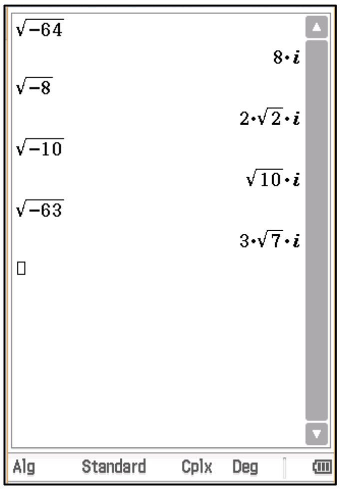
SADLER UNIT 3 MATHEMATICS SPECIALIST

WORKED SOLUTIONS

Chapter 1: Complex numbers, a reminder.

Exercise 1A

Question 1



a $\sqrt{-64} = \sqrt{64i^2} = 8i$

b $\sqrt{-8} = \sqrt{8i^2} = 2\sqrt{2}\,i$

c $\sqrt{-10} = \sqrt{10i^2} = \sqrt{10}\,i$

d $\sqrt{-63} = \sqrt{63i^2} = 3\sqrt{7}\,i$

Question 2

a Given $z = -5 + 3i$, $\text{Re}(z) = -5$

b Given $z = -5 + 3i$, $\text{Im}(z) = 3$

Question 3

a Given $z = 12 - 5i$, $\text{Re}(z) = 12$

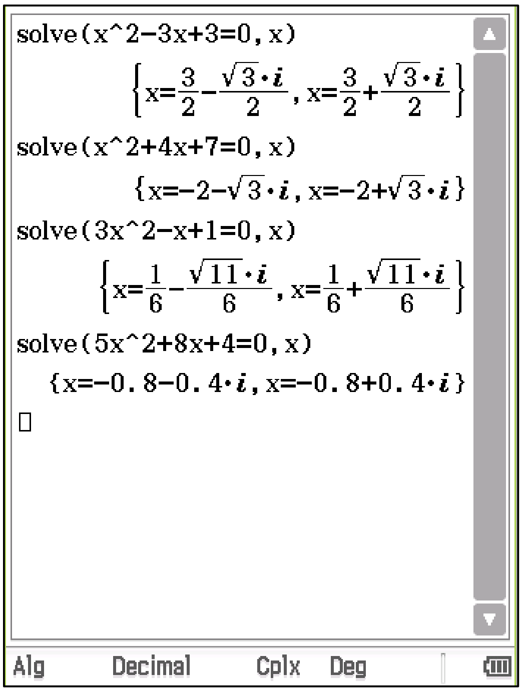
b Given $z = 12 - 5i$, $\text{Im}(z) = -5$

Question 4

Given $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

a If $x^2 - 3x + 3 = 0$, then $a = 1$, $b = -3$ and $c = 3$.

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \times 1 \times 3}}{2 \times 1} = \frac{3 \pm \sqrt{-3}}{2}$$
$$= \frac{3 \pm \sqrt{3i^2}}{2} = \frac{3 \pm \sqrt{3}\,i}{2} = \frac{3}{2} \pm \frac{\sqrt{3}}{2}i$$
$$x = \frac{3}{2} + \frac{\sqrt{3}}{2}i,\ \frac{3}{2} - \frac{\sqrt{3}}{2}i$$



b If $x^2 + 4x + 7 = 0$, then $a = 1$, $b = 4$ and $c = 7$.

$$x = \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times 7}}{2 \times 1} = \frac{-4 \pm \sqrt{-12}}{2}$$
$$= \frac{-4 \pm \sqrt{12i^2}}{2} = \frac{-4 \pm 2\sqrt{3}\,i}{2} = -2 \pm \sqrt{3}i$$
$$x = -2 + \sqrt{3}i,\ -2 - \sqrt{3}i$$

c If $3x^2 - x + 1 = 0$, then $a = 3$, $b = -1$ and $c = 1$.

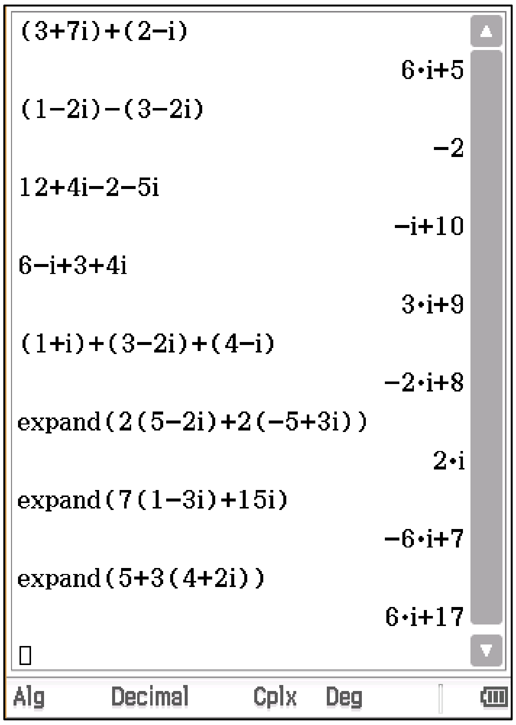
$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 3 \times 1}}{2 \times 3} = \frac{1 \pm \sqrt{-11}}{6}$$
$$= \frac{1 \pm \sqrt{11i^2}}{6} = \frac{1 \pm \sqrt{11}\,i}{6} = \frac{1}{6} \pm \frac{\sqrt{11}}{6}i$$
$$x = \frac{1}{6} + \frac{\sqrt{11}}{6}i,\ \frac{1}{6} - \frac{\sqrt{11}}{6}i$$

d If $5x^2 + 8x + 4 = 0$, then $a = 5$, $b = 8$ and $c = 4$.

