SADLER UNIT 3 MATHEMATICS METHODS

WORKED SOLUTIONS

Chapter 2: Complex numbers, a reminder.

Exercise 2A

Question 1

a $|z|=\sqrt{4^2+(-3)^2}=\sqrt{16+9}=\sqrt{25}=5$

b $|z|=\sqrt{12^2+5^2}=\sqrt{144+25}=\sqrt{169}=13$

c $|z|=\sqrt{3^2+2^2}=\sqrt{9+4}=\sqrt{13}$

d $|z|=\sqrt{3^2+(-2)^2}=\sqrt{9+4}=\sqrt{13}$

e $|z|=\sqrt{1^2+5^2}=\sqrt{1+25}=\sqrt{26}$

f $|z|=\sqrt{0^2+5^2}=\sqrt{25}=5$

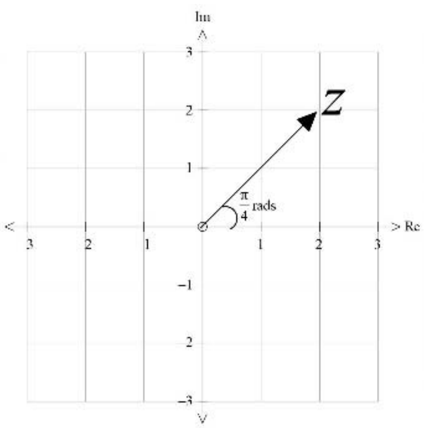
Question 2

a $z=2+2i$

$\tan\theta=\frac{2}{2}=1$

$\theta=\frac{\pi}{4}$

$\arg z=\frac{\pi}{4}$



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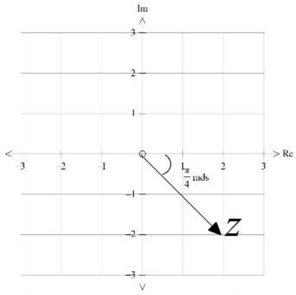
b $z = 2 - 2i$

$$\tan\theta = \frac{-2}{2} = -1$$

$$\theta = -\frac{\pi}{4}$$

$$\arg z = -\frac{\pi}{4}$$

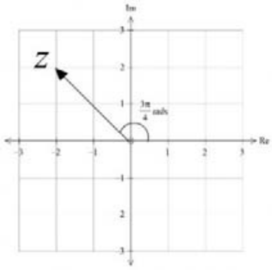


c $z = -2 - 2i$

$$\tan\theta = \frac{2}{-2} = -1$$

$$\theta = \frac{3\pi}{4}$$

$$\arg z = \frac{3\pi}{4}$$

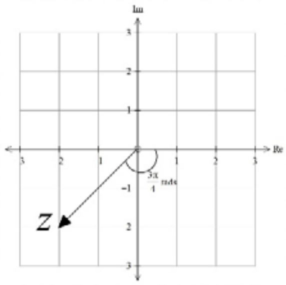


d $z = -2 - 2i$

$$\tan\theta = \frac{-2}{-2} = 1$$

$$\theta = -\frac{3\pi}{4}$$

$$\arg z = -\frac{3\pi}{4}$$

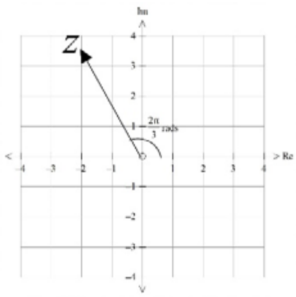


e $z = -2 + 2\sqrt{3}i$

$$\tan\theta = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$$

$$\theta = \frac{2\pi}{3}$$

$$\arg z = \frac{2\pi}{3}$$

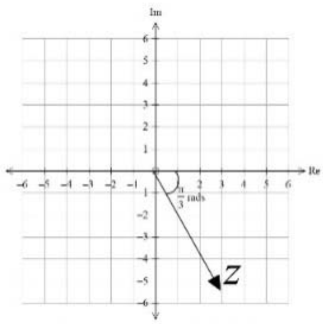


f

$$\tan\theta = \frac{-3\sqrt{3}}{3} = -\sqrt{3}$$

$$\theta = -\frac{\pi}{3}$$

$$\arg z = -\frac{\pi}{3}$$



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Question 3

z_1 has an angle of $\frac{13\pi}{6}$ which is $\frac{12\pi}{6}+\frac{\pi}{6}=2\pi+\frac{\pi}{6}$, this is equivalent to $\frac{\pi}{6}$

$z_1=3\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

z_2 has an angle of 3π which is $2\pi+\pi$, this is equivalent to π

$z_2=3(\cos\pi+i\sin\pi)$

z_3 has an angle of $\frac{5\pi}{4}$ which is equivalent to $-\frac{3\pi}{4}$

$z_3=4\left[\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right]$

z_4 has an angle of $-\pi$ which is not in the domain but is equivalent to π, which is in the domain.

$z_4=2\left[\cos(\pi)+i\sin(\pi)\right]$

$z_5=6(\cos 1+i\sin 1)$

z_6 has a length of 5 units and angle $180°-45°=135°$ ($\frac{3}{4}\pi$ in radians)

$z_6=5\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

z_7 has a length of 8 units and angle of $-150°$ ($-\frac{5\pi}{6}$ in radians)

$z_7=8\left[\cos\left(-\frac{5\pi}{6}\right)+i\sin\left(-\frac{5\pi}{6}\right)\right]$

$z_8=5\left[\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right]$

$z_9=6(\cos 2+i\sin 2)$

$z_{10}=4(\cos\pi+i\sin\pi)$

z_{11} is 5 units in length and $-\pi+\frac{\pi}{4}=\frac{-3\pi}{4}$

$z_{11}=5\left[\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right]$

z_{12} is 7 units in length and $-\frac{\pi}{2}+\frac{\pi}{3}=-\frac{\pi}{6}$

$z_{12}=7\left[\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right]$

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Question 4

z_{13} has $r=\sqrt{5^2+5^2}=\sqrt{50}=5\sqrt{2}$, $\tan\theta=\frac{5}{5}=1$ so $\theta=\frac{\pi}{4}$

$z_{13}=5\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

z_{14} has $r=\sqrt{3^2+4^2}=\sqrt{9+16}=\sqrt{25}=5$, $\tan\theta=\frac{4}{3}$ so $\theta=\pi-0.9273=2.2143$

$z_{14}=5\left[\cos(2.2143)+i\sin(2.2143)\right]$

z_{15} has $r=\sqrt{(-4)^2+(-5)^2}=\sqrt{16+25}=\sqrt{41}$, $\tan\theta=\frac{-5}{-4}$ so $\theta=(-\pi+0.8961)=-2.2455$

$z_{15}=\sqrt{41}\left[\cos(-2.2455)+i\sin(-2.2455)\right]$

z_{16} has $r=\sqrt{5^2+5^2}=\sqrt{50}=5\sqrt{2}$, $\tan\theta=\frac{-5}{5}=-1$ so $\theta=-\frac{\pi}{4}$

$z_{16}=5\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right]$

For z_{17}, $r=\sqrt{5^2+12^2}=\sqrt{25+144}=\sqrt{169}=13$

and $\tan\theta=\frac{12}{5}$ so $\theta=1.1760$

$z_{17}=13\left[\cos(1.1760)+i\sin(1.1760)\right]$

For z_{18}, $r=\sqrt{1^2+7^2}=\sqrt{1+49}=\sqrt{50}=5\sqrt{2}$

and $\tan\theta=\frac{7}{1}$ so $\theta=1.4289$

$z_{18}=5\sqrt{2}\left[\cos(1.4289)+i\sin(1.4289)\right]$

For z_{19}, $r=\sqrt{1^2+(-7)^2}=\sqrt{1+49}=\sqrt{50}=5\sqrt{2}$

and $\tan\theta=\frac{-7}{1}$ so $\theta=-1.4289$

$z_{19}=5\sqrt{2}\left[\cos(-1.4289)+i\sin(-1.4289)\right]$

For z_{20}, $r=\sqrt{(-7)^2+1^2}=\sqrt{49+1}=\sqrt{50}=5\sqrt{2}$

and $\tan\theta=\frac{1}{-7}$ so $\theta=-0.1419+\pi=2.999$

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$z_{20} = 5\sqrt{2}\left[\cos(2.9997) + i\sin(2.9997)\right]$

For z_{21}, $r = \sqrt{(5\sqrt{3})^2 + 5^2} = \sqrt{75+25} = \sqrt{100} = 10$

and $\tan\theta = \dfrac{5}{5\sqrt{3}} = \dfrac{1}{\sqrt{3}}$ so $\theta = \dfrac{\pi}{6}$

$$z_{21} = 10\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$$

For z_{22} no calculation required as line would be straight up at an angle of $\dfrac{\pi}{2}$ with a length of 4.

$$z_{22} = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

For z_{23} no calculation required as line is at angle of 0 radians with a length of 4.

