SADLER UNIT 4 MATHEMATICS SPECIALIST

WORKED SOLUTIONS

Chapter 8: Differentiation techniques and applications

Exercise 8A

Question 1

Given $xy + 8x = 10 - 2y$

Differentiate with respect to *x*:

$$\frac{d}{dx}(xy) + \frac{d}{dx}(8x) = \frac{d}{dx}(10) - \frac{d}{dx}(2y)$$

$$x\frac{d}{dx}(y) + y\frac{d}{dx}(x) + \frac{d}{dx}(8x) = \frac{d}{dx}(10) - \frac{d}{dx}(2y)$$

$$x\frac{d}{dy}(y)\frac{dy}{dx} + y\frac{d}{dx}(x) + \frac{d}{dx}(8x) = \frac{d}{dx}(10) - \frac{d}{dy}(2y)\frac{dy}{dx}$$

$$x\frac{dy}{dx} + y + 8 = 0 - 2\frac{dy}{dx}$$

$$(x+2)\frac{dy}{dx} = -y - 8$$

$$\frac{dy}{dx} = \frac{-y-8}{(x+2)}$$

$$\frac{dy}{dx} = -\frac{y+8}{x+2}$$

Question 2

Given $xy + y - 4x = 3x^2 - 5$

Differentiate with respect to *x*:

$$\frac{d}{dx}(xy) + \frac{d}{dx}(y) - \frac{d}{dx}(4x) = \frac{d}{dx}(3x^2) - \frac{d}{dx}(5)$$

$$x\frac{d}{dx}(y) + y\frac{d}{dx}(x) + \frac{d}{dx}(y) - \frac{d}{dx}(4x) = \frac{d}{dx}(3x^2) - \frac{d}{dx}(5)$$

$$x\frac{d}{dy}(y)\frac{dy}{dx} + y\frac{d}{dx}(x) + \frac{d}{dy}(y)\frac{dy}{dx} - \frac{d}{dx}(4x) = \frac{d}{dx}(3x^2) - \frac{d}{dx}(5)$$

$$x\frac{dy}{dx} + y + \frac{dy}{dx} - 4 = 6x$$

$$\frac{dy}{dx}(x+1) = 6x - y + 4$$

$$\frac{dy}{dx} = \frac{6x - y + 4}{x+1}$$

Question 3

Given $y^3 - 2x = 3x^2 y$

Differentiate with respect to *x*:

$$\frac{d}{dx}(y^3) - \frac{d}{dx}(2x) = \frac{d}{dx}(3x^2 y)$$

$$\frac{d}{dy}(y^3)\frac{dy}{dx} - \frac{d}{dx}(2x) = 3x^2\frac{d}{dy}(y)\frac{dy}{dx} + y\frac{d}{dx}(3x^2)$$

$$3y^2\frac{dy}{dx} - 2 = 3x^2\frac{dy}{dx} + y \times 6x$$

$$\frac{dy}{dx}\left(3y^2 - 3x^2\right) = 6xy + 2$$

$$\frac{dy}{dx} = \frac{6xy + 2}{3y^2 - 3x^2}$$

$$= \frac{2(1+3xy)}{3(y^2 - x^2)}$$

Question 4

Given $y^2 = 2x^3y + 5x$

Differentiate with respect to *x*:

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(2x^3y) + \frac{d}{dx}(5x)$$

$$\frac{d}{dy}(y^2)\frac{dy}{dx} = 2x^3\frac{d}{dy}(y)\frac{dy}{dx} + y\frac{d}{dx}(2x^3) + \frac{d}{dx}(5x)$$

$$2y\frac{dy}{dx} = 2x^3\frac{dy}{dx} + y6x^2 + 5$$

$$\frac{dy}{dx}(2y - 2x^3) = 6x^2y + 5$$

$$\frac{dy}{dx} = \frac{6x^2y + 5}{2(y - x^3)}$$

Question 5

Given $5y^2 = x^2 + 2xy - 3x$

Differentiate with respect to *x*:

$$\frac{d}{dx}(5y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(2xy) - \frac{d}{dx}(3x)$$

