SADLER UNIT 3 MATHEMATICS SPECIALIST

WORKED SOLUTIONS

Chapter 4: Vector equation of a line in the *x*-*y* plane

Exercise 4A

Question 1

At t hours after 8 a.m.:

$\mathbf{r}_A(t) = (5\mathbf{i}+4\mathbf{j}) + (10\mathbf{i}-\mathbf{j})t = (5+10t)\mathbf{i} + (4-t)\mathbf{j}$

$\mathbf{r}_B(t) = (6\mathbf{i}-8\mathbf{j}) + (2\mathbf{i}+8\mathbf{j})t = (6+2t)\mathbf{i} + (8t-8)\mathbf{j}$

$\mathbf{r}_C(t) = (2\mathbf{i}+3\mathbf{j}) + (-4\mathbf{i}+3\mathbf{j})t = (2-4t)\mathbf{i} + (3+3t)\mathbf{j}$

Body D has position vector $(9+10)\mathbf{i}+(-10+6)\mathbf{j}$ at 8 a.m.

$\mathbf{r}_D(t) = (19\mathbf{i}-4\mathbf{j}) + (10\mathbf{i}+6\mathbf{j})t = (19+10t)\mathbf{i} + (6t-4)\mathbf{j}$

Body E has position vector $[16-(-4)]\mathbf{i}+(7-3)\mathbf{j}$ at 8 a.m.

$\mathbf{r}_E(t) = (20\mathbf{i}+4\mathbf{j}) + (-4\mathbf{i}+3\mathbf{j})t = (20-4t)\mathbf{i} + (4+3t)\mathbf{j}$, $t \geq 1$.

Body F has position vector $[2-(12\div 2)]\mathbf{i}+[3-(-8\div 2)]\mathbf{j}$ at 8 a.m.

$\mathbf{r}_F(t) = (-4\mathbf{i}+7\mathbf{j}) + (12\mathbf{i}-8\mathbf{j})t = (12t-4)\mathbf{i} + (7-8t)\mathbf{j}$, $t \geq 0.5$.

Question 2

a 6 a.m. is one hour after 5 a.m.

$$\begin{aligned}\mathbf{r}(t) &= (7\mathbf{i}+10\mathbf{j})+(3\mathbf{i}+4\mathbf{j})t \\ &= (7+3t)\mathbf{i}+(10+4t)\mathbf{j}\end{aligned}$$

$$\mathbf{r}(1) = (10\mathbf{i}+14\mathbf{j})\text{ km}$$

b 7 a.m. is two hours after 5 a.m.

$$\mathbf{r}(t) = (7+3t)\mathbf{i}+(10+4t)\mathbf{j}$$

$$\mathbf{r}(2) = (13\mathbf{i}+18\mathbf{j})\text{ km}$$

c 9 a.m. is four hours after 5 a.m.

$$\mathbf{r}(t) = (7+3t)\mathbf{i}+(10+4t)\mathbf{j}$$

$$\mathbf{r}(4) = (19\mathbf{i}+26\mathbf{j})\text{ km}$$

d Speed $= \sqrt{3^2+4^2} = 5\text{ km/h}$

e 8 a.m. is three hours after 5 a.m.

$$\mathbf{r}(t) = (7+3t)\mathbf{i}+(10+4t)\mathbf{j}$$

$$\mathbf{r}(3) = 16\mathbf{i}+22\mathbf{j}$$

Distance from $16\mathbf{i}+22\mathbf{j}$ to $21\mathbf{i}+20\mathbf{j}$ is $\sqrt{(21-16)^2+(20-22)^2} = \sqrt{29}$ km

Question 3

$9\mathbf{i}+36\mathbf{j}-(2\mathbf{i}+12\mathbf{j}) = (7\mathbf{i}+24\mathbf{j})\text{ km}$

$7\mathbf{i}+24\mathbf{j}$ was the position vector at 9 a.m.

a At 9 a.m. the boat was $\sqrt{7^2+24^2} = 25$ km from O.

b At 8 a.m. the position vector of the boat was $9\mathbf{i}+36\mathbf{j}-2(2\mathbf{i}+12\mathbf{j}) = 5\mathbf{i}+12\mathbf{j}$

At 8 a.m. the boat was $\sqrt{5^2+12^2} = 13$ km from O.

Question 4

a At 3 p.m. the distance between the boats is $|(25\mathbf{i}-6\mathbf{j})-(21\mathbf{i}+7\mathbf{j})|=\sqrt{4^2+(-13)^2}=\sqrt{185}$ km

b At 4 p.m. $t=1$

$\mathbf{r}_A(1)=(21\mathbf{i}+7\mathbf{j})+(10\mathbf{i}+5\mathbf{j})1=31\mathbf{i}+12\mathbf{j}$

$\mathbf{r}_B(1)=(25\mathbf{i}-6\mathbf{j})+(7\mathbf{i}+10\mathbf{j})1=32\mathbf{i}+4\mathbf{j}$

$|\mathbf{i}-8\mathbf{j}|=\sqrt{1+64}=\sqrt{65}$ km

At 4 p.m. the distance between the boats is $\sqrt{65}$ km.

c At 5 p.m. $t=2$

$\mathbf{r}_A(2)=(21\mathbf{i}+7\mathbf{j})+(10\mathbf{i}+5\mathbf{j})2=41\mathbf{i}+17\mathbf{j}$

$\mathbf{r}_B(1)=(25\mathbf{i}-6\mathbf{j})+(7\mathbf{i}+10\mathbf{j})2=39\mathbf{i}+14\mathbf{j}$

$|-2\mathbf{i}-3\mathbf{j}|=\sqrt{(-2)^2+(-3)^2}=\sqrt{13}$ km

Question 5

a At 9 a.m. $t=1$

$\mathbf{r}_A(1)=(-5\mathbf{i}+13\mathbf{j})+(7\mathbf{i}-2\mathbf{j})1=2\mathbf{i}+11\mathbf{j}$

$\mathbf{r}_B(1)=(-3\mathbf{j})+(-3\mathbf{i}+2\mathbf{j})1=-3\mathbf{i}-1\mathbf{j}$

$|-5\mathbf{i}-12\mathbf{j}|=\sqrt{(-5)^2+(-12)^2}=\sqrt{169}=13$ km

At 4 p.m. the distance between the boats is 13 km.

b At 10 a.m. $t=2$

$\mathbf{r}_A(2)=(-5\mathbf{i}+13\mathbf{j})+(7\mathbf{i}-2\mathbf{j})2=9\mathbf{i}+9\mathbf{j}$

$\mathbf{r}_B(1)=(-3\mathbf{j})+(-3\mathbf{i}+2\mathbf{j})2=-6\mathbf{i}+\mathbf{j}$

$|-15\mathbf{i}-8\mathbf{j}|=\sqrt{(-15)^2+(-8)^2}=\sqrt{289}=17$ km

At 5 p.m. the distance between the boats is 17 km.

Question 6

a At t hours after 8 a.m.:

$$\mathbf{r}_A(t) = (28\mathbf{i} - 5\mathbf{j}) + (-8\mathbf{i} + 4\mathbf{j})t = (28 - 8t)\mathbf{i} + (4t - 5)\mathbf{j}$$

$$\mathbf{r}_B(t) = (24\mathbf{j}) + (6\mathbf{i} + 2\mathbf{j})t = (6t)\mathbf{i} + (24 + 2t)\mathbf{j}$$

b $\sqrt{(14t - 28)^2 + (-2t + 29)^2} = 25$

$t = 2, 2.5$

The ships will be 25 km apart at 10 a.m. and 10:30 a.m.

Question 7

$$\mathbf{r}_A(t) = (12\mathbf{i} + 61\mathbf{j}) + (7\mathbf{i} - 8\mathbf{j})t = (12 + 7t)\mathbf{i} + (61 - 8t)\mathbf{j}$$

$$\mathbf{r}_B(t) = (57\mathbf{i} - 29\mathbf{j}) + (-2\mathbf{i} + 10\mathbf{j})t = (57 - 2t)\mathbf{i} + (-29 + 10t)\mathbf{j}$$

Equating the parts:

$$\begin{aligned} 12 + 7t &= 57 - 2t \\ 9t &= 45 \\ t &= 5 \end{aligned}$$

$$\begin{aligned} 61 - 8t &= -29 + 10t \\ 18t &= 90 \\ t &= 5 \end{aligned}$$

$$\begin{aligned} \mathbf{r} &= (12 + 7 \times 5)\mathbf{i} + (61 - 8 \times 5)\mathbf{j} \\ &= (47\mathbf{i} + 21\mathbf{j})\text{ km} \end{aligned}$$

The $\mathbf{i}$ components and $\mathbf{j}$ components are the same position at the same time for both ships, therefore the ships collide after 5 hours at 1 p.m.

Question 8

$\mathbf{r}_A(t) = (-11\mathbf{i} - 8\mathbf{j}) + (7\mathbf{i} - 1\mathbf{j})t = (-11 + 7t)\mathbf{i} + (-8 - t)\mathbf{j}$

$\mathbf{r}_B(t) = (-2\mathbf{i} - 4\mathbf{j}) + (4\mathbf{i} + 5\mathbf{j})t = (-2 + 4t)\mathbf{i} + (-4 + 5t)\mathbf{j}$

Equating the parts:

$$\begin{aligned} -11 + 7t &= -2 + 4t \\ 3t &= 9 \\ t &= 3 \end{aligned}$$

$$\begin{aligned} -8 - t &= -4 + 5t \\ 6t &= -4 \\ t &= -\frac{2}{3} \end{aligned}$$

The $\mathbf{i}$ components and $\mathbf{j}$ components are not the same at the same time for both ships, therefore the ships will not collide.

Question 9

At 8 a.m.:

$\mathbf{r}_A(t) = (24\mathbf{i} - 25\mathbf{j}) + (-3\mathbf{i} + 4\mathbf{j})t = (-11 + 7t)\mathbf{i} + (-8 - t)\mathbf{j}$

At 9 a.m.:

$\mathbf{r}_B(t) = (-9\mathbf{i} + 33\mathbf{j}) + (2\mathbf{i} - 5\mathbf{j})t = (-9 + 2t)\mathbf{i} + (33 - 5t)\mathbf{j}$

At t hours past 8 a.m.:

$$\mathbf{r}_A(t) = \begin{pmatrix} 24 \\ -25 \end{pmatrix} + t\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 24 - 3t \\ -25 + 4t \end{pmatrix}$$

At t hours past 9 a.m.:

$$\begin{aligned} \mathbf{r}_B(t) &= \begin{pmatrix} -9 \\ 33 \end{pmatrix} + (t-1)\begin{pmatrix} 2 \\ -5 \end{pmatrix} \\ &= \begin{pmatrix} -9 + 2t - 2 \\ 33 - 5t + 5 \end{pmatrix} \\ &= \begin{pmatrix} -11 + 2t \\ 38 - 5t \end{pmatrix} \end{aligned}$$

Position vectors of A and B will have the same $\mathbf{i}$ component when $24 - 3t = -11 + 2t$

i.e. when $t = 7$.

Position vectors of A and B will have the same $\mathbf{j}$ component when $-25 + 4t = 38 - 5t$

i.e. when $t = 7$.

A and B will collide at 3 p.m. at position vector $(3\mathbf{i} + 3\mathbf{j})$ km.

Question 10

At 9:30 a.m.:

$\mathbf{r}_A(t) = (-6\mathbf{i} + 44\mathbf{j}) + (4\mathbf{i} - 6\mathbf{j})t = (-6 + 4t)\mathbf{i} + (44 - 6t)\mathbf{j}$

At 9 a.m.:

$\mathbf{r}_B(t) = (2\mathbf{i} - 18\mathbf{j}) + (2\mathbf{i} + 7\mathbf{j})t = (2 + 2t)\mathbf{i} + (-18 + 7t)\mathbf{j}$

At t hours past 9 a.m.:

$$\begin{aligned}\mathbf{r}_A(t) &= \begin{pmatrix}-6\\44\end{pmatrix} + (t - 0.5)\begin{pmatrix}4\\-6\end{pmatrix}\\ &= \begin{pmatrix}-6 + 4t - 2\\44 - 6t + 3\end{pmatrix}\\ &= \begin{pmatrix}-8 + 4t\\47 - 6t\end{pmatrix}\end{aligned}$$

At t hours past 9 a.m.:

$$\begin{aligned}\mathbf{r}_B(t) &= \begin{pmatrix}2\\-18\end{pmatrix} + (t)\begin{pmatrix}2\\7\end{pmatrix}\\ &= \begin{pmatrix}2 + 2t\\-18 + 7t\end{pmatrix}\end{aligned}$$

Position vectors of A and B will have the same $\mathbf{i}$ component when $-8 + 4t = 2 + 2t$

i.e. when $t = 5$

Position vectors of A and B will have the same $\mathbf{j}$ component when $47 - 6t = -18 + 7t$

i.e. when $t = 5$

A and B will collide at 2 p.m. at position vector $(12\mathbf{i} + 17\mathbf{j})$ km.

