-at}}\left[\frac{1}{1} - \frac{b}{b + ke^{-at}}\right]$
$= \frac{a^2}{b + ke^{-at}}\left[\frac{b + ke^{-at} - b}{b + ke^{-at}}\right]$

$= \frac{a^2ke^{-at}}{(b + ke^{-at})^2} = \frac{a^2\left(\frac{a}{P} - b\right)}{\left(\frac{a}{P}\right)^2} = aP - bP^2$

2 If $\frac{dP}{dt} = 0.2P - 0.002P^2$, determine P as a function of t from question 1 above given that when $t = 0, P = 5$.

$\frac{0.2}{0.002 + ke^0} = 5 \rightarrow k = 0.038$

$\therefore P = \frac{0.2}{0.002 + 0.038e^{-0.2t}}$

VECTOR AND MOTION CALCULUS

Displacement, Velocity and Acceleration

Displacement	$r(t)$
Velocity	$v(t) = r'(t)$
Acceleration	$a(t) = v'(t) = r''(t)$

1 A particle is moving in m/s along a straight line and the acceleration of the particle is modelled by $a(t) = 2 - e^{\frac{-x}{2}}$. When $v = 4, x = 0$. Find v^2 in terms of x.

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = a(t) = 2 - e^{\frac{-x}{2}}$
$\frac{1}{2}v^2 = \int 2 - e^{\frac{-x}{2}}dx = 2x + 2e^{\frac{-x}{2}} + c$
When $v = 4, x = 0$ hence,
$\frac{1}{2}(16) = 0 + 2 + c, c = 6$
$\therefore \frac{1}{2}v^2 = 2x + 2e^{\frac{-x}{2}} + 6v^2 = 4x + 4e^{\frac{-x}{2}} + 12$

The position vector of a particle is initially at $r = -j$ cm and is moving horizontally with velocity in cm/s according to the equation $v = (3cost)i + (sint)j$

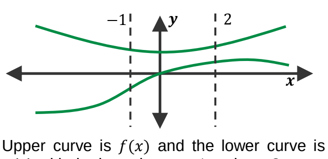
2 What is the initial acceleration?
$a(t) = v'(t) = (-3sint)i + (cost)j$

3 Find the displacement function.
$r(t) = \int v(t)dt = (3sint)i - (cost)j + c$
As initially $r = -j$, $c = 0$ hence:
$r(t) = (3sint)i - (cost)j$

3 Determine the cartesian equation of the path of the particle.
$sint = \frac{x}{3}$ and $cost = -y$
$\sin^2 t + \cos^2 t = \left(\frac{x}{3}\right)^2 + (-y)^2 = 1$
$\frac{x^2}{9} + y^2 = 1$

AREA BETWEEN CURVES

1 Determine the area between the two curves $f(x) = x^2 + 2$ and $g(x) = \sin(x)$ with the condition $-1 \le x \le 2$



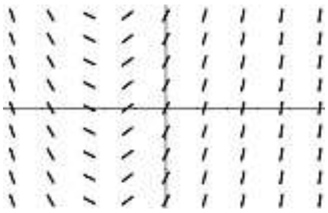
Upper curve is $f(x)$ and the lower curve is $g(x)$ with the bounds $x = -1$ and $x = 2$.

Hence, $A = \int_{-1}^2 f(x) - g(x)dx$
$= \int_{-1}^2 (x^2 + 2) - (\sin(x))dx$
$= \left[\frac{1}{3}x^3 + 2x + \cos(x)\right]_{-1}^2$
$= \left[\left(\frac{1}{3}(2)^3 + 2(2) + \cos(2)\right) - \left(\frac{1}{3}(-1)^3 + 2(-1) + \cos(-1)\right)\right] = 8.04$

CALCULUS

SLOPE (GRADIENT FIELDS)

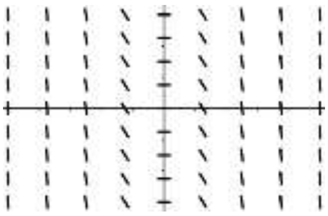
1 Determine a general differential equation for the following slope field and explain your reasoning.



$$\frac{dy}{dx} = ax + b$$

- Quadratic equation formed by isoclines.
- Convex nature, hence a is positive.
- x-intercept on the negative x-axis, hence b is positive.