$$\frac{d}{dy}(5y^2)\frac{dy}{dx} = 2x + 2x\frac{d}{dy}(y)\frac{dy}{dx} + y \times 2 - 3$$

$$10y\frac{dy}{dx} = 2x + 2x\frac{dy}{dx} + 2y - 3$$

$$\frac{dy}{dx}(10y - 2x) = 2x + 2y - 3$$

$$\frac{dy}{dx} = \frac{2x + 2y - 3}{2(5y - x)}$$

Question 6

Given $x+3y^2=5+x^2+2xy$

Differentiate with respect to *x*:

$$\frac{d}{dx}(x)+\frac{d}{dx}(3y^2)=\frac{d}{dx}(5)+\frac{d}{dx}(x^2)+\frac{d}{dx}(2xy)$$

$$1+\frac{d}{dy}(3y^2)\frac{dy}{dx}=0+2x+2x\frac{d}{dy}(y)\frac{dy}{dx}+y\frac{d}{dx}(2x)$$

$$1+6y\frac{dy}{dx}=2x+2x\frac{dy}{dx}+y\times 2$$

$$\frac{dy}{dx}(6y-2x)=2x+2y-1$$

$$\frac{dy}{dx}=\frac{2x+2y-1}{2(3y-x)}$$

Question 7

Given $x^2+y^2=9x$

Differentiate with respect to *x*:

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=\frac{d}{dx}(9x)$$

$$2x+\frac{d}{dy}(y^2)\frac{dy}{dx}=9$$

$$2x+2y\frac{dy}{dx}=9$$

$$\frac{dy}{dx}=\frac{9-2x}{2y}$$

Question 8

Given $x^2 + y^2 = 9y$

Differentiate with respect to *x*:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(9y)$$

$$2x + \frac{d}{dy}(y^2)\frac{dy}{dx} = \frac{d}{dy}(9y)\frac{dy}{dx}$$

$$2x + 2y\frac{dy}{dx} = 9\frac{dy}{dx}$$

$$\frac{dy}{dx}(9 - 2y) = 2x$$

$$\frac{dy}{dx} = \frac{2x}{9 - 2y}$$

Question 9

Given $x^2 + y^2 = 9xy$

Differentiate with respect to *x*:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(9xy)$$

$$2x + \frac{d}{dy}(y^2)\frac{dy}{dx} = 9x\frac{d}{dy}(y)\frac{dy}{dx} + y\frac{d}{dx}(9x)$$

$$2x + 2y\frac{dy}{dx} = 9x\frac{dy}{dx} + y \times 9$$

$$\frac{dy}{dx}(2y - 9x) = 9y - 2x$$

$$\frac{dy}{dx} = \frac{9y - 2x}{2y - 9x}$$

Question 10

Given $x^2 + y^2 = 9xy + x + y$

Differentiate with respect to *x*:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(9xy) + \frac{d}{dx}(x) + \frac{d}{dx}(y)$$

$$2x + \frac{d}{dy}(y^2)\frac{dy}{dx} = 9x\frac{d}{dy}(y)\frac{dy}{dx} + y\frac{d}{dx}(9x) + \frac{d}{dx}(x) + \frac{d}{dy}(y)\frac{dy}{dx}$$

$$2x + 2y\frac{dy}{dx} = 9x\frac{dy}{dx} + y \times 9 + 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx}(2y - 9x - 1) = 9y + 1 - 2x$$

$$\frac{dy}{dx} = \frac{9y + 1 - 2x}{2y - 9x - 1}$$

Question 11

Given $\sin x + \cos y = 10$

Differentiate with respect to *x*:

$$\frac{d}{dx}(\sin x) + \frac{d}{dy}(\cos y)\frac{dy}{dx} = \frac{d}{dx}(10)$$

$$\cos x - \sin y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{\cos x}{\sin y}$$