Question 11

At noon:

$\mathbf{r}_A(t) = (-11\mathbf{i} + 4\mathbf{j}) + (10\mathbf{i} - 4\mathbf{j})t = (-11 + 10t)\mathbf{i} + (4 - 4t)\mathbf{j}$

At 12:30 p.m.:

$\mathbf{r}_B(t) = (3\mathbf{i} - 5\mathbf{j}) + (7\mathbf{i} + 5\mathbf{j})t = (3 + 7t)\mathbf{i} + (-5 - 5t)\mathbf{j}$

At t hours past noon:

$$\mathbf{r}_A(t) = \begin{pmatrix} -11 \\ 4 \end{pmatrix} + t\begin{pmatrix} 10 \\ -4 \end{pmatrix}$$
$$= \begin{pmatrix} -11 + 10t \\ 4 - 4t \end{pmatrix}$$

At t hours past 12:30 p.m.:

$$\mathbf{r}_B(t) = \begin{pmatrix} 3 \\ -5 \end{pmatrix} + (t - 0.5)\begin{pmatrix} 7 \\ 5 \end{pmatrix}$$
$$= \begin{pmatrix} 3 + 7t - 3.5 \\ -5 + 5t - 2.5 \end{pmatrix}$$
$$= \begin{pmatrix} -0.5 + 7t \\ -7.5 + 5t \end{pmatrix}$$

Position vectors of A and B will have the same $\mathbf{i}$ component when $-11 + 10t = -0.5 + 7t$

i.e. when $t = 3.5$

Position vectors of A and B will have the same $\mathbf{j}$ component when $4 - 4t = -7.5 + 5t$

i.e. when $t = 1.28$

The same $\mathbf{i}$ component of position vector occurs at 3:30 p.m., but the same $\mathbf{j}$ component occurs at 1:17 p.m. so A and B do not collide.

Question 12

a At 8 a.m.:

$\mathbf{r}_P(t) = (-23\mathbf{i} + 3\mathbf{j}) + (18\mathbf{i} + 4\mathbf{j})t = (-23 + 18t)\mathbf{i} + (3 + 4t)\mathbf{j}$

$\mathbf{r}_Q(t) = (7\mathbf{i} + 30\mathbf{j}) + (12\mathbf{i} - 10\mathbf{j})t = (7 + 12t)\mathbf{i} + (30 - 10t)\mathbf{j}$

$\mathbf{r}_R(t) = (32\mathbf{i} - 30\mathbf{j}) + (2\mathbf{i} + 14\mathbf{j})t = (32 + 2t)\mathbf{i} + (-30 + 14t)\mathbf{j}$

At t hours past 8 a.m.:

$$\mathbf{r}_P(t) = \begin{pmatrix} -23 \\ 3 \end{pmatrix} + t\begin{pmatrix} 18 \\ 4 \end{pmatrix} = \begin{pmatrix} -23 + 18t \\ 3 + 4t \end{pmatrix}$$

$$\mathbf{r}_Q(t) = \begin{pmatrix} 7 \\ 30 \end{pmatrix} + t\begin{pmatrix} 12 \\ -10 \end{pmatrix} = \begin{pmatrix} 7 + 12t \\ 30 - 10t \end{pmatrix}$$

$$\mathbf{r}_R(t) = \begin{pmatrix} 32 \\ -30 \end{pmatrix} + t\begin{pmatrix} 2 \\ 14 \end{pmatrix} = \begin{pmatrix} 32 + 2t \\ -30 + 14t \end{pmatrix}$$

Position vectors of P and Q will have the same $\mathbf{i}$ component when $-23 + 18t = 7 + 12t$
i.e. when $t = 5$

Position vectors of P and Q will have the same $\mathbf{j}$ component when $3 + 4t = 30 - 10t$
i.e. when $t = 1.92$

The same $\mathbf{i}$ component of the position vector occurs at 1:00 p.m. but the same $\mathbf{j}$ component occurs at 9:58 a.m., so P and Q do not collide.

Position vectors of *P* and *R* will have the same $\mathbf{i}$ component when $-23 + 18t = 32 + 2t$

i.e. when $t = 3.44$

Position vectors of P and R will have same $\mathbf{j}$ component when $3 + 4t = -30 + 14t$

i.e. when $t = 3.3$

The same $\mathbf{i}$ component of the position vector occurs at 11:26 a.m. but the same $\mathbf{j}$ component occurs at 11:18 a.m., so P and R do not collide.

Position vectors of Q and R will have the same $\mathbf{i}$ component when $7 + 12t = 32 + 2t$

i.e. when $t = 2.5$

Position vectors of Q and R will have the same $\mathbf{j}$ component when $30 - 10t = -30 + 14t$

i.e. when $t = 2.5$

Q and R will collide at 10:30 a.m. at position vector $(37\mathbf{i} + 5\mathbf{j})$ km.

When $t = 2.5$

$\mathbf{r}_P(2.5) = (-23\mathbf{i} + 3\mathbf{j}) + (18\mathbf{i} + 4\mathbf{j})2.5 = (-23 + 45)\mathbf{i} + (3 + 10)\mathbf{j} = 22\mathbf{i} + 13\mathbf{j}$

b The distance from the collision to boat P is $\sqrt{(37-22)^2 + (5-13)^2} = \sqrt{15^2 + (-8)^2} = 17$ km.

Exercise 4B

Question 1

The line through the point with position vector $\mathbf{a}$ and parallel to $\mathbf{b}$ has equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ so

$$\begin{aligned}\mathbf{r} &= 2\mathbf{i} + 3\mathbf{j} + \lambda(5\mathbf{i} - \mathbf{j}) \\ &= (2 + 5\lambda)\mathbf{i} + (3 - \lambda)\mathbf{j}\end{aligned}$$

Question 2

The line through the point with position vector $\mathbf{a}$ and parallel to $\mathbf{b}$ has equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ so

$$\begin{aligned}\mathbf{r} &= 3\mathbf{i} - 2\mathbf{j} + \lambda(\mathbf{i} + \mathbf{j}) \\ &= (3 + \lambda)\mathbf{i} + (\lambda - 2)\mathbf{j}\end{aligned}$$

Question 3

The line through the point with position vector $\mathbf{a}$ and parallel to $\mathbf{b}$ has equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ so

$$\begin{aligned}\mathbf{r} &= 5\mathbf{i} + 3\mathbf{j} + \lambda(0\mathbf{i} - 2\mathbf{j}) \\ &= 5\mathbf{i} + (3 - 2\lambda)\mathbf{j}\end{aligned}$$

Question 4

The line through the point with position vector $\mathbf{a}$ and parallel to $\mathbf{b}$ has equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ so

$$\begin{aligned}\mathbf{r} &= 0\mathbf{i} + 5\mathbf{j} + \lambda(3\mathbf{i} - 10\mathbf{j}) \\ &= 3\lambda\mathbf{i} + (5 - 10\lambda)\mathbf{j}\end{aligned}$$

Question 5

The line through the point with position vector $\mathbf{a}$ and parallel to $\mathbf{b}$ has equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ so

$$\mathbf{r} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 + \lambda \\ -3 + 4\lambda \end{pmatrix}$$

Question 6

The line through the point with position vector $\mathbf{a}$ and parallel to $\mathbf{b}$ has equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ so

$$\mathbf{r} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 5\lambda \\ 5 \end{pmatrix}$$

Question 7

The line that passes through point *A*, position vector $\mathbf{a}$, and point *B*, position vector $\mathbf{b}$ is parallel to $\overrightarrow{AB}$.

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} \\ &= -(5\mathbf{i} + 3\mathbf{j}) + 2\mathbf{i} - \mathbf{j} \\ &= -3\mathbf{i} - 4\mathbf{j} \end{aligned}$$

The line is parallel to $-3\mathbf{i} - 4\mathbf{j}$ and passes through point *A*, position vector $\mathbf{a}$. Thus the vector equation of the line is $\mathbf{r} = 5\mathbf{i} + 3\mathbf{j} + \lambda(-3\mathbf{i} - 4\mathbf{j})$.

i.e. $\mathbf{r} = (5 - 3\lambda)\mathbf{i} + (3 - 4\lambda)\mathbf{j}$

Question 8

The line that passes through point *A*, position vector $\mathbf{a}$, and point *B*, position vector $\mathbf{b}$ is parallel to $\overrightarrow{AB}$.

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} \\ &= -(6\mathbf{i} + 7\mathbf{j}) + (-5\mathbf{i} + 2\mathbf{j}) \\ &= -11\mathbf{i} - 5\mathbf{j} \end{aligned}$$

The line is parallel to $-11\mathbf{i} - 5\mathbf{j}$ and passes through point *A*, position vector $\mathbf{a}$. Thus the vector equation of the line is $\mathbf{r} = 6\mathbf{i} + 7\mathbf{j} + \lambda(-11\mathbf{i} - 5\mathbf{j})$.

i.e. $\mathbf{r} = (6 - 11\lambda)\mathbf{i} + (7 - 5\lambda)\mathbf{j}$

Question 9

The line that passes through point A, position vector $\mathbf{a}$, and point B, position vector $\mathbf{b}$ is parallel to $\overrightarrow{AB}$.

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} \\ &= -\begin{pmatrix} -6 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 8 \\ 1 \end{pmatrix}\end{aligned}$$

The line is parallel to $\begin{pmatrix} 8 \\ 1 \end{pmatrix}$ and passes through point A, position vector $\mathbf{a}$. Thus the vector equation of the line is $\mathbf{r} = \begin{pmatrix} -6 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ 1 \end{pmatrix}$.

i.e. $\mathbf{r} = \begin{pmatrix} -6 + 8\lambda \\ 3 + \lambda \end{pmatrix}$

Question 10

The line that passes through point A, position vector $\mathbf{a}$, and point B, position vector $\mathbf{b}$ is parallel to $\overrightarrow{AB}$.

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} \\ &= -\begin{pmatrix} 1 \\ -3 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -4 \\ 4 \end{pmatrix}\end{aligned}$$

The line is parallel to $\begin{pmatrix} -4 \\ 4 \end{pmatrix}$ and passes through point A, position vector $\mathbf{a}$. Thus the vector equation of the line is $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 4 \end{pmatrix}$.

i.e. $\mathbf{r} = \begin{pmatrix} 1 - 4\lambda \\ -3 + 4\lambda \end{pmatrix}$

Question 11

The line that passes through point A, position vector $\mathbf{a}$, and point B, position vector $\mathbf{b}$ is parallel to $\overrightarrow{AB}$.

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO}+\overrightarrow{OB}\\ &= -\overrightarrow{OA}+\overrightarrow{OB}\\ &= -\begin{pmatrix}1\\4\end{pmatrix}+\begin{pmatrix}-1\\9\end{pmatrix}\\ &= \begin{pmatrix}-2\\5\end{pmatrix}\end{aligned}$$

The line is parallel to $\begin{pmatrix}-2\\5\end{pmatrix}$ and passes through point A, position vector $\mathbf{a}$. Thus the vector equation of the line is $\mathbf{r}=\begin{pmatrix}1\\4\end{pmatrix}+\lambda\begin{pmatrix}-2\\5\end{pmatrix}$.

i.e. $\mathbf{r}=\begin{pmatrix}1-2\lambda\\4+5\lambda\end{pmatrix}$

Question 12

The line that passes through point A, position vector $\mathbf{a}$, and point B, position vector $\mathbf{b}$ is parallel to $\overrightarrow{AB}$.

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO}+\overrightarrow{OB}\\ &= -\overrightarrow{OA}+\overrightarrow{OB}\\ &= -\begin{pmatrix}5\\0\end{pmatrix}+\begin{pmatrix}-1\\-4\end{pmatrix}\\ &= \begin{pmatrix}-6\\-4\end{pmatrix}\end{aligned}$$

The line is parallel to $\begin{pmatrix}-6\\-4\end{pmatrix}$ and passes through point A, position vector $\mathbf{a}$. Thus the vector equation of the line is $\mathbf{r}=\begin{pmatrix}5\\0\end{pmatrix}+\lambda\begin{pmatrix}-6\\-4\end{pmatrix}$.

i.e. $\mathbf{r}=\begin{pmatrix}5-6\lambda\\-4\lambda\end{pmatrix}$

Question 13

$\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + (-1)(\mathbf{i} - 4\mathbf{j})$
$= \mathbf{i} + 7\mathbf{j}$

$\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + 1(\mathbf{i} - 4\mathbf{j})$
$= 3\mathbf{i} - \mathbf{j}$

$\mathbf{c} = 2\mathbf{i} + 3\mathbf{j} + 2(\mathbf{i} - 4\mathbf{j})$
$= 4\mathbf{i} - 5\mathbf{j}$

a $\overrightarrow{AB} = 2\mathbf{i} - 8\mathbf{j}$

b $\overrightarrow{BC} = \mathbf{i} - 4\mathbf{j}$

$\left|\overrightarrow{BC}\right| = \sqrt{17}$

c
$$\begin{aligned}\overrightarrow{AB} : \overrightarrow{BC} &= (2\mathbf{i} - 8\mathbf{j}) : (\mathbf{i} - 4\mathbf{j}) \\ &= 2(\mathbf{i} - 4\mathbf{j}) : (\mathbf{i} - 4\mathbf{j}) \\ &= 2 : 1\end{aligned}$$

Question 14

a The line passing through point A, position vector $5\mathbf{i} - \mathbf{j}$, and parallel to $7\mathbf{i} + 2\mathbf{j}$ has vector equation $\mathbf{r} = 5\mathbf{i} - \mathbf{j} + \lambda(7\mathbf{i} + 2\mathbf{j})$.

i.e. $\mathbf{r} = (5 + 7\lambda)\mathbf{i} + (2\lambda - 1)\mathbf{j}$

b For the line passing through point A, position vector $5\mathbf{i} - \mathbf{j}$, and parallel to $7\mathbf{i} + 2\mathbf{j}$

Considering the general point, position vector $\begin{pmatrix} x \\ y \end{pmatrix}$: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 + 7\lambda \\ -1 + 2\lambda \end{pmatrix}$

