SADLER UNIT 4 MATHEMATICS SPECIALIST

WORKED SOLUTIONS

Chapter 12 – Sample means

Exercise 12A

Question 1

Standard deviation of sample means $= \frac{1.71}{\sqrt{50}} = 0.2418$.

The sample means will be approximately normally distributed with a mean of 3.5 and a standard deviation of 0.24.

If instead a sample size of 150 were used the distribution would still be approximately normal with a mean of 3.5 but with a smaller standard deviation than before, now $= \frac{1.71}{\sqrt{150}} = 0.14$.

Question 2

Standard deviation of sample means $= \frac{0.696}{\sqrt{60}} = 0.08985$.

The sample means will be approximately normally distributed with a mean of 2.375 and a standard deviation of 0.09.

If instead a sample size of 100 were used the distribution would still be approximately normal with a mean of 2.375 but with a smaller standard deviation than before, now $= \frac{0.696}{\sqrt{100}} = 0.07$.

Question 3

Standard deviation of sample means $= \dfrac{2.415}{\sqrt{36}} = 0.4025$.

The 100 sample means will be approximately normally distributed with a mean of 7 and a standard deviation of 0.40.

If instead a sample size of 120 were used the distribution would still be approximately normal with a mean of 7 but with a smaller standard deviation than before, now $= \dfrac{2.415}{\sqrt{120}} = 0.22$.

Question 4

Standard deviation $= 132 \div \sqrt{50}$

```
normCDf(-∞, 2175, 18.67, 2145)
                  0.9459570109
```

$Y \sim \mathrm{N}(2145, 18.67^2)$
$\mathrm{P}(Y < 2175) = 0.946$

Question 5

Standard deviation $= 3 \cdot 2 \div \sqrt{64} = 0.4$

```
normCDf(17.5, ∞, 0.4, 16.8)
                 0.04005915686
```

$Y \sim \mathrm{N}(16.8, 0.4^2)$
$\mathrm{P}(Y > 17.5) = 0.040$

Question 6

Standard deviation $= 20 \div \sqrt{100} = 2$

```
normCDf(144, 150, 2, 145)
                  0.6852527959
```

$Y \sim \mathrm{N}(145, 2^2)$
$\mathrm{P}(144 < Y < 150) = 0.685$

Question 7

a Y will be normally distributed with mean 5 and standard deviation 0.2.

b Standard deviation $= 1 \div \sqrt{25} = 0.2$

$Y \sim \mathrm{N}(5, 0.2^2)$

$$\begin{aligned} \mathrm{P}(Y \geq 5.5) &= 1 - \mathrm{P}(Y < 5.5) \\ &= 1 - 0.9938 \\ &= 0.006 \end{aligned}$$

Question 8

Y is normally distributed with mean 30 and standard deviation $\sqrt{0.24}$.

Mean $= 50 \times 0.6 = 30$

Standard deviation $= \sqrt{0.6 \times 0.4} = \sqrt{0.24}$

$Y \sim \mathrm{N}(30, 0.24)$

Question 9

a $X \sim \mathrm{N}(508, 3^2)$

```
normCDf(-∞,500,3,508)
                3.830380568E-3
```

$\mathrm{P}(X < 500) = 0.004$

0.4% of packets will contain less than 500 g.

b $X \sim \mathrm{N}(508, 3^2)$

```
normCDf(510,∞,3,508)
                0.2524925375
```

$\mathrm{P}(X > 510) = 0.252$

25% of packets will contain more than 510g.

c $\mathrm{Y} \sim \mathrm{N}(508, \left(\frac{3}{\sqrt{10}}\right)^2)$

```
normCDf(507.5,508.5,3/√10,508)
                0.4018385473
```

$\mathrm{P}(507.5 < \mathrm{Y} < 508.5) = 0.402$

Question 10

a $X \sim \mathrm{N}(1.015, 0.006^2)$

```
normCDf(-∞,1,6E-3,1.015)
                6.209665326E-3
```

$\mathrm{P}(X < 1) = 0.006$

b $X \sim \mathrm{N}(1.015, 0.006^2)$

```
normCDf(1.02,∞,6E-3,1.015)
                0.202328381
```

