SADLER UNIT 3 MATHEMATICS SPECIALIST

WORKED SOLUTIONS

Chapter 6: Systems of linear equations

Exercise 6A

Question 1

$x = 3$
$y = -2$
$z = 5$

Question 2

$x = 4$
$y = 7$
$z = -2$

Question 3

$z = 1$
$2y + 5z = 15$
$y = 5$
$2x + y + z = 4$
$x = -1$

Question 4

$$
\begin{aligned}
-z &= 3 \\
z &= -3 \\
3y + 2z &= 6 \\
y &= 4 \\
x - 2z &= 7 \\
x &= 1
\end{aligned}
$$

Question 5

$$
\begin{aligned}
2z &= 12 \\
z &= 6 \\
5y - 3z &= 2 \\
y &= 4 \\
x + 3y + 2z &= 27 \\
x &= 3
\end{aligned}
$$

Question 6

$$
\begin{aligned}
-3z &= 9 \\
z &= -3 \\
3y - 2z &= 0 \\
y &= -2 \\
2x + y + z &= -3 \\
x &= 1
\end{aligned}
$$

Question 7

$$
\begin{bmatrix} 3 & 2 & 10 \\ 1 & -4 & 8 \end{bmatrix}
$$

Question 8

$$
\begin{bmatrix} -1 & 5 & 12 \\ 2 & 3 & 2 \end{bmatrix}
$$

Question 9

$$\begin{bmatrix} 1 & 4 & 3 & 18 \\ 3 & 1 & 2 & 11 \\ 5 & 2 & 1 & 12 \end{bmatrix}$$

Question 10

$$\begin{bmatrix} 2 & 0 & 3 & 14 \\ 4 & 1 & -1 & 0 \\ 2 & 1 & 6 & 26 \end{bmatrix}$$

Question 11

$$\begin{bmatrix} 3 & 2 & 0 & 8 \\ 1 & 0 & 2 & 8 \\ 0 & 2 & -1 & -1 \end{bmatrix}$$

Question 12

$$\begin{bmatrix} 1 & 3 & -5 & 2 \\ 2 & 1 & 7 & 37 \\ -1 & 0 & 1 & 3 \end{bmatrix}$$

Question 13

$x + 3y = 34 \quad \leftarrow \text{Eq}^{\text{n}}①$

$2x + 5y = 59 \quad \leftarrow \text{Eq}^{\text{n}}②$

$\text{Eq}^{\text{n}}① \times 2 \qquad 2x + 6y = 68$

$\text{Eq}^{\text{n}}② \qquad 2x + 5y = 59$

$\text{Eq}^{\text{n}}① - \text{Eq}^{\text{n}}② \qquad y = 9$

$x + 3y = 34$ (Substitute $y = 9$ into equation)

$x = 7$

Question 14

$2x+3y=4 \quad \leftarrow \text{Eq}^{\text{n}}①$

$4x+9y=2 \quad \leftarrow \text{Eq}^{\text{n}}②$

$\text{Eq}^{\text{n}}①\times 2 \qquad 4x+6y=8$

$\text{Eq}^{\text{n}}② \qquad 4x+9y=2$

$\text{Eq}^{\text{n}}①-\text{Eq}^{\text{n}}② \qquad -3y=6$

$y=-2$

$4x+9y=2 \quad$ (Substitute $y=-2$ into equation)

$x=5$

Question 15

$$\begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \begin{bmatrix} 1 & 2 & 1 & 7 \\ 0 & 1 & 3 & 7 \\ 3 & 3 & 1 & 14 \end{bmatrix}$$

$$\begin{matrix} r_1 \\ r_2 \\ r_3-3r_1 \end{matrix} \begin{bmatrix} 1 & 2 & 1 & 7 \\ 0 & 1 & 3 & 7 \\ 0 & -3 & -2 & -7 \end{bmatrix}$$

$$\begin{matrix} r_1 \\ r_2 \\ r_3-3r_2 \end{matrix} \begin{bmatrix} 1 & 2 & 1 & 7 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 7 & 14 \end{bmatrix}$$

$7z=14$

$z=2$

$y+3z=7$

$y=1$

$x+2y+z=7$

$x=3$

Question 16

$$\begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \quad \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 4 & 6 \\ 2 & 3 & -3 & 20 \end{bmatrix}$$

$$\begin{matrix} r_1 \\ r_2 - r_1 \\ r_3 - 2r_1 \end{matrix} \quad \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & -5 & 8 \end{bmatrix}$$

$$\begin{matrix} r_1 \\ \text{new } r_2 \\ r_3 - r_2 \end{matrix} \quad \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -8 & 8 \end{bmatrix}$$

$$\begin{aligned} -8z &= 8 \\ z &= -1 \\ y + 3z &= 0 \\ y &= 3 \\ x + y + z &= 6 \\ x &= 4 \end{aligned}$$

Question 17

$x + 4z = -1$ Eqⁿ ①

$2x + y + 3z = 8$ Eqⁿ ②

$5x + y = 35$ Eqⁿ ③

$5 \times$ Eqⁿ ① $\quad 5x + 20z = -5 \quad$ new Eqⁿ ①

Eqⁿ ② $-$ 2Eqⁿ ① $\quad y - 5z = 10 \quad$ new Eqⁿ ②

Eqⁿ ③ $-$ Eqⁿ ① $\quad y - 20z = 40 \quad$ new Eqⁿ ③

Eqⁿ ③ $-$ Eqⁿ ② $\quad -15z = 30$

$z = -2$

From new equation ③ $y - 20(-2) = 40$

$y = 0$

From equation ① $\quad x + 4(-2) = -1$

$x = 7$

Question 18

$$\begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 3 & 2 & -1 \\ 3 & 7 & -2 & 6 \end{bmatrix}$$

$$\begin{matrix} r_1 \\ r_2 - 2r_1 \\ r_3 - 3r_1 \end{matrix} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 4 & -7 \\ 0 & 1 & 1 & -3 \end{bmatrix}$$

$$\begin{matrix} r_1 \\ -r_2 \\ r_3 + r_2 \end{matrix} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -4 & 7 \\ 0 & 0 & 5 & -10 \end{bmatrix}$$

$$\begin{aligned} z &= -2 \\ y - 4z &= 7 \\ y &= -1 \\ x + 2y - z &= 3 \\ x &= 3 \end{aligned}$$