$$x = \frac{-8 \pm \sqrt{8^2 - 4 \times 5 \times 4}}{2 \times 5} = \frac{-8 \pm \sqrt{-16}}{10} = \frac{-8 \pm \sqrt{16i^2}}{10}$$
$$= \frac{-8 \pm 4i}{10} = \frac{-8}{10} \pm \frac{4}{10}i$$
$$x = -\frac{4}{5} + \frac{2}{5}i,\ -\frac{4}{5} - \frac{2}{5}i$$
$$x = -0.8 + 0.4i,\ -0.8 - 0.4i$$

Question 5

$$(3+7i)+(2-i) = 3+2+7i-i$$
$$= 5+6i$$



Question 6

$$(1-2i)-(3-2i) = 1-3-2i-2i$$
$$= -2$$

Question 7

$$12+4i-2-5i = 12-2+4i-5i$$
$$= 10-i$$

Question 8

$$6-i+3+4i = 6+3-i+4i$$
$$= 9+3i$$

Question 9

$$(1+i)+(3-2i)+(4-i) = 8-2i$$

Question 10

$$2(5-2i)+2(-5+3i) = 10-4i-10+6i$$
$$= 2i$$

Question 11

$$7(1-3i)+15i = 7-21i+15i = 7-6i$$

Question 12

$$\begin{aligned}5+3(4+2i) &= 5+12+6i \\ &= 17+6i\end{aligned}$$

Question 13

$$\begin{aligned}\mathrm{Re}(5+2i)+\mathrm{Re}(-3+4i) &= 5-3 \\ &= 2\end{aligned}$$

Question 14

$$\begin{aligned}\mathrm{Im}(-1-7i)+\mathrm{Im}(3+2i) &= -7+2 \\ &= -5\end{aligned}$$

Question 15

$$\begin{aligned}(5-2i)(2+3i) &= 10+15i-4i-6i^2 \\ &= 10+11i+6 \\ &= 16+11i\end{aligned}$$

```
expand((5−2i)(2+3i))
                         −6•i^2+11•i+10
−6•(−1)+11•i+10
                               11•i+16
expand((3+i)(3+2i))
                              2•i^2+9•i+9
2•(−1)+9•i+9
                                  9•i+7
expand((2+i)(2−i))
                                 −i^2+4
−(−1)+4
                                      5
expand((−2+7i)(7−2i))
                        −14•i^2+53•i−14
−14•(−1)+53•i−14
                                   53•i
```

Question 16

$$\begin{aligned}(3+i)(3+2i) &= 9+6i+3i+2i^2 \\ &= 9+9i-2 \\ &= 7+9i\end{aligned}$$

Question 17

$$\begin{aligned}(2+i)(2-i) &= 4-2i+2i-i^2 \\ &= 4-i^2 \\ &= 4-(-1) \\ &= 4+1 \\ &= 5\end{aligned}$$

Question 18

$$\begin{aligned}(-2+7i)(7-2i) &= -14+4i+49i-14i^2 \\ &= -14+53i-14(-1) \\ &= -14+53i+14 \\ &= 53i\end{aligned}$$

Question 19

$$\frac{2-3i}{1+2i}=\frac{2-3i}{1+2i}\times\frac{1-2i}{1-2i}=\frac{2-4i-3i+6i^2}{1-2i+2i-4i^2}$$
$$=\frac{2-7i+6(-1)}{1-4(-1)}=\frac{-4-7i}{5}$$
$$=-0.8-1.4i$$

Question 20

$$\frac{2-3i}{2+3i}=\frac{2-3i}{2+3i}\times\frac{2-3i}{2-3i}=\frac{4-6i-6i+9i^2}{4-6i+6i-9i^2}$$
$$=\frac{4-12i+9(-1)}{4-9(-1)}=\frac{-5-12i}{13}$$
$$=-\frac{5}{13}-\frac{12}{13}i$$

Question 21

$$\frac{5-2i}{3+4i}=\frac{5-2i}{3+4i}\times\frac{3-4i}{3-4i}=\frac{15-20i-6i+8i^2}{9-12i+12i-16i^2}$$
$$=\frac{15-26i+8(-1)}{9-16(-1)}=\frac{7}{25}-\frac{26i}{25}$$
$$=\frac{7-26i}{25}$$