$\mathrm{P}(X > 1.02) = 0.202$

c $\mathrm{Y} \sim \mathrm{N}(1.015, \left(\frac{0.006}{\sqrt{5}}\right)^2)$

```
normCDf(1.01,1.02,6E-3/√5,1.015)
                0.9375925814
```

$\mathrm{P}(1.01 < X < 1.02) = 0.938$

Question 11

$X \sim \mathrm{N}(17.4, 2.1^2)$

$Y \sim \mathrm{N}(17.4, 0.664^2)$

Standard deviation of sample means is $= 2.1 \div \sqrt{10} \approx 0.664$.

$2 \div 0.664 = 3.01$ which means the sample mean is more than 3 standard deviations above expected mean.

We would expect the mean length of samples of ten adult male lizards of this species to be normally distributed with mean 17.4 and standard deviation of approximately 0.664. A sample mean of 19.4 is more than 3 standard deviations above what was expected and this is very unlikely, although not impossible.

$\mathrm{P}(Y > 19.4) = 0.0013$

```
normCDf(19.4,∞,0.664,17.4)
                1.297456987E-3
```

Hence while it is possible that the sample of ten may be a "freakish" sample we would be wise to consider other possible reasons for the surprising sample means. Consider whether the sample was really random. Perhaps the lizards were caught in a region where larger than normal lizards of this species were found. Perhaps the scientists' confidence in the assumption of a normal distribution or in the given population mean and standard deviation was misplaced. Many reasons could exist for this higher than expected sample mean.

Question 12

a $P(88 < X < 90) = \dfrac{(90-88)\times\frac{1}{6}}{(90-84)\times\frac{1}{6}} = \dfrac{2}{6} = \dfrac{1}{3}$

b Standard deviation of X $= \dfrac{b-a}{\sqrt{12}} = \dfrac{90-84}{\sqrt{12}} = \dfrac{6}{2\sqrt{3}} = \sqrt{3}$

c $Z \sim \left(87, \dfrac{1}{4}\right)$

```
normCDf(87.5,∞,0.25,87)
                0.02275013195
```

$P(Z > 87.5) \approx 0.023$

Question 13

a Sample means are normally distributed with mean 513 and standard deviation $\dfrac{26}{8}$.

$513 - 1.96\times\dfrac{26}{8} = 506.63$

$513 + 1.96\times\dfrac{26}{8} = 519.37$

The interval 1.96 standard deviations above and below the mean is 506.63 to 519.37.

505 is not in this interval so for the first sample taken there is a significant difference at the 5% level.

b Sample means are normally distributed with mean 513 and standard deviation $\dfrac{26}{10}$.

$513 - 1.96\times\dfrac{26}{10} = 507.90$

$513 + 1.96\times\dfrac{26}{10} = 518.10$

The interval 1.96 standard deviations above and below the mean is 507.90 to 518.10.

510 is in this interval so for the second sample taken there is not a significant difference at the 5% level.

Exercise 12B

Question 1

The 90% confidence interval has the smaller width.

(If you want to be more confident of catching the population mean you need a bigger net).

Question 2

The 95% confidence interval has the smaller width.

Question 3

The bigger size sample will give the narrower 95% confidence interval.

Question 4

Standard deviation $= \dfrac{48}{\sqrt{100}} = 4.8$

$$573 - 4.8 \times 1.645 = 565.10$$
$$573 + 4.8 \times 1.645 = 580.90$$

The 90% confidence interval is $565\,\text{cm} \le \mu \le 581\,\text{cm}$.

Question 5

Standard deviation $= \dfrac{3.67}{\sqrt{50}} = 0.519$

$$26.14 - 0.519 \times 1.96 = 25.12$$
$$26.14 + 0.519 \times 1.96 = 27.16$$

The 90% confidence interval is $25.12\,\text{kg} \le \mu \le 27.16\,\text{kg}$.

Question 6

Standard deviation $=\dfrac{\sqrt{5.76}}{\sqrt{80}}=0.268$

$17.2-0.268\times 2.576=16.51$

$17.2+0.268\times 2.576=17.89$

The 99% confidence interval is $16.51\,\text{cm}\le\mu\le 17.89\,\text{cm}$.