Question 19

$$\begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \begin{bmatrix} 2 & 1 & 0 & 11 \\ 1 & 2 & -1 & 15 \\ 3 & 9 & 1 & 16 \end{bmatrix}$$

$$\begin{matrix} r_1 \leftrightarrow r_2 \\ r_1 \leftrightarrow r_2 \\ r_3 - 3r_1 \end{matrix} \begin{bmatrix} 1 & 2 & -1 & 15 \\ 2 & 1 & 0 & 11 \\ 0 & 3 & 4 & -29 \end{bmatrix}$$

$$\begin{matrix} r_1 \\ r_2 - 2r_1 \\ r_3 + r_2 \end{matrix} \begin{bmatrix} 1 & 2 & -1 & 15 \\ 0 & -3 & 2 & -19 \\ 0 & 0 & 6 & -48 \end{bmatrix}$$

$$\begin{aligned} z &= -8 \\ -3y + 2z &= -19 \\ y &= 1 \\ x + 2y - z &= 15 \\ x &= 5 \end{aligned}$$

Question 20

$$\begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \begin{bmatrix} 2 & 4 & -3 & 1 \\ 2 & 5 & -2 & 5 \\ 3 & 7 & -3 & 7 \end{bmatrix}$$

$$\begin{array}{c} r_1 \div 2 \\ r_2 - r_1 \\ r_3 - 3r_1 \end{array} \begin{bmatrix} 1 & 2 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 1 & 4 \\ 0 & 1 & \frac{3}{2} & 5\frac{1}{2} \end{bmatrix}$$

$$\begin{array}{c} r_1 \\ r_2 - 2r_1 \\ r_3 + r_2 \end{array} \begin{bmatrix} 1 & 2 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 1 & 4 \\ 0 & 0 & \frac{1}{2} & 1\frac{1}{2} \end{bmatrix}$$

$$z = 3$$

$$y + z = 4$$

$$y = 1$$

$$x + 2y - \frac{3}{2}z = \frac{1}{2}$$

$$x = 3$$

Question 21

$$\begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \begin{bmatrix} 3 & 4 & 5 & 14 \\ 5 & 7 & 6 & 13 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

$$\begin{array}{c} r_1 - 3r_3 \\ r_2 - 5r_3 \\ r_3 \end{array} \begin{bmatrix} 0 & 1 & 2 & 5 \\ 0 & 2 & 1 & -2 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

$$\begin{array}{c} r_1 \\ r_2 - 2r_1 \\ r_3 \end{array} \begin{bmatrix} 0 & 1 & 2 & 5 \\ 0 & 0 & -3 & -12 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

$$-3z = -12$$

$$z = 4$$

$$y + 2z = 5$$

$$y = -3$$

$$x + y + z = 3$$

$$x = 2$$

Question 22

$$\begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \begin{bmatrix} 2 & 0 & 1 & 4 \\ 2 & 3 & 3 & 3 \\ 5 & 1 & 3 & 10 \end{bmatrix}$$

$$\begin{matrix} r_1 \div 2 \\ r_2 - r_1 \\ r_3 - \frac{5}{2} r_1 \end{matrix} \begin{bmatrix} 1 & 0 & \frac{1}{2} & 2 \\ 0 & 3 & 2 & -1 \\ 0 & 1 & \frac{1}{2} & 0 \end{bmatrix}$$

$$\begin{matrix} r_1 \\ r_2 - 3r_3 \\ r_3 - r_2 \end{matrix} \begin{bmatrix} 1 & 0 & \frac{1}{2} & 2 \\ 0 & 0 & \frac{1}{2} & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$y = 1$$

$$\frac{1}{2} z = -1 \quad \Rightarrow \quad z = -2$$

$$x + \frac{1}{2} z = 2 \quad \Rightarrow \quad x = 3$$

Question 23

$$\begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \begin{bmatrix} 1 & 1 & 2 & 6 \\ 3 & 2 & 1 & 7 \\ 5 & 4 & 4 & 19 \end{bmatrix}$$

$$\begin{matrix} r_1 \\ r_2 - 3r_1 \\ r_3 - 5r_1 \end{matrix} \begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -5 & -11 \\ 0 & -1 & -6 & -11 \end{bmatrix}$$

$$\begin{matrix} r_1 \\ -r_2 \\ r_3 - r_2 \end{matrix} \begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & 1 & 5 & 11 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$z = 0$$