Question 22

$$\frac{i}{2-i}=\frac{i}{2-i}\times\frac{2+i}{2+i}=\frac{2i+i^2}{4+2i-2i-i^2}$$
$$=\frac{2i-1}{4-(-1)}=\frac{2i-1}{5}$$
$$=-0.2+0.4i$$

Question 23

a

$$\begin{aligned} w+z &= 2+3i+5-i \\ &= 7+2i \end{aligned}$$

b

$$\begin{aligned} w-z &= 2+3i-(5-i) \\ &= 2+3i-5+i \\ &= -3+4i \end{aligned}$$

c

$$\begin{aligned} 5w-4z &= 5(2+3i)-4(5-i) \\ &= 10+15i-20+4i \\ &= -10+19i \end{aligned}$$

d

$$\begin{aligned} wz &= (2+3i)(5-i) \\ &= 10-2i+15i-3i^2 \\ &= 10+13i-3(-1) \\ &= 10+13i+3 \\ &= 13+3i \end{aligned}$$

e

$$\begin{aligned} z^2 &= (5-i)^2 \\ &= (5-i)(5-i) \\ &= 25-5i-5i+i^2 \\ &= 25-10i-1 \\ &= 24-10i \end{aligned}$$

f

$$\begin{aligned} \frac{w}{z} &= \frac{2+3i}{5-i} = \frac{(2+3i)}{(5-i)} \times \frac{(5+i)}{(5+i)} \\ &= \frac{10+2i+15i+3i^2}{25-5i+5i-i^2} \\ &= \frac{10+17i+3(-1)}{25-(-1)} \\ &= \frac{10+17i-3}{26} \\ &= \frac{7+17i}{26} \\ &= \frac{7}{26}+\frac{17}{26}i \end{aligned}$$

Question 24

Given $z = 4 - 7i$,

a $\quad \overline{z} = 4 + 7i$

b $\quad z + \overline{z} = 4 - 7i + 4 + 7i = 8$

c $\quad z\overline{z} = (4 - 7i)(4 + 7i) = 16 + 28i - 28i - 49i^2$

$= 16 - 49(-1) = 65$

d

$$\frac{z}{\overline{z}} = \frac{4-7i}{4+7i} = \frac{4-7i}{4+7i} \times \frac{4-7i}{4-7i} = \frac{16-28i-28i+49i^2}{16-28i+28i-49i^2}$$

$$= \frac{16-56i+49(-1)}{16-49(-1)} = \frac{16-56i-49}{16+49} = \frac{-33-56i}{65}$$

$$= -\frac{33}{65} - \frac{56}{65}i$$

Question 25

Given that $z = 5 + ai,\ w = b - 34i$, a and b are real numbers and $z = w$

Real parts are the same so $5 = b$.

Imaginary parts are equal so $a = -34$

$\therefore a = -34,\ b = 5$

Question 26

Given $(a + 5i)(2 - i) = b$, where a and b are real numbers.

$(a + 5i)(2 - i) = 2a - ai + 10i - 5i^2$

$= 2a - ai + 10i - 5(-1)$

$= 2a - ai + 10i + 5$

Real part is $2a + 5$, which is equal to b

Imaginary part is $-a + 10$, which is equal to 0 as there is no imaginary part on the right hand side of the equation.

$-a + 10 = 0$

So $\quad a = 10$

$2a + 5 = b$

$2(10) + 5 = b$

$25 = b$

$\therefore a = 10,\ b = 25$

Question 27

a The quadratic formula is: $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

In order for a quadratic to have a non-real root, b^2-4ac must be negative

Let $b^2-4ac=-k$, given that k is a real number and is positive.

Then $\sqrt{-k}=\sqrt{(-1)k}=\sqrt{i^2k}=\sqrt{ki^2}=\sqrt{k}\ i$

$$
\begin{aligned}
x &= \frac{-b\pm\sqrt{k}\ i}{2a} \\
&= \frac{-b+\sqrt{k}\,i}{2a}, \frac{-b-\sqrt{k}\ i}{2a}
\end{aligned}
$$

So one solution is the complex conjugate of the other.

b $x=2+3i$ is one solution so the other solution must be $x=2-3i$

$$
\begin{aligned}
(x-2-3i)(x-2+3i) &= 0 \\
x^2-2x+3ix-2x+4-6i-3ix+6i-9i^2 &= 0 \\
x^2-4x+4-9(-1) &= 0 \\
x^2-4x+4+9 &= 0 \\
x^2-4x+13 &= 0 \\
\text{Given } x^2+px+q &= 0
\end{aligned}
$$

$p=-4,\ q=13.$

c $x=3-2i$ is one solution so the other solution must be $x=3+2i$

$$
\begin{aligned}
(x-3+2i)(x-3-2i) &= 0 \\
x^2-3x-2ix-3x+9+6i+2ix-6i-4i^2 &= 0 \\
x^2-6x+9-4(-1) &= 0 \\
x^2-6x+9+4 &= 0 \\
x^2-6x+13 &= 0 \\
\text{Given } x^2+dx+e &= 0
\end{aligned}
$$