Question 7

Note that we can assume that the sample mean is from a normal distribution of sample means because, thought the sample is small, the population the sample is taken from is normally distributed.

Standard deviation $=\dfrac{1.4}{\sqrt{10}}=0.443$

$74.6-0.443\times 1.96=73.73$

$74.6+0.443\times 1.96=75.47$

The 95% confidence interval is $73.73\,\text{cm}\le\mu\le 75.47\,\text{cm}$.

We can be 95% confident that the mean length of 12 month old baby girls lies in between 73.73cm and 75.47cm (because 95% of the 95% confidence intervals constructed in this way will contain the population mean).

Question 8

Standard deviation $=\dfrac{2.4}{\sqrt{40}}=0.379$

$17.18-0.379\times 1.645=17.18$

$17.18+0.379\times 1.645=18.42$

The 90% confidence interval is $17.18\,\text{cm}\le\mu\le 18.42\,\text{cm}$.

We can be 90% confident that the mean length of three month old seedlings of the particular plant type will lie between 17.18 cm and 18.42 cm (because 90% of the 90% confidence intervals constructed in this way will contain the population mean).

Question 9

For the sample of 200 birds:

Standard deviation $= \frac{2.7}{\sqrt{200}} = 0.191$

$18.3 - 0.191 \times 1.96 = 17.93$
$18.3 + 0.191 \times 1.96 = 18.67$

The 95% confidence interval is $17.93\,\text{cm} \le \mu \le 18.67\,\text{cm}$.

For the sample of 300 birds:

Standard deviation $= \frac{2.7}{\sqrt{300}} = 0.156$

$18.3 - 0.156 \times 1.96 = 17.99$
$18.3 + 0.156 \times 1.96 = 18.61$

The 95% confidence interval is $17.99\,\text{cm} \le \mu \le 18.61\,\text{cm}$.

Exercise 12C

Question 1

$$1.96 \times \frac{8}{\sqrt{n}} = 1.5$$
$$n = 109.27$$

The sample size would need to be 110 to be 95% confident that the mean of the sample is within 1.5 units of the population mean.

Question 2

$$2.576 \times \frac{18.7}{\sqrt{n}} = 2.5$$
$$n = 371.27$$

The sample size would need to be 372 to be 99% confident that the mean of the sample is within 2.5 units of the population mean.

Question 3

$$1.96 \times \frac{7.3}{\sqrt{n}} = 3$$
$$n = 22.75$$

The sample size would need to be 23 to be 95% confident that the mean of the sample is within 2.5 units of the population mean.

Question 4

$$1.645 \times \frac{8.4}{\sqrt{n}} = 2$$
$$n = 47.73$$

The sample size would need to be 48 to be 90% confident that the mean of this second sample is within 2 units of the population mean.

Question 5

$$1.96\times\frac{3.6}{\sqrt{n}}=0.5$$
$$n=199.14$$

The sample size would need to be 200 to be 95% confident that the mean of this second sample is within 0.5 units of the population mean.

Question 6

$$1.96\times\frac{3}{\sqrt{n}}=1$$
$$n=34.57$$

The sample size would need to be 35 to be 95% confident that the mean of this sample is within 1 mL of the sample mean.

Miscellaneous Exercise 12

Question 1

a $\dfrac{dC}{dx} = 7$

b $\dfrac{dC}{dx} = \dfrac{3x^2}{12} - 24x + 800 = \dfrac{x^2}{4} - 24x + 800$

c $\dfrac{dC}{dx} = 2x + 4$

d $\dfrac{dC}{dx} = \dfrac{2}{\sqrt{x}} + \dfrac{1000}{x^2}$

Question 2

$$\frac{d}{dx}\left(x^3 + 2x^2 y + y^3\right) = \frac{d}{dx}(10)$$

$$3x^2 + 2x^2\frac{dy}{dx} + 4xy + 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-4xy - 3x^2}{2x^2 + 3y^2}$$