$$y + 5z = 11$$

$$y = 11$$

$$x + y + 2z = 6$$

$$x = -5$$

Question 24

$w + x - y + 3z = -1 \quad \leftarrow \text{Eq}^{\text{n}} ①$

$x + 2y - 3z = -2 \quad \leftarrow \text{Eq}^{\text{n}} ②$

$w + 2x + 2y + z = 0 \quad \leftarrow \text{Eq}^{\text{n}} ③$

$2w + 3x + 2y + 7z = 4 \quad \leftarrow \text{Eq}^{\text{n}} ④$

$\text{Eq}^{\text{n}} ① \quad w + x - y + 3z = -1$

$\text{Eq}^{\text{n}} ② \quad x + 2y - 3z = -2$

$\text{Eq}^{\text{n}} ③ - \text{Eq}^{\text{n}} ① \quad x + 3y - 2z = 1$

$\text{Eq}^{\text{n}} ④ - 2\text{Eq}^{\text{n}} ① \quad x + 4y + z = 6$

$\text{Eq}^{\text{n}} ① \quad w + x - y + 3z = -1$

$\text{Eq}^{\text{n}} ② \quad x + 2y - 3z = -2$

$\text{Eq}^{\text{n}} ③ - \text{Eq}^{\text{n}} ② \quad y + z = 3$

$\text{Eq}^{\text{n}} ④ - \text{Eq}^{\text{n}} ② \quad 2y + 4z = 8$

$\text{Eq}^{\text{n}} ① \quad w + x - y + 3z = -1$

$\text{Eq}^{\text{n}} ② \quad x + 2y - 3z = -2$

$\text{Eq}^{\text{n}} ③ \quad y + z = 3$

$\text{Eq}^{\text{n}} ④ - 2\text{Eq}^{\text{n}} ③ \quad 2z = 2$

$z = 1$

$y = 2$

$x = -3$

$w = 1$

Question 25

$5x + 3y = 270 \quad \leftarrow \text{Eq}^{\text{n}} ①$

$2x + 4y = 220 \quad \leftarrow \text{Eq}^{\text{n}} ②$

$4 \times \text{Eq}^{\text{n}} ① \quad 20x + 12y = 1080$

$3 \times \text{Eq}^{\text{n}} ② \quad 6x + 12y = 660$

$\text{Eq}^{\text{n}} ① - \text{Eq}^{\text{n}} ② \quad 14x = 420$

$x = 30$

$y = 40$

Question 26

$250p + 500q + 200r = 8000 \quad \leftarrow \text{Eq}^{\text{n}} ①$

$10p + 5q + 20r = 470 \quad \leftarrow \text{Eq}^{\text{n}} ②$

$50p + 100q + 100r = 2800 \quad \leftarrow \text{Eq}^{\text{n}} ③$

$\text{Eq}^{\text{n}} ① \qquad 250p + 500q + 200r = 8000$

$5 \times \text{Eq}^{\text{n}} ② \qquad 50p + 25q + 100r = 2350$

$\text{Eq}^{\text{n}} ③ - \text{Eq}^{\text{n}} ② \qquad 75q = 450$

$q = 6$

$\text{Eq}^{\text{n}} ① - 5\text{Eq}^{\text{n}} ② \qquad 375q - 300r = -3750$

$r = 20$

From $\text{Eq}^{\text{n}} ② \quad 10p + 5(6) + 20(20) = 470$

$p = 4$

The vet used 4 of tablet P, 6 of tablet Q and 20 of tablet R.

Question 27

a

$0.5x + 0.3y + 0.8z = 610 \quad \leftarrow \text{Eq}^{\text{n}} ①$

$0.1x + 0.5y + 0.1z = 180 \quad \leftarrow \text{Eq}^{\text{n}} ②$

$0.4x + 0.2y + 0.1z = 210 \quad \leftarrow \text{Eq}^{\text{n}} ③$

$5x + 3y + 8z = 6100 \quad \leftarrow \text{Eq}^{\text{n}} ①$

$1x + 5y + 1z = 1800 \quad \leftarrow \text{Eq}^{\text{n}} ②$

$4x + 2y + 1z = 2100 \quad \leftarrow \text{Eq}^{\text{n}} ③$

b

$-22y + 3z = -2900 \quad \leftarrow \text{Eq}^{\text{n}} ①$

$x + 5y + z = 1800 \quad \leftarrow \text{Eq}^{\text{n}} ②$

$-18y - 3z = -5100 \quad \leftarrow \text{Eq}^{\text{n}} ③$

$-40y = -8000$

$y = 200$

$-18(200) - 3z = -5100$

$z = 500$

$x + 5(200) + 500 = 1800$

$x = 300$

Exercise 6B

Question 1

If $k = 0$ there is no solution as $0 \neq 5$.

Question 2

If $k = 2$ there is no solution as $0 \neq 3$.

Question 3

If $k = -\frac{1}{2}$ there is no solution as $0 \neq 2$.

Question 4

Adding the two equations:

$$y + kz = 2 \text{ and} - y + 3z = 5$$
$$(3 + k)z = 7$$

If $k = -3$ there is no solution as $0 \neq 7$.

Question 5

By subtracting 1.5 times the last equation from the second equation:

$$3y + \quad kz = 4 \quad -$$
$$\underline{3y + 1.5z = 4.5}$$
$$(k - 1.5)z = -0.5$$

If $k = 1.5$ there is no solution as $0 \neq 1.5$.

Question 6

By adding twice the second equation to the last equation:

$$\begin{array}{rl} 2y-6z=2k & + \\ \underline{-2y+6z=-4} & \\ 0=2k-4 & \end{array}$$

If $k = 2$ there is a solution, for all other values of k there is no solution.

Hence $k \neq 2$.

Question 7

If $k = 2$ there is no solution as $0 \neq 4$.

Question 8

If $k = -1$ there is no solution as $0 \neq 5$.

Question 9

Twice the first equation minus the second equation will give:

$$\begin{array}{rl} 2x+4y+2kz=2 & - \\ \underline{2x-3y+z=5} & \\ 7y+\left(2k-1\right)z=-3 & \end{array}$$

Third equation minus three times the first equation:

$$\begin{array}{rl} 3x-y+4z=3 & - \\ \underline{3x+6y+3kz=3} & \\ -7y+(4-3k)z=0 & \end{array}$$

Add the two results found:

$$\begin{array}{rl} 7y+(2k-1)z=-3 & + \\ \underline{-7y+(4-3k)z=0} & \\ (2k-1+4-3k)z=-3 & \\ (-k+3)z=-3 & \end{array}$$

If $k = 3$ there is no solution as $0 \neq -3$.

Question 10

Second equation minus first equation:

$$\begin{array}{l} x + y + z = 0 \quad - \\ \underline{x + 3y - 6z = 3} \\ \quad -2y + 7z = -3 \end{array}$$

Last equation minus three times the first equation:

$$\begin{array}{l} 3x + 5y + (k+1)z = 2 \quad - \\ \underline{3x + 9y - \quad 18z = 9} \\ \quad -4y + (k+19)z = -7 \end{array}$$

Twice the first result minus the second:

$$\begin{array}{l} -4y + \quad 14z = -6 \quad - \\ \underline{-4y + (k+19)z = -7} \\ \left[14 - (k+19)\right] z = 1 \\ \qquad (-k-5)z = 1 \end{array}$$

If $k = -5$, there is no solution as $0 \neq 1$.