$d=-6,\ e=13.$

Question 28

a (5, 1) + (–3, 2) = (2, 3)

Note: $5+i-3+2i=2+3i$

b (–2, 3) – (1, 3) = (–3, 0)

Note: $-2+3i-(1+3i)=-2+3i-1-3i=-3$

c (2, 0) × (2, 1) = (4, 2) as this is the real number 2 times the complex number (2, 1)

Note: $2\times(2+i)=4+2i$

d (5, –1) ÷ (–5, 12) $=(-\frac{37}{169},-\frac{55}{169})$

Note: $\frac{5-i}{-5+12i}=\frac{5-i}{-5+12i}\times\frac{-5-12i}{-5-12i}=\frac{-25-60i+5i+12i^2}{25+60i-60i-144i^2}=\frac{-37-55i}{169}=-\frac{37}{169}-\frac{55}{169}i$

Question 29

$\frac{14-5i}{a-4i}=2+bi$

$14-5i=(2+bi)(a-4i)=2a-8i+abi-4bi^2$

$14-5i+8i=2a+abi+4b$

$14+3i=2a+4b+abi$

$2a+4b=14$

$ab=3$

$a=\frac{b}{3}$

$2\left(\frac{b}{3}\right)+4b=14$

$\frac{14}{3}b=14$

$b=3$

$a=1$

(or $a=6,\ b=0.5$)

Exercise 1B

Question 1

$$(x-1)(ax^2+bx+c)=2x^3+x^2+px+35$$
$$a=2$$
$$c=-35$$
$$\begin{aligned}(x-1)(2x^2+bx-35)&=2x^3+bx^2-35x-2x^2-bx+35\\&=2x^3+(b-2)x^2+(-35-b)x+35\end{aligned}$$
$$\begin{aligned}b-2&=1\\b&=3\end{aligned}$$
$$\begin{aligned}-35-b&=-35-3\\&=-38\end{aligned}$$

The coefficient of x^2 is 1 so $b = 3$

If $b = 3$, the co-efficient of x is –38

So $p=-38$

Question 2

$$\begin{aligned}x^3+3x^2-2x-16&=(x-a)(bx^2+cx+d)\\&=bx^3+cx^2+dx-abx^2-acx-ad\\&=bx^3+(c-ab)x^2+(d-ac)x-ad\end{aligned}$$

$$\begin{aligned}b&=1\\ad&=16\\c-ab&=3\\d-ac&=-2\end{aligned}$$

Given that 2 is a solution as $f(2) = 0$, let $a=2$

$d=8$

$c=5$

$\therefore a=2,\ b=1,\ c=5,\ d=8$

Question 3

a By 'algebraic juggling':

$$\frac{x^2-7x+3}{x-1}=\frac{x(x-1)-6(x-1)-3}{x-1}=\frac{(x-1)(x-6)}{x-1}+\frac{-3}{x-1}x$$
$$=x-6+\frac{-3}{x-1}$$

So the remainder is –3.

b Using the remainder theorem:

$$f(1)=(1)^2-7(1)+3=1-7+3$$
$$=-3$$

Question 4

a By 'algebraic juggling':

$$\frac{2x^3+3x^2-4x+3}{x+1}=\frac{2x^2(x+1)+x(x+1)-5(x+1)+8}{x+1}$$
$$=\frac{(x+1)(2x^2+x-5)}{x+1}+\frac{8}{x+1}$$
$$=2x^2+x-5+\frac{8}{x+1}$$

So the remainder is 8.

b Using the remainder theorem:

$$f(-1)=2(-1)^3+3(-1)^2-4(-1)+3$$
$$=-2+3+4+3$$
$$=8$$

Question 5

$$f(x)=x^2+3x-6$$
$$f(2)=2^2+3(2)-6=4$$

So by the remainder theorem the remainder is 4.

Question 6

$$
\begin{aligned}
f(x) &= x^3 - 5x^2 - 8x + 7 \\
f(-2) &= (-2)^3 - 5(-2)^2 - 8(-2) + 7 \\
&= -8 - 20 + 16 + 7 \\
&= -5
\end{aligned}
$$

So by the remainder theorem the remainder is –5.

Question 7

If $2x-1$ is a factor then

$$
f\left(\frac{1}{2}\right) = 0
$$

$$
\begin{aligned}
f(x) &= 2x^3 + ax^2 + bx - 2 \\
f\left(\frac{1}{2}\right) &= 2\left(\frac{1}{2}\right)^3 + a\left(\frac{1}{2}\right)^2 + b\left(\frac{1}{2}\right) - 2 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{2}{8} + \frac{a}{4} + \frac{b}{2} - 2 &= 0 \\
2 + 2a + 4b - 16 &= 0 \\
2a + 4b &= 14 \\
a + 2b &= 7
\end{aligned}
$$