Question 12

Given $3+x^2\cos y=10xy$

Differentiate with respect to *x*:

$$\frac{d}{dx}(3)+\frac{d}{dx}(x^2\cos y)=\frac{d}{dx}(10xy)$$

$$0+x^2\frac{d}{dy}(\cos y)\frac{dy}{dx}+\cos y\frac{d}{dx}(x^2)=10x\frac{d}{dy}(y)\frac{dy}{dx}+y\frac{d}{dx}(10x)$$

$$x^2(-\sin y)\frac{dy}{dx}+\cos y\times 2x=10x\frac{dy}{dx}+y\times 10$$

$$\frac{dy}{dx}(-10x-x^2\sin y)+2x\cos y=10y$$

$$\frac{dy}{dx}=\frac{2(5y-x\cos y)}{-(10x+x^2\sin y)}$$

$$\frac{dy}{dx}=\frac{2(x\cos y-5y)}{10x+x^2\sin y}$$

Question 13

Given $6x+xy+20+2y=0$

Differentiate with respect to *x*:

$$\frac{d}{dx}(6x)+\frac{d}{dx}(xy)+\frac{d}{dx}(20)+\frac{d}{dy}(2y)\frac{dy}{dx}=0$$

$$6+x\frac{d}{dy}(y)\frac{dy}{dx}+y\frac{d}{dx}(x)+0+2\frac{dy}{dx}=0$$

$$6+x\frac{dy}{dx}+y+2\frac{dy}{dx}=0$$

$$\frac{dy}{dx}(x+2)=-y-6$$

$$\frac{dy}{dx}=\frac{-y-6}{x+2}$$

At the point (–3, 2), $\frac{dy}{dx}=\frac{-2-6}{-3+2}=\frac{-8}{-1}=8$

The gradient of $6x+xy+20+2y=0$ is 8.

Question 14

Given $6y + xy = 10 + 3x$

Differentiate with respect to *x*:

$$\frac{d}{dy}(6y)\frac{dy}{dx} + \frac{d}{dx}(xy) = \frac{d}{dx}(10) + \frac{d}{dx}(3x)$$

$$\frac{d}{dy}(6y)\frac{dy}{dx} + \frac{d}{dx}(xy) = \frac{d}{dx}(10) + \frac{d}{dx}(3x)$$

$$6\frac{dy}{dx} + x\frac{d}{dy}(y)\frac{dy}{dx} + y\frac{d}{dx}(x) = 0 + 3$$

$$6\frac{dy}{dx} + x\frac{dy}{dx} + y = 3$$

$$\frac{dy}{dx} = \frac{3 - y}{x + 6}$$

At the point (2, 2), $\frac{dy}{dx} = \frac{1}{8} = 0.125$

The gradient of $6x + xy + 20 + 2y = 0$ is 0.125.

Question 15

Given $5 + x^3 = xy + y^2$

Differentiate with respect to *x*:

$$\frac{d}{dx}(5) + \frac{d}{dx}(x^3) = \frac{d}{dx}(xy) + \frac{d}{dy}(y^2)\frac{dy}{dx}$$

$$0 + 3x^2 = x\frac{d}{dy}(y)\frac{dy}{dx} + y\frac{d}{dx}(x) + 2y\frac{dy}{dx}$$

$$3x^2 = x\frac{dy}{dx} + y + 2y\frac{dy}{dx}$$

$$\frac{dy}{dx}(x + 2y) = 3x^2 - y$$

$$\frac{dy}{dx} = \frac{3x^2 - y}{x + 2y}$$

At the point (1, –3), $\frac{dy}{dx} = \frac{3(1)^2 - (-3)}{1 + 2(-3)} = -\frac{6}{5} = -1.2$

Question 16

Given $y^2 + 3xy = 4x$

Differentiate with respect to *x*:

$$\frac{d}{dy}(y^2)\frac{dy}{dx} + \frac{d}{dx}(3xy) = \frac{d}{dx}(4x)$$

$$2y\frac{dy}{dx} + 3x\frac{d}{dy}(y)\frac{dy}{dx} + y\frac{d}{dx}(3x) = 4$$

$$2y\frac{dy}{dx} + 3x\frac{dy}{dx} + y \times 3 = 4$$

$$\frac{dy}{dx}(3x + 2y) = 4 - 3y$$

$$\frac{dy}{dx} = \frac{4 - 3y}{3x + 2y}$$

At the point (1, –4), $\frac{dy}{dx} = \frac{4 - 3(-4)}{3(1) + 2(-4)} = -\frac{16}{5} = -3.2$.