Question 11

The first equation minus twice the second equation:

$$\begin{array}{l} 2x + y + kz = -1 \quad - \\ \underline{2x + 8y + 4z = 10} \\ \quad -7y + (k-4)z = -11 \end{array}$$

The third equation minus three times the third:

$$\begin{array}{l} 3x - 2y + 4z = 1 \quad - \\ \underline{3x + 12y + 6z = -21} \\ \quad -14y - 2z = 22 \end{array}$$

The second result minus twice the first:

$$\begin{array}{l} -14y - 2z \qquad = 22 \quad - \\ \underline{-14y + 2(k-4)z = -22} \\ \left[-2 - 2(k-4)\right] z = 44 \\ \qquad (6-2k)z = 44 \end{array}$$

If $k = 3$, there is no solution as $0 \neq 44$.

Question 12

Add the first two equations:

$$\begin{array}{r} x+\ \ y+\qquad 3z=4 \qquad + \\ \underline{-x+5y+(k+1)z=6} \\ 6y+(k+4)z=10 \end{array}$$

The last equation minus twice the first equation:

$$\begin{array}{r} 2x-\ \ y+\ \ z=5 \qquad - \\ \underline{2x+2y+6z=8} \\ -3y-5z=-3 \end{array}$$

The first result add twice the second result:

$$\begin{array}{r} 6y+(k+4)z=10 \qquad + \\ \underline{-6y-\qquad 10z=-6} \\ (k+4-10)z=4 \\ (k-6)z=4 \end{array}$$

If k = 6, there is no solution as $0 \neq 4$.

Question 13

If k = 0, there are infinitely many solutions.

Question 14

If $k = -\frac{1}{2}$, there are infinitely many solutions.

Question 15

If k = –2, there are infinitely many solutions.

Question 16

If k = 0, there are infinitely many solutions.

Question 17

If $k = 0$, there are infinitely many solutions.

Question 18

Add twice the first to the second equation:

$$\begin{array}{r} 2x-4y+6z=-2 \quad + \\ \underline{2x-4y+6z=-2} \\ 0=0 \end{array}$$

So regardless of the value of k, there are infinitely many solutions.

k can take any value.

Question 19

The second equation minus twice the first equation:

$$\begin{array}{l} 2x+3y+kz=2 \quad - \\ \underline{2x-2y+2z=6} \\ 5y+(k-2)z=-4 \end{array}$$

The last equation minus four times the first equation:

$$\begin{array}{r} 4x+11y-5z=0 \quad - \\ \underline{4x-\ 4y+4z=12} \\ 15y-9z=-12 \end{array}$$

The second result minus three times the first result:

$$\begin{array}{r} 15y-\qquad 9z=-12 \quad - \\ \underline{15y+3(k-2)z=-12} \\ \left[-9-3(k-2)\right]z=0 \\ \left(-3-3k\right)z=0 \end{array}$$

If $k = -1$, there are infinitely many solutions.

Question 20

$$x+3y-2z=4 \quad \leftarrow \text{Eq}^{\text{n}}①$$
$$x+5y+(k-2)z=3 \quad \leftarrow \text{Eq}^{\text{n}}②$$
$$2x+(k+1)y-7z=9 \quad \leftarrow \text{Eq}^{\text{n}}③$$

$\text{Eq}^{\text{n}}② - \text{Eq}^{\text{n}}①$ $\quad 2y+kz=-1 \leftarrow \text{Eq}^{\text{n}}②'$

$\text{Eq}^{\text{n}}③ - 2\text{Eq}^{\text{n}}①$ $(k-5)y-3z=1 \quad \leftarrow \text{Eq}^{\text{n}}③'$

If $k=3, 2y+3z=-1 \leftarrow \text{Eq}^{\text{n}}②'$

and $\quad -2y-3z=1 \quad \leftarrow \text{Eq}^{\text{n}}③'$

$$0=0 \quad \leftarrow \text{Eq}^{\text{n}}②' + \text{Eq}^{\text{n}}③'$$

If $k = 3$, there are infinitely many solutions.

Question 21

Given $\begin{cases} x+py=5 \\ 2x+3y=q \end{cases}$

The second equation minus twice the first gives $3-2p=q-10$

a There are infinitely many solutions when $p=1.5$ and $q=10$.

b There are no solutions when $p=1.5$ and $q \neq 10$

c The system will have a unique solution when $p \neq 1.5$, there is no restriction on the value of q.

Question 22

Given $\begin{cases} px+4y=6 \\ 9x+6y=q \end{cases}$

The second equation minus one and a half times the first gives $9-1.5p=q-9$

a There are infinitely many solutions when $p=6$ and $q=9$.

b There are no solutions when $p=6$ and $q \neq 9$

c The system will have a unique solution when $p \neq 6$, there is no restriction on the value of q.

Question 23

$x \ +2y+ \ \ z=3 \quad \leftarrow \text{Eq}^{\text{n}}①$

$x \ +3y-2z=7 \quad \leftarrow \text{Eq}^{\text{n}}②$

$3x+4y+pz=q \quad \leftarrow \text{Eq}^{\text{n}}③$

$\text{Eq}^{\text{n}}②-\text{Eq}^{\text{n}}① \qquad y-3z=4 \quad \leftarrow \text{Eq}^{\text{n}}②'$

$\text{Eq}^{\text{n}}③-3\text{Eq}^{\text{n}}①-2y+(p-3)z=q-9 \leftarrow \text{Eq}^{\text{n}}③'$

$\text{Eq}^{\text{n}}③'+2\text{Eq}^{\text{n}}②' \qquad (p-9)z=q-1$

There are infinitely many solutions when $p=9$ and $q=1$.