By the remainder theorem

$$
\begin{aligned}
f(-1) &= -6 \\
2(-1)^3 + a(-1)^2 + b(-1) - 2 &= -6 \\
-2 + a - b - 2 &= -6 \\
a - b &= -2
\end{aligned}
$$

So solving the two equations with two unknowns:

$$
\begin{aligned}
a + 2b &= 7 \\
a - b &= -2 \\
3b &= 9 \\
b &= 3 \\
a &= 1
\end{aligned}
$$

Question 8

a

$$f(x) = x^3 - 3x^2 + 7x - 5$$

$$\begin{aligned} f(-1) &= (-1)^3 - 3(-1)^2 + 7(-1) - 5 \\ &= -1 - 3 - 7 - 5 \\ &= -16 \end{aligned}$$

$$\begin{aligned} f(1) &= 1^3 - 3(1)^2 + 7(1) - 5 \\ &= 1 - 3 + 7 - 5 \\ &= 0 \end{aligned}$$

b

$$x^3 - 3x^2 + 7x - 5 = (x-1)(x^2 + bx + 5)$$

$$\begin{aligned} bx^2 - 1x^2 &= -3x^2 \\ b - 1 &= -3 \\ b &= -2 \end{aligned}$$

$$x^3 - 3x^2 + 7x - 5 = (x-1)(x^2 - 2x + 5)$$

For $x^2 - 2x + 5,\ a = 1,\ b = -2,\ c = 5$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm \sqrt{-16}}{2} \\ &= \frac{2 \pm 4i}{2} = 1 \pm 2i \end{aligned}$$

$$\therefore\ x^3 - 3x^2 + 7x - 5 = (x-1)(x-1+2i)(x-1-2i)$$

$$x = 1,\ 1 + 2i,\ 1 - 2i$$

c

$$x^4 - 3x^3 + 7x^2 - 5x = 0$$

$$x(x^3 - 3x^2 + 7x - 5) = 0$$

So this equation has the same solutions as part b but with one additional solution at $x = 0$.

$$x = 0,\ 1,\ 1 + 2i,\ 1 - 2i$$

Question 9

a
$$\begin{aligned} f(x) &= x^4 - 5x^3 - x^2 + 11x - 30 \\ f(-2) &= 16 + 40 - 4 - 22 - 30 \\ &= 0 \\ f(2) &= 16 - 40 - 4 + 22 - 30 \\ &= -36 \\ f(-5) &= 625 + 625 - 25 - 55 - 30 \\ &= 1140 \\ f(5) &= 625 - 625 - 25 + 55 - 30 \\ &= 0 \end{aligned}$$

b
$$x^4 - 5x^3 - x^2 + 11x - 30 = (x+2)(x-5)(x^2 + bx + 3)$$
$$x = -2,\ 5$$

$$x = \frac{2 \pm \sqrt{4-12}}{2} = \frac{2 \pm \sqrt{-8}}{2} = \frac{2 \pm 2\sqrt{2}\,i}{2} = 1 \pm \sqrt{2}\ i$$

So the solutions are $x = -2,\ 5,\ 1 + \sqrt{2}\,i,\ 1 - \sqrt{2}\,i$

Question 10

a
$$\begin{aligned} f(x) &= 2x^3 - x^2 + 2x - 1 \\ f(1) &= 2 - 1 + 2 - 1 = 2 \\ f(0.5) &= 2 \times 0.125 - 0.25 + 1 - 1 = 0 \end{aligned}$$

b
$$2x^3 - x^2 + 2x - 1 = (x - 0.5)(2x^2 + bx + 2)$$
$$bx^2 - x^2 = -1$$
$$b = 0$$

$$\begin{aligned} 2x^2 + 2 &= 0 \\ 2x^2 &= -2 \\ x^2 &= -1 \\ x^2 &= i^2 \\ x &= \pm i \end{aligned}$$

Solutions are $x = 0.5,\ i,\ -i$

Question 11

$x^2+2x+2=0$

$$x=\frac{-2\pm\sqrt{4-8}}{2}=\frac{-2\pm\sqrt{-4}}{2}=\frac{-2\pm 2i}{2}=-1\pm i$$

$x^2+2x-5=0$

$$x=\frac{-2\pm\sqrt{4-20}}{2}=\frac{-2\pm\sqrt{-16}}{2}=\frac{-2\pm 4i}{2}=-1\pm 2i$$

$x=-1+i,\ -1-i,\ 1-2i,\ 1+2i$

Question 12

Given

$f(x)=2x^3-3x^2+9x-8$

$f(1)=2-3+9-8=0$

So $x=1$ is a solution

$$\begin{aligned}2x^3-3x^2+9x-8&=(x-1)(2x^2+bx+8)\\ bx^2-2x^2&=-3\\ b-2&=-3\\ b&=-1\end{aligned}$$

$(x-1)(2x^2-x+8)$

$$x=\frac{1\pm\sqrt{1-64}}{4}=\frac{1\pm\sqrt{-63}}{4}=\frac{1\pm 3\sqrt{7}\ i}{4}$$