$$= -\frac{x(3x + 4y)}{2x^2 + 3y^2}$$

Question 3

a $\dfrac{d}{dx}\left(y^4\right) = \dfrac{d}{dx}\left(x^4 - 4xy - 7\right)$

$$4y^3\frac{dy}{dx} = 4x^3 - 4x\frac{dy}{dx} - 4y$$

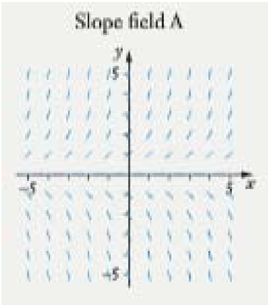
$$\frac{dy}{dx} = \frac{4x^3 - 4y}{4x + 4y^3} = \frac{4(x^3 - y)}{4(x + y^3)} = \frac{x^3 - y}{x + y^3}$$

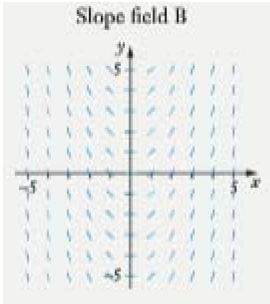
b At the point (2, 1)

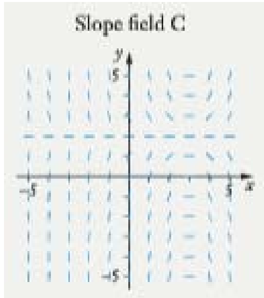
$$\frac{dy}{dx} = \frac{4(2^3) - 4(1)}{4(2) + 4(1^3)} = \frac{28}{12} = \frac{7}{3}$$

Question 4

Slope field A	Slope field B	Slope field C
$\dfrac{dy}{dx} = y$	$\dfrac{dy}{dx} = x$	$\dfrac{dy}{dx} = (x-3)(y-2)$







Question 5

$$\int \frac{3x^3+6x^2-4x-8}{(x+1)(x^2-2)}\,dx$$

$$= \int \left[\frac{3(x^3+x^2-2x-2)}{x^3+x^2-2x-2} + \frac{3x^2+2x-2}{(x+1)(x^2-2)} \right] dx$$

$$= \int 3dx + \int \left[\frac{A}{x+1} + \frac{B}{x-\sqrt{2}} + \frac{C}{x+\sqrt{2}} \right] dx$$

$$= 3x + c + \int \frac{A(x^2-2) + B(x+1)(x+\sqrt{2}) + C(x+1)(x-\sqrt{2})}{(x+1)(x^2-2)}\,dx$$

$$= 3x + c + \int \left[\frac{1}{x+1} + \frac{1}{x-\sqrt{2}} + \frac{1}{x+\sqrt{2}} \right] dx$$

$$= 3x + \ln|x+1| + \ln|x-\sqrt{2}| + \ln|x+\sqrt{2}| + c$$

$$= 3x + \ln|x+1| + \ln|x^2-2| + c$$

$$A + B + C = 3$$

$$-2A + \sqrt{2}B - \sqrt{2}C = -2$$

$$(1+\sqrt{2})B + (1-\sqrt{2})C = 2$$

Solving gives $A = 1$, $B = 1$ and $C = 1$.

Question 6

a Let $u = \sin x$ $\qquad \dfrac{du}{dx} = \cos x$

$$\int_0^k 3\sin^2 x \cos x \, dx = \int_0^k 3u^2 \cos x \frac{dx}{du} du$$

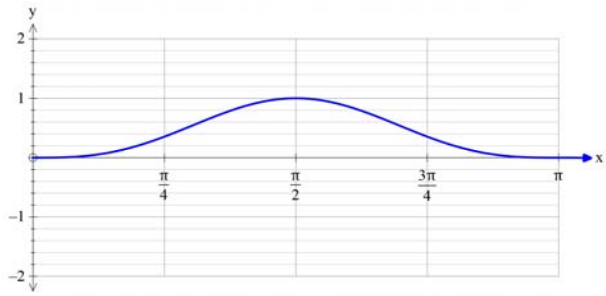
$$= \int_{x=0}^{x=k} 3u^2 \cos x \frac{1}{\cos x} du$$

$$= \int_{u=0}^{u=\sin k} 3u^2 \, du$$

$$= \left[u^3 \right]_0^{\sin k}$$

$$= \sin^3 k \text{ units}^2, 0 < k < \frac{\pi}{2}$$

b $\displaystyle\int_0^k 3\sin^2 x \cos x \, dx = 2 - \sin^3 k, \frac{\pi}{2} < k < \pi.$