$z_{23} = 4(\cos 0 + i\sin 0)$

For z_{24} no calculation required as line is at angle of π radians with a length of 4.

$z_{24} = 4(\cos\pi + i\sin\pi)$

For z_{25} no calculation required as line is at angle of $-\dfrac{\pi}{2}$ radians with a length of 3.

$$z_{25} = 3\left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]$$

For z_{26} no calculation required as line is at angle of 0 radians with a length of 3.

$z_{26} = 3(\cos 0 + i\sin 0)$

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Question 5

For z_{27}, $a = 2\cos\left(\frac{\pi}{4}\right) = \sqrt{2}$

$b = 2\sin\left(\frac{\pi}{4}\right) = \sqrt{2}$

$z_{27} = \sqrt{2} + \sqrt{2}i$

For z_{28}, $a = 4\cos\left(\frac{5\pi}{6}\right) = -2\sqrt{3}$

$b = 4\sin\left(\frac{5\pi}{6}\right) = 2$

$z_{28} = -2\sqrt{3} + 2i$

For z_{29}, $a = 4\cos\left(\frac{-\pi}{3}\right) = 2$

$b = 4\sin\left(\frac{-\pi}{3}\right) = -2\sqrt{3}$

$z_{29} = 2 - 2\sqrt{3}i$

For z_{30}, $a = 6\cos\left(\frac{-2\pi}{3}\right) = -3$

$b = 6\sin\left(\frac{-2\pi}{3}\right) = -3\sqrt{3}$

$z_{30} = -3 - 3\sqrt{3}i$

For z_{31} the angle is 2π
$a = 5$, $b = 0$ (no calculation required)
$z_{31} = 5 + 0i$

For z_{32} the angle is $\frac{7\pi}{2}$ which is equivalent to $-\frac{\pi}{2}$
$a = 0$, $b = -1$ (no calculation required)
$z_{32} = 0 - i$

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Exercise 2B

Question 1

For z_1, $r = 3$ and $\theta = 60°$, which is equivalent to $\frac{\pi}{3}$ radians.

$z_1 = 3\,\text{cis}\frac{\pi}{3}$

For z_2, $r = 5$ and $\theta = 120°$, which is equivalent to $\frac{2\pi}{3}$ radians.

$z_2 = 5\,\text{cis}\frac{2\pi}{3}$

For z_3, $r = 4$ and $\theta = -150°$, which is equivalent to $-\frac{5\pi}{6}$ radians.

$z_3 = 4\text{cis}\left(-\frac{5\pi}{6}\right)$

For z_4, $r = 5$ and $\theta = -90°$, which is equivalent to $-\frac{\pi}{2}$ radians.

$z_4 = 5\,\text{cis}\left(-\frac{\pi}{2}\right)$

For z_5, $r = 4$ and $\theta = 0°$, which is equivalent to 0 radians.
$z_5 = 4\text{cis}(0)$

For z_6, $r = 5$ and $\theta = 90°$, which is equivalent to $\frac{\pi}{2}$ radians.

$z_6 = 5\,\text{cis}\left(\frac{\pi}{2}\right)$

For z_7, $r = 5$ and $\theta = 135°$, which is equivalent to $\frac{3\pi}{4}$ radians.

$z_7 = 5\,\text{cis}\left(\frac{3\pi}{4}\right)$

For z_8, $r = 3$ and $\theta = -135°$, which is equivalent to $-\frac{3\pi}{4}$ radians.

$z_8 = 3\,\text{cis}\left(-\frac{3\pi}{4}\right)$

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Question 2

$$2\left(\cos\frac{\pi}{10}+i\sin\frac{\pi}{10}\right)=2\text{cis}\frac{\pi}{10}$$

Question 3

$$7\left(\cos\frac{5\pi}{8}+i\sin\frac{5\pi}{8}\right)=7\text{cis}\frac{5\pi}{8}$$

Question 4

$9(\cos 30°+i\sin 30°)=9\,\text{cis}\frac{\pi}{6}$ (as $30°$ is equivalent to $\frac{\pi}{6}$ in radians)

Question 5

$330°$ is not in the domain but is equivalent to $-30°$ or $-\frac{\pi}{6}$ and in the domain.

$$3(\cos 330°+i\sin 330°)=3\,\text{cis}\left(-\frac{\pi}{6}\right)$$

Question 6

$\frac{3\pi}{2}$ is not in the domain but is equivalent to $-\frac{\pi}{2}$ in the domain.

$$5\left(\cos\left(\frac{3\pi}{2}\right)+i\sin\left(\frac{3\pi}{2}\right)\right)=5\,\text{cis}\left(-\frac{\pi}{2}\right)$$

Question 7

$\frac{8\pi}{3}$ is not in the domain but is equivalent to $\frac{2\pi}{3}$ in the domain.

$$4\left(\cos\frac{8\pi}{3}+i\sin\frac{8\pi}{3}\right)=4\text{cis}\frac{2\pi}{3}$$

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Question 8

$-\frac{5\pi}{3}$ is not in the domain but is equivalent to $\frac{\pi}{3}$ in the domain.

$$2\left[\cos\left(-\frac{5\pi}{3}\right)+i\sin\left(-\frac{5\pi}{3}\right)\right]=2\text{cis}\left(\frac{\pi}{3}\right)$$

Question 9

-3π is not in the domain but is equivalent to π in the domain.

$$2\left[\cos(-3\pi)+i\sin(-3\pi)\right]=2\text{cis}(\pi)$$

Question 10

$$7\text{cis}\frac{\pi}{2}=7\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=7(0+i)=7i$$

Question 11

$$5\,\text{cis}\left(-\frac{\pi}{2}\right)=5\left[\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right]=5(0-i)=-5i$$

Question 12

$$\text{cis}\pi=\cos\pi+i\sin\pi=-1+0i=-1$$

Question 13

$$3\,\text{cis}\,2\pi=3(\cos 2\pi+i\sin 2\pi)=3+0i=3$$

Question 14

$$10\,\text{cis}\frac{\pi}{4}=10\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=10\times\frac{\sqrt{2}}{2}+10\times\frac{\sqrt{2}}{2}i=5\sqrt{2}+5\sqrt{2}\,i$$

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Question 15

$$4\operatorname{cis}\frac{2\pi}{3} = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = -2 + 2\sqrt{3}\,i$$

Question 16

$$4\operatorname{cis}\left(-\frac{2\pi}{3}\right) = 4\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right] = -2 - 2\sqrt{3}\,i$$

Question 17

$$12\operatorname{cis}\left(-\frac{4\pi}{3}\right) = 12\left[\cos\left(-\frac{4\pi}{3}\right) + i\sin\left(-\frac{4\pi}{3}\right)\right] = -6 + 6\sqrt{3}\,i$$