Question 24

$x \ +3y- \ \ z=2 \quad \leftarrow \text{Eq}^{\text{n}}①$

$2x+8y-2z=q \quad \leftarrow \text{Eq}^{\text{n}}②$

$x \ -3y+ \ pz=-1 \leftarrow \text{Eq}^{\text{n}}③$

$\text{Eq}^{\text{n}}②-2\text{Eq}^{\text{n}}① \qquad 2y=q-4 \quad \leftarrow \text{Eq}^{\text{n}}②'$

$\text{Eq}^{\text{n}}③-\text{Eq}^{\text{n}}① \ \ -6y+(p+1)z=-3 \quad \leftarrow \text{Eq}^{\text{n}}③'$

$\text{Eq}^{\text{n}}③'+3\text{Eq}^{\text{n}}②' \qquad (p+1)z=3q-15$

There are infinitely many solutions when $p=-1$ and $q=5$.

Question 25

$x \ -2y+ \ \ z=-2 \quad \leftarrow \text{Eq}^{\text{n}}①$

$x \ + \ \ y+ \ \ z=7 \quad \leftarrow \text{Eq}^{\text{n}}②$

$-x+5y+ \ pz=-4 \quad \leftarrow \text{Eq}^{\text{n}}③$

$\text{Eq}^{\text{n}}②-\text{Eq}^{\text{n}}① \qquad 3y=9 \quad \leftarrow \text{Eq}^{\text{n}}②'$

$\text{Eq}^{\text{n}}③+\text{Eq}^{\text{n}}① \ \ 3y+(p+1)z=-6 \quad \leftarrow \text{Eq}^{\text{n}}③'$

$\text{Eq}^{\text{n}}③'-\text{Eq}^{\text{n}}②' \qquad (p+1)z=-15$

There is a unique solution when $p \neq -1$.

Question 26

$$5x + 2y - z = 2 \quad \leftarrow \text{Eq}^{\text{n}} ①$$
$$2x + y = 1 \quad \leftarrow \text{Eq}^{\text{n}} ②$$
$$-3x + y + pz = 1 \quad \leftarrow \text{Eq}^{\text{n}} ③$$
$$\text{Eq}^{\text{n}} ② - \frac{2}{5}\text{Eq}^{\text{n}} ① \qquad \frac{1}{5}y + \frac{2}{5}z = \frac{1}{5} \quad \leftarrow \text{Eq}^{\text{n}} ②'$$
$$\text{Eq}^{\text{n}} ③ + \frac{3}{5}\text{Eq}^{\text{n}} ① \quad \frac{11}{5}y + \left(p - \frac{3}{5}\right)z = \frac{11}{5} \quad \leftarrow \text{Eq}^{\text{n}} ③'$$
$$\text{Eq}^{\text{n}} ③' - 11\text{Eq}^{\text{n}} ②' \qquad (p-5)z = 0$$

There is a unique solution when $p \neq 5$.

Question 27

$$x + 3y - z = 5 \quad \leftarrow \text{Eq}^{\text{n}} ①$$
$$-x + 3y + z = 5 \quad \leftarrow \text{Eq}^{\text{n}} ②$$
$$2x + 6y - 2z = 10 \quad \leftarrow \text{Eq}^{\text{n}} ③$$
$$\text{Eq}^{\text{n}} ② + \text{Eq}^{\text{n}} ① \qquad 6y = 10 \quad \leftarrow \text{Eq}^{\text{n}} ②'$$
$$\text{Eq}^{\text{n}} ③ - 2\text{Eq}^{\text{n}} ① \qquad 0 = 0 \quad \leftarrow \text{Eq}^{\text{n}} ③'$$

This system of equations has infinitely many solutions.

Question 28

$$x + 2y + z = 4 \quad \leftarrow \text{Eq}^{\text{n}} ①$$
$$y - 3z = 1 \quad \leftarrow \text{Eq}^{\text{n}} ②$$
$$(2k-1)y = m + 1 \quad \leftarrow \text{Eq}^{\text{n}} ③$$

a There is a unique solution when $k \neq \frac{1}{2}$, there is no restriction on the value of *m*.

b There is no solution when $k = \frac{1}{2}$ and $m \neq -1$.

c There are infinitely many solutions when $k = \frac{1}{2}$ and $m = -1$.

Question 29

$x \; - \; y + 2z = 12 \;\; \leftarrow \text{Eq}^{\text{n}}①$

$-x \; -2y + \; z = \; 3 \;\; \leftarrow \text{Eq}^{\text{n}}②$

$8x \; +7y + \; pz = q \;\; \leftarrow \text{Eq}^{\text{n}}③$

$\text{Eq}^{\text{n}}② + \text{Eq}^{\text{n}}① \qquad -3y + 3z = 15 \quad \leftarrow \text{Eq}^{\text{n}}②'$

$\text{Eq}^{\text{n}}③ - 8\text{Eq}^{\text{n}}① 15y + (p-16)z = q - 96 \quad \leftarrow \text{Eq}^{\text{n}}③'$

$\text{Eq}^{\text{n}}③' + 5\text{Eq}^{\text{n}}②' \qquad (p-1)z = \text{q} - 21$

a When $p = 1$ and $q = 10$, $0 = -11$, so there is no solution.

b When $p = 1$ and $q = 21$, $0 = 0$, so there are infinitely many solutions.

c When $p = 7$ and $q = 45$, $6z = 24$, so there is one solution, (3, –1, 4).

Question 30

$x - \; y = m \;\; \leftarrow \text{Eq}^{\text{n}}①$

$x + ky - 3z = 7 \;\; \leftarrow \text{Eq}^{\text{n}}②$

$4x - y - 3z = 3 \;\; \leftarrow \text{Eq}^{\text{n}}③$

$\text{Eq}^{\text{n}}② - \text{Eq}^{\text{n}}① \qquad (k+1)y - 3z = 7 - m \quad \leftarrow \text{Eq}^{\text{n}}②'$

$\text{Eq}^{\text{n}}③ - 4\text{Eq}^{\text{n}}① \qquad 3y - 3z = 3 - 4m \quad \leftarrow \text{Eq}^{\text{n}}③'$

$\text{Eq}^{\text{n}}②' - \text{Eq}^{\text{n}}③' \qquad (k-2)y = 4 + 3m$

a When $k \neq 2$, for any value of m, there is a unique solution.

b When $k = 2$ and $m \neq -\dfrac{4}{3}$, there is no solution.

c When $k = 2$ and $m = -\dfrac{4}{3}$, there are infinitely many solutions.