So the solutions are $x=1,\ \frac{1+3\sqrt{7}\ i}{4},\frac{1-3\sqrt{7}\ i}{4}$

Question 13

$f(x)=3x^4-3x^3-2x^2+4x$

$f(-1)=3+3-2-4=0$

$$\begin{aligned}3x^4-3x^3-2x^2+4x&=x(x+1)(3x^2+bx+4)\\ bx^3+3x^3&=-3x^3\\ b+3=-3,\qquad b&=-6\end{aligned}$$

$3x^2-6x+4=0$

$$x=\frac{6\pm\sqrt{36-48}}{6}=\frac{6\pm\sqrt{-12}}{6}=\frac{6\pm 2\sqrt{3}\ i}{6}=\frac{3\pm\sqrt{3}}{3}i$$

So the solutions are $x=0,\ -1,\ \frac{3+\sqrt{3}}{3}i,\ \frac{3-\sqrt{3}}{3}i$

Miscellaneous Exercise 1

Question 1

a $(7+3i)(7-3i) = 49-21i+21i-9i^2 = 49-9i^2$

$= 49-9(-1) = 49+9 = 58$

b $(5+i)(5-i) = 25-5i+5i-i^2 = 25-i^2$

$= 25-(-1) = 26$

c $(3+2i)(2-3i) = 6-9i+4i-6i^2$

$= 6-5i-6(-1)$

$= 12-5i$

d $(1-5i)^2 = (1-5i)(1-5i) = 1-5i-5i+25i^2$

$= 1-10i+25(-1) = -24-10i$

e $\dfrac{3-2i}{2+i} = \dfrac{3-2i}{2+i} \times \dfrac{2-i}{2-i} = \dfrac{6-3i-4i+2i^2}{4-2i+2i-i^2}$

$= \dfrac{6-7i+2(-1)}{4-(-1)} = \dfrac{4-7i}{5}$

$= \dfrac{4}{5} - \dfrac{7}{5}i$

f $\dfrac{1+2i}{3-4i} = \dfrac{1+2i}{3-4i} \times \dfrac{3+4i}{3+4i} = \dfrac{3+4i+6i+8i^2}{9+12i-12i-16i^2}$

$= \dfrac{3+10i+8(-1)}{9-16(-1)} = \dfrac{-5+10i}{25} = \dfrac{5(-1+2i)}{5\times 5} = \dfrac{-1+2i}{5}$

$= -\dfrac{1}{5} + \dfrac{2}{5}i$

Question 2

Given $z = 3 - 4i$ and $w = -4 + 5i$

a $z + w = 3 - 4i + (-4) + 5i = -1 + i$

b $zw = (3 - 4i)(-4 + 5i)$

$= -12 + 15i + 16i - 20i^2$

$= -12 + 31i - 20(-1)$

$= 8 + 31i$

c $\overline{z} = 3 + 4i$

d $z^2 = (3 - 4i)^2 = (3 - 4i)(3 - 4i)$

$= 9 - 12i - 12i + 16i^2$

$= 9 - 24i + 16(-1)$

$= -7 - 24i$

e $zw = 8 + 31i$ (above in Question 2 **b**)

$\overline{zw} = 8 - 31i$

f $\overline{zw} = (3 + 4i)(-4 - 5i)$

$= -12 - 15i - 16i - 20i^2$

$= -12 - 31i - 20(-1)$

$= 8 - 31i$

g $\text{Re}(q) = \text{Re}(\overline{w}) = -4$

$\text{Im}(q) = \text{Im}(\overline{z}) = 4i$

$q = -4 + 4i$

Question 3

To find $(1+i)^5$, first find $(1+i)^2$ as ($(1+i)^5 = (1+i)^2(1+i)^2(1+i)$ using index laws.

$(1+i)^2 = (1+i)(1+i) = 1 + i + i + i^2 = 1 + 2i - 1 = 2i$

$(1+i)^2(1+i)^2 = 2i \times 2i = 4i^2 = 4(-1) = -4$

So $(1+i)^4 = -4$

$(1+i)^5 = (1+i)^4(1+i) = -4(1+i) = -4 - 4i$

Question 4

To find $\text{Im}[(1-3i)^3]$, first find $(1-3i)^3$.

$$\begin{aligned}(1-3i)^3 &= (1-3i)^2(1-3i) \\ &= (1-3i-3i+9i^2)(1-3i) \\ &= (1-6i+9(-1))(1-3i) \\ &= (-8-6i)(1-3i) \\ &= -8+24i-6i+18i^2 \\ &= -8+18i+18(-1) \\ &= -26+18i\end{aligned}$$

$\text{Im}(-26+18i) = 18$

Question 5

a $\text{Re}(3-2i)\times\text{Re}(2+i) = 3\times 2 = 6$

b $$\begin{aligned}\text{Re}[(3-2i)(2+i)] &= \text{Re}(6+3i-2i-2i^2) \\ &= \text{Re}(6+i-2(-1)) \\ &= \text{Re}(8+i) \\ &= 8\end{aligned}$$

Question 6

Given $(x-5)$ is a factor of $x^4+qx^3-14x^2-45x-50$,

when $x=5$, $x^4+qx^3-14x^2-45x-50$ must equal 0.