Thus the parametric equations are $\begin{cases} x = 5 + 7\lambda \\ y = 2\lambda - 1 \end{cases}$

c Eliminating λ from the parametric equations (as $\lambda = \dfrac{y+1}{2}$)

$$\begin{aligned} x &= 5 + 7\left(\frac{y+1}{2}\right) \\ x - 5 &= \frac{7y + 7}{2} \\ 2x - 10 &= 7y + 7 \\ 7y &= 2x - 17 \end{aligned}$$

Question 15

a

The line passing through point A, position vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$, and parallel to $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ has vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \end{pmatrix}$.

i.e. $\quad \mathbf{r} = \begin{pmatrix} 2-3\lambda \\ -1+4\lambda \end{pmatrix}$

b

For the line passing through point A, position vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$, and parallel to $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$

Considering the general point, position vector $\begin{pmatrix} x \\ y \end{pmatrix}$: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2-3\lambda \\ -1+4\lambda \end{pmatrix}$

Thus the parametric equations are $\begin{cases} x = 2-3\lambda \\ y = -1+4\lambda \end{cases}$

c

Eliminating λ from the parametric equations (as $\lambda = \dfrac{y+1}{4}$)

$$x = 2 - 3\left(\frac{y+1}{4}\right)$$

$$x - 2 = \frac{-3y-3}{4}$$

$$4x - 8 = -3y - 3$$
$$4x + 3y = 5$$

Question 16

a

The line passing through point A, position vector $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$, and parallel to $\begin{pmatrix} 7 \\ -8 \end{pmatrix}$ has vector equation $\mathbf{r} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ -8 \end{pmatrix} = \begin{pmatrix} 7\lambda \\ 3-8\lambda \end{pmatrix}$.

b

For the line passing through point A, position vector $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$, and parallel to $\begin{pmatrix} 7 \\ -8 \end{pmatrix}$

Considering the general point, position vector $\begin{pmatrix} x \\ y \end{pmatrix}$: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0+7\lambda \\ 3-8\lambda \end{pmatrix}$

Thus the parametric equations are $\begin{cases} x = 7\lambda \\ y = 3-8\lambda \end{cases}$

c

Eliminating λ from the parametric equations (as $\lambda = \frac{x}{7}$)

$$y = 3 - 8\left(\frac{x}{7}\right)$$
$$y = \frac{21-8x}{7}$$
$$7y = 21 - 8x$$
$$8x + 7y = 21$$

Question 17

Given the parametric equations $\begin{cases} x = 2-3\lambda \\ y = -5+2\lambda \end{cases}$

a

The vector equation is $\mathbf{r} = \begin{pmatrix} 2 \\ -5 \end{pmatrix} + \begin{pmatrix} -\lambda \\ 2\lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda \\ -5+2\lambda \end{pmatrix}$

b

From $y = -5 + 2\lambda$, $\lambda = \frac{y+5}{2}$

$$x = 2 - 3\lambda = 2 - 3\left(\frac{y+5}{2}\right)$$
$$x - 2 = \frac{-3y-15}{2}$$
$$2x - 4 = -3y - 15$$
$$3y = -2x - 11$$
$$2x + 3y + 11 = 0$$

Question 18

$\mathbf{r} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$D = \begin{pmatrix} 2 \\ -1 \end{pmatrix} - 1\begin{pmatrix} -1 \\ 3 \end{pmatrix} \Rightarrow$ Point D has position vector $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$

$E = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 3 \end{pmatrix} \Rightarrow$ Point E has position vector $\begin{pmatrix} 0 \\ 5 \end{pmatrix}$

$F = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 3 \end{pmatrix} \Rightarrow$ Point E has position vector $\begin{pmatrix} -1 \\ 8 \end{pmatrix}$

a $\overrightarrow{EF} = \begin{pmatrix} -1-0 \\ 8-5 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

b $\overrightarrow{ED} = \begin{pmatrix} 3-0 \\ -4-5 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix}$

c $\overrightarrow{DE} = \begin{pmatrix} 0-3 \\ 5-(-4) \end{pmatrix} = \begin{pmatrix} -3 \\ 9 \end{pmatrix}$

$\left|\overrightarrow{DE}\right| = \sqrt{3^2 + (-9)^2} = \sqrt{90} = 3\sqrt{10}$

d $\overrightarrow{DE} : \overrightarrow{EF} = \begin{pmatrix} -3 \\ 9 \end{pmatrix} : \begin{pmatrix} -1 \\ 3 \end{pmatrix} = 3:1$

e $\overrightarrow{DE} : \overrightarrow{FE} = \begin{pmatrix} -3 \\ 9 \end{pmatrix} : \begin{pmatrix} 1 \\ -3 \end{pmatrix} = 3:-1$

f $\left|\overrightarrow{FE}\right| = \sqrt{1^2 + (-3)^2} = \sqrt{10}$

$$\begin{aligned} \left|\overrightarrow{DE}\right| : \left|\overrightarrow{FE}\right| &= 3\sqrt{10} : \sqrt{10} \\ &= 3:1 \end{aligned}$$

Question 19

The line passing through point *A*, position vector $7\mathbf{i}-2\mathbf{j}$, and parallel to $-2\mathbf{i}+6\mathbf{j}$ has vector equation $\mathbf{r}=7\mathbf{i}-2\mathbf{j}+\lambda(-2\mathbf{i}+6\mathbf{j})$.

i.e. $\mathbf{r}=(7-2\lambda)\mathbf{i}+(6\lambda-2)\mathbf{j}$

If *B*, position vector $\mathbf{i}+16\mathbf{j}$, lies on $\mathbf{r}=(7-2\lambda)\mathbf{i}+(6\lambda-2)\mathbf{j}$ there must exist some λ for which

$$\mathbf{i}+16\mathbf{j}=(7-2\lambda)\mathbf{i}+(6\lambda-2)\mathbf{j}$$

i.e. $1=7-2\lambda$ and $16=6\lambda-2$

$\lambda=3$ $\lambda=3$

Thus a suitable value of λ does exist.

Point *B* lies on $\mathbf{r}=(7-2\lambda)\mathbf{i}+(6\lambda-2)\mathbf{j}$.

If *C*, position vector $2\mathbf{i}+13\mathbf{j}$, lies on $\mathbf{r}=(7-2\lambda)\mathbf{i}+(6\lambda-2)\mathbf{j}$, there must exist some λ for which

$2\mathbf{i}+13\mathbf{j}=(7-2\lambda)\mathbf{i}+(6\lambda-2)\mathbf{j}$

i.e. $2=7-2\lambda$ and $13=6\lambda-2$

$\lambda=\frac{5}{2}$ $\lambda=\frac{15}{6}=\frac{5}{2}$

Thus a suitable value of λ exists.

Point *C* lies on $\mathbf{r}=(7-2\lambda)\mathbf{i}+(6\lambda-2)\mathbf{j}$.

If *D*, position vector $8\mathbf{i}-7\mathbf{j}$, lies on $\mathbf{r}=(7-2\lambda)\mathbf{i}+(6\lambda-2)\mathbf{j}$, there must exist some λ for which

$8\mathbf{i}-7\mathbf{j}=(7-2\lambda)\mathbf{i}+(6\lambda-2)\mathbf{j}$

i.e. $8=7-2\lambda$ and $-7=6\lambda-2$

$\lambda=-\frac{1}{2}$ $\lambda=\frac{-5}{6}$

Point *D* does <u>not</u> lie on $\mathbf{r}=(7-2\lambda)\mathbf{i}+(6\lambda-2)\mathbf{j}$.

If *E*, position vector $-2\mathbf{i}+5\mathbf{j}$, lies on $\mathbf{r}=(7-2\lambda)\mathbf{i}+(6\lambda-2)\mathbf{j}$, there must exist some λ for which

$-2\mathbf{i}+5\mathbf{j}=(7-2\lambda)\mathbf{i}+(6\lambda-2)\mathbf{j}$

i.e. $-2=7-2\lambda$ and $5=6\lambda-2$

$\lambda=\frac{9}{2}$ $\lambda=\frac{7}{6}$

Point *E* does <u>not</u> lie on $\mathbf{r}=(7-2\lambda)\mathbf{i}+(6\lambda-2)\mathbf{j}$.

Question 20

The line passing through point *F*, position vector $\begin{pmatrix} 4 \\ -9 \end{pmatrix}$, and parallel to $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ has vector equation

$\mathbf{r} = \begin{pmatrix} 4 \\ -9 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4-\lambda \\ -9+2\lambda \end{pmatrix}$.

If *G*, position vector $\begin{pmatrix} 5 \\ 9 \end{pmatrix}$, lies on $\mathbf{r} = \begin{pmatrix} 4-\lambda \\ -9+2\lambda \end{pmatrix}$, there must exist some λ for which $\begin{pmatrix} 5 \\ 9 \end{pmatrix} = \begin{pmatrix} 4-\lambda \\ -9+2\lambda \end{pmatrix}$,

i.e. $5 = 4-\lambda$ and $9 = -9+2\lambda$

$\lambda = -1$ $\lambda = 9$

Point *G* does <u>not</u> lie on $\mathbf{r} = \begin{pmatrix} 4-\lambda \\ -9+2\lambda \end{pmatrix}$.

If *H*, position vector $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, lies on $\mathbf{r} = \begin{pmatrix} 4-\lambda \\ -9+2\lambda \end{pmatrix}$, there must exist some λ for which

$\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 4-\lambda \\ -9+2\lambda \end{pmatrix}$

i.e. $0 = 4-\lambda$ and $-1 = -9+2\lambda$

$\lambda = 4$ $\lambda = 4$

Thus a suitable value of λ exists.

Point *H* lies on $\mathbf{r} = \begin{pmatrix} 4-\lambda \\ -9+2\lambda \end{pmatrix}$.

If *I*, position vector $\begin{pmatrix} -3 \\ 5 \end{pmatrix}$, lies on $\mathbf{r} = \begin{pmatrix} 4-\lambda \\ -9+2\lambda \end{pmatrix}$, there must exist some λ for which

$\begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 4-\lambda \\ -9+2\lambda \end{pmatrix}$

i.e. $-3 = 4-\lambda$ and $5 = -9+2\lambda$

$\lambda = 7$ $\lambda = 7$

Thus a suitable value of λ exists.

Point *I* lies on $\mathbf{r} = \begin{pmatrix} 4-\lambda \\ -9+2\lambda \end{pmatrix}$.

Question 21

Points A to F lie on the line $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ 8 \end{pmatrix}$.

A, position vector $-3\mathbf{i} + a\mathbf{j}$

$-3 = 3 + 6\lambda$

$\lambda = -1$

$a = -1 + 8\lambda = -1 - 8 = -9$

B, position vector $b\mathbf{i} + 23\mathbf{j}$

$23 = -1 + 8\lambda \quad \Rightarrow \quad \lambda = 3$

$b = 3 + 6\lambda = 3 + 6 \times 3 = 21$

C, position vector $\langle -9, c \rangle$

$-9 = 3 + 6\lambda \quad \Rightarrow \quad \lambda = -2$

$c = -1 + 8\lambda = -1 + 8(-2) = -17$

D, position vector $\langle d, -21 \rangle$

$-21 = -1 + 8\lambda \quad \Rightarrow \quad \lambda = \frac{-5}{2}$

$d = 3 + 6\lambda = 3 + 6\left(-\frac{5}{2}\right) = -12$

E, position vector $\begin{pmatrix} 12 \\ e \end{pmatrix}$

$12 = 3 + 6\lambda$

$\lambda = \frac{3}{2}$

$e = -1 + 8\lambda = -1 + 8\left(\frac{3}{2}\right) = 11$

F, position vector $\begin{pmatrix} f \\ f \end{pmatrix}$

$f = 3 + 6\lambda$

$f = -1 + 8\lambda$

$3 + 6\lambda = -1 + 8\lambda$

$2\lambda = 4 \quad \Rightarrow \quad \lambda = 2$

$f = 3 + 6 \times 2 = 3 + 12 = 15$

$\therefore a = -9,\ b = 21,\ c = -17,\ d = -12,\ e = 11,\ f = 15.$

Question 22

The vector equation of the line passing through the point with position vector $5\mathbf{i}-6\mathbf{j}$ and parallel to the line $\mathbf{r}=(2+\lambda)\mathbf{i}+(3-\lambda)\mathbf{j}$ is $(5+\lambda)\mathbf{i}+(-6-\lambda)\mathbf{j}=(5+\lambda)\mathbf{i}-(6+\lambda)\mathbf{j}$.

Question 23

The vector equation of the line passing through the point with position vector $\begin{pmatrix}6\\5\end{pmatrix}$ and parallel to the line $\mathbf{r}=\begin{pmatrix}2+3\lambda\\1-4\lambda\end{pmatrix}$ is $\begin{pmatrix}6+3\lambda\\5-4\lambda\end{pmatrix}$.

Question 24

$$x=2+6\lambda$$
$$y=12-10\lambda$$
$$\lambda=\frac{x-2}{6}$$

$$y=12-10\left(\frac{x-2}{6}\right)=12-\frac{10x+20}{6}=12-\frac{5}{3}x+\frac{10}{3}$$
$$=\frac{46}{3}-\frac{5}{3}x$$

So the Cartesian equation is

$$3y=46-5x$$
$$5x+3y=46$$

Question 25

When $\lambda=4$, the line $\mathbf{r}=2\mathbf{i}+8\mathbf{j}+\lambda(\mathbf{i}-2\mathbf{j})$ cuts the *x*-axis at *A*.

The position vector for *A* is $6\mathbf{i}$.

When $\lambda=-2$, the line $\mathbf{r}=2\mathbf{i}+8\mathbf{j}+\lambda(\mathbf{i}-2\mathbf{j})$ cuts the *y*-axis at *B*.

The position vector for *B* is $12\mathbf{j}$.