Question 1

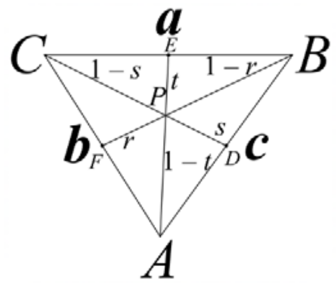
$\overrightarrow{AB} = \boldsymbol{c}$, $\overrightarrow{BC} = \boldsymbol{a}$, $\overrightarrow{CA} = \boldsymbol{b}$

D is the midpoint of AB

E is the midpoint of BC

F is the midpoint of CA

P is the point where the medians of the triangle meet.



$\overrightarrow{FB} = \frac{1}{2}\boldsymbol{b} + \boldsymbol{c}$

$\overrightarrow{FP} = r\left(\frac{1}{2}\boldsymbol{b} + \boldsymbol{c}\right)$

$\overrightarrow{CD} = \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}$

$\overrightarrow{CP} = (1-s)\left(\boldsymbol{b} + \frac{1}{2}\boldsymbol{c}\right)$

$\overrightarrow{FP} = -\frac{1}{2}\boldsymbol{b} + (1-s)\left(\boldsymbol{b} + \frac{1}{2}\boldsymbol{c}\right)$

Comparing the coefficients of $\boldsymbol{b}$ and $\boldsymbol{c}$ in the two equations for $\overrightarrow{FP}$ gives:

$\frac{1}{2}r = -\frac{1}{2} + 1 - s \quad \Rightarrow \quad r = 1 - 2s$

and

$r = \frac{1}{2} - \frac{1}{2}s$

Substitute $r = 1 - 2s$ into $r = \frac{1}{2} - \frac{1}{2}s$

$1 - 2s = \frac{1}{2} - \frac{1}{2}s$

$-\frac{3}{2}s = -\frac{1}{2}$

So $s = \frac{1}{3}$ and $r = \frac{1}{3}$.

r is not the length of the median closest to the vertex, so the median of the triangle closest to the vertex will be $1 - \frac{1}{3} = \frac{2}{3}$.

Which means that the medians of a triangle intersect at a point that is two thirds of the way along each median, measured from the end of the median that is a vertex of the triangle.

Question 2

Find the extreme points for the range of values

$$x-a=5+x+3$$
$$-a=8$$
$$a=-8$$

OR

$$-x+a=5-x-3$$
$$a=2$$

a For there to be exactly two solutions $\{a\in\mathbb{R}:-8<a<2\}$.

b There are more than two solutions when $a=-8,\ a=2$.

Question 3

a P_1 is the y-intercept of the function with equation $y=|x-a|$. $P_1(0,a)$.

P_2 is the y-intercept of the function with equation $y=|0.5x-b|$. $P_2=(0,b)$.

b P_1 is clearly above P_2, so $a>b$.

c P_4 is the x-intercept of the function with equation $y=|x-a|$. $P_4=(a,0)$.

P_6 is the x-intercept of the function with equation $y=|0.5x-b|$. $P_6=(2b,0)$.

d $|x-a|=|0.5x-b|$

$$x-a=0.5x-b$$
$$0.5x=a-b$$
$$x=2a-2b$$

$y=|x-a|=|2a-2b-a|=|a-2b|=2b-a$ in the first quadrant.

OR

$$x-a=-(0.5x-b)$$
$$1.5x=a+b$$
$$x=\frac{2}{3}a+\frac{2}{3}b$$

$y=|x-a|=\left|\frac{2}{3}a-\frac{2}{3}b-a\right|=\left|-\frac{a}{3}-\frac{2}{3}b\right|=\frac{2b-a}{3}$ in the first quadrant.

The solutions are $(2a-2b,2b-a)$ and $\left(\frac{2}{3}a+\frac{2}{3}b,\ \frac{2}{3}b-\frac{1}{3}a\right)$

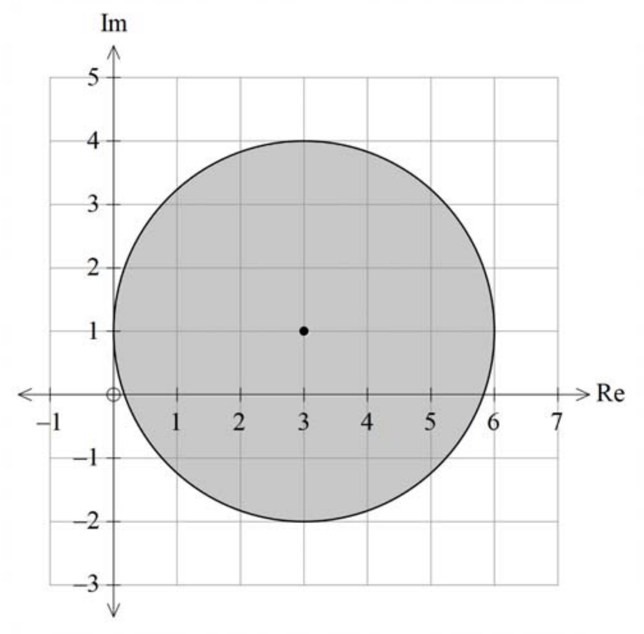
e $\{(x,y):y<|x-a|\text{ and } y<|0.5x-b|\}$ is made up of the regions A, E and G.

f $\{(x,y):y>|x-a|\text{ and } y>|0.5x-b|\}$ is the region C.

g $\{(x,y):y<|x-a|\text{ and } y>|0.5x-b|\}$ is made up of the regions B and F.

h $\{(x,y):y>|x-a|\text{ and } y<|0.5x-b|\}$ is the region D.