Question 17

Given $x^2 + \frac{y}{x} = 2y$

Differentiate with respect to *x*:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}\left(\frac{y}{x}\right) = \frac{d}{dx}(2y)$$

$$2x + \frac{x\frac{d}{dy}(y)\frac{dy}{dx} - y\frac{d}{dx}(x)}{x^2} = \frac{d}{dy}(2y)\frac{dy}{dx}$$

$$2x + \frac{x\frac{dy}{dx} - y}{x^2} = 2\frac{dy}{dx}$$

$$2x^3 + x\frac{dy}{dx} - y = 2x^2\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(2x^2 - x\right) = 2x^3 - y$$

$$\frac{dy}{dx} = \frac{2x^3 - y}{2x^2 - x}$$

At the point (1, 1), $\frac{dy}{dx} = 1$

The tangent to the curve has gradient of 1 and by substituting (1, 1) into the equation $y = 1x + c$ it follows that $y = x$ is the equation of the tangent to the curve.

Question 18

Given $5x^2 + \sqrt{xy} = 5 + y^2$

Differentiate with respect to *x*:

$$\frac{d}{dx}(5x^2) + \frac{d}{dx}\left(\sqrt{xy}\right) = \frac{d}{dx}(5) + \frac{d}{dy}(y^2)\frac{dy}{dx}$$

$$10x + \sqrt{x}\frac{d}{dy}\left(\sqrt{y}\right)\frac{dy}{dx} + \sqrt{y}\frac{d}{dx}\left(\sqrt{x}\right) = 0 + 2y\frac{dy}{dx}$$

$$10x + \left(\frac{\sqrt{x}}{2\sqrt{y}}\right)\frac{dy}{dx} + \frac{\sqrt{y}}{2\sqrt{x}} = 2y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{10x + \frac{\sqrt{y}}{2\sqrt{x}}}{\left(2y - \frac{\sqrt{x}}{2\sqrt{y}}\right)}$$

At the point (4, 9), $\frac{dy}{dx} = \frac{10(4) + \frac{\sqrt{9}}{2\sqrt{4}}}{\left(2(9) - \frac{\sqrt{4}}{2\sqrt{9}}\right)} = \frac{\frac{163}{4}}{\frac{53}{3}} = \frac{489}{212}$

The gradient, at point (4, 9) is $\frac{489}{212}$.

Question 19

Given

$$\frac{dy}{dx} = x^2 y$$

$$\frac{d^2y}{dx^2} = x^2\frac{dy}{dx} + y(2x)$$

$$\frac{d^2y}{dx^2} = x^2(x^2 y) + y(2x)$$

$$= x^4 y + 2xy$$

Question 20

Given $x^2 + 4y^2 - 2x + 6y = 17$

Differentiate with respect to x:

$$\frac{d}{dx}(x^2) + \frac{d}{dy}(4y^2)\frac{dy}{dx} - \frac{d}{dx}(2x) + \frac{d}{dy}(6y)\frac{dy}{dx} = \frac{d}{dx}(17)$$

$$2x + 8y\frac{dy}{dx} - 2 + 6\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2(1-x)}{2(4y+3)} = \frac{1-x}{4y+3}$$

When the gradient is zero, $x = 1$.

By substituting $x = 1$ into the original equation, find $y = -3$ or $y = 1.5$.

The tangent to the graph is horizontal at (1, –3) and (1, 1.5).

Question 21

Given $x^2 + y^2 - 4x + 6y + 12 = 0$

Differentiate with respect to x:

$$\frac{d}{dx}(x^2) + \frac{d}{dy}(y^2)\frac{dy}{dx} - \frac{d}{dx}(4x) + \frac{d}{dy}(6y)\frac{dy}{dx} + \frac{d}{dx}(12) = 0$$

$$2x + 2y\frac{dy}{dx} - 4 + 6\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{4-2x}{2y+6}$$

When the gradient is undefined, the tangent to the curve is vertical. This occurs when $2y + 6 = 0$, so at $y = -3$.