Question 18

$$r = \sqrt{(-7)^2 + 24^2} = \sqrt{49 + 576} = \sqrt{625} = 25$$

$$\tan\theta = -\frac{24}{7}$$

$$\theta = \pi - 1.2870$$

$$-7 + 24i = 25\operatorname{cis}(\pi - 1.2870) = 25\operatorname{cis}(1.8546)$$

Question 19

$$r = \sqrt{(-5)^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

$$\tan\theta = -\frac{12}{5}$$

$$\theta = \pi - 1.1760$$

$$-5 + 12i = 13\operatorname{cis}(1.9656)$$

Question 20

$$r = \sqrt{1^2 + 2^2} = \sqrt{1 + 4} = \sqrt{5}$$

$$\tan\theta = \frac{2}{1}$$

$$\theta = 1.1071$$

$$1 + 2i = \sqrt{5}\operatorname{cis}(1.1071)$$

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Question 21

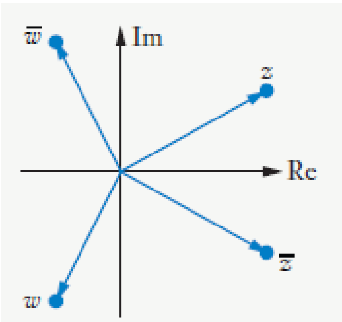
$r = \sqrt{0^2 + 5^2} = \sqrt{25} = 5$

$\theta = \frac{\pi}{2}$

$5i = 5\,\text{cis}\left(\frac{\pi}{2}\right)$

Question 22

a



b $\overline{z} = r_1\,\text{cis}(-\alpha)$, $\overline{w} = r_2\,\text{cis}(-\beta)$

Question 23

The conjugate of $2\text{cis}\,30°$ is $2\,\text{cis}(-30°)$.

Question 24

The conjugate of $7\,\text{cis}\,120°$ is $7\,\text{cis}(-120°)$.

Question 25

$4\,\text{cis}\,390° = 4\,\text{cis}\,30°$

The conjugate of $4\,\text{cis}\,30°$ is $4\,\text{cis}(-30°)$.

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Question 26

$10\,\text{cis}\,(-200°) = 10\,\text{cis}\,160°$

The conjugate of $10\,\text{cis}\,160°$ is $10\,\text{cis}\,(-160°)$.

Question 27

The conjugate of $2\,\text{cis}\,\frac{\pi}{2}$ is $2\,\text{cis}\left(-\frac{\pi}{2}\right)$.

Question 28

The conjugate of $5\,\text{cis}\left(-\frac{3\pi}{4}\right)$ is $5\,\text{cis}\left(\frac{3\pi}{4}\right)$.

Question 29

The conjugate of $5\,\text{cis}\,0.5$ is $5\,\text{cis}\,(-0.5)$.

Question 30

$5\,\text{cis}\,\frac{7\pi}{2} = 5\,\text{cis}\left(-\frac{\pi}{2}\right)$

The conjugate of $5\,\text{cis}\left(-\frac{\pi}{2}\right)$ is $5\,\text{cis}\left(\frac{\pi}{2}\right)$.

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Exercise 2C

Question 1

$$zw = (2+3i)(5-2i) = 10-4i+15i-6i^2$$
$$= 10+11i-6(-1) = 16+11i$$

Question 2

$$zw = (3+2i)(-1+2i) = -3+6i-2i+4i^2$$
$$= -3+4i+4(-1) = -7+4i$$

Question 3

$$z = 3\,\text{cis}\,60°,\ w = 5\,\text{cis}\,20°$$
$$zw = 3\times 5\,\text{cis}\,(60°+20°) = 15\,\text{cis}\ 80°$$

Question 4

$$z = 3\,\text{cis}\,120°,\ w = 3\,\text{cis}\,150°$$
$$zw = 3\times 3\,\text{cis}\,(120°+150°) = 9\,\text{cis}\ 270° = 9\,\text{cis}\,(-90°)$$

Question 5

$$z = 3\,\text{cis}\,30°,\ w = 3\,\text{cis}\,(-80°)$$
$$zw = 3\times 3\,\text{cis}\,(30°-80°) = 9\,\text{cis}\,(-50°)$$

Question 6

$$z = 5\,\text{cis}\,\frac{\pi}{3},\ w = 2\,\text{cis}\,\frac{\pi}{4}$$
$$zw = 5\times 2\,\text{cis}\left(\frac{\pi}{3}+\frac{\pi}{4}\right) = 10\,\text{cis}\,\frac{7\pi}{12}$$

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Question 7

$$z = 4\,\text{cis}\,\frac{\pi}{4},\ w = 2\,\text{cis}\,\frac{-3\pi}{4}$$
$$zw = 4 \times 2\,\text{cis}\left(\frac{\pi}{4} - \frac{3\pi}{4}\right) = 8\,\text{cis}\left(-\frac{\pi}{2}\right)$$

Question 8

$$z = 2(\cos 50° + i\sin 50°),\ w = \cos 60° + i\sin 60°$$
$$zw = 2(\cos 110° + i\sin 110°)$$

Question 9

$$z = 2(\cos 170° + i\sin 170°),\ w = 3(\cos 150° + i\sin 150°)$$
$$zw = 6(\cos 320° + i\sin 320°) = 6\left[\cos(-40°) + i\sin(-40°)\right]$$

Question 10

$$z = 6 - 3i,\ w = 3 - 4i$$

$$\frac{z}{w} = \frac{6-3i}{3-4i} = \frac{6-3i}{3-4i} \times \frac{3+4i}{3+4i} = \frac{18+24i-9i-12i^2}{9+12i-12i-16i^2}$$
$$= \frac{30+15i}{25} = \frac{6+3i}{5} = 1.2 + 0.6i$$

Question 11

$$z = -6 + 3i,\ w = -3 + 4i$$

$$\frac{z}{w} = \frac{-6+3i}{-3+4i} = \frac{-6+3i}{-3+4i} \times \frac{-3-4i}{-3-4i} = \frac{18+24i-9i-12i^2}{9+12i-12i-16i^2}$$
$$= \frac{30+15i}{25} = \frac{6+3i}{5} = 1.2 + 0.6i$$

Question 12

$$\frac{z}{w} = \frac{8\,\text{cis}\,60°}{2\,\text{cis}\,40°} = 4\,\text{cis}\,(60° - 40°) = 4\,\text{cis}\,20°$$

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Question 13

$$\frac{z}{w} = \frac{5\,\text{cis}\,120°}{\text{cis}\,150°} = 5\,\text{cis}\,(-30°)$$

Question 14

$$\frac{z}{w} = \frac{3\,\text{cis}\,(-150°)}{3\,\text{cis}\,80°} = \text{cis}\,(-150° - 80°) = \text{cis}\,(-230°) = \text{cis}\,130°$$

Question 15

$$\frac{z}{w} = \frac{2\text{cis}\,\frac{3\pi}{5}}{2\text{cis}\,\frac{2\pi}{5}} = \text{cis}\left(\frac{3\pi}{5} - \frac{2\pi}{5}\right) = \text{cis}\,\frac{\pi}{5}$$

Question 16

$$\frac{z}{w} = \frac{4\text{cis}\,\frac{\pi}{4}}{2\text{cis}\left(-\frac{3\pi}{4}\right)} = 2\text{cis}\left[\frac{\pi}{4} - \left(-\frac{3\pi}{4}\right)\right] = 2\,\text{cis}\,\pi$$

Question 17

$$\frac{z}{w} = \frac{5\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]}{2\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right]} = 2.5\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$$

Question 18

$$\frac{z}{w} = \frac{2\,(\cos 50° + i\sin 50°)}{5\,(\cos 50° + i\sin 50°)} = 0.4(\cos 0 + i\sin 0)$$

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Question 19

$z = \text{cis}\,30°,\ zw = 2\,\text{cis}\,70°$
$w = 2\,\text{cis}\,(70° - 30°) = 2\,\text{cis}\,40°$

Question 20

$z = \text{cis}\,30°,\ zw = 3\,\text{cis}\,130°$
$w = 3\,\text{cis}\,(130° - 30°) = 3\,\text{cis}\,100°$

Question 21

$z = \text{cis}\,30°,\ zw = 2\,\text{cis}\,(-60°)$
$w = 2\,\text{cis}\,(-60° - 30°) = 2\,\text{cis}\,(-90°)$