Question 7

a $v = 3 + 0.1x,\ x > 0$

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt}$$
$$= 0.1 \times v$$
$$= 0.1(0.1x + 3)$$
$$= 0.01x + 0.3$$

When $x = 2,\ a = 0.32 \text{ m/s}^2$

b $\dfrac{dx}{dt} = 3 + 0.1x$

$$\int \frac{1}{3 + 0.1x} dx = \int dt$$

$$10\int \frac{0.1}{3 + 0.1x} dx = t + c$$

$$10 \ln|3 + 0.1x| = t + c$$

When $t = 0, x = 20$

$c = 10 \ln(5)$

When $t = 5$

$$\ln|3 + 0.1x| = 0.5 + \ln(5)$$

$$e^{0.5 + \ln(5)} = 3 + 0.1x$$

$$x = 10\left(e^{0.5} e^{\ln(5)} - 3\right) = 50\sqrt{e} - 30$$

Question 8

a $\frac{dA}{dt} = 6e^{2t}$

$\int dA = \int 6e^{2t}\,dt$

$A = 3e^{2t} + c$

$4 = 3 + c$

$c = 1$

$A = 3e^{2t} + 1$

b When $t = 0.5$

$A = 3e + 1$

c $\frac{dA}{dt} \approx \frac{\delta A}{\delta t}$

$\frac{\delta A}{\delta t} \approx 6e^{2t}$

$\delta A \approx 6e^{2t}\delta t$

$\delta A \approx 6e^{2(0)}0.01 \approx 0.06$

Question 9

a $\frac{dP}{dt} = 0.1P$

$\int \frac{1}{P}\,dP = \int 0.1\,dt$

$\ln P = 0.1t + c$

$P = P_0 e^{0.1t}$

$20000 = 10000e^{0.1t}$

$t \approx 6.93$ years

b $40000 = 10000e^{0.1t}$

$t \approx 13.86$ years

c $80000 = 10000e^{0.1t}$

$t \approx 20.79$ years

Question 10

a Standard deviation $= 5.12 \div \sqrt{64} = 0.64$

$53.24 - 1.645 \times 0.64 = 52.19$

$53.24 + 1.645 \times 0.64 = 54.29$

We can have 90% confidence that the mean will lie between 52.19 and 54.29.

b Standard deviation $= 0.64$

$53.24 - 1.96 \times 0.64 = 51.99$

$53.24 + 1.96 \times 0.64 = 54.49$

We can have 95% confidence that the mean will lie between 51.99 and 54.49.

c Standard deviation $= 0.64$

$53.24 - 2.576 \times 0.64 = 51.59$

$53.24 + 2.576 \times 0.64 = 54.89$

We can have 99% confidence that the mean will lie between 51.59 and 54.89.

Question 11

Standard deviation $= 18 \div \sqrt{40} = 2.85$

$223 - 1.96 \times 2.85 = 217.41$

$223 + 1.96 \times 2.85 = 228.59$

We can have 95% confidence that the mean of the population lies between 217.4 and 228.6.

Question 12

a $x = A\cos kt$

$\dot{x} = -Ak\sin kt$

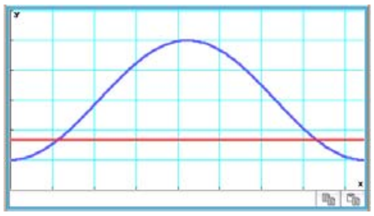
$\ddot{x} = -Ak^2\cos kt = -k^2(A\cos kt) = -k^2x$

The particle is in simple harmonic motion and as the displacement is a cos function the object is initially at its extreme position.

b Low tide is 3 m and high tide is 15 m, a difference of 12 m.

The equation of the function displaying the water level would be a trigonometric function with an amplitude of 6. Given that we are starting at low tide at 7 a.m., we can model the function as a cosine function, amplitude 6, reflected about the x-axis and translated up 9 as the low tide mark is at 3 m, which is 9 m above the value of –6, the value of –6 cos (0).