$x=5$

$$\begin{aligned}x^4+qx^3-14x^2-45x-50 &= 5^4+q(5)^3-14(5)^2-45(5)-50 \\ &= 625+125q-350-225-50 \\ &= 125q\end{aligned}$$

So if $x^4+qx^3-14x^2-45x-50=0$, then $125q=0$

Therefore $q=0$.

Question 7

Given $2x^3 - x^2 + 3x + 6 = (x-a)(bx^2 + cx + d)$

When $x = -1$, $2x^3 - x^2 + 3x + 6 = 0$ so $(x+1)$ is a factor of $2x^3 - x^2 + 3x + 6$.

$$2x^3 - x^2 + 3x + 6 = (x+1)(2x^2 + bx + 6)$$

$$\begin{aligned} 6x + bx &= 3x \\ 6 + b &= 3 \\ b &= -3 \end{aligned}$$

$$\begin{aligned} 2x^3 - x^2 + 3x + 6 &= (x+1)(2x^2 - 3x + 6) \\ &= (x-(-1))(2x^2 - 3x + 6) \end{aligned}$$

$a = -1,\ b = 2,\ c = -3,\ d = 6.$

Question 8

When $x = 3$, $x^4 + 3x^3 + px^2 + qx - 30 = 0$

$$\begin{aligned} (3)^4 + 3(3)^3 + p(3)^2 + q(3) - 30 &= 0 \\ 81 + 81 + 9p + 3q - 30 &= 0 \\ 9p + 3q &= -132 \end{aligned}$$

When $x = 1$, $x^4 + 3x^3 + px^2 + qx - 30 = -48$

$$\begin{aligned} (1)^4 + 3(1)^3 + p(1)^2 + q(1) - 30 &= -48 \\ 1 + 3 + p + q - 30 &= -48 \\ p + q &= -22 \end{aligned}$$

Solving simultaneously gives

$p = -11,\ q = -11$

Question 9

a $2(3\mathbf{i}-\mathbf{j}) = 6\mathbf{i}-2\mathbf{j}$

b $|\mathbf{a}| = \sqrt{3^2+(-1)^2} = \sqrt{10}$

$|\mathbf{b}| = \sqrt{2^2+4^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$

To find the vector in the same direction as $\mathbf{b}$ but with the same magnitude as $\mathbf{a}$, divide vector $\mathbf{b}$ by the magnitude of vector $\mathbf{b}$ and then multiply by the magnitude of vector $\mathbf{a}$.

Solution:

$$(2\mathbf{i}+4\mathbf{j}) \div 2\sqrt{5} \times \sqrt{10} = \frac{\sqrt{2}(2\mathbf{i}+4\mathbf{j})}{2} = \frac{2\sqrt{2}(\mathbf{i}+2\mathbf{j})}{2}$$
$$= \sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} = \sqrt{2}(\mathbf{i}+2\mathbf{j})$$

c $3\mathbf{a} = 3(3\mathbf{i}-\mathbf{j}) = 9\mathbf{i}-3\mathbf{j}$

$|3\mathbf{a}| = \sqrt{9^2+(-3)^2} = \sqrt{90}$

If $|\mathbf{c}| = |3\mathbf{a}| = \sqrt{90}$,

$$\sqrt{d^2+(-9)^2} = \sqrt{90}$$
$$d^2+81 = 90$$
$$d^2 = 9$$
$$d = \pm 3$$

d $\mathbf{a}\bullet\mathbf{b} = (3\mathbf{i}-\mathbf{j})\bullet(2\mathbf{i}+4\mathbf{j}) = 3\times 2 + (-1)\times 4 = 2$

e Let θ be the angle between vectors $\mathbf{a}$ and $\mathbf{b}$

$$\cos\theta = \frac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{2}{2\sqrt{5}\times\sqrt{10}} = \frac{1}{5\sqrt{2}} = \frac{\sqrt{2}}{10}$$
$$\theta = 81.87°$$

The angle between vector $\mathbf{a}$ and vector $\mathbf{b}$ is $82°$ (to the nearest degree).

Question 10

$\mathbf{b}$ is in the same magnitude as $\mathbf{a}$ but in the opposite direction, so $\mathbf{b} = -\mathbf{a}$

$\mathbf{c}$ is in the same direction as $\mathbf{a}$ but twice the magnitude, so $\mathbf{c} = 2\mathbf{a}$

$\mathbf{d}$ is in the same direction as $\mathbf{a}$ but half the magnitude, so $\mathbf{d} = \frac{1}{2}\mathbf{a}$

$\mathbf{e}$ is in the opposite direction to $\mathbf{a}$ and half the magnitude, so $\mathbf{e} = -\frac{1}{2}\mathbf{a}$

$\mathbf{f}$ is in the same direction as $\mathbf{a}$ but one and a half times the magnitude, so $\mathbf{f} = \frac{3}{2}\mathbf{a}$

$\mathbf{g}$ is in the opposite direction to $\mathbf{a}$ but one and a half times the magnitude, so $\mathbf{g} = -\frac{3}{2}\mathbf{a}$

Question 11

$\mathbf{r} = \mathbf{p} + \mathbf{q}$

$\mathbf{s} = \frac{1}{2}\mathbf{p} + \mathbf{q}$

$\mathbf{t} = \mathbf{p} + 2\mathbf{q}$

$\mathbf{u} = -\frac{3}{2}\mathbf{p} - \mathbf{q}$

Question 12

By substitution it is easy to find that $x = 2$ is one solution for $x^3 + 6x^2 + 4x - 40 = 0$.