Question 22

$z = \text{cis}\,110°,\ zw = 2\,\text{cis}\,(-130°)$
$w = 2\,\text{cis}\,(-130° - 110°) = 2\,\text{cis}\,(-240°) = 2\,\text{cis}\,(120°)$

Question 23

$z = \text{cis}\,110°,\ zw = \text{cis}\,(-90°)$
$w = \text{cis}\,(-90° - 110°) = \text{cis}\,(-200°) = \text{cis}\,160°$

Question 24

$z = \text{cis}\,110°,\ zw = 2\,\text{cis}\,(-30°)$
$w = 2\text{cis}\,(-30° - 110°) = 2\text{cis}\,(-140°)$

Question 25

$z = 2\,\text{cis}\ 150°,\ \dfrac{z}{w} = 2\,\text{cis}\,(30°)$
$w = \text{cis}\,(150° - 30°) = \text{cis}\,(120°)$

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Question 26

$z = 2\,\text{cis}\,150°,\ \frac{z}{w} = \text{cis}\,(70°)$

$w = 2\,\text{cis}\,(150° - 70°) = 2\,\text{cis}\,(80°)$

Question 27

$z = 2\,\text{cis}\,150°,\ \frac{z}{w} = \text{cis}\,(-110°)$

$w = 2\,\text{cis}\,(150° + 110°) = 2\,\text{cis}\,(260°) = 2\,\text{cis}\,(-100°)$

Question 28

$z = 6\,\text{cis}\,40°,\ w = 2\,\text{cis}\,30°$

a $2z = 12\,\text{cis}\,40°$

b $3w = 6\,\text{cis}\,30°$

c $zw = 12\,\text{cis}\,70°$

d $wz = 12\,\text{cis}\,70°$

e $iz = i(6\,\text{cis}\,40°) = 6\,\text{cis}\,(40°+90°) = 6\,\text{cis}\,130°$

f $iw = i(2\,\text{cis}\,30°) = 2\,\text{cis}\,(30° + 90°) = 2\,\text{cis}\,120°$

g $\frac{w}{z} = \frac{2\,\text{cis}\,30°}{6\,\text{cis}\,40°} = \frac{1}{3}\text{cis}\,(-10°)$

h $\frac{1}{z} = \frac{1}{6}\text{cis}\,(-40°)$

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Question 29

$z = 8\,\text{cis}\,\frac{2\pi}{3},\ w = 4\,\text{cis}\,\frac{3\pi}{4}$

a $zw = 32\,\text{cis}\left(\frac{2\pi}{3} + \frac{3\pi}{4}\right) = 32\,\text{cis}\left(\frac{17\pi}{12}\right) = 32\,\text{cis}\left(-\frac{7\pi}{12}\right)$

b $wz = 32\,\text{cis}\left(\frac{3\pi}{4} + \frac{2\pi}{3}\right) = 32\,\text{cis}\left(\frac{17\pi}{12}\right) = 32\,\text{cis}\left(-\frac{7\pi}{12}\right)$

c $\frac{w}{z} = \frac{4\,\text{cis}\,\frac{3\pi}{4}}{8\,\text{cis}\,\frac{2\pi}{3}} = \frac{1}{2}\,\text{cis}\,\frac{\pi}{12}$

d $\frac{z}{w} = \frac{8\,\text{cis}\,\frac{2\pi}{3}}{4\,\text{cis}\,\frac{3\pi}{4}} = 2\,\text{cis}\left(-\frac{\pi}{12}\right)$

e $\bar{z} = 8\,\text{cis}\left(-\frac{2\pi}{3}\right)$

f $\bar{w} = 4\,\text{cis}\left(-\frac{3\pi}{4}\right)$

g $\frac{1}{z} = \frac{1}{8}\,\text{cis}\left(-\frac{2\pi}{3}\right)$

h $\frac{i}{w} = \frac{1}{4}\,\text{cis}\left(\frac{\pi}{2} - \frac{3\pi}{4}\right) = \frac{1}{4}\,\text{cis}\left(-\frac{\pi}{4}\right)$

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Exercise 2D

Question 1

D

Question 2

A

Question 3

E

Question 4

H

Question 5

K

Question 6

L

Question 7

M

Question 8

P

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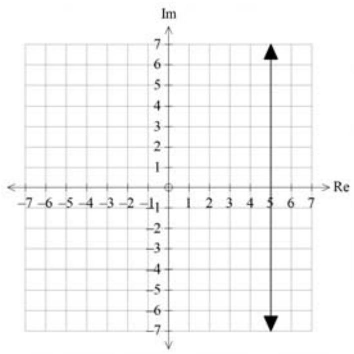
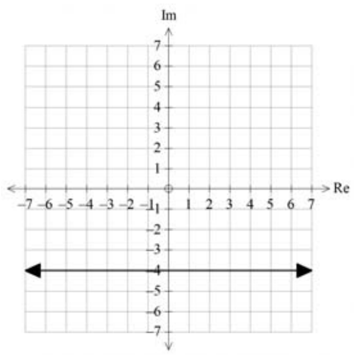
Question 9



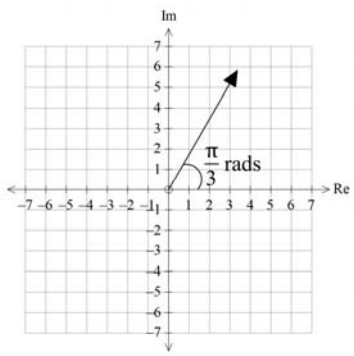
$x = 5$

Question 10



$y = -4$

Question 11



$y = \sqrt{3}x, x > 0$

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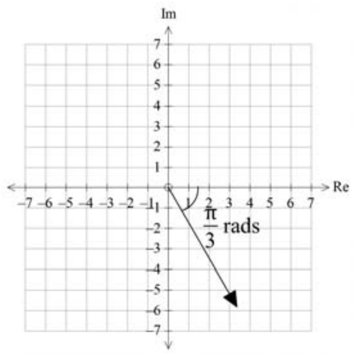
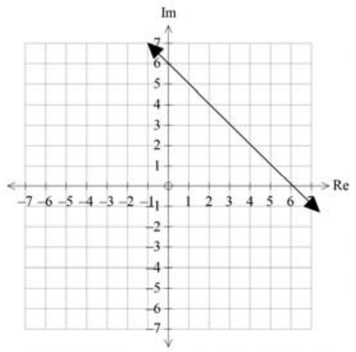
Question 12



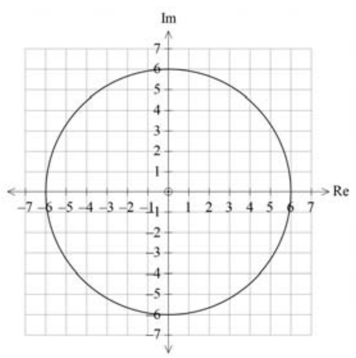
$y = -\sqrt{3}x,\ x > 0$

Question 13



$x + y = 6$

Question 14



$x^2 + y^2 = 36$

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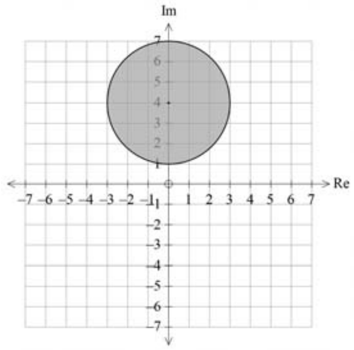
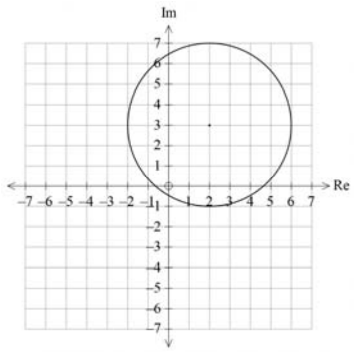
Question 15