Question 26

When $\lambda = -4$, the line $\mathbf{r} = \begin{pmatrix} 5 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ cuts the *x*-axis at *A*.

The position vector for *A* is $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$.

For point *B*, $5 + 2\lambda = 11$, so $\lambda = 3$.

$c = -4 - 3 = -7$

Question 27

Line through $2\mathbf{i} + 3\mathbf{j}$ and parallel to vector from position vector $2\mathbf{i} + 3\mathbf{j}$ to position vector $5\mathbf{i} - 4\mathbf{j}$.

Line through $2\mathbf{i} + 3\mathbf{j}$ and parallel to $3\mathbf{i} - 7\mathbf{j}$ has vector equation $\mathbf{r} = (2 + 3\lambda)\mathbf{i} + (3 - 7\lambda)\mathbf{j}$.

For point *B* with position vector $b\mathbf{i} + 7\mathbf{j}$

$$\begin{aligned} 3 - 7\lambda &= 7 \\ -7\lambda &= 4 \\ \lambda &= -\frac{4}{7} \end{aligned}$$

$$b = 2 + 3\lambda = 2 + 3\left(-\frac{4}{7}\right) = \frac{2}{7}$$

For point *D* with position vector $-2\mathbf{i} + d\mathbf{j}$

$$\begin{aligned} 2 + 3\lambda &= -2 \\ 3\lambda &= -4 \\ \lambda &= -\frac{4}{3} \end{aligned}$$

$$d = 3 - 7\lambda = 3 - 7\left(-\frac{4}{3}\right) = \frac{37}{3}$$

Question 28

Given $\mathbf{r} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 9 \\ d \end{pmatrix} + \mu \begin{pmatrix} 2 \\ c \end{pmatrix}$ represent the same straight line,

$\begin{pmatrix} 2 \\ c \end{pmatrix}$ must be a multiple of $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ so $c = 4 \times 2 = 8$.

$$\begin{aligned} 5 + \lambda &= 9 + 2\mu \Rightarrow \lambda = 4 + 2\mu \\ 3 + 4\lambda &= d + 8\mu \end{aligned}$$

By substitution, this becomes

$$\begin{aligned} 3 + 4(4 + 2\mu) &= d + 8\mu \\ 3 + 16 + 8\mu &= d + 8\mu \\ d &= 19 \end{aligned}$$

Question 29

$3\mathbf{i} + 4\mathbf{j}$ must be a multiple of $\mathbf{i} + f\,\mathbf{j}$.

$f = 4 \div 3 = \frac{4}{3}$.

$$\begin{aligned} 1 + 3\lambda &= e + \mu \\ -3 + 4\lambda &= 5 + \frac{4}{3}\mu \Rightarrow \lambda = 2 + \frac{1}{3}\mu \end{aligned}$$

By substitution, this becomes

$$\begin{aligned} 1 + 3\left(2 + \frac{1}{3}\mu\right) &= e + \mu \\ 1 + 6 + \mu &= e + \mu \\ e &= 7 \\ f &= \frac{4}{3} \end{aligned}$$

Question 30

The Cartesian equation for set ① is shown below:

$\lambda = y - 3$

$x = 1 + 2\lambda$

$x = 1 + 2(y - 3) = 2y - 5$

$2y = x + 5$

The Cartesian equation for set ② is shown below:

$\lambda = y - 1$

$x = 2\lambda - 2$

$x = 2(y - 1) - 2$

$2y = x + 4$

The Cartesian equation for set ③ is shown below:

$\lambda = y - 6$

$x = 8 + 2\lambda$

$x = 8 + 2(y - 6)$

$x = 8 + 2y - 12$

$2y = x + 4$

So set ①is the odd one out as the other two have the same Cartesian equation.

Question 31

L_1 is a line with equation $\mathbf{r} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ is a line through $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and parallel to $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

L_2 is a line with equation $\mathbf{r} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 6 \\ 2 \end{pmatrix}$ is a line through $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ and parallel to $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$.

The scalar product of $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$ is $-1 \times 6 + 2 \times 3 = 0$, so L_1 is perpendicular to L_2.

Question 32

$3 \times a + 2 \times b = 0,\ a, b \in \mathbb{R}$

$a = -2, b = 3$

So the line with equation $\mathbf{r} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ is perpendicular to L_1.

Question 33

$$\begin{aligned}
\mathbf{a} \bullet \mathbf{b} &= |\mathbf{a}| \times |\mathbf{b}| \times \cos\theta \\
1 \times 2 + (-4) \times 1 &= \sqrt{1^2 + (-4)^2} \times \sqrt{2^2 + 1^2} \times \cos\theta \\
-2 &= \sqrt{85}\cos\theta \\
\cos\theta &= -\frac{2}{\sqrt{85}} \\
\theta &= 102.53^\circ \approx 103^\circ \text{(obtuse)} \\
\alpha &= 180 - 103 = 77^\circ
\end{aligned}$$

The acute angle between the L_1 and L_2 is approximately 77°.

Exercise 4C

Question 1

L_1: $\mathbf{r} = 14\mathbf{i} - \mathbf{j} + \lambda(5\mathbf{i} - 4\mathbf{j})$

L_2: $\mathbf{r} = 9\mathbf{i} - 4\mathbf{j} + \mu(-4\mathbf{i} + 6\mathbf{j})$

The point common to both lines will be such that

$14 + 5\lambda = 9 - 4\mu$
$-1 - 4\lambda = -4 + 6\mu$
Solving simultaneously gives $\lambda = -3, \mu = 2.5$

With $\lambda = -3$ line L_1 gives $\mathbf{r} = 14\mathbf{i} - \mathbf{j} - 3(5\mathbf{i} - 4\mathbf{j})$ i.e. $\mathbf{r} = -\mathbf{i} + 11\mathbf{j}$

With $\mu = 2.5$ line L_1 gives $\mathbf{r} = 9\mathbf{i} - 4\mathbf{j} + 2.5(-4\mathbf{i} + 6\mathbf{j})$ i.e. $\mathbf{r} = -\mathbf{i} + 11\mathbf{j}$

Lines L_1 and L_2 intersect at the point with position vector $-\mathbf{i} + 11\mathbf{j}$.

Question 2

L_1: $\mathbf{r} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

L_2: $\mathbf{r} = \begin{pmatrix} -10 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

The point common to both lines will be such that

$-3 + \lambda = -10 - 4\mu$
$4 - \lambda = 2 + \mu$
Solving simultaneously gives $\lambda = 5, \mu = -3$

With $\lambda = 5$, line L_1 gives $\mathbf{r} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} + 5\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ i.e. $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

With $\mu = -3$, line L_1 gives $\mathbf{r} = \begin{pmatrix} -10 \\ 2 \end{pmatrix} - 3\begin{pmatrix} -4 \\ 1 \end{pmatrix}$ i.e. $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Lines L_1 and L_2 intersect at the point with position vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

Question 3

L_1: $\mathbf{r} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -10 \end{pmatrix}$

L_2: $\mathbf{r} = \begin{pmatrix} -5 \\ -9 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 7 \end{pmatrix}$

The point common to both lines will be such that

$-1 + 4\lambda = -5 + \mu$

$-10\lambda = -9 + 7\mu$

Solving simultaneously gives $\lambda = -0.5, \mu = 2$

With $\lambda = -0.5$, line L_1 gives $\mathbf{r} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} - 0.5 \begin{pmatrix} 4 \\ -10 \end{pmatrix}$ i.e. $\mathbf{r} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$

With $\mu = 2$, line L_1 gives $\mathbf{r} = \begin{pmatrix} -5 \\ -9 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 7 \end{pmatrix}$ i.e. $\mathbf{r} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$

Lines L_1 and L_2 intersect at the point with position vector $\begin{pmatrix} -3 \\ 5 \end{pmatrix}$.

Question 4

At time t_1, $t_2 > 0$, particle *A* will have position vector $\mathbf{r} = 16\mathbf{i} + t_1(3\mathbf{i} + 2\mathbf{j})$

At time t_1, $t_2 > 0$, particle *B* will have position vector $\mathbf{r} = -\mathbf{i} + 6\mathbf{j} + t_2(2\mathbf{i} - 3\mathbf{j})$

For these position vectors to be equal, $16\mathbf{i} + t_1(3\mathbf{i} + 2\mathbf{j}) = -\mathbf{i} + 6\mathbf{j} + t_2(2\mathbf{i} - 3\mathbf{j})$

i.e. $\begin{cases} 16 + 3t_1 = -1 + 2t_2 \\ 2t_1 = 6 - 3t_2 \end{cases}$

Solving simultaneously gives $t_1 = -3,\ t_2 = 4$

With $t_1 = -3$, particle *A* has position vector $\mathbf{r} = 16\mathbf{i} - 3(3\mathbf{i} + 2\mathbf{j})$, i.e. $\mathbf{r} = 7\mathbf{i} - 6\mathbf{j}$

With $t_2 = 4$, particle *B* has position vector $\mathbf{r} = -\mathbf{i} + 6\mathbf{j} + 4(2\mathbf{i} - 3\mathbf{j})$, i.e. $\mathbf{r} = 7\mathbf{i} - 6\mathbf{j}$

Thus if particle *A* was moving with the given velocity prior to $t = 0$ then, when $t = -3$ particle *A* was at $7\mathbf{i} - 6\mathbf{j}$ and particle *B* reaches that point at $t = 4$. Paths of particles *A* and *B* do not cross in the subsequent motion.

Question 5

At time $t_1, t_2 > 0$, particle A will have position vector $\mathbf{r} = \mathbf{i} + 4\mathbf{j} + t_1(4\mathbf{i} + \mathbf{j})$

At time $t_1, t_2 > 0$, particle B will have position vector $\mathbf{r} = 37\mathbf{i} - 20\mathbf{j} + t_2(-2\mathbf{i} + 5\mathbf{j})$

For these position vectors to be equal $\mathbf{i} + 4\mathbf{j} + t_1(4\mathbf{i} + \mathbf{j}) = 37\mathbf{i} - 20\mathbf{j} + t_2(-2\mathbf{i} + 5\mathbf{j})$

i.e. $$\begin{cases} 1 + 4t_1 = 37 - 2t_2 \\ 4 + t_1 = -20 + 5t_2 \end{cases}$$

Solving simultaneously gives $t_1 = 6,\ t_2 = 6$

With $t_1 = 6$, particle A has position vector $\mathbf{r} = \mathbf{i} + 4\mathbf{j} + 6(4\mathbf{i} + \mathbf{j})$ i.e. $\mathbf{r} = 25\mathbf{i} + 10\mathbf{j}$

With $t_2 = 6$, particle B has position vector $\mathbf{r} = 37\mathbf{i} - 20\mathbf{j} + 6(-2\mathbf{i} + 5\mathbf{j})$ i.e. $\mathbf{r} = 25\mathbf{i} + 10\mathbf{j}$

Thus particles A and B are each at the point with position vector $25\mathbf{i} + 10\mathbf{j}$ at time $t = 6$ seconds. A collision is involved.

Question 6

At time $t_1, t_2 > 0$, particle A will have position vector $\mathbf{r} = \mathbf{i} + 19\mathbf{j} + t_1(2\mathbf{i} - \mathbf{j})$

At time $t_1, t_2 > 0$, particle B will have position vector $\mathbf{r} = 3\mathbf{i} + 8\mathbf{j} + t_2(3\mathbf{i} + \mathbf{j})$

For these position vectors to be equal $\mathbf{i} + 19\mathbf{j} + t_1(2\mathbf{i} - \mathbf{j}) = 3\mathbf{i} + 8\mathbf{j} + t_2(3\mathbf{i} + \mathbf{j})$

i.e. $$\begin{cases} 1 + 2t_1 = 3 + 3t_2 \\ 19 - t_1 = 8 + t_2 \end{cases}$$

Solving simultaneously gives $t_1 = 7,\ t_2 = 4$

With $t_1 = 6$, particle A has position vector $\mathbf{r} = \mathbf{i} + 19\mathbf{j} + 7(2\mathbf{i} - \mathbf{j})$ i.e. $\mathbf{r} = 15\mathbf{i} + 12\mathbf{j}$

With $t_2 = 6$, particle B has position vector $\mathbf{r} = 3\mathbf{i} + 8\mathbf{j} + 4(3\mathbf{i} + \mathbf{j})$ i.e. $\mathbf{r} = 15\mathbf{i} + 12\mathbf{j}$

Thus particles A and B each pass through the point with position vector $15\mathbf{i} + 12\mathbf{j}$, but at different times, both greater than zero. Thus in the subsequent motion the paths of the particles cross at the point with position vector $15\mathbf{i} + 12\mathbf{j}$, but a collision is not involved.

For questions **7** to **12**, the line in the answer will be parallel to $\overrightarrow{AB}$ and passes through point A.