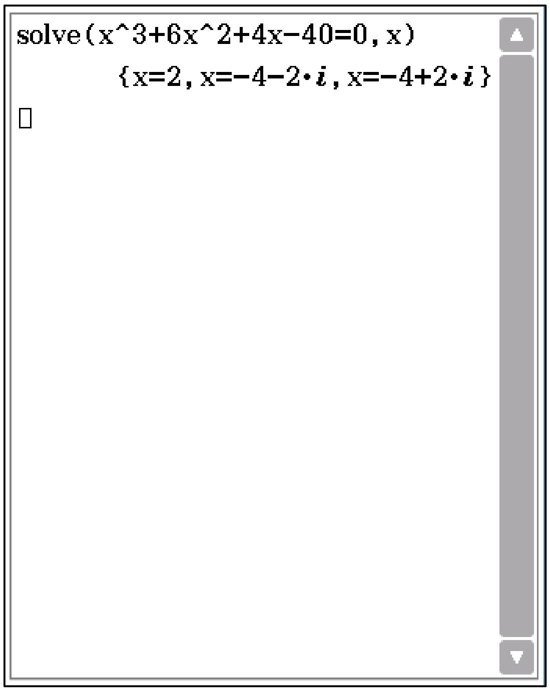
$$x^3 + 6x^2 + 4x - 40 = (x-2)(x^2 + bx + 20)$$
$$20x - 2bx = 4x$$
$$20 - 2b = 4$$
$$-2b = -16$$
$$b = 8$$
$$x^3 + 6x^2 + 4x - 40 = (x-2)(x^2 + 8x + 20)$$

Using the quadratic formula to solve $x^2 + 8x + 20 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-8 \pm \sqrt{64 - 4(1)(20)}}{2} = \frac{-8 \pm \sqrt{-16}}{2}$$
$$= \frac{-8 \pm \sqrt{16i^2}}{2} = \frac{-8 \pm 4i}{2} = \frac{2(-4 \pm 2i)}{2}$$
$$= -4 + 2i, -4 - 2i$$

So $(x + 4 - 2i)$ and $(x + 4 + 2i)$ are factors.

$$x^3 + 6x^2 + 4x - 40 = (x-2)(x+4-2i)(x+4+2i)$$



So when $x^3 + 6x^2 + 4x - 40 = 0$

$x = 2,\ -4 + 2i,\ -4 - 2i$

Question 13

p is perpendicular to **q**, so $\mathbf{p} \bullet \mathbf{q} = 0$, hence $2a - ab = 0$

The magnitude of **p** equals the magnitude of **q**, so $\sqrt{a^2 + a^2} = \sqrt{2^2 + (-b)^2}$

$\sqrt{2a^2} = \sqrt{4 + b^2} \quad \Rightarrow \quad 2a^2 = 4 + b^2$

We know

$2a - ab = 0$, so $a(2 - b) = 0$

$\therefore\ a = 0$ or $b = 2$, but if $a = 0$ the magnitude of **p** is not equal to the magnitude of **q**.

So $b = 2$.

$|\mathbf{q}| = \sqrt{2^2 + 2^2} = \sqrt{8}$

$|\mathbf{r}| = \sqrt{a^2 + a^2}$

$2a^2 = 8 \quad \Rightarrow \quad a^2 = 4 \quad \Rightarrow \quad a = \pm 2$

$$\mathbf{q} - 3\mathbf{r} = 23\mathbf{i} - 5\mathbf{j}$$

$$2\mathbf{i} - b\mathbf{j} - 3(c\mathbf{i} + d\mathbf{j}) = 23\mathbf{i} - 5\mathbf{j}$$

We know that $b = 2$

$2 - 3c = 23$

$c = -7$

$-2 - 3d = -5$

$d = 1$

$e = -f$

$|\mathbf{r}| = \sqrt{50}$

$\sqrt{e^2 + f^2} = \sqrt{(-f)^2 + f^2} = \sqrt{2f^2}$

$\sqrt{2f^2} = \sqrt{50}$

$2f^2 = 50$

$f^2 = 25$

$f = \pm 5$

but **s** is in the same direction as **q**, so $f = -5$.

Hence $e = 5$.

$a = \pm 2,\ b = 2,\ c = -7,\ d = 1,\ e = 5,\ f = -5.$