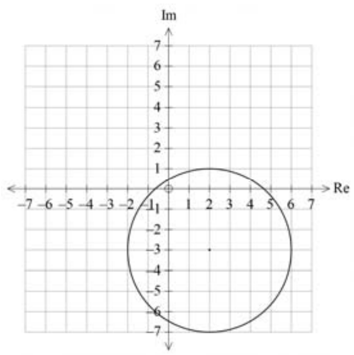
$x^2 + (y-4)^2 \le 9$

Question 16



$(x-2)^2 + (y-3)^2 = 16$

Question 17



$(x-2)^2 + (y+3)^2 = 16$

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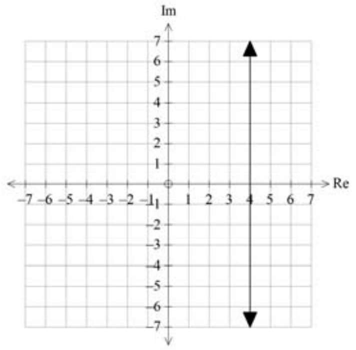
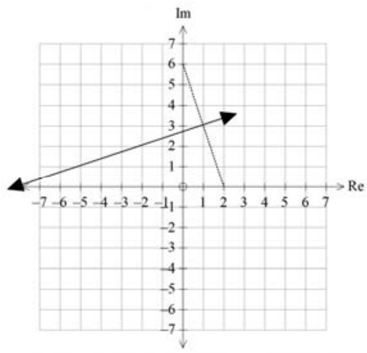
Question 18



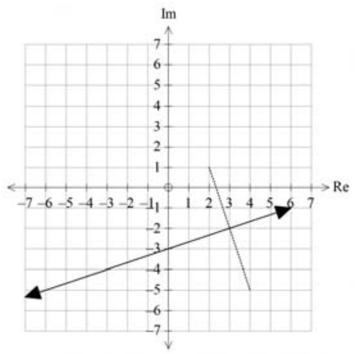
$x = 4$

Question 19



$3y = x + 8$

Question 20



$3y = x - 9$

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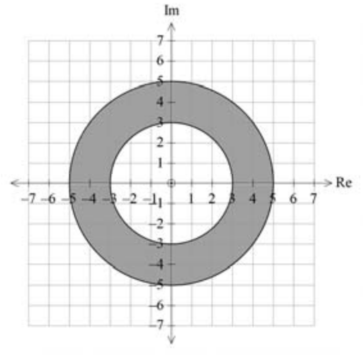
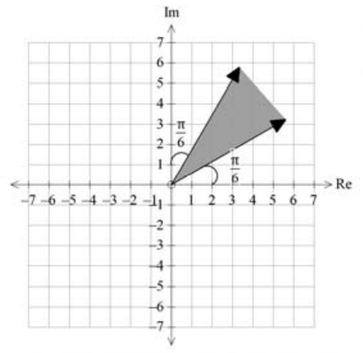
Question 21



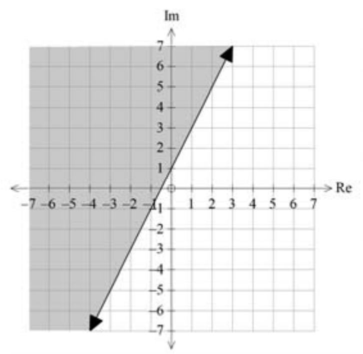
$9 \le x^2 + y^2 \le 25$

Question 22



$\frac{1}{\sqrt{3}}x \le y \le \sqrt{3}x,\ x > 0$

Question 23



$y \ge 2x + 1$

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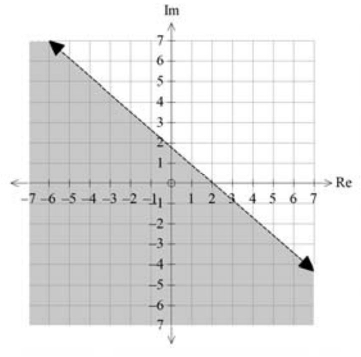
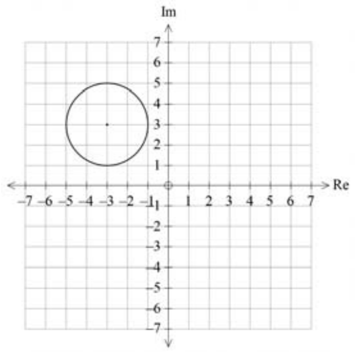
Question 24



$x + y < 2$

Question 25



a The minimum possible value of $\text{Im}(z)$ is 1.

b The maximum possible value of $|\text{Re}(z)|$ is 5.

c The minimum possible value of $|z|$ is $\sqrt{3^2+3^2}-2=3\sqrt{2}-2$.

d The maximum possible value of $|z|$ is $\sqrt{3^2+3^2}+2=3\sqrt{2}+2$.

e The maximum possible value of $|\overline{z}|$ is $3\sqrt{2}+2$.

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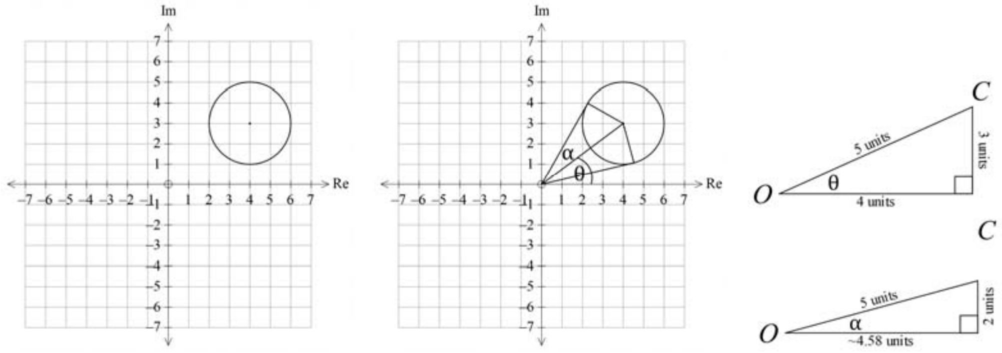
Question 26



a The minimum possible value of $\mathrm{Im}(z)$ is 1.

b The maximum possible value of $\mathrm{Re}(z)$ is 6.

c The maximum possible value of $|z|$ is $\sqrt{4^2+3^2}+2=7$.

d The minimum possible value of $|z|$ is $\sqrt{4^2+3^2}-2=3$.

e The minimum possible value of $\arg(z)$ is found by looking at the tangents to the circle.

$$\tan\theta=\frac{3}{4}$$
$$\theta\approx 0.6435$$
$$\tan\alpha=\frac{2}{\sqrt{21}}$$
$$\alpha\approx 0.4115$$

$\arg(z)\approx 0.6435-0.4115\approx 0.23$ radians

f The maximum possible value of $\arg(z)$ is found by looking at the tangent to the circle.

$\arg(z)=\theta+\alpha\approx 0.6435+0.4115\approx 1.06$ radians

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Question 27

Given $|z-(2+3i)|=2|z-(5-3i)|$

Thus if $z=x+y\text{i}$ $|x+yi-(2+3i)|=2|x+yi-(5-3i)|$

$$(x-2)^2+(y-3)^2=2^2\left[(x-5)^2+(y+3)^2\right]$$
$$x^2-4x+4+y^2-6y+9=2^2\left[x^2-10x+25+y^2+6y+9\right]$$
$$x^2-4x+4+y^2-6y+9=4x^2-40x+100+4y^2+24y+36$$

$$0=3x^2-36x+3y^2+30y+123$$
$$0=x^2-12x+y^2+10y+41$$

$$(x-6)^2-36+(y+5)^2-25+41=0$$

$$(x-6)^2+(y+5)^2=20$$

The set of points form a circle with centre (6, –5) and radius $\sqrt{20}=2\sqrt{5}$ units.

Question 28

Given $|z-(10+5i)|=3|z-(2-3i)|$

Thus if $z=x+yi$ $|x+yi-(10+5i)|=3|x+yi-(2-3i)|$

$$(x-10)^2+(y-5)^2=3^2\left[(x-2)^2+(y+3)^2\right]$$
$$x^2-20x+100+y^2-10y+25=3^2\left[x^2-4x+4+y^2+6y+9\right]$$
$$x^2-20x+100+y^2-10y+25=9x^2-36x+36+9y^2+54y+81$$

$$0=8x^2-16x+8y^2+64y-8$$
$$0=x^2-2x+y^2+8y-1$$
$$0=(x-1)^2-1+(y+4)^2-16-1$$
$$18=(x-1)^2+(y+4)^2$$

The set of points form a circle with centre (1, –4) and radius $\sqrt{18}=3\sqrt{2}$ units.