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB}$

Exercise 4D

Question 1

The vector equation of a line passing through the point with position vector $\mathbf{a}$ and perpendicular to the vector $\mathbf{n}$ is: $\mathbf{r} \bullet \mathbf{n} = \mathbf{a} \bullet \mathbf{n}$

Thus the vector equation of a line passing through the point with a position vector of $2\mathbf{i}+3\mathbf{j}$ and perpendicular to $3\mathbf{i}+4\mathbf{j}$ is:

$$\begin{aligned}\mathbf{r} \bullet (3\mathbf{i}+4\mathbf{j}) &= (2\mathbf{i}+3\mathbf{j}) \bullet (3\mathbf{i}+4\mathbf{j}) \\ &= (2)(3)+(3)(4) \\ &= 18\end{aligned}$$

Thus the vector equation of a line perpendicular to $3\mathbf{i}+4\mathbf{j}$ and passing through the point with position vector $2\mathbf{i}+3\mathbf{j}$, is $\mathbf{r} \bullet (3\mathbf{i}+4\mathbf{j}) = 18$

Question 2

The vector equation of a line passing through the point with position vector $\mathbf{a}$ and perpendicular to the vector $\mathbf{n}$ is: $\mathbf{r} \bullet \mathbf{n} = \mathbf{a} \bullet \mathbf{n}$

Thus the vector equation of a line passing through the point with a position vector of $-\mathbf{i}+7\mathbf{j}$ and perpendicular to $5\mathbf{i}-\mathbf{j}$ is:

$$\begin{aligned}\mathbf{r} \bullet (5\mathbf{i}-\mathbf{j}) &= (-\mathbf{i}+7\mathbf{j}) \bullet (5\mathbf{i}-\mathbf{j}) \\ &= (-1)(5)+(7)(-1) \\ &= -12\end{aligned}$$

Thus the vector equation of a line perpendicular to $5\mathbf{i}-\mathbf{j}$ and passing through the point with position vector $-\mathbf{i}+7\mathbf{j}$, is $\mathbf{r} \bullet (5\mathbf{i}-\mathbf{j}) = -12$.

Question 3

$$\mathbf{r}\bullet(\mathbf{i}+2\mathbf{j})=12$$
$$(x\mathbf{i}+y\mathbf{j})\bullet(\mathbf{i}+2\mathbf{j})=12$$
$$x+2y=12$$

A general point on the line has position vector $\mathbf{r}=x\mathbf{i}+y\mathbf{j}$ and a point lying on the line must have $x+2y=12$.

For point *A*, position vector $6\mathbf{j}$, $x+2y=0+2\times6=12$, so point *A* lies on the line.

For point *B*, position vector $6\mathbf{i}+3\mathbf{j}$, $x+2y=6+2\times3=12$, so point *B* lies on the line.

For point *C*, position vector $10\mathbf{i}$, $x+2y=10+2\times0=10\neq12$, so point *C* does <u>not</u> lie on the line.

For point *D*, position vector $3\mathbf{i}+6\mathbf{j}$, $x+2y=3+2\times6=15\neq12$, so point *D* does <u>not</u> lie on the line.

For point *E*, position vector $-4\mathbf{i}+8\mathbf{j}$, $x+2y=-4+2\times8=12$, so point *E* lies on the line.

For point *F*, position vector $14\mathbf{i}-\mathbf{j}$, $x+2y=14+2\times(-1)=12$, so point *F* lies on the line.

Points *A*, *B*, *E* and *F* lie on the line, *C* and *D* do not.

Question 4

Given that all points lie on the line with vector equation $\mathbf{r}\bullet\begin{pmatrix}2\\3\end{pmatrix}=10$

For point *U*,

$$\begin{pmatrix}u\\2\end{pmatrix}\bullet\begin{pmatrix}2\\3\end{pmatrix}=10$$
$$u\times2+2\times3=10$$
$$2u+6=10$$
$$u=2$$

For point *X*,

$$\begin{pmatrix}x\\-2\end{pmatrix}\bullet\begin{pmatrix}2\\3\end{pmatrix}=10$$
$$x\times2-2\times3=10$$
$$2x-6=10$$
$$x=8$$

For point *V*,

$$\begin{pmatrix}-10\\v\end{pmatrix}\bullet\begin{pmatrix}2\\3\end{pmatrix}=10$$
$$-10\times2+v\times3=10$$
$$3v-20=10$$
$$v=10$$

For point *Y*,

$$\begin{pmatrix}5\\y\end{pmatrix}\bullet\begin{pmatrix}2\\3\end{pmatrix}=10$$
$$5\times2+y\times3=10$$
$$3y+10=10$$
$$y=0$$

For point W,

$$\begin{pmatrix} w \\ -4 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 10$$
$$w \times 2 - 4 \times 3 = 10$$
$$2w - 12 = 10$$
$$w = 11$$

For point Z,

$$\begin{pmatrix} z \\ 6 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 10$$
$$z \times 2 + 6 \times 3 = 10$$
$$2z + 18 = 10$$
$$z = -4$$

Question 5

a The vector equation of a line passing through point A, with a position vector of $\mathbf{i} + \mathbf{j}$ and perpendicular to $5\mathbf{i} + 2\mathbf{j}$ is:

$$\begin{aligned} \mathbf{r} \bullet (5\mathbf{i} + 2\mathbf{j}) &= (\mathbf{i} + \mathbf{j}) \bullet (5\mathbf{i} + 2\mathbf{j}) \\ &= (1)(5) + (1)(2) \\ &= 7 \end{aligned}$$

Thus the vector equation of a line perpendicular to $5\mathbf{i} + 2\mathbf{j}$ and passing through the point with position vector $\mathbf{i} + \mathbf{j}$, is $\mathbf{r} \bullet (5\mathbf{i} + 2\mathbf{j}) = 7$.

b If a general point on the line has position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ then:

$$\begin{aligned} (x\mathbf{i} + y\mathbf{j}) \bullet (5\mathbf{i} + 2\mathbf{j}) &= 7 \\ 5x + 2y &= 7 \end{aligned}$$

Thus the Cartesian equation of the line is $5x + 2y = 7$.

Question 6

a The vector equation of a line passing through point A, with a position vector of $2\mathbf{i} - \mathbf{j}$ and perpendicular to $2\mathbf{i} + 5\mathbf{j}$ is:

$$\begin{aligned} \mathbf{r} \bullet (2\mathbf{i} + 5\mathbf{j}) &= (2\mathbf{i} - \mathbf{j}) \bullet (2\mathbf{i} + 5\mathbf{j}) \\ &= (2)(2) + (-1)(5) \\ &= -1 \end{aligned}$$

Thus the vector equation of a line perpendicular to $2\mathbf{i} + 5\mathbf{j}$ and passing through the point with position vector $2\mathbf{i} - \mathbf{j}$, is $\mathbf{r} \bullet (2\mathbf{i} + 5\mathbf{j}) = -1$.

b If a general point on the line has position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ then:

$$\begin{aligned} (x\mathbf{i} + y\mathbf{j}) \bullet (2\mathbf{i} + 5\mathbf{j}) &= -1 \\ 2x + 5y &= -1 \end{aligned}$$

Thus the Cartesian equation of the line is $2x + 5y = -1$.

Question 7

From $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \lambda(\mathbf{i} - 4\mathbf{j})$

$x = 2 + \lambda \Rightarrow \lambda = x - 2$
$y = 3 - 4\lambda$
$y = 3 - 4(x - 2)$
$y = 11 - 4x$, the gradient of the line is -4.

From $\mathbf{r} \cdot (8\mathbf{i} + 2\mathbf{j}) = 5$

$$\begin{aligned}(x\mathbf{i} + y\mathbf{j}) \cdot (8\mathbf{i} + 2\mathbf{j}) &= 5 \\ 8x + 2y &= 5 \\ 2y &= -8x + 5 \\ y &= -4x + \frac{5}{2}\end{aligned}$$

, the gradient of the line is -4.

Both lines have gradient = –4 so the lines are parallel.

Question 8

Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ be the vector equation of the line perpendicular to $8\mathbf{i} + 5\mathbf{j}$.
$\mathbf{r} \cdot (8\mathbf{i} + 5\mathbf{j}) = 0$

$(x\mathbf{i} + y\mathbf{j}) \cdot (8\mathbf{i} + 5\mathbf{j}) = 0$
$8x + 5y = 0$
A line parallel to $\mathbf{r}$ would also be perpendicular to $8\mathbf{i} + 2\mathbf{j}$
$8x + 5y = k$
This line would pass through the point $(-1, 3)$ at $8(-1) + 5(3) = k = 7$

Hence the line perpendicular to the vector $8\mathbf{i} + 5\mathbf{j}$ and passing through the point (–1, 3) has Cartesian equation $8x + 5y = 7$.

Question 9

L_1: $\mathbf{r} = 5\mathbf{i} + 2\mathbf{j} + \lambda(3\mathbf{i} - 2\mathbf{j})$

L_2: $\mathbf{r} \bullet (6\mathbf{i} - 4\mathbf{j}) = -4$

From L_1

$$x = 5 + 3\lambda \Rightarrow \lambda = \frac{x-5}{3}$$
$$y = 2 - 2\lambda$$
$$y = 2 - 2\left(\frac{x-5}{3}\right)$$
$$y - 2 = -2\left(\frac{x-5}{3}\right)$$
$$3y - 6 = -2x + 10$$
$3y = -2x + 16$, the gradient of this line is $\frac{-2}{3}$.

From L_2

$\mathbf{r} \bullet (6\mathbf{i} - 4\mathbf{j}) = -4$

$6x - 4y = -4$

$4y = 6x + 4$, the gradient of this line is $\frac{3}{2}$.

$m_1 \times m_2 = -1$ so the two lines are perpendicular.

Exercise 4E

Question 1

a

$$\begin{cases} x = 4 + t \\ y = 2t \end{cases}$$

$t = x - 4$

$y = 2(x - 4)$

$y = 2x - 8$

b

$$\begin{cases} x = t \\ y = \dfrac{1}{t} \end{cases}$$

$y = \dfrac{1}{x}$

c

$$\begin{cases} x = t^2 \\ y = 2t \end{cases}$$

$t = \dfrac{y}{2}$

$x = \dfrac{y^2}{4}$

$y^2 = 4x$

d

$$\begin{cases} x = \sqrt{t-1} \\ y = t^2 \end{cases}$$

$t = \sqrt{y}$

$x = \sqrt{\sqrt{y} - 1}, [\sqrt{y} - 1 \geq 0, \text{ so } \sqrt{y} \geq 1]$

$x^2 = \sqrt{y} - 1$

$x^2 + 1 = \sqrt{y}$

$y = (x^2 + 1)^2,\ x \geq 0$

Question 2

a

$\mathbf{r} = (3 - t)\mathbf{i} + (4 + 2t)\mathbf{j}$

$x = 3 - t \Rightarrow t = 3 - x$

$y = 4 + 2t$

$y = 4 + 2(3 - x)$

$y = 10 - 2x$

b

$\mathbf{r} = (t - 1)\mathbf{i} + \dfrac{1}{t}\mathbf{j}$

$x = t - 1 \Rightarrow t = x + 1$

$y = \dfrac{1}{t}$

$y = \dfrac{1}{x + 1}$

c

$\mathbf{r} = (t - 1)\mathbf{i} + (t^2 + 4)\mathbf{j}$

$x = t - 1 \Rightarrow t = x + 1$

$y = t^2 + 4$

$y = (x + 1)^2 + 4$

$y = x^2 + 2x + 5$

d

$\mathbf{r} = (2 + \cos\theta)\mathbf{i} + (1 + 2\sin\theta)\mathbf{j}$

$x = 2 + \cos\theta$

$y = 1 + 2\sin\theta$

$x - 2 = \cos\theta$

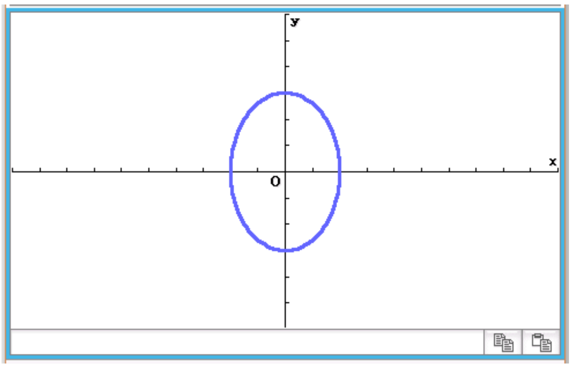
$\dfrac{y - 1}{2} = \sin\theta$

$(x - 2)^2 + \left(\dfrac{y - 1}{2}\right)^2 = \sin^2\theta + \cos^2\theta = 1$

Question 3

Given $\mathbf{r} = (2\cos\theta)\mathbf{i} + (3\sin\theta)\mathbf{j}$

$$\begin{cases} x = 2\cos\theta \\ y = 3\sin\theta \end{cases}$$



From the parametric equations:

$$\frac{x}{2} = \cos\theta, \qquad \frac{y}{3} = \sin\theta$$

$$\frac{x^2}{4} = \cos^2\theta, \qquad \frac{y^2}{9} = \sin^2\theta$$

$$\frac{x^2}{4} + \frac{y^2}{9} = \sin^2\theta + \cos^2\theta$$
$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$
$$9x^2 + 4y^2 = 36$$

Question 4

Given $\mathbf{r} = (-3\sec\theta)\mathbf{i} + (2\tan\theta)\mathbf{j}$

$$\begin{cases} x = -3\sec\theta \\ y = 2\tan\theta \end{cases}$$

From the parametric equations:

$$\frac{x}{-3} = \sec\theta, \qquad \frac{y}{2} = \tan\theta$$
$$\frac{x^2}{9} = \sec^2\theta, \qquad \frac{y^2}{4} = \tan^2\theta$$
$$\frac{x^2}{9} - \frac{y^2}{4} = \sec^2\theta - \tan^2\theta$$
$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$
$$4x^2 - 9y^2 = 36$$

Question 5

B $|\mathbf{r}| = 6$ is a circle, centre (0, 0) with radius = 6 units.

D $|\mathbf{r} - (5\mathbf{i} - 4\mathbf{j})| = 24$ is a circle, centre (5, 4) with radius = 24 units.

E $x^2 + y^2 + 4x - 8y = 5$ is a circle, centre (–2, 4) with radius = 5 units.

Question 6

a $|\mathbf{r}| = 25$

b To find out if point *A* lies inside, on or outside the circle, first find the magnitude of *OA*.