Question 4



Question 5

a $-5\left(\sqrt{3}+i\right)=-5\sqrt{3}-5i$

$$r=\sqrt{\left(-5\sqrt{3}\right)^2+(-5)^2}=10$$

$$\tan\theta=\frac{-5}{-5\sqrt{3}}$$

$$\theta=-\frac{5\pi}{6}$$

$$-5\left(\sqrt{3}+i\right)=10\operatorname{cis}\left(-\frac{5\pi}{6}\right)$$

b

$$6\operatorname{cis}\frac{3\pi}{4}=6\cos\frac{3\pi}{4}+6\sin\frac{3\pi}{4}i$$

$$=-3\sqrt{2}+3\sqrt{2}i$$

Question 6

$$|z-5| = |x+iy-5| = |x-5+iy| = \sqrt{(x-5)^2+y^2}$$

$$|z+5i| = |x+yi+5i| = |x+i(y+5)| = \sqrt{x^2+(y+5)^2}$$

$$\begin{aligned}
3|z-5| &= 2|z+5i| \\
3\sqrt{(x-5)^2+y^2} &= 2\sqrt{x^2+(y+5)^2} \\
9\left[(x-5)^2+y^2\right] &= 4\left[x^2+(y+5)^2\right] \\
9(x^2-10x+25)+9y^2 &= 4x^2+4(y^2+10y+25) \\
9x^2-90x+225+9y^2 &= 4x^2+4y^2+40y+100 \\
5x^2-90x+5y^2-40y &= -125 \\
x^2-18x+y^2-8y &= -25 \\
(x-9)^2-81+(y-4)^2-16 &= -25 \\
(x-9)^2+(y-4)^2 &= 72
\end{aligned}$$

Question 7

a $z = 1-i$

$r = \sqrt{1^2+1^2} = \sqrt{2}$

$\tan\theta = -1$

$\theta = -\frac{\pi}{4}$

$z = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

b $z^{14} = \sqrt{2}^{14}\,\text{cis}\left[14\times\left(-\frac{\pi}{4}\right)\right] = 128\,\text{cis}\left(-\frac{7\pi}{2}\right)$

$= 128\,\text{cis}\frac{\pi}{2} = 128\cos\frac{\pi}{2} + 128i\sin\frac{\pi}{2}$

$= 0+128i = 128i$

Question 8

$2\mathbf{i}+2\mathbf{k}+\lambda(2\mathbf{i}+3\mathbf{j}-2\mathbf{k})=-2\mathbf{i}+\mathbf{j}+6\mathbf{k}+\mu(-2\mathbf{i}-\mathbf{j}+2\mathbf{k})$

$2+2\lambda=-2-2\mu$

$0+3\lambda=1-\mu$

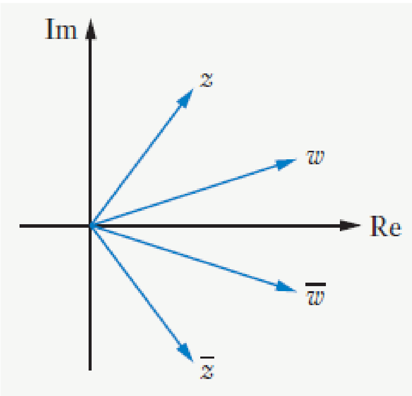
$2-2\lambda=6+2\mu$

Solving gives $\lambda=\frac{3}{2}$ and $\mu=-\frac{7}{2}$.

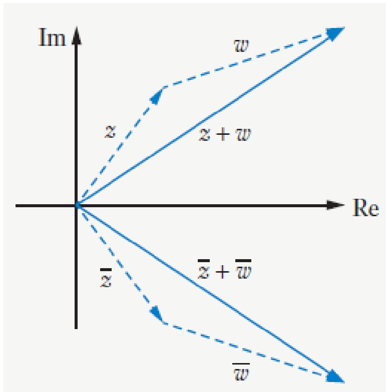
The lines intersect at $5\mathbf{i}+\frac{9}{2}\mathbf{j}-\mathbf{k}$

Question 9

a



b

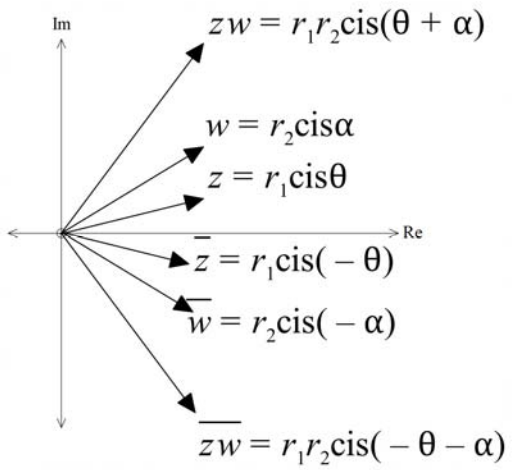


$\overline{z+w}$ is the reflection of $(z+w)$ in the real axis.

But, from the diagram, $\bar{z}+\bar{w}$ is a reflection of $(z+w)$ in the real axis.

Hence $\overline{z+w}=\bar{z}+\bar{w}$.

c



$\overline{zw}$ is the reflection of zw in the real axis.

From the labelled diagram, $\overline{zw}=\bar{z}\times\bar{w}$.

Question 10

$360° \div 4 = 90°$

Given $z_1 = 2\,\text{cis}\,40°$

$z^4 = 2^4\,\text{cis}\,(4 \times 40°) = 16\,\text{cis}\,160°$

$z_2 = 2\,\text{cis}\,130°$

$z_3 = 2\,\text{cis}\,(-140°)$

$z_4 = 2\,\text{cis}\,(-50°)$

Question 11

a Given $f(x) = \dfrac{1}{\sqrt{x-3}} + 4$

Domain $\{x \in \mathbb{R} : x > 3\}$ so that the denominator is the square root of a number ≥ 0.