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Exercise 2E

Question 1

$z^6 = 1$

$z = 1$ is one solution

$\frac{2\pi}{6} = \frac{\pi}{3}$

Solutions are:

$z = 1\operatorname{cis} 0$ (i.e. $z = 1$), $z = 1\operatorname{cis}\frac{\pi}{3}$, $z = 1\operatorname{cis}\frac{2\pi}{3}$

$z = 1\operatorname{cis}\pi$, $z = 1\operatorname{cis}\left(-\frac{\pi}{3}\right)$, $z = \operatorname{cis}\left(-\frac{2\pi}{3}\right)$

Question 2

$z^8 = 1$

$z = 1$ is one solution

Another solution every $360° \div 8 = 45°$

Solutions are:

$z = 1\operatorname{cis} 0°$, $z = 1\operatorname{cis} 45°$, $z = 1\operatorname{cis} 90°$, $z = 1\operatorname{cis} 135°$

$z = 1\operatorname{cis} 180°$, $z = 1\operatorname{cis}(-135°)$, $z = 1\operatorname{cis}(-90°)$, $z = 1\operatorname{cis}(-45°)$

Question 3

$z^7 = 1$

$z = 1$ is one solution

Another solution every $2\pi \div 7 = \frac{2\pi}{7}$

Solutions are:

$z = 1\operatorname{cis} 0$, $z = 1\operatorname{cis}\frac{2\pi}{7}$, $z = 1\operatorname{cis}\frac{4\pi}{7}$, $z = 1\operatorname{cis}\frac{6\pi}{7}$

$z = 1\operatorname{cis}\left(-\frac{2\pi}{7}\right)$, $z = 1\operatorname{cis}\left(-\frac{4\pi}{7}\right)$, $z = 1\operatorname{cis}\left(-\frac{6\pi}{7}\right)$

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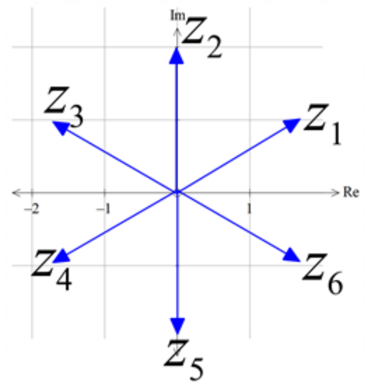
Question 4

$\left(\sqrt{3}+i\right)^6 = -64$

Placing $\sqrt{3}+i$ on an Argand diagram and dividing the complex plane into six equal size regions allows the six roots to be determined.



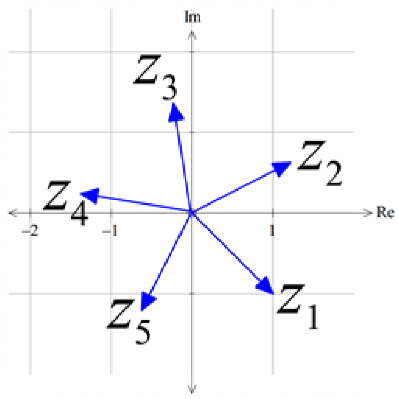
$$z_1 = 2\text{cis}\frac{\pi}{6}, \quad z_2 = 2\text{cis}\frac{\pi}{2}, \quad z_3 = 2\text{cis}\frac{5\pi}{6}$$

$$z_4 = 2\text{cis}\left(-\frac{5\pi}{6}\right), \quad z_5 = 2\text{cis}\left(-\frac{\pi}{2}\right), \quad z_6 = 2\text{cis}\left(-\frac{\pi}{6}\right)$$

Question 5

$(1-i)^5 = -4+4i$

Placing $1-i$ on an Argand diagram and dividing the complex plane into five equal size regions allows the six roots to be determined.



$$r = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\tan\theta = -\frac{1}{1}$$

$$\theta = -45°$$

$$z_1 = \sqrt{2}\,\text{cis}(-45°), \quad z_2 = \sqrt{2}\,\text{cis}\,27°, \quad z_3 = \sqrt{2}\,\text{cis}\,99°$$

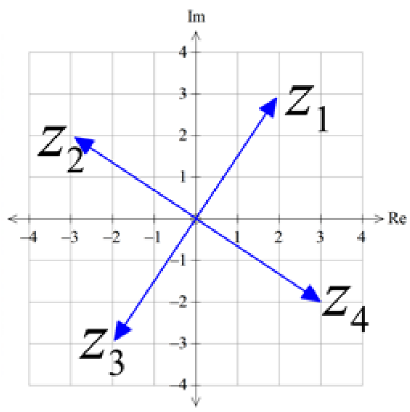
$$z_4 = \sqrt{2}\,\text{cis}\,171°, \quad z_5 = \sqrt{2}\,\text{cis}(-117°)$$

Question 6

$(2+3i)^4 = -119-120i$

$r = \sqrt{(-119)^2+(-120)^2} = 169$

Placing $2+3i$ on an Argand diagram and dividing the complex plane into four equal size regions allows the six roots to be determined.



$$r = \sqrt{2^2+3^2} = \sqrt{13}$$

$$\tan\theta = \frac{3}{2}$$

$$\theta = 56.3°$$

$$z_1 = 2+3i, \quad z_2 = -3+2i, \quad z_3 = -2-3i, \quad z_4 = 3-2i$$

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Question 7

a $(2+i)^2 = (2+i)(2+i)$

$= 4 + 2i + 2i + i^2$

$= 4 + 4i - 1$

$= 3 + 4i$

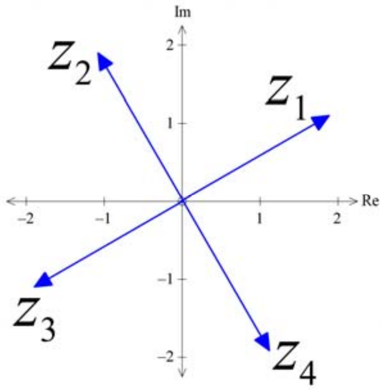
b $(2+i)^4 = [(2+i)^2]^2$

$= (3+4i)^2$

$= 9 + 12i + 12i + 16i^2$

$= -7 + 24i$

c



d $z = 2 + i$

$z = -1 + 2i$

$z = -2 - i$

$z = 1 - 2i$

Question 8

$360° \div 5 = 72°$

$k = 2^5 \operatorname{cis}(5 \times 20)° = 32 \operatorname{cis} 100°$

$z_1 = 2 \operatorname{cis} 20°$

$z_2 = 2 \operatorname{cis}(20° + 72°) = 2 \operatorname{cis} 92°$

$z_3 = 2 \operatorname{cis}(92° + 72°) = 2 \operatorname{cis} 164°$

$z_4 = 2 \operatorname{cis}(164° + 72°) = 2 \operatorname{cis} 236° = 2 \operatorname{cis}(20° - 2 \times 72°) = 2 \operatorname{cis}(-124°)$

$z_5 = 2 \operatorname{cis}(20° - 72°) = 2 \operatorname{cis}(-52°)$

Question 9

$z_1 = 2 + 4i$

$z_2 = -4 + 2i$

$z_3 = -2 - 4i$

$z_4 = 4 - 2i$

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Exercise 2F

Question 1

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

When $n = -1$

$$\text{LHS} = (\cos\theta + i\sin\theta)^{-1}$$
$$= \frac{1}{(\cos\theta + i\sin\theta)} = \frac{1}{(\cos\theta + i\sin\theta)} \times \frac{(\cos\theta - i\sin\theta)}{(\cos\theta - i\sin\theta)}$$
$$= \frac{\cos\theta - i\sin\theta}{\cos^2\theta - i\sin\theta\cos\theta + i\sin\theta\cos\theta - i^2\sin^2\theta} = \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta}$$
$$= \cos\theta - i\sin\theta$$
$$= \cos(-\theta) + i\sin(-\theta)\ [\text{as } \cos(-\theta) = \cos\theta \text{ and } \sin(-\theta) = -\sin\theta]$$
$$= \text{RHS}$$