$|19\mathbf{i} - 18\mathbf{j}| = \sqrt{19^2 + (-18)^2} = \sqrt{685} \approx 26.17$

The magnitude is larger than the radius so point *A* is outside the circle.

To find out if point *B* lies inside, on or outside the circle, first find the magnitude of *OB*.

$|-20\mathbf{i} + 15\mathbf{j}| = \sqrt{(-20)^2 + (15)^2} = \sqrt{625} = 25$

The magnitude is equal to the radius so point *B* lies on the circle.

To find out if point *C* lies inside, on or outside the circle, first find the magnitude of *OC*.

$|14\mathbf{i} + 17\mathbf{j}| = \sqrt{14^2 + 17^2} = \sqrt{485} \approx 22.02$

The magnitude is less than the radius so point *C* is inside the circle.

To find out if point *D* lies inside, on or outside the circle, first find the magnitude of *OD*.

$|-24\mathbf{i} - 7\mathbf{j}| = \sqrt{(-24)^2 + (-7)^2} = \sqrt{625} = 25$

The magnitude is equal to the radius so point *D* lies on the circle.

Question 7

$|\mathbf{r}| = 65$ is a circle, centre (0, 0) and radius 65.

Cartesian equation for this situation is $x^2 + y^2 = 65^2$

Substitute Point *A* (–52, *a*) into the equation $x^2 + y^2 = 65^2$

$(-52)^2 + a^2 = 65^2$ and solve to get *a* = ±39, given that *a* is positive we know that *a* = 39.

Substitute Point *A* (*b*, 25) into the equation $x^2 + y^2 = 65^2$

$b^2 + 25^2 = 65^2$ and solve to get *b* = ±60, given that *b* is negative we know that *b* = –60.

Question 8

A circle has centre *C* with position vector $-7\mathbf{i}+4\mathbf{j}$, this is the point (–7, 4) on the Cartesian plane.

$(x+7)^2+(y-4)^2=\left(4\sqrt{5}\right)^2$

$(x+7)^2+(y-4)^2=80$

The distance from *C*, position vector $-7\mathbf{i}+4\mathbf{j}$, to point *A* with position vector $\mathbf{i}+8\mathbf{j}$ is given by $\sqrt{(1-(-7))^2+(8-4)^2}=4\sqrt{5}$, which is equal to the radius of the circle.

$|\mathbf{r}+7\mathbf{i}-4\mathbf{j}|=4\sqrt{5}$

Point *A* lies on the circle.

Question 9

a Circle centre (1, –5) with radius = 9 has vector equation $|\mathbf{r}-\mathbf{i}+5\mathbf{j}|=9$.

b Circle centre (–3, 4) with radius = 10 has vector equation $|\mathbf{r}+3\mathbf{i}-4\mathbf{j}|=10$.

c Circle centre (–12, 3) with radius = $2\sqrt{3}$ has vector equation $|\mathbf{r}+12\mathbf{i}-3\mathbf{j}|=2\sqrt{3}$.

d Circle centre (–13,–2) with radius = 4 has vector equation $|\mathbf{r}+13\mathbf{i}+2\mathbf{j}|=4$.

Question 10

a Circle, centre has position vector $2\mathbf{i}+3\mathbf{j}$ and radius 5.

$$(x-2)^2+(y-3)^2=25$$
$$x^2-4x+4+y^2-6y+9=25$$
$$x^2+y^2-4x-6y+13=25$$
$$x^2+y^2-4x-6y=12$$

b Circle, centre has position vector $-4\mathbf{i}+2\mathbf{j}$ and radius $\sqrt{7}$.

$$(x+4)^2+(y-2)^2=\sqrt{7}^2$$
$$x^2+8x+16+y^2-4y+4=7$$
$$x^2+y^2+8x-4y+20=7$$
$$x^2+y^2+8x-4y=-13$$

c Circle, centre has position vector $4\mathbf{i}-3\mathbf{j}$ and radius 7.

$$(x-4)^2+(y+3)^2=7^2$$
$$x^2-8x+16+y^2+6y+9=49$$
$$x^2+y^2-8x+6y+25=49$$
$$x^2+y^2-8x+6y=24$$

Question 11

a Given $|\mathbf{r}-(6\mathbf{i}+3\mathbf{j})|=5$

Position vector of centre is $6\mathbf{i}+3\mathbf{j}$ and radius of the circle is 5.

b Given $|\mathbf{r}-2\mathbf{i}+3\mathbf{j}|=6$

Position vector of centre is $2\mathbf{i}-3\mathbf{j}$ and radius of the circle is 6.

c Given $|(x-3)\mathbf{i}+(y+4)\mathbf{j})|=3$

Position vector of centre is $3\mathbf{i}-4\mathbf{j}$ and radius of the circle is 3.

d Given $|\mathbf{r}|=20$

Position vector of centre is $0\mathbf{i}+0\mathbf{j}$ and radius of the circle is 20.

e Given $16x^2+16y^2=25$, then $x^2+y^2=\frac{25}{16}$

Position vector of centre is $0\mathbf{i}+0\mathbf{j}$ and radius of the circle is $\sqrt{\frac{25}{16}}=\frac{5}{4}=1.25$.

f Given $(x-2)^2+(y+3)^2=49$

The centre of the circle is (2, –3)

Position vector of centre is $2\mathbf{i}-3\mathbf{j}$ and radius of the circle $\sqrt{49}=7$.

g Given $x^2+y^2-6x-18y+65=0$

$(x-3)^2-9+(y-9)^2-81+65=0$

$(x-3)^2+(y-9)^2=25$

The centre of the circle is (3, 9)

Position vector of centre is $3\mathbf{i}+9\mathbf{j}$ and radius of the circle $\sqrt{25}=5$.

h Given $x^2+y^2+20x-2y=20$

$(x+10)^2-100+(y-1)^2-1=20$

$(x+10)^2+(y-1)^2=121$

The centre of the circle is (–10, 1)

Position vector of centre is $-10\mathbf{i}+\mathbf{j}$ and radius of the circle $\sqrt{121}=11$.

Question 12

The circle $|\mathbf{r}-(\mathbf{i}-\mathbf{j})|=6$ has centre (1, –1).

The circle $|\mathbf{r}-6\mathbf{i}-11\mathbf{j}|=7$ has centre (6, 11).

The distance between (1, –1) and (6, 11) is

$\sqrt{(6-1)^2+(11-(-1))^2}=\sqrt{5^2+12^2}=\sqrt{169}=13$ units

Question 13

The circle $|\mathbf{r}-(2\mathbf{i}-5\mathbf{j})|=5$ has centre $A(2, -5)$.

The circle $|\mathbf{r}-(5\mathbf{i}+2\mathbf{j})|=3$ has centre $B(5, 2)$.

The horizontal distance from 2 to 5 is 3 and moving vertically from –5 to 2 is 7 units, hence the vector equation from A to B is $\mathbf{r}=3\mathbf{i}+7\mathbf{j}$.

The straight line through A and B has equation $\mathbf{r}=2\mathbf{i}-5\mathbf{j}+\lambda(3\mathbf{i}+7\mathbf{j})$

$\mathbf{r}=(2+3\lambda)\mathbf{i}+(-5+7\lambda)\mathbf{j}$

Question 14

The circle $|\mathbf{r}-(3\mathbf{i}-2\mathbf{j})|=3$ has centre $A(3, -2)$.

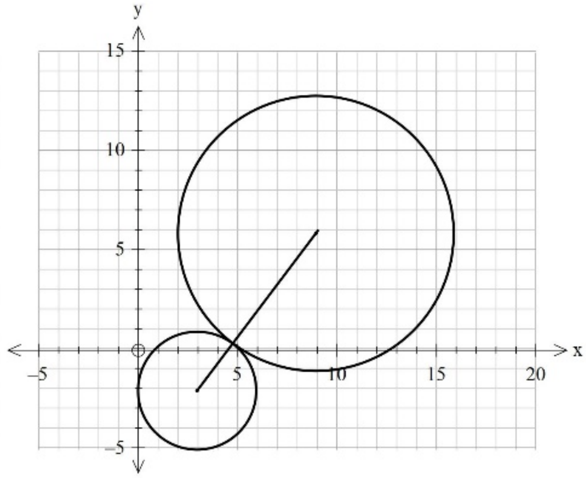
The circle $|\mathbf{r}-(9\mathbf{i}+6\mathbf{j})|=7$ has centre $B(9, 6)$.

$\overrightarrow{AB}=6\mathbf{i}-8\mathbf{j}$

$|\overrightarrow{AB}|=\sqrt{6^2+(-8)^2}$

$=10\text{ units}$

The distance between the centres of the two circles is equal to the sum of the two radii, hence the circles touch at only one point.



Question 15

The circle $|\mathbf{r}-(3\mathbf{i}-\mathbf{j})|=3$ has centre $A(3, -1)$.

The circle $|\mathbf{r}-(13\mathbf{i}+\mathbf{j})|=7$ has centre $B(13, 1)$.

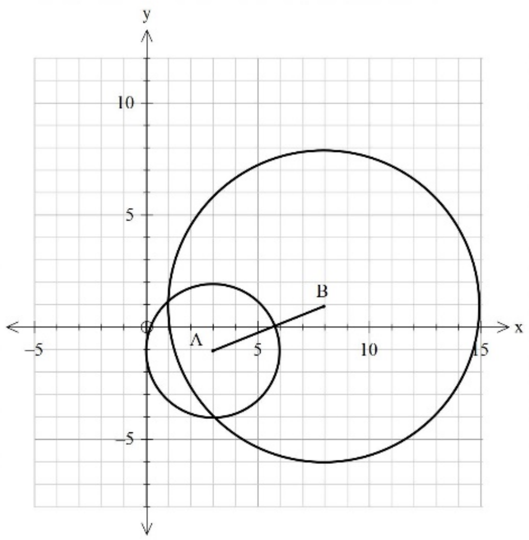
$\overrightarrow{AB}=10\mathbf{i}+2\mathbf{j}$

$|\overrightarrow{AB}|=\sqrt{10^2+2^2}$

$=2\sqrt{26}$

$\approx 10.20\text{ units}$

The distance between the centres of the two circles is greater than the sum of the two radii, therefore the circles do not touch.



Question 16

If point A, position vector $\mathbf{r}_A$, lies on both the line and the circle then

$$\mathbf{r}_A = \begin{pmatrix} -10 \\ 15 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ -3 \end{pmatrix} \text{ and } \left| \mathbf{r} - \begin{pmatrix} -1 \\ 7 \end{pmatrix} \right| = \sqrt{29}.$$

Substituting the first expression into the second gives:

$$\begin{aligned}
\left| \begin{pmatrix} -10 \\ 15 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ -3 \end{pmatrix} - \begin{pmatrix} -1 \\ 7 \end{pmatrix} \right| &= \sqrt{29} \\
\left| \begin{pmatrix} 7\lambda - 9 \\ 8 - 3\lambda \end{pmatrix} \right| &= \sqrt{29} \\
(7\lambda - 9)^2 + (8 - 3\lambda)^2 &= 29 \\
49\lambda^2 - 126\lambda + 81 + 64 - 48\lambda + 9\lambda^2 &= 29 \\
58\lambda^2 - 174\lambda + 145 - 29 &= 0 \\
58\lambda^2 - 174\lambda + 116 &= 0 \\
\lambda^2 - 3\lambda + 2 &= 0 \\
(\lambda - 1)(\lambda - 2) &= 0 \\
\lambda &= 1,\ 2
\end{aligned}$$

If $\lambda = 1$, $\mathbf{r}_A = \begin{pmatrix} -3 \\ 12 \end{pmatrix}$

If $\lambda = 2$, $\mathbf{r}_A = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$

So the line meets the circle at the points with position vectors $\begin{pmatrix} -3 \\ 12 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 9 \end{pmatrix}$.

Question 17

If point A, position vector $\mathbf{r}_A$, lies on both the line and the circle then

$\mathbf{r}_A = 10\mathbf{i} - 9\mathbf{j} + \lambda(4\mathbf{i} - 5\mathbf{j})$ and $|\mathbf{r} + 7\mathbf{i} - 2\mathbf{j}| = \sqrt{41}$.

Substituting the first expression into the second gives:

$$|10\mathbf{i} - 9\mathbf{j} + \lambda(4\mathbf{i} - 5\mathbf{j}) + 7\mathbf{i} - 2\mathbf{j}| = \sqrt{41}$$
$$|(17 + 4\lambda)\mathbf{i} + (-11 - 5\lambda)\mathbf{j}| = \sqrt{41}$$
$$(17 + 4\lambda)^2 + (-11 - 5\lambda)^2 = 41$$

$$289 + 136\lambda + 16\lambda^2 + 121 + 110\lambda + 25\lambda^2 = 41$$
$$41\lambda^2 + 246\lambda + 369 = 0$$
$$\lambda^2 + 6\lambda + 9 = 0$$
$$(\lambda + 3)^2 = 0$$
$\lambda = -3$ is the only solution.

If $\lambda = -3$, $\mathbf{r}_A = -2\mathbf{i} + 6\mathbf{j}$

So the tangent to the circle meets the circle at the point with position vector $\begin{pmatrix} -2 \\ 6 \end{pmatrix}$.

Exercise 4F

Question 1

Let A be the point where the ship is closest to the drilling platform, at point P.

$$\overrightarrow{PA} = \overrightarrow{PO} + \overrightarrow{OA}$$
$$= -25\mathbf{i} - 15\mathbf{j} + t(10\mathbf{i} + 5\mathbf{j})$$
$$= (10t - 25)\mathbf{i} + (5t - 15)\mathbf{j}$$

$\overrightarrow{PA}$ is perpendicular to $\overrightarrow{PO}$ at the point when the ship is closest to the platform.