Range $\{y \in \mathbb{R} : y > 4\}$ as the lowest value of $\dfrac{1}{\sqrt{x-3}}$ approaches but does not equal 0.

b Function: $x \to x - 3 \to \dfrac{1}{\sqrt{x-3}} \to \dfrac{1}{\sqrt{x-3}} + 4$

Inverse: $x \to x - 4 \to \dfrac{1}{(x-4)^2} \to \dfrac{1}{(x-4)^2}$

$f^{-1}(x) = \dfrac{1}{(x-4)^2} + 3$

Domain $\{x \in \mathbb{R} : x > 4\}$

Range $\{y \in \mathbb{R} : y > 3\}$

Question 12

By de Moivre's theorem

$$\cos 4\theta + i\,\sin 4\theta = (\cos\theta + i\sin\theta)^4$$
$$= \cos^4\theta + 4\cos^3\theta\, i\sin\theta + 6\cos^2\theta\, i^2\sin^2\theta + 4\cos\theta\, i^3\sin^3\theta + i^4\sin^4\theta$$

Equating the real parts $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

Equating the imaginary parts $\sin 4\theta = 4\cos^3\theta\,\sin\theta - 4\cos x\sin^3\theta$

Question 13

The plane $\mathbf{r} \bullet \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = -14$ is perpendicular to the line with vector equation $\mathbf{r} = \lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$.

The plane $\mathbf{r} \bullet \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = 42$ is perpendicular to the line with vector equation $\mathbf{r} = \lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$.

The normal to the first plane is a scalar multiple of the normal to the second plane, therefore the planes are parallel.

The plane $\mathbf{r} \bullet \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = -14$ meets the line $\mathbf{r} = \lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$ at point *A*.

Substitute the equation of the line into the $\mathbf{r}$ value of the plane to get:

$$\lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = -14$$

$$4\lambda + 9\lambda + 36\lambda = -14$$

$$49\lambda = -14 \quad \Rightarrow \quad \lambda = -\frac{2}{7}$$

Point *A* has position vector $\mathbf{r} = \begin{pmatrix} -\frac{4}{7} \\ \frac{6}{7} \\ -\frac{12}{7} \end{pmatrix}$. The plane $\mathbf{r} \bullet \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = 42$ meets the line $\mathbf{r} = \lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$ at point *B*.

Substitute the equation of the line into the $\mathbf{r}$ value of the plane to get:

$$\left(\lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \right) \bullet \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = 42$$

$$4\lambda + 9\lambda + 36\lambda = 42$$

$$49\lambda = 42 \quad \Rightarrow \quad \lambda = \frac{6}{7}$$

Point *B* has position vector $\mathbf{r} = \begin{pmatrix} \frac{12}{7} \\ -\frac{18}{7} \\ \frac{36}{7} \end{pmatrix}$.

$$\overrightarrow{AB} = \begin{pmatrix} \frac{12}{7} \\ -\frac{18}{7} \\ \frac{36}{7} \end{pmatrix} - \begin{pmatrix} -\frac{4}{7} \\ \frac{6}{7} \\ -\frac{12}{7} \end{pmatrix} = \begin{pmatrix} \frac{16}{7} \\ -\frac{24}{7} \\ \frac{48}{7} \end{pmatrix}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{\left(\frac{16}{7}\right)^2 + \left(-\frac{24}{7}\right)^2 + \left(\frac{48}{7}\right)^2} = 8 \text{ units}$$

Question 14

$$\overrightarrow{OX} = \begin{pmatrix} 2+5\lambda \\ 3+2\lambda \\ -1+\lambda \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2+5\lambda \\ 3+2\lambda \\ -1+\lambda \end{pmatrix} = \mathbf{b}$$

$$\mathbf{a} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$$

Let X be the point on the line that is closest to the origin, O is the origin.

$$\mathbf{a} \bullet \mathbf{b} = 0$$

$$\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2+5\lambda \\ 3+2\lambda \\ -1+\lambda \end{pmatrix} = 0$$

$$10 + 25\lambda + 6 + 4\lambda - 1 + \lambda = 0$$

$$30\lambda + 15 = 0$$

$$\lambda = -\frac{1}{2}$$

$$\overrightarrow{OX} = \begin{pmatrix} -\frac{1}{2} \\ 2 \\ -\frac{3}{2} \end{pmatrix}$$

$$\left|\overrightarrow{OX}\right| = \sqrt{\left(-\frac{1}{2}\right)^2 + 2^2 + \left(-\frac{3}{2}\right)^2} = \frac{\sqrt{26}}{2}$$

The shortest distance from the line to the origin is $\frac{\sqrt{26}}{2}$ units.

Question 15

$$pz - 3y = 1 + x \quad \leftarrow \text{Eq}^n \text{①}$$

$$x + 3y = 3 + 2z \quad \leftarrow \text{Eq}^n \text{②}$$

$$2x + 6y - 6z = 5 \quad \leftarrow \text{Eq}^n \text{③}$$

Eq^n ① $\quad -x - 3y + p = 1$

Eq^n ② $\quad x + 3y - 2z = 3$

Eq^n ③ $\quad 2x + 6y - 2z = 5$

Transfer this information to a matrix for ease of solving.

$$\begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \begin{bmatrix} -1 & -3 & p & 1 \\ 1 & 3 & -2 & 3 \\ 2 & 6 & -6 & 5 \end{bmatrix}$$

$$\begin{matrix} -r_1 \\ r_2 + r_1 \\ r_3 + 2r_1 \end{matrix} \begin{bmatrix} 1 & 3 & p & 1 \\ 0 & 0 & p-2 & 4 \\ 0 & 0 & 2p-6 & 7 \end{bmatrix}$$

$$\begin{matrix} r_1 \\ r_2 \\ r_3 - 2r_2 \end{matrix} \begin{bmatrix} 1 & 3 & p & 1 \\ 0 & 0 & p-2 & 4 \\ 0 & 0 & -2 & -1 \end{bmatrix}$$

This system of equations will have no solution when $p = 2$, as the second row will become 0 = 4, which is not possible.

If row 3 was multiplied by –4, then the matrix would become:

$$\begin{bmatrix} 1 & 3 & p & 1 \\ 0 & 0 & p-2 & 4 \\ 0 & 0 & 8 & 4 \end{bmatrix}$$

It follows that there will be infinitely many solutions when $p - 2 = 8$, so when $p = 10$.