Question 2

$$z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$$
$$z^4 = \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^4 = \cos\frac{4\pi}{6} + i\sin\frac{4\pi}{6} \text{ (by de Moivre's Theorem)}$$
$$= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$

Question 3

$$z = 2\text{cis}\frac{\pi}{6}$$
$$z^5 = 2^5\text{cis}\frac{5\pi}{6} = 32\text{cis}\frac{5\pi}{6} \text{ (by de Moivre's Theorem)}$$

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Question 4

$$z = 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$z^5 = \left[3\left((\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^5 = 3^5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^5 \text{ (by de Moivre's Theorem)}$$

$$= 243\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = 243\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$$

Question 5

$$\cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)^2 \text{ (by de Moivre's Theorem)}$$

$$= (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)$$

$$= \cos^2\theta + i\sin\theta\cos\theta + i\sin\theta\cos\theta + i^2\sin^2\theta$$

$$= \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta$$

Real parts are equal so $\cos 2\theta = \cos^2\theta - \sin^2\theta$

Imaginary parts are equal so $\sin 2\theta = 2\sin\theta\cos\theta$

Question 6

$$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3 \text{ (by de Moivre's Theorem)}$$

$$= i^3\sin^3\theta + 3i^2\sin^2\theta\cos\theta + 3i\sin\theta\cos^2\theta + \cos^3\theta$$

$$= -i\sin^3\theta - 3\sin^2\theta\cos\theta + 3i\sin\theta\cos^2\theta + \cos^3\theta$$

$$= -3\sin^2\theta\cos\theta + \cos^3\theta - i\sin^3\theta + 3i\sin\theta\cos^2\theta$$

Real parts are equal so $\cos 3\theta = -3\sin^2\theta\cos\theta + \cos^3\theta$

$$= \cos\theta\left(-3\sin^2\theta - 3\cos^2\theta + 4\cos^2\theta\right)$$

$$= \cos\theta\left(-3 + 4\cos^2\theta\right)$$

$$= 4\cos^3\theta - 3\cos\theta$$

Imaginary parts are equal so $\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta$

Question 7

$$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5 \text{ (by de Moivre's Theorem)}$$

$$= i^5\sin^5\theta + 5i^4\sin^4\theta\cos\theta + 10i^3\sin^3\theta\cos^2\theta + 10i^2\sin^2\theta\cos^3\theta + 5i\sin\theta\cos^4\theta + \cos^5\theta$$

$$= i\sin^5\theta + 5\sin^4\theta\cos\theta - 10i\sin^3\theta\cos^2\theta - 10\sin^2\theta\cos^3\theta + 5i\sin\theta\cos^4\theta + \cos^5\theta$$

Real parts are equal so $\cos 5\theta = 5\sin^4\theta\cos\theta - 10\sin^2\theta\cos^3\theta + \cos^5\theta$

Imaginary parts are equal so $\sin 5\theta = \sin^5\theta - 10\sin^3\theta\cos^2\theta + 5\sin\theta\cos^4\theta$

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Question 8

Change $1+i$ to polar form

$r=\sqrt{1^2+1^2}=\sqrt{2}$

$\tan\theta=1$

$\theta=\frac{\pi}{4}$

$1+i=\sqrt{2}\,\text{cis}\frac{\pi}{4}$

$\left(\sqrt{2}\,\text{cis}\frac{\pi}{4}\right)^6=8\,\text{cis}\frac{6\pi}{4}=8\,\text{cis}\frac{3\pi}{2}=8\,\text{cis}\left(-\frac{\pi}{2}\right)$

Question 9

Change $\sqrt{3}+i$ to polar form

$r=\sqrt{\sqrt{3}^2+1^2}=\sqrt{10}$

$\tan\theta=\frac{1}{\sqrt{3}}$

$\theta=\frac{\pi}{6}$

$\sqrt{3}+i=2\text{cis}\frac{\pi}{6}$

$\left(2\text{cis}\frac{\pi}{6}\right)^5=32\text{cis}\frac{5\pi}{6}$

Question 10

Change $-3+3\sqrt{3}\,i$ to polar form

$r=\sqrt{(-3)^2+(3\sqrt{3})^2}=\sqrt{36}=6$

$\tan\theta=-\frac{3\sqrt{3}}{3}$

$\theta=-\frac{\pi}{3}$

$-3+3\sqrt{3}\,i=6\,\text{cis}\left(-\frac{\pi}{3}\right)=6\,\text{cis}\left(\frac{2\pi}{3}\right)$

$\left(6\,\text{cis}\frac{2\pi}{3}\right)^4=6^4\,\text{cis}\frac{8\pi}{3}=6^4\,\text{cis}\frac{2\pi}{3}$

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Question 11

$4-4\sqrt{3}\,i=8\operatorname{cis}\left(-\frac{\pi}{3}\right)$

$z^3=8\operatorname{cis}\left(-\frac{\pi}{3}+2k\pi\right)$

$z=\sqrt[3]{8}\operatorname{cis}\left(-\frac{\pi}{9}+\frac{2k\pi}{3}\right)$

Solutions occur at $k=0,\ k=1,\ k=2$

$z_1=2\operatorname{cis}\left(-\frac{\pi}{9}\right),\quad z_2=2\operatorname{cis}\left(\frac{5\pi}{9}\right),\quad z_3=2\operatorname{cis}\left(-\frac{7\pi}{9}\right)$

Question 12

$z^4=16i$

$z^4=16\operatorname{cis}\left(\frac{\pi}{2}+2k\pi\right)$

$z=\sqrt[4]{16}\operatorname{cis}\left(\frac{\pi}{8}+\frac{2k\pi}{4}\right)=2\operatorname{cis}\left(\frac{\pi}{8}+\frac{k\pi}{2}\right)$

Solutions occur at $k=0, k=1, k=2, k=3$

$z_1=2\operatorname{cis}\frac{\pi}{8},\quad z_2=2\operatorname{cis}\frac{5\pi}{8},\quad z_3=2\operatorname{cis}\left(-\frac{7\pi}{8}\right),\quad z_4=2\operatorname{cis}\left(-\frac{3\pi}{8}\right)$

Question 13

$z^4=-8\sqrt{2}+8\sqrt{2}\,i$

$z^4=16\operatorname{cis}\left(\frac{3\pi}{4}+2k\pi\right)$

$z=\sqrt[4]{16}\operatorname{cis}\left(\frac{3\pi}{16}+\frac{2k\pi}{4}\right)=2\operatorname{cis}\left(\frac{3\pi}{16}+\frac{k\pi}{2}\right)$

Solutions occur at $k=0, k=1, k=2, k=3$

$z_1=2\operatorname{cis}\frac{3\pi}{16},\quad z_2=2\operatorname{cis}\frac{11\pi}{16},\quad z_3=2\operatorname{cis}\left(-\frac{13\pi}{16}\right),\quad z_4=2\operatorname{cis}\left(-\frac{5\pi}{16}\right)$

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Question 14

$z^4 + 4 = 0$

$z^4 = -4$

$z^4 = 4\text{cis}(\pi + 2k\pi)$

$z = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4} + \frac{2k\pi}{4}\right) = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)$

Solutions occur at $k = 0, k = 1, k = 2, k = 3$

$z_1 = \sqrt{2}\,\text{cis}\frac{\pi}{4}, \quad z_2 = \sqrt{2}\,\text{cis}\frac{3\pi}{4}, \quad z_3 = \sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right), \quad z_4 = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

Question 15

$z_1 = \frac{\sqrt{2} + \sqrt{6}\,i}{2} = \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i = \sqrt{2}\,\text{cis}\frac{\pi}{3}$