$x = -6\cos kt + 9$



The value of the constant k, is found by considering the period of the function, t = 760 minutes, hence $k = \frac{9}{19}$.

$x = -6\cos\frac{9}{19}t + 9$

The tide is above 5 m for t between 101 min and 658 min after 7 a.m.

That is approximately 8:41 am to 5:58 p.m.

To the nearest 5 minutes, that is between 8:40 a.m. and 6 p.m.

Question 13

$\frac{dy}{dx} = xe^x$

$\delta y \approx xe^x\delta x$

$\delta y \approx 1.25e^{1.25}(0.01) \approx 0.044$

Question 14

$\overline{AC} = \sqrt{y^2 + 12^2}$

$\overline{AC}$ is equal to 20 m when $y^2 + 144 = 400$, $y = 16$ m.

$$\frac{\overline{CE}}{\overline{CE} + \sqrt{y^2 + 144}} = \frac{1}{5}$$

$$5\overline{CE} = \overline{CE} + \sqrt{y^2 + 144}$$

$$\overline{CE} = \frac{1}{4}\sqrt{y^2 + 144}$$

$$\frac{d\overline{CE}}{dy} = \frac{y}{4\sqrt{y^2 + 144}}$$

$$\frac{dy}{dt} = 2\ \text{m/s}$$

$$\frac{d\overline{CE}}{dt} = \frac{d\overline{CE}}{dy} \times \frac{dy}{dt} = \frac{y}{2\sqrt{y^2 + 144}}$$

When $y = 16$ m

$$\frac{d\overline{AE}}{dt} = \frac{8}{\sqrt{16^2 + 144}} = 0.4\ \text{m/s}$$

Question 15

Let x be the horizontal distance AB at time t.

$$\tan\theta = \frac{800}{x}$$

$$\frac{1}{\cos^2\theta}\frac{d\theta}{dt} = \frac{800}{x^2}\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{800\cos^2\theta}{x^2}\frac{dx}{dt}$$

From the diagram:

$$\cos\theta = \frac{1000}{\sqrt{800^2+1000^2}} = \frac{5}{\sqrt{41}}$$

$$\cos^2\theta = \frac{25}{41}$$

$$\frac{dx}{dt} = -200\,\text{m/s}$$

$$\frac{d\theta}{dt} = \frac{800\times\frac{25}{41}}{1000^2}(-200) = \frac{4}{41}\,\text{radians/s}$$

Question 16

a $X \sim \mathrm{N}(78,12^2)$

$\mathrm{P}(X > 100) = 0.0334$

b $X \sim \mathrm{N}(78,12^2)$

$\mathrm{P}(70 < X < 90) = 0.5889$

c Standard deviation $= 12 \div \sqrt{4} = 6$

Even though the number of samples is small you would expect normal distribution as the population that the samples are drawn from is known to be normally distributed.

$\mathrm{Y} \sim \mathrm{N}(78,6^2)$

$P(Y < 75) = 0.3085$

Question 17

$$\int_0^h \pi x^2 dy = \pi\int_0^h (y+4)dy = \left[\frac{1}{2}\pi y^2 + 4\pi y\right]_0^h = \frac{1}{2}\pi h^2 + 4\pi h$$

$$2500 = \frac{1}{2}\pi h^2 + 4\pi h$$

Solving gives $h \approx 36\,\text{cm}$.

$$\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}$$

$$5 = (\pi h + 4\pi)\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{5}{\pi h + 4\pi}$$

When $h = 6\text{cm}$

$$\frac{dh}{dt} = \frac{5}{10\pi} = \frac{1}{2\pi}\,\text{cm/s}$$

Question 18

$$\frac{dT}{dt} = kT$$

$$\int \frac{dT}{T} = \int k\,dt$$

$$\ln T = kt + c$$

$$T = e^{kt+c}$$

$$T = T_0 e^{kt}$$

$$(27.7-22) = (28.6-22)e^{1k}$$

$$5.7 = 6.6e^{k}$$

$$k = \ln\frac{5.7}{6.6} = -0.4166$$

$$(28.6-22) = (37-22)e^{-0.4166t}$$

$$t = \frac{\ln\left(\frac{6.6}{15}\right)}{-0.4166} = 5.6\,\text{hours}$$

So the suspicious death would have occurred approximately 5 h and 36 minutes before 1:30 p.m., so at 7:54 a.m.