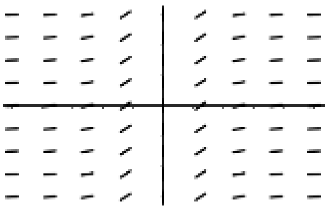
2 Determine a general differential equation for the following slope field and explain your reasoning.



$$\frac{dy}{dx} = -ax^2 - b$$

- Isoclines are all have negative gradient, hence cubic function.
- Point of inflection is on the y-axis.
- Consistent negative isoclines indicate negative gradient.

3 Determine a general differential equation for the following slope field and explain your reasoning.



$$\frac{dy}{dx} = \frac{a}{x^2} + b$$

- Hyperbolic function formed by isoclines.
- Gradient is ∞ at $x = 0$, hence vertical asymptote at $x = 0$
- Power of x must be even as gradient of positive x-values is positive as well as negative x-values.

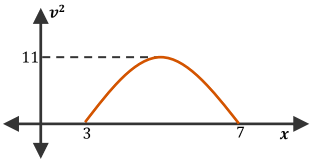
SIMPLE HARMONIC MOTION

Simple Harmonic Motion Rules

- $\boldsymbol{A}$: amplitude of the motion
- $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$: angles of phase
- $\boldsymbol{v}$: velocity and $\boldsymbol{x}$: displacement

$\frac{d^2x}{dt^2} = -k^2x$	
$x = Asin(kt+\alpha)$	$x = Acos(kt+\beta)$
$v^2 = k^2(A^2 - x^2)$	

1 A particle is moving in m/s along the x-axis in simple harmonic motion. The parabola below shows v^2 as a function of x.



Determine the values of a, c and n in the equation $v^2 = n^2(a^2 - (x - c)^2)$.

$c = 5$ as the particle oscillated about $x = 5$
$a = 2$ as the amplitude is $5 - 3 = 2$ or $7 - 5 = 2$
Hence, $11 = n^2(4 - (x-5)^2)$
As $v^2 = 11$ when $x = 5$, $11 = 4n^2 \rightarrow n = \frac{\sqrt{11}}{2}$

INCREMENTAL FORMULA

Incremental Formula (small change)

$$\delta y \approx \frac{dy}{dx} \times \delta x$$

1 A differential equation has a point at $(5, 6)$ and $\frac{dy}{dx} = xy - x^2$. Determine an estimate for y when $x = 5.2$.

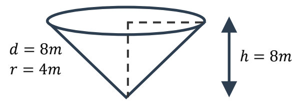
Using Euler's method with $\delta x = 0.1$

x	y	$\frac{dy}{dx}$	$\delta y \approx \frac{dy}{dx} \times \delta x$
5	6	5	0.5
5.1	6.5	7.14	0.714
5.2	7.214		

Estimate is $y = 7.214$

RELATED RATES

1 An inverted cone $8m$ tall has an upper diameter of $8m$ and is filling with water at a rate of $2m^3/min$. At what rate is the water level rising in the container when the depth of water is exactly $3.5m$?



From the question, substitute $r = \frac{h}{2}$ into volume

$\therefore V = \frac{1}{3}\pi r^2 h = \frac{1}{12}\pi h^3 \rightarrow \frac{dV}{dh} = \frac{1}{4}\pi h^2$

To find $\frac{dh}{dt}$ when $h = 3.5$:

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{4}{\pi h^2} \times 2 = \frac{4}{\pi(3.5)^2} \times 2$

$= \frac{8}{\pi(3.5)^2} = \frac{32}{49\pi} m/min$

VOLUMES OF REVOLUTION

Revolution about the x-axis

- $\boldsymbol{a}$ and $\boldsymbol{b}$: are bounds on the x-axis

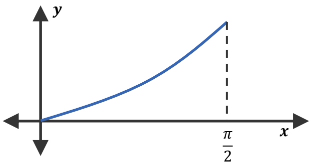
$$V = \pi \int_a^b y^2 dx$$

Revolution about the y-axis

- $\boldsymbol{a}$ and $\boldsymbol{b}$: are bounds on the y-axis

$$V = \pi \int_a^b x^2 dy$$

1 Determine the region bounded by the line $x = \frac{\pi}{2}$ and $y = 3tan\left(\frac{x}{3}\right)$ rotated around the x-axis.