$z_2 = \frac{\sqrt{6} + \sqrt{2}\,i}{2} = \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i = \sqrt{2}\,\text{cis}\frac{\pi}{6}$

$z_3 = 2\text{cis}\frac{\pi}{8}$

$$\frac{z_1^6\, z_2^3}{z_3^4} = \frac{\sqrt{2}^6\,\text{cis}\,2\pi \times \sqrt{2}^3\,\text{cis}\frac{\pi}{2}}{2^4\,\text{cis}\frac{\pi}{2}} = \sqrt{2}\,\text{cis}\,2\pi = \sqrt{2}$$

Question 16

$z = r\,\text{cis}\,\theta$ so $\overline{z} = r\,\text{cis}(-\theta)$

a $\quad -\overline{z} = -r\,\text{cis}(-\theta) = r\,\text{cis}(\pi - \theta)$

b $\quad \frac{1}{z} = \frac{1}{r\,\text{cis}\,\theta} = \frac{1}{r}\text{cis}(-\theta)$

c $\quad -\frac{1}{z} = -\frac{1}{r}\text{cis}(-\theta) = \frac{1}{r}\text{cis}(\pi - \theta)$

d $\quad -\frac{1}{z^2} = -\frac{1}{r^2}\text{cis}(-2\theta) = \frac{1}{r^2}\text{cis}(\pi - 2\theta)$

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Miscellaneous Exercise 2

Question 1

$z = 3 - 4i$ and $w = 2 + 3i$

a $z + w = 3 - 4i + 2 + 3i = 5 - i$

b $z - w = 3 - 4i - (2 + 3i) = 1 - 7i$

c $zw = (3 - 4i)(2 + 3i) = 6 + 9i - 8i - 12i^2 = 18 + i$

d $z^2 = (3 - 4i)(3 - 4i) = 9 - 12i - 12i + 16i^2 = 9 - 24i - 16 = -7 - 24i$

e $\frac{z}{w} = \frac{3 - 4i}{2 + 3i} = \frac{3 - 4i}{2 + 3i} \times \frac{2 - 3i}{2 - 3i} = \frac{6 - 9i - 8i + 12i^2}{4 - 6i + 6i - 9i^2} = \frac{-6 - 17i}{13} = -\frac{6}{13} - \frac{17}{13}i$

f $\frac{w}{z} = \frac{2 + 3i}{3 - 4i} \times \frac{3 + 4i}{3 + 4i} = \frac{6 + 8i + 9i + 12i^2}{9 + 12i - 12i - 16i^2} = \frac{-6 + 17i}{25} = -\frac{6}{25} + \frac{17}{25}i$

Question 2

a $\overrightarrow{AB} = \mathbf{c}$

b $\overrightarrow{AD} = \frac{1}{4}\mathbf{c}$

c $\overrightarrow{DB} = \frac{3}{4}\mathbf{c}$

d $\overrightarrow{DE} = \frac{3}{4}\mathbf{c} + \frac{1}{2}\mathbf{c} = \frac{5}{4}\mathbf{c}$

e $\overrightarrow{OB} = \mathbf{a} + \mathbf{c}$

f $\overrightarrow{OD} = \mathbf{a} + \frac{1}{4}\mathbf{c}$

g $\overrightarrow{CE} = \mathbf{a} + \frac{1}{2}\mathbf{c}$

h $\overrightarrow{OE} = \mathbf{a} + \frac{3}{2}\mathbf{c}$

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Question 3

a $r = \sqrt{(-3)^2 + (-3\sqrt{3})^2} = \sqrt{9+27} = \sqrt{36} = 6$

$\tan\theta = \dfrac{-3\sqrt{3}}{-3} = \sqrt{3}$

$\theta = \dfrac{\pi}{3}$

$-3 - 3\sqrt{3}\,i = 6\,\text{cis}\left(-\dfrac{2\pi}{3}\right)$

b $8\cos\left(\dfrac{-5\pi}{6}\right) = -4\sqrt{3}$

$8\sin\left(\dfrac{-5\pi}{6}\right) = -4$

$8\,\text{cis}\dfrac{-5\pi}{6} = -4\sqrt{3} - 4i$

Question 4

a $2\cos\dfrac{\pi}{2} = 0$

$2\sin\dfrac{\pi}{2} = 2$

$2\,\text{cis}\dfrac{\pi}{2} = (0, 2)$

b $5\cos\pi = -5$

$5\sin\pi = 0$

$5\text{cis}\,\pi = (-5, 0)$

c $4\cos\left(\dfrac{-3\pi}{4}\right) = -2\sqrt{2}$

$4\sin\left(\dfrac{-3\pi}{4}\right) = -2\sqrt{2}$

$4\text{cis}\left(\dfrac{-3\pi}{4}\right) = (-2\sqrt{2}, -2\sqrt{2})$

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Question 5

$z = 1 + i$ and $w = -1 + i$

For z:

$r = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\tan\theta = \frac{1}{1}, \ \theta = \frac{\pi}{4}$

$z = \sqrt{2}\,\text{cis}\frac{\pi}{4}$

For w:

$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\tan\theta = \frac{1}{-1}, \ \theta = \frac{3\pi}{4}$

$w = \sqrt{2}\text{cis}\frac{3\pi}{4}$

$zw = \sqrt{2}\text{cis}\frac{\pi}{4} \times \sqrt{2}\text{cis}\frac{3\pi}{4} = 2\text{cis}\frac{\pi}{4}\text{cis}\frac{3\pi}{4} = 2\text{cis}\pi$

$$\frac{z}{w} = \frac{\sqrt{2}\text{cis}\frac{\pi}{4}}{\sqrt{2}\text{cis}\frac{3\pi}{4}} = \frac{\text{cis}\frac{\pi}{4}}{\text{cis}\frac{3\pi}{4}} = \text{cis}\left(-\frac{\pi}{2}\right)$$

Question 6

a $\text{cis}\,0 = \cos 0 + i\sin 0 = 1 + i \times 0 = 1$

b
$$\begin{aligned}\text{cis}\,\alpha \times \text{cis}\,\beta &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\ &= \cos\alpha\cos\beta + i\sin\beta\cos\alpha + i\sin\alpha\cos\beta + i^2\sin\alpha\sin\beta \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\beta\cos\alpha + \sin\alpha\cos\beta) \\ &= \cos(\alpha+\beta) + i\sin(\alpha+\beta) \\ &= \text{cis}(\alpha+\beta)\end{aligned}$$

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Question 7

$f(x)=4x^3-18x^2+22x-12$

a $f(-3)=4(-3)^3-18(-3)^2+22(-3)-12=-108-54-66-12=-348$

b $f(3)=4\times 3^3-18\times 3^2+22(-3)-12=0$

c 3 is a solution so $x-3$ is a factor of $4x^3-18x^2+22x-12$.

$$
\begin{array}{r}
4x^2-6x+4 \\
(x-3)\overline{\smash{)}4x^3-18x^2+22x-12} \\
\underline{4x^3-12x^2} \\
-6x^2+22x \\
\underline{-6x^2+18x} \\
4x-12 \\
\underline{4x-12} \\
0
\end{array}
$$

$4x^2-6x+4$

$a=4,\ b=-6,\ c=4$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{6\pm\sqrt{(-6)^2-4\times4\times4}}{2\times4}=\frac{6\pm\sqrt{36-64}}{8}$$

$$=\frac{6\pm\sqrt{-28}}{8}=\frac{6\pm2\sqrt{7}\,i}{8}=\frac{3\pm\sqrt{7}\,i}{4}=\frac{3}{4}\pm\frac{\sqrt{7}}{4}i$$

$$
\begin{aligned}
f(x)&=4x^3-18x^2+22x-12\\
&=(x-3)\left(x-\frac{3}{4}-\frac{\sqrt{7}}{4}i\right)\left(x-\frac{3}{4}+\frac{\sqrt{7}}{4}i\right)
\end{aligned}
$$

So when $f(x)=0$, $x=3,\ \frac{3}{4}+\frac{\sqrt{7}}{4}i,\ \frac{3}{4}-\frac{\sqrt{7}}{4}i$

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