Question 19

a $1.96\times\frac{80}{\sqrt{n}}=20$

$n\approx 61.47$

The sample should be 62 (or more) to be 95% confident that the mean value of h is within 20 hours of the population mean.

b Standard deviation of sample means is = 65 ÷ 10 = 6.5.

$-50\div 6.5=7.69$ which means the sample mean is more than 7 standard deviations below expected mean.

We would expect the mean hours of samples of 100 electrical components to be normally distributed with mean 1850 and standard deviation of approximately 6.5. A sample mean of 1800 is more than 7 standard deviations below what was expected and this is very unlikely, although not impossible.

Another way of looking at this is to consider the expected mean, with a 95% confidence interval.

With mean of 1800 and standard deviation of 6.5, 95% confident that the mean will be between:

$1800-1.96\times 6.5=1787.26$

$1800+1.96\times 6.5=1812.74$

1850 is well above this range so the claimed mean value is unlikely.

Question 20

$$\begin{aligned}\int\frac{2x^3+3x^2-24x-29}{x^3-7x-6}dx&=\int\frac{2(x^3-7x-6)}{x^3-7x-6}dx+\int\frac{3x^2-10x-17}{x^3-7x-6}dx\\&=\int 2dx+\int\left(\frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{x-3}\right)dx\\&=2x+c+\int\left(\frac{A(x^2-x-6)+B(x^2-2x-3)+C(x^2+3x+2)}{x^3-7x-6}\right)dx\end{aligned}$$

$$A+B+C=3$$

$$-A-2B+3C=-10$$

$$-6A-3B+2C=-17$$

Solving gives $A=1,B=3,C=-1$.

$$2x+c+\int\left(\frac{1}{x+1}+\frac{3}{x+2}-\frac{1}{x-3}\right)dx=2x+\ln|x+1|+3\ln|x+2|-\ln|x-3|+c$$

Question 21

a $$\frac{dN}{dt}=\frac{2N}{5}-\frac{N^2}{15625}=\frac{6250N-N^2}{15625}$$

$$\int\frac{15625}{N(6250-N)}dN=\int dt$$

$$\int\left[\frac{A}{N}+\frac{B}{6250-N}\right]dN=\int dt$$

$$\int\left[\frac{A(6250-N)+BN}{6250N-N^2}\right]=\int dt$$

$$\int\left[\frac{2.5}{N}+\frac{2.5}{N-6250}\right]dN=\int dt$$

$$2.5\ln\left|N\right|+2.5\ln\left|N-6250\right|=t+c$$

$$\frac{5}{2}\left(\ln\left|N\right|+\ln\left|N-6250\right|\right)=t+c$$

$$\ln\left|N^2-6250N\right|=\frac{2}{5}t+c$$

$$N=\frac{6250}{1+ce^{-0.4t}}$$

When $t=0,\ N=250$

$c=24$

$$N=\frac{6250}{1+24e^{-0.4t}}$$

b As $t\to\infty, e^{-0.04t}=\dfrac{1}{e^{0.04t}}\to 0$ and so $N\to 6250.$

Question 22

$$\begin{aligned}-\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x+c&=\cos x\left(-1+\frac{2}{3}\cos^2 x-\frac{1}{5}\cos^4 x\right)+c\\&=\cos x\left[-1+\frac{2}{3}(1-\sin^2 x)-\frac{1}{5}(1-\sin^2 x)^2\right]+c\\&=\cos x\left[-1+\frac{2}{3}-\frac{2}{3}\sin^2 x-\frac{1}{5}(1-2\sin^2 x+\sin^4 x)\right]+c\\&=\cos x\left(-\frac{1}{5}\sin^4 x-\frac{4}{15}\sin^2 x-\frac{8}{15}\right)+c\end{aligned}$$