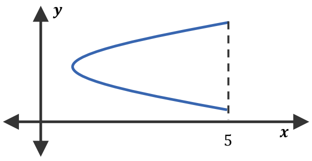
$V = \pi \int_a^b y^2 dx = \pi \int_0^{\pi/2} \left(3tan\left(\frac{x}{3}\right)\right)^2 dx$

$\left(3tan\left(\frac{x}{3}\right)\right)^2 = 9\tan^2\left(\frac{x}{3}\right) = 9\sec^2\left(\frac{x}{3}\right) + 9$

$= \pi \int_0^{\frac{\pi}{2}} 9\sec^2\left(\frac{x}{3}\right) - 9\,dx$

$= \pi\left[27tan\left(\frac{x}{3}\right) - 9x\right]_0^{\frac{\pi}{2}} = \frac{-9\pi}{2} + 9\sqrt{3}\pi$

2 Determine the volume of the region in between the functions $x = y^2 - 6y + 10$ and $x = 5$ rotated around the y-axis.



Determine the points of intersection:
$5 = y^2 - 6y + 10 \rightarrow 0 = y^2 - 6y + 5$
$0 = (y-5)(y-1) \rightarrow y = 1, 5$
Hence, points of intersection are $(5,1)$ and $(5,5)$

Inner radius $= y^2 - 6y + 10$
Outer radius $= 5$
Revolution around y-axis $= \pi \int_a^b x^2 dy$

Hence, this question can be treated as an area between two curves question with respect to the y-axis.

$\therefore x^2 = [(outer\ radius)^2 - (inner\ radius)^2]$
$= [(5)^2 - (y^2 - 6y + 10)^2]$
$= [-75 + 120y - 56y^2 + 12y^3 - y^4]$

Finding volume:

$V = \pi \int_1^5 -75 + 120y - 56y^2 + 12y^3 - y^4 dy$

$= \pi\left[-75y + 60y^2 - \frac{56}{3}y^3 + 3y^4 - \frac{1}{5}y^5\right]_1^5$

$= \frac{1088}{15}\pi = 227.87\ units^2$

STATISTICAL INFERENCE

RANDOM SAMPLES

Population Notation

- $\boldsymbol{\mu}$: population mean
- $\boldsymbol{\sigma}$: population standard deviation
- $\boldsymbol{\sigma^2}$: variance

Sample Notation

- $\bar{X}$: sample mean
- $\boldsymbol{n}$: sample size
- If $n \geq 30$, regardless of the prior distribution, the sample data will become normally distributed with parameters:
  - Mean: $\bar{X}$
  - Standard Deviation: $\frac{\sigma}{\sqrt{n}}$

Z-Score $Z \sim N(0, 1)$

$$Z = \frac{X - \mu}{\sigma}$$

Sample Size

- $\boldsymbol{d}$: value of the difference from the mean.

$$n = \left(\frac{z \times \sigma}{d}\right)^2$$

CONFIDENCE INTERVALS

Confidence Intervals

- $\boldsymbol{z}$: z-score for a given confidence interval

$$\bar{X} - z\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + z\frac{\sigma}{\sqrt{n}}$$

Common Confidence Intervals (z-scores)

99% CI	2.58
95% CI	1.96
90% CI	1.645

- Custom Confidence Interval: ClassPad → Main → Action → Distribution → Inverse → invNormCDf

$z_c = -1\times$ invNormCDf("C", c, 1, 0)

Where c is the CI% as a decimal.

STATISTICAL INFERENCE EXAMPLES

1 Determine a 95% confidence interval of a sample of 25 results with mean of 20 and variance of 4.

$20 - 1.96\left(\frac{2}{\sqrt{25}}\right) \leq \mu \leq 20 + 1.96\left(\frac{2}{\sqrt{25}}\right)$

Hence, the 95% CI is $[19.216, 20.784]$

2 What size sample is needed to ensure that sample mean is within 1.5 of the population mean with 99% confidence, given the standard deviation is 13.

$n = \left(\frac{z \times \sigma}{d}\right)^2 = \left(\frac{2.58 \times 13}{1.5}\right)^2 = 499.96 \approx 500$

3 How large of a sample is needed to be 95% confident that the sample mean is within 10 of the population mean, given the standard deviation is 15.

$10 = 1.96\left(\frac{15}{\sqrt{n}}\right) \rightarrow n = 8.6436 \approx 9$

4 45 samples of mean 94 and standard deviation 12 was taken. Determine the parameters of the normal distribution.

$X \sim N\left(94, \left(\frac{12}{\sqrt{45}}\right)^2\right)$

YOUR NOTES AND EXAMPLES