By substituting $y = -3$ into the original equation, find $x = 1$ or $x = 3$.

The tangent to the graph is vertical at (1, –3) and (3, –3).

Question 22

Given $y - y^3 = x^2 + x - 2$

Differentiate with respect to *x*:

$$\frac{d}{dy}(y)\frac{dy}{dx} - \frac{d}{dy}(y^3)\frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(x) - \frac{d}{dx}(2)$$

$$\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 2x + 1 - 0$$

$$\frac{dy}{dx} = \frac{2x+1}{1-3y^2}$$

At (1, 0), $\frac{dy}{dx} = 3$.

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{(1-3y^2)(2) - (2x+1)(-6y)\frac{dy}{dx}}{(1-3y^2)^2} \\ &= \frac{(1-3y^2)(2) - (2x+1)(-6y)\frac{2x+1}{1-3y^2}}{(1-3y^2)^2} \\ &= \frac{2(1-3y^2)^2 + 6y(2x+1)^2}{(1-3y^2)^3}\end{aligned}$$

At (1, 0), $\frac{d^2y}{dx^2} = 2$.

Question 23

Given $x^2 = 2\sin y$

Differentiate with respect to x: $\frac{d}{dx}(x^2) = \frac{d}{dy}(2\sin y)\frac{dy}{dx}$

$$2x = 2\cos y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{x}{\cos y}$$

At $\left(1, \frac{\pi}{6}\right)$, $\frac{dy}{dx} = \frac{1}{\cos\left(\frac{\pi}{6}\right)} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

$$\left(y - \frac{\pi}{6}\right) = \frac{2\sqrt{3}}{3}(x-1)$$

$$y = \frac{2\sqrt{3}}{3}x - \frac{2\sqrt{3}}{3} + \frac{\pi}{6}$$

$$6y = 4\sqrt{3}x - 4\sqrt{3} + \pi$$

Question 24

Given $y^2 + \cos x = 3y + 1$

Differentiate with respect to x:

$$\frac{d}{dy}(y^2)\frac{dy}{dx} + \frac{d}{dx}(\cos x) = \frac{d}{dy}(3y)\frac{dy}{dx} + \frac{d}{dx}(1)$$

$$2y\frac{dy}{dx} - \sin x = 3\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{\sin x}{2y-3}$$

$$\frac{d^2y}{dx^2} = \frac{(2y-3)\cos x - 2\sin x\frac{dy}{dx}}{(2y-3)^2}$$

$$= \frac{(2y-3)\cos x - 2\sin x\frac{\sin x}{2y-3}}{(2y-3)^2}$$

$$= \frac{(2y-3)^2\cos x - 2\sin^2 x}{(2y-3)^3}$$

Question 25

Given $2\sin y - x^2 = 2x + 1$

Differentiate with respect to x: $\frac{d}{dy}(2\sin y)\frac{dy}{dx} - \frac{d}{dx}(x^2) = \frac{d}{dx}(2x) + \frac{d}{dx}(1)$

$$2\cos y\frac{dy}{dx} - 2x = 2 + 0$$

$$\frac{dy}{dx} = \frac{x+1}{\cos y}$$

At $(-2, \frac{\pi}{6})$, $\frac{dy}{dx} = \frac{-2+1}{\cos\left(\frac{\pi}{3}\right)} = \frac{-1}{\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$

$$\frac{d^2y}{dx^2} = \frac{\cos y\left[\frac{d}{dx}(x) + \frac{d}{dx}(1)\right] - (x+1)(-\sin y)\frac{dy}{dx}}{\cos^2 y}$$

$$= \frac{\cos y(1+0) + (x+1)(\sin y)\frac{x+1}{\cos y}}{\cos^2 y}$$

$$= \frac{\cos^2 y + (x+1)^2(\sin y)}{\cos^3 y}$$

At $(-2, \frac{\pi}{6})$, $\frac{d^2y}{dx^2} = \frac{10\sqrt{3}}{9}$

Question 26

Given $3x^2 + y^2 = 9$

Differentiate with respect to *x*:

$$\frac{d}{dx}(3x^2) + \frac{d}{dy}(y^2)\frac{dy}{dx} = \frac{d}{dx}(9)$$

$$6x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{3x}{y}$$

$$\frac{dy}{dx} = -1 \text{ when } -\frac{3x}{y} = -1$$

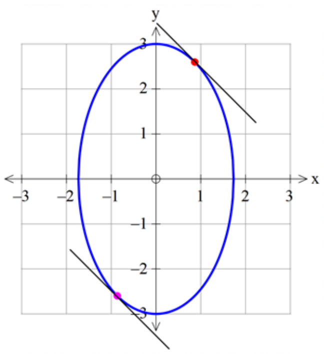
$$y = 3x$$

Substitute this back into the original equation to get:

$$\begin{aligned} 3x^2 + (3x)^2 &= 9 \\ 3x^2 + 9x^2 &= 9 \\ 12x^2 &= 9 \\ x^2 &= \frac{3}{4} \\ x &= \pm\frac{\sqrt{3}}{2} \end{aligned}$$

Substituting these values into the original equation gives $\left(-\frac{\sqrt{3}}{2}, -\frac{3\sqrt{3}}{2}\right)$ and $\left(\frac{\sqrt{3}}{2}, \frac{3\sqrt{3}}{2}\right)$.

The diagram shows the tangent lines with gradient of -1 and the two points where the tangent lines meet the ellipse.



Exercise 8B

Question 1

a $x = 3\sin 2t$

$$\frac{dx}{dt} = 6\cos 2t$$

b $y = 2\cos 5t$

$$\frac{dy}{dt} = -10\sin 5t$$

c $$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = -\frac{5\sin 5t}{3\cos 2t}$$

Question 2

a $x = \sin^2 t$

$$\frac{dx}{dt} = 2\sin t\cos t$$

b $y = \cos 3t$

$$\frac{dy}{dt} = -3\sin 3t$$

c $$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = -\frac{3\sin 3t}{2\sin t\cos t}$$

Using trigonometric identities $2\sin t\cos t = \sin 2t$

Hence $\frac{dy}{dx} = -\frac{3\sin 3t}{\sin 2t}$

Question 3

$x = 2 + 3t \quad \Rightarrow \quad \frac{dx}{dt} = 3$

$y = t^2 \quad \Rightarrow \quad \frac{dy}{dt} = 2t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{2t}{3}$$

Question 4

$x = t^2 \quad \Rightarrow \quad \frac{dx}{dt} = 2t$

$y = 2 + 3t \quad \Rightarrow \quad \frac{dy}{dt} = 3$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{3}{2t}$

Question 5

$x = 5t^3 \quad \Rightarrow \quad \frac{dx}{dt} = 15t^2$

$y = t^2 + 2t \quad \Rightarrow \quad \frac{dy}{dt} = 2t + 2$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{2(t+1)}{15t^2}$

Question 6

$x = 3t^2 + 6t \quad \Rightarrow \quad \frac{dx}{dt} = 6t + 6$

$y = \frac{1}{t+1} \quad \Rightarrow \quad \frac{dy}{dt} = -\frac{1}{(t+1)^2}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = -\frac{1}{6(t+1)^2(t+1)} = -\frac{1}{6(t+1)^3}$

Question 7

$x = t^2 - 1 \quad \Rightarrow \quad \frac{dx}{dt} = 2t$

$y = (t-1)^2 \quad \Rightarrow \quad \frac{dy}{dt} = 2(t-1)$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{2(t-1)}{2t} = \frac{t-1}{t}$

Question 8

$x = \frac{t}{t-1} \quad \Rightarrow \quad \frac{dx}{dt} = \frac{-1}{(t-1)^2}$

$y = \frac{2}{t+1} \quad \Rightarrow \quad \frac{dy}{dt} = \frac{-2}{(t+1)^2}$

$\frac{dy}{dx} = \frac{2(t-1)^2}{(t+1)^2}$

Question 9

$x = t^2 + 2 \quad \Rightarrow \quad \frac{dx}{dt} = 2t$

$y = t^3 \quad \Rightarrow \quad \frac{dy}{dt} = 3t^2$

$\frac{dy}{dx} = \frac{3}{2}t$

When $t = -1$, $\frac{dy}{dx} = -\frac{3}{2}$

Question 10

$x = \frac{1}{t+1} \quad \Rightarrow \quad \frac{dx}{dt} = -\frac{1}{(t+1)^2}$

$y = t^2 + 1 \quad \Rightarrow \quad \frac{dy}{dt} = 2t$

$\frac{dy}{dx} = -2t(t+1)^2$

When $t = 2$, $\frac{dy}{dt} = -36$.