Thus

$$(25\mathbf{i} + 15\mathbf{j}) \bullet \left[(10t - 25)\mathbf{i} + (5t - 15)\mathbf{j}\right] = 0$$
$$25(10t - 25) + 15(5t - 15) = 0$$
$$325t + 850 = 0$$
$$t = \frac{34}{13}$$

The ship is closest to the platform $\frac{34}{13}$ minutes after 8 a.m., at approximately 10:36 a.m.

When $t = \frac{34}{13}$, $\overrightarrow{PA} = \frac{15}{13}\mathbf{i} - \frac{25}{13}\mathbf{j}$. Distance from the drilling platform = $\sqrt{\left(\frac{15}{13}\right)^2 + \left(\frac{25}{13}\right)^2} \approx \sqrt{5}$

Question 2

$$\overrightarrow{AB} = \begin{pmatrix} -16 \\ 13 \end{pmatrix}$$

$${}_A\mathbf{v}_B = \mathbf{v}_A - \mathbf{v}_B = \begin{pmatrix} -10 \\ -2 \end{pmatrix} - \begin{pmatrix} -2 \\ -6 \end{pmatrix} = \begin{pmatrix} -8 \\ 4 \end{pmatrix}$$

Minimum distance between A and B is given by CB.

Now $\sin(\phi - \theta) = \frac{CB}{AB}$

$$CB = AB \times \sin(\phi - \theta) = \sqrt{(-16)^2 + 13^2} \times \sin(\phi - \theta)$$
$$= \sqrt{425} \times \sin(\phi - \theta) = 5\sqrt{17} \times \sin(\phi - \theta)$$

But $\tan\theta = -\frac{13}{16}$ and $\tan\phi = -\frac{4}{8}$

By determining θ and ϕ and hence $(\phi - \theta)$, we obtain $CB = 2\sqrt{5}$ m, which occurs at $t = 2.25$.

Question 3

Let M be the point where the mouse is closest to the snake, at point S.

$$\begin{aligned}\overrightarrow{SM} &= \overrightarrow{SO} + \overrightarrow{OM} \\ &= -5\mathbf{i} - 6\mathbf{j} + t(\mathbf{i} + 2\mathbf{j}) \\ &= (t-5)\mathbf{i} + (2t-6)\mathbf{j}\end{aligned}$$

$\overrightarrow{SM}$ is perpendicular to $\overrightarrow{SO}$ at the point where the mouse is closest to the snake.

Thus

$$(5\mathbf{i} + 6\mathbf{j}) \bullet \left[(t-5)\mathbf{i} + (2t-6)\mathbf{j}\right] = 0$$

$$5(t-5) + 6(2t-6) = 0$$

$$17t - 61 = 0$$

$$t = \frac{61}{17}$$

When $t = \frac{61}{17}$, $\overrightarrow{SM} = -\frac{24}{17}\mathbf{i} - \frac{20}{17}\mathbf{j}$

Distance from snake to mouse is approximately $\sqrt{\left(\frac{24}{17}\right)^2 + \left(\frac{20}{17}\right)^2} \approx 1.8\,\text{m}$.

So the snake is more likely to catch the mouse than miss it.

Question 4

$$\overrightarrow{AB} = 40\mathbf{i} + 5\mathbf{j}$$

$${}_A\mathbf{v}_B = \mathbf{v}_A - \mathbf{v}_B = 3\mathbf{i} + 4\mathbf{j} - (-3\mathbf{i}) = 6\mathbf{i} + 4\mathbf{j}$$

$$\sin(\phi - \theta) = \frac{CB}{AB}$$

$$CB = AB\sin(\phi - \theta) = \sqrt{40^2 + 5^2}\,\sin(\phi - \theta)$$

$$\tan\theta = \frac{5}{40},\ \tan\phi = \frac{4}{6}$$

By determining θ and ϕ and hence $(\phi - \theta)$, we obtain $CB = 5\sqrt{13}$ cm.

$$\begin{aligned}6(40-6t) + 4(5-4t) &= 0 \\ 240 - 36t + 20 - 16t &= 0 \\ 52t &= 260 \\ t &= 5\text{seconds}\end{aligned}$$

Question 5

$$\mathbf{r}_A = \begin{pmatrix} 30+10t \\ 10-5t \end{pmatrix}, \qquad \mathbf{r}_B = \begin{pmatrix} 54-8t \\ -19+7t \end{pmatrix}$$

$$\mathbf{r}_A - \mathbf{r}_B = \begin{pmatrix} -24+18t \\ 29-12t \end{pmatrix}$$

$$|\mathbf{r}_A - \mathbf{r}_B| = \sqrt{(-24+18t)^2 + (29-12t)^2} = \sqrt{468t^2 - 1560t + 1417}$$

By viewing the graph of this function, the minimum value is $\sqrt{117} = 3\sqrt{13}$.

The minimum distance is $3\sqrt{13}$ km .

Question 6

$$\mathbf{r}_A = (20+4t)\mathbf{i} + (-10+5t)\mathbf{j}, \qquad \mathbf{r}_B = (16+6t)\mathbf{i} + (23-3t)\mathbf{j}$$

$$\mathbf{r}_A - \mathbf{r}_B = (4-2t)\mathbf{i} + (-33+8t)\mathbf{j}$$

$$|\mathbf{r}_A - \mathbf{r}_B| = \sqrt{(4-2t)^2 + (-33+8t)^2} = \sqrt{68t^2 - 544t + 1105}$$

By viewing the graph of this function, the minimum value is $\sqrt{17}$.

The minimum distance is $\sqrt{17}$ km .

Question 7

Suppose that the perpendicular from A to the line L meets the line at P. Suppose also that at P the value of λ is λ_1.

Then $\overrightarrow{OP} = -5\mathbf{i} + 22\mathbf{j} + \lambda_1(5\mathbf{i} - 2\mathbf{j})$

Now $\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP}$

$$\begin{aligned}\overrightarrow{AP} &= -(14\mathbf{i} - 3\mathbf{j}) - 5\mathbf{i} + 22\mathbf{j} + \lambda_1(5\mathbf{i} - 2\mathbf{j}) \\ &= (-19 + 5\lambda_1)\mathbf{i} + (25 - 2\lambda_1)\mathbf{j}\end{aligned}$$

Line L is parallel to $5\mathbf{i} - 2\mathbf{j}$ and so $(5\mathbf{i} - 2\mathbf{j}) \bullet \overrightarrow{AP} = 0$

$\therefore 5(-19 + 5\lambda_1) - 2(25 - 2\lambda_1) = 0$

giving $\lambda_1 = 5$

Hence $\overrightarrow{AP} = 6\mathbf{i} + 15\mathbf{j}$ and so $|\overrightarrow{AP}| = 3\sqrt{29}$ units.

Question 8

Suppose that the perpendicular from A to the line L meets the line at P. Suppose also that at P the value of λ is λ_1.

Then $\overrightarrow{OP} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

Now $\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP}$

$$\overrightarrow{AP} = -\begin{pmatrix} 11 \\ 18 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -8+3\lambda_1 \\ -19+4\lambda_1 \end{pmatrix}$$

Line L is parallel to $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and so $\begin{pmatrix} 3 \\ 4 \end{pmatrix} \bullet \overrightarrow{AP} = 0$

$\therefore 3(-8+3\lambda_1) + 4(-19+4\lambda_1) = 0$

giving $\lambda_1 = 4$

Hence $\overrightarrow{AP} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and so $\left|\overrightarrow{AP}\right| = 5$ units.

Question 9

Suppose that the perpendicular from A to the line L meets the line at P. Suppose also that at P the value of λ is λ_1.

Then $\overrightarrow{OP} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

Now $\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP}$

$$\overrightarrow{AP} = -\begin{pmatrix} -3 \\ 8 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2\lambda_1 \\ -8-2\lambda_1 \end{pmatrix}$$

Line L is parallel to $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ and so $\begin{pmatrix} 2 \\ -2 \end{pmatrix} \bullet \overrightarrow{AP} = 0$

$\therefore 2(2\lambda_1) - 2(-8-2\lambda_1) = 0$

giving $\lambda_1 = -2$

Hence $\overrightarrow{AP} = \begin{pmatrix} -4 \\ -4 \end{pmatrix}$ and so $\left|\overrightarrow{AP}\right| = 4\sqrt{2}$ units.

Miscellaneous Exercise 4

Question 1

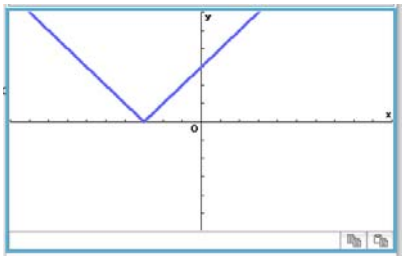
The scalar product of the position vector of L_1 and the position vector of L_2 is

$10(-2)+4\times 5=0$.

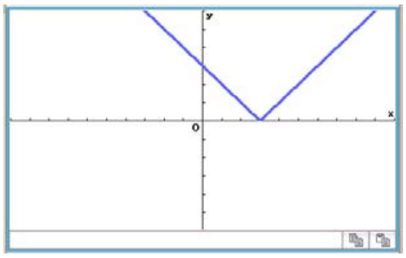
Hence L_1 is perpendicular to L_2.

Question 2

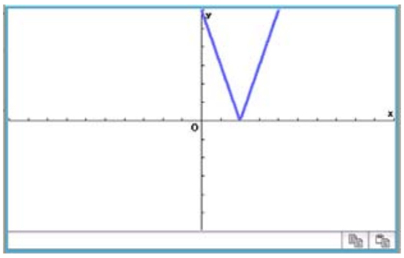
a $y=|x+3|$



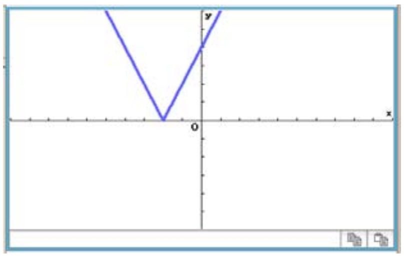
b $y=|x-3|$



c $y=3|x-2|$



d $y=2|x+2|$



Question 3

a $|\mathbf{r}-(7\mathbf{i}-\mathbf{j})|=5$ is a circle with centre (7, –1) and radius 5 units.

b $|\mathbf{r}-7\mathbf{i}-\mathbf{j}|=6$ is a circle with centre (7, 1) and radius 6 units.

c $x^2+y^2=18$ is a circle with centre (0, 0) and radius $\sqrt{18}=3\sqrt{2}$ units .

d $(x-1)^2+(y+8)^2=75$ is a circle with centre (1, –8) and radius $\sqrt{75}=5\sqrt{3}$ units.

e $x^2+y^2+2x=14y+50$ is a circle with centre (–1, 7) and radius 10 units (see working below).

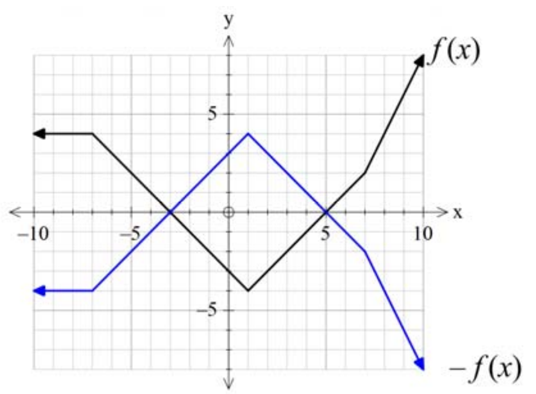
$$(x+1)^2-1+(y-7)^2-49=50$$
$$(x+1)^2+(y-7)^2=100$$

f $(x+5)^2+(y-7)^2=225$ is a circle with centre (–5, 7) and radius 15 units (see working below).

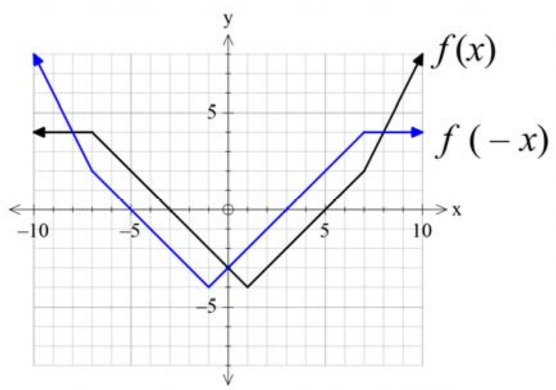
$$x^2+10x+y^2=151+14y$$
$$(x+5)^2-25+(y-7)^2-49=151$$

Question 4

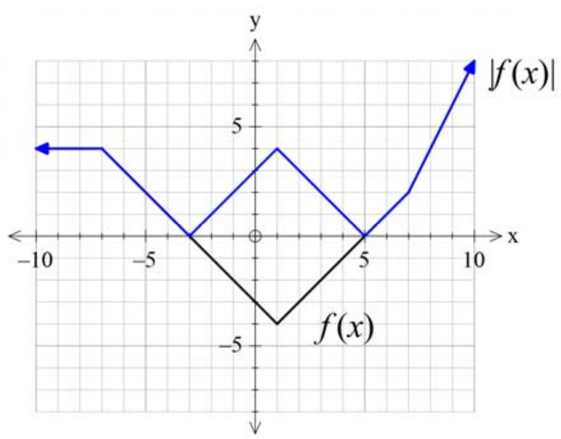
a



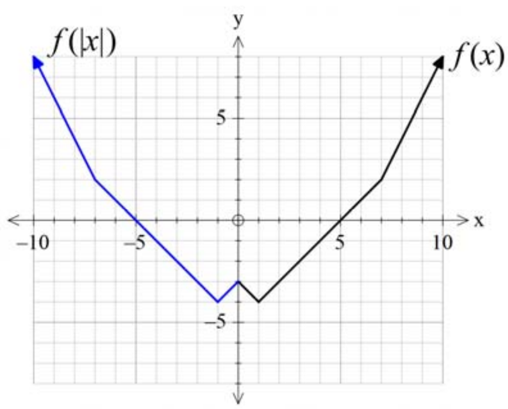
b



c



d



Question 5

a $3x^3 - 11x^2 + 25x - 25 = (ax - b)(x^2 + cx + 5)$

$a = 3,\ b = 5$

$$
\begin{aligned}
3x^3 - 11x^2 + 25x - 25 &= (3x - 5)(x^2 + cx + 5) \\
&= 3x^3 - 5x^2 + 3cx^2 - 5cx + 15x - 25 \\
&= 3x^3 + (3c - 5)x^2 + (15 - 5c)x - 25
\end{aligned}
$$

From this $3c - 5 = -11$

$3c = -6$

$c = -2$

b $3x^3 - 11x^2 + 25x - 25 = 0$

$(3x - 5)(x^2 - 2x + 5) = 0$

So $3x - 5 = 0$

$3x = 5$

$x = 1\frac{2}{3}$

$x^2 - 2x + 5 = 0$

$$
\begin{aligned}
x &= \frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm \sqrt{-16}}{2} \\
&= \frac{2 \pm \sqrt{16i^2}}{2} = \frac{2 \pm 4i}{2} \\
&= 1 \pm 2i
\end{aligned}
$$

Solutions are $\frac{5}{3}$, $1 + 2i$, $1 + 2i$.