Question 23

a $x = \sin u \qquad \frac{dx}{du} = \cos x$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-\sin^2 u}}\frac{dx}{du}\,du = \int \frac{1}{\sqrt{\cos^2 u}} \times \cos u\,du$$

$$= \int 1du = u + c = \arcsin x + c$$

b $x = 5\sin u \qquad \frac{dx}{du} = 5\cos u$

$$\int \frac{1}{\sqrt{25-x^2}}\,dx = \int \frac{1}{\sqrt{25-25\sin^2 u}}\frac{dx}{du}\,du = \int \frac{1}{\sqrt{25(1-\sin^2 u)}}\cos u\,du$$

$$= \int \frac{1}{\sqrt{25\cos^2 u}}\cos u\,du = \int \frac{1}{5\cos u}\cos u\,du = \int \frac{1}{5}\,du$$

$$= \frac{u}{5} + c = \arcsin\frac{x}{5} + c$$

c $x = \frac{3}{2}\sin u \qquad \frac{dx}{du} = \frac{3}{2}\cos u$

$$\int \frac{1}{\sqrt{9-4x^2}}\,dx = \int \frac{1}{\sqrt{9-4\times\frac{9}{4}\sin^2 u}}\frac{dx}{du}\,du = \int \frac{1}{\sqrt{9(1-\sin^2 u)}}\frac{3}{2}\cos u\,du$$

$$= \int \frac{1}{\sqrt{9\cos^2 u}}\frac{3}{2}\cos u du = \int \frac{1}{3\cos u}\frac{3}{2}\cos u du$$

$$= \int \frac{1}{2}du = \frac{u}{2} + c = \frac{1}{2}\arcsin\frac{2x}{3} + c$$

d $x = \sin u \qquad \frac{dx}{du} = \cos u$

$$\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2 u}\,\frac{dx}{du}\,du = \int \sqrt{\cos^2 u}\,\cos u\,du$$

$$= \int \cos^2 u\,du = \int\left(\frac{\cos 2u + 1}{2}\right)du = \frac{\sin 2u}{4} + \frac{u}{2} + c$$

$$= \frac{2\sin u\cos u}{4} + \frac{u}{2} + c = \frac{\sin u\sqrt{1-\sin^2 u}}{2} + \frac{u}{2} + c$$

$$= \frac{x\sqrt{1-x^2}}{2} + \frac{\arcsin x}{2} + c$$

e $x = 2\sin u \qquad \frac{dx}{du} = 2\cos u$

$$\int \sqrt{4-x^2}\,dx = \int \sqrt{4-4\sin^2 u}\,\frac{dx}{du}\,du = \int \sqrt{4(1-\sin^2 u)}\,2\cos u\,du$$
$$= 2\int \sqrt{4\cos^2 u}\,\cos u\,du = 2\int 2\cos^2 u\,du$$
$$= 2\int 2\left(\frac{\cos 2u + 1}{2}\right)du = \sin 2u + 2u + c$$
$$= 2\sin u\cos u + 2u + c = 2\sin u\sqrt{1-\sin^2 u} + 2u + c$$
$$= x\sqrt{1-\frac{x^2}{4}} + 2\arcsin\frac{x}{2} + c$$
$$= \frac{x\sqrt{4-x^2}}{2} + 2\arcsin\frac{x}{2} + c$$

f $x = 2\cos u \qquad \frac{dx}{du} = -2\sin u$

$$\int \sqrt{4-x^2}\,dx = \int \sqrt{4-4\cos^2 u}\,\frac{dx}{du}\,du = -\int \sqrt{4(1-\cos^2 u)}\,2\sin u\,du$$
$$= -2\int \sqrt{4\sin^2 u}\,\sin u\,du = -\int 4\sin^2 u\,du$$
$$= 4\int \frac{\cos 2u - 1}{2}\,du = 2\left(\frac{\sin 2u}{2} - u\right) + c$$
$$= 2\sin u\cos u - 2u + c = \sqrt{1-\cos^2 u}\,2\cos u - 2u + c$$
$$= x\sqrt{1-\left(\frac{x}{2}\right)^2} - 2\arccos\frac{x}{2} + c = \frac{x}{2}\sqrt{4-x^2} - 2\arccos\frac{x}{2} + c$$