Question 11

$x = 2t^2 + 3t \quad \Rightarrow \quad \frac{dx}{dt} = 4t + 3$

$y = t^3 - 12t \quad \Rightarrow \quad \frac{dy}{dt} = 3t^2 - 12$

$\frac{dy}{dx} = \frac{3t^2 - 12}{4t + 3}$

When $\frac{dy}{dx} = 0$, $t = \pm 2$.

The coordinates of the graph when $t = 2$ are (14, –16).

The coordinates of the graph when $t = -2$ are (2, 16).

Question 12

a

$x = 4\sin t \quad \Rightarrow \quad \frac{dx}{dt} = 4\cos t$

$y = 2\sin 2t \quad \Rightarrow \quad \frac{dy}{dt} = 4\cos 2t$

$\frac{dy}{dx} = \frac{\cos 2t}{\cos t}$

An expression for $\frac{dy}{dx}$ in terms of t is $\frac{\cos 2t}{\cos t}$.

b

When $t = \frac{\pi}{6}$, $x = 4\sin\left(\frac{\pi}{6}\right) = 2$ and $y = 2\sin\left(\frac{\pi}{3}\right) = \sqrt{3}$

The coordinates at $t = \frac{\pi}{6}$ are $(2, \sqrt{3})$.

When $t = \frac{\pi}{6}$, $\frac{\cos\frac{\pi}{3}}{\cos\frac{\pi}{6}} = \frac{1}{\sqrt{3}}$

The gradient at $t = \frac{\pi}{6}$ is $\frac{1}{\sqrt{3}}$.

c

When $\frac{dy}{dx} = 0$, $\frac{\cos 2t}{\cos t} = 0$

$2t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ (for $0 \le t \le 2\pi$).

Question 13

a

$$y = t + \frac{2}{t}$$

$$\frac{dy}{dt} = 1 - \frac{2}{t^2} = \frac{t^2 - 2}{t^2}$$

$$x = 2t - \frac{1}{t}$$

$$\frac{dx}{dt} = 2 + \frac{1}{t^2} = \frac{2t^2 + 1}{t^2}$$

$$\frac{dy}{dx} = \frac{t^2 - 2}{t^2} \times \frac{t^2}{2t^2 + 1} = \frac{t^2 - 2}{2t^2 + 1}$$

b

$$\frac{d^2 y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$$

$$= \frac{(2t^2 + 1)2t - (t^2 - 2)4t}{(2t^2 + 1)^2} \times \frac{t^2}{2t^2 + 1}$$

$$= \frac{10t^3}{(2t^2 + 1)^3}$$

Exercise 8C

Question 1

$$y = 3x^2 + 4x$$

$$\frac{d}{dt}(y) = \frac{d}{dt}(3x^2 + 4x)$$

$$\frac{dy}{dt} = (6x + 4)\frac{dx}{dt}$$

Given $\frac{dx}{dt} = 5$

$$\frac{dy}{dt} = (6x + 4) \times 5 = 30x + 20$$

When $x = 6$, $\frac{dy}{dt} = 200$.

Question 2

$$A = 8p^3$$

$$\frac{d}{dt}(A) = \frac{d}{dt}(8p^3)$$

$$\frac{dA}{dt} = 24p^2\frac{dp}{dt}$$

Given $\frac{dp}{dt} = 0.25$

$$\frac{dA}{dt} = 6p^2$$

When $p = 0.5$, $\frac{dA}{dt} = \frac{3}{2}$.

Question 3

$X = \sin 2p$