Using calculator to find the solutions:

solve$(3x^3 - 11x^2 + 25x - 25 = 0, x)$

$\left\{x = \frac{5}{3}, x = 1 - 2 \cdot i, x = 1 + 2 \cdot i\right\}$

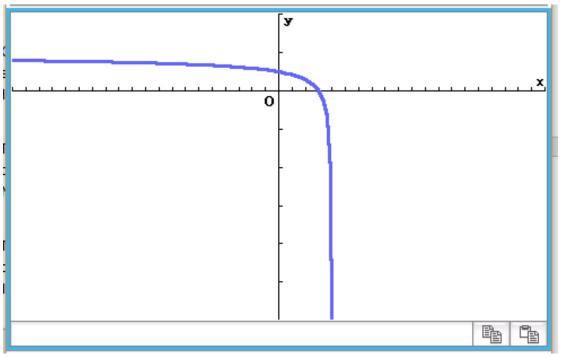
Question 6

$$f(x)=1-\frac{1}{\sqrt{4-x}}$$

a $f(-21)=1-\frac{1}{\sqrt{4-(-21)}}=1-\frac{1}{5}=\frac{4}{5}$

b $f(3)=0$

$f[f(3)]=f(0)=1-\frac{1}{\sqrt{4-0}}=1-\frac{1}{2}=\frac{1}{2}$



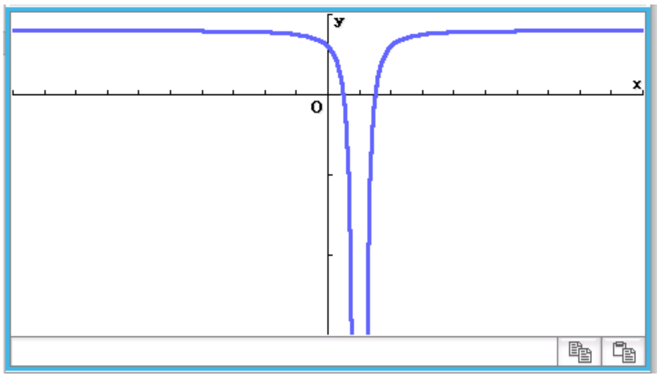
c Domain $\{x\in\mathbb{R}:x\le 4\}$

d Range $\{y\in\mathbb{R}:y<1\}$

e $x\xrightarrow{\times(-1)}-x\xrightarrow{+4}4-x\xrightarrow{\sqrt{x}}\sqrt{4-x}\xrightarrow{\frac{1}{x}}\frac{1}{\sqrt{4-x}}\xrightarrow{\times(-1)}-\frac{1}{\sqrt{4-x}}\xrightarrow{+1}1-\frac{1}{\sqrt{4-x}}$

$-\frac{1}{(1-x)^2}+4\xleftarrow{\times(-1)}\frac{1}{(1-x)^2}-4\xleftarrow{-4}\frac{1}{(1-x)^2}\xleftarrow{x^2}\frac{1}{1-x}\xleftarrow{\frac{1}{x}}1-x\xleftarrow{\div(-1)}x-1\xleftarrow{-1}x$

$$f^{-1}(x)=4-\frac{1}{(1-x)^2}$$



Domain $\{x\in\mathbb{R}:x<1\}$

Range $\{y\in\mathbb{R}:y<4\}$

Question 7

a $z = 2\text{cis}\dfrac{\pi}{6}$

$z = a + \text{i}b$

$a = 2\cos\dfrac{\pi}{6} = 2\times\dfrac{\sqrt{3}}{2} = \sqrt{3}, \qquad b = 2\sin\dfrac{\pi}{6} = 2\times\dfrac{1}{2} = 1$

$z = \sqrt{3} + i$

b $w = -1 - \sqrt{3}\,i$

$w = r\,\text{cis}\,\theta$

$r = \sqrt{(-1)^2 + \left(-\sqrt{3}\right)^2} = \sqrt{10}$

$\tan\theta = \sqrt{3}$

$\theta = -\dfrac{2\pi}{3}$

$w = 2\text{cis}\left(-\dfrac{2\pi}{3}\right)$

c $zw = 2\,\text{cis}\dfrac{\pi}{6}\times 2\,\text{cis}\left(-\dfrac{2\pi}{3}\right)$

In polar form:

$zw = 4\text{cis}\left(\dfrac{\pi}{6} - \dfrac{2\pi}{3}\right) = 4\text{cis}\left(-\dfrac{\pi}{2}\right)$

In Cartesian form:

$zw = a + \text{i}b = 4\cos\left(-\dfrac{\pi}{2}\right) + 4i\sin\left(-\dfrac{\pi}{2}\right) = 4\times 0 + 4i(-1) = -4i$

d $\dfrac{z}{w} = \dfrac{2\text{cis}\dfrac{\pi}{6}}{2\text{cis}\left(-\dfrac{2\pi}{3}\right)}$

In polar form:

$\dfrac{z}{w} = \text{cis}\left(\dfrac{\pi}{6} - \left(-\dfrac{2\pi}{3}\right)\right) = \text{cis}\left(\dfrac{5\pi}{6}\right)$

In Cartesian form:

$\dfrac{z}{w} = \cos\dfrac{5\pi}{6} + i\,\sin\dfrac{5\pi}{6} = -\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$

Question 8

a When line $\mathbf{r} = -10\mathbf{i} + 24\mathbf{j} + \lambda(5\mathbf{i} + \mathbf{j})$ meets circle $|\mathbf{r} - (34\mathbf{i} + 12\mathbf{j})| = 2\sqrt{130}$

$$\left|\begin{pmatrix} -10+5\lambda \\ 24+\lambda \end{pmatrix} - \begin{pmatrix} 34 \\ 12 \end{pmatrix}\right| = 2\sqrt{130}$$

$$\left|\begin{pmatrix} -44+5\lambda \\ 12+\lambda \end{pmatrix}\right| = 2\sqrt{130}$$

$$\sqrt{(-44+5\lambda)^2 + (12+\lambda)^2} = 2\sqrt{130}$$

Solving gives $\lambda = 6,\ \lambda = 10$.

The position vectors of the points of intersection are:

$\mathbf{r} = (-10 + 5\times 6)\mathbf{i} + (24+6)\mathbf{j} = 20\mathbf{i} + 30\mathbf{j}$

and

$\mathbf{r} = (-10 + 5\times 10)\mathbf{i} + (24+10)\mathbf{j} = 40\mathbf{i} + 34\mathbf{j}$

b When line $\mathbf{r} = -\mathbf{i} + 5\mathbf{j} + \lambda(3\mathbf{i} + \mathbf{j})$ meets circle $|\mathbf{r} - (3\mathbf{i} + \mathbf{j})| = \sqrt{5}$

$$\left|\begin{pmatrix} -1+3\lambda \\ 24+\lambda \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right| = \sqrt{5}$$

$$\left|\begin{pmatrix} -4+3\lambda \\ 23+\lambda \end{pmatrix}\right| = \sqrt{5}$$

$$\sqrt{(-4+3\lambda)^2 + (23+\lambda)^2} = \sqrt{5}$$

There are no real solutions so the line never touches the circle.

c When line $\mathbf{r} = -\mathbf{i} + 7\mathbf{j} + \lambda(\mathbf{i} + 3\mathbf{j})$ meets circle $|\mathbf{r} - (4\mathbf{i} + 2\mathbf{j})| = 2\sqrt{10}$

$$\left|\begin{pmatrix} -1+\lambda \\ 7+3\lambda \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix}\right| = 2\sqrt{10}$$

$$\left|\begin{pmatrix} -5+\lambda \\ 5+3\lambda \end{pmatrix}\right| = 2\sqrt{10}$$

$$\sqrt{(-5+\lambda)^2 + (5+3\lambda)^2} = 2\sqrt{10}$$

Solving gives $\lambda = -1$.

The position vector of the point of intersection is:

$\mathbf{r} = [-1 + (-1)]\mathbf{i} + [7 + 3(-1)]\mathbf{j} = -2\mathbf{i} + 4\mathbf{j}$

Question 9

$$\frac{1}{\text{cis}\,\theta}=\frac{1}{\cos\theta+i\sin\theta}$$
$$=\frac{1}{\cos\theta+i\sin\theta}\times\frac{\cos\theta-i\sin\theta}{\cos\theta-i\sin\theta}$$
$$=\frac{\cos\theta-i\sin\theta}{\cos^2\theta-i^2\sin^2\theta}$$
$$=\frac{\cos\theta-i\sin\theta}{\cos^2\theta-(-1)\sin^2\theta}$$
$$=\frac{\cos\theta-i\sin\theta}{\cos^2\theta+\sin^2\theta}$$
$$=\frac{\cos\theta-i\sin\theta}{1}$$
$$=\cos(-\theta)-i(-1)\sin(-\theta)$$
$$=\cos(-\theta)+i\sin(-\theta)$$
$$=\text{cis}(-\theta)$$

Question 10

a Given $z=\cos\theta+i\sin\theta$,

$$z^k+\frac{1}{z^k}=(\cos\theta+i\sin\theta)^k+\frac{1}{(\cos\theta+i\sin\theta)^k}\text{, for some constant } k$$
$$=\cos(k\theta)+i\sin(k\theta)+\frac{1}{\cos(k\theta)+i\sin(k\theta)}$$
$$=\cos(k\theta)+i\sin(k\theta)+\left[\cos(k\theta)+i\sin(k\theta)\right]^{-1},$$
by applying De Moivre's theorem
$$=\cos(k\theta)+i\sin(k\theta)+\cos(-k\theta)+i\sin(-k\theta)$$
$$=\cos(k\theta)+i\sin(k\theta)+\cos(k\theta)-i\sin(k\theta)$$
$$=2\cos(k\theta)$$

b **i** Prove that $\cos^3\theta = \dfrac{\cos(3\theta)+3\cos\theta}{4}$.

Since $z+\dfrac{1}{z}=2\cos\theta$, then $\left(z+\dfrac{1}{z}\right)^3 = 8\cos^3\theta$

$$\left(z+\frac{1}{z}\right)^3 = \left(z+\frac{1}{z}\right)^2\left(z+\frac{1}{z}\right)$$
$$= \left(z^2+2+\frac{1}{z^2}\right)\left(z+\frac{1}{z}\right)$$
$$= (2\cos 2\theta + 2)\times 2\cos\theta \quad \text{(by the result from part } \mathbf{a}\text{)}$$
$$= 4\cos\theta\cos 2\theta + 4\cos\theta$$
$$= 2[\cos\theta + \cos 3\theta] + 4\cos\theta$$
$$= 6\cos\theta + 2\cos 3\theta$$

Hence, $6\cos\theta + 2\cos 3\theta = 8\cos^3\theta$, and dividing both sides by 8, we get

$\dfrac{3\cos\theta+\cos 3\theta}{4} = \cos^3\theta$, as required.

ii Since $z+\dfrac{1}{z}=2\cos\theta$, then $\left(z+\dfrac{1}{z}\right)^4 = 16\cos^4\theta$.

$$\left(z+\frac{1}{z}\right)^4 = \left[\left(z+\frac{1}{z}\right)^2\right]^2$$
$$= \left(z^2+2+\frac{1}{z^2}\right)^2$$
$$= (2\cos 2\theta + 2)^2 \quad \text{(by the result from part } \mathbf{a}\text{)}$$
$$= 4\cos^2 2\theta + 8\cos 2\theta + 4$$
$$= 4\left[\frac{1}{2}(1+\cos 4\theta)\right] + 8\cos 2\theta + 4$$
$$= 2 + 2\cos 4\theta + 8\cos 2\theta + 4$$
$$= 6 + 2\cos 4\theta + 8\cos 2\theta$$

Hence, $6 + 2\cos 4\theta + 8\cos 2\theta = 16\cos^4\theta$, and dividing both sides by 16, we get

$\dfrac{3+\cos 4\theta + 4\cos 2\theta}{8} = \cos^4\theta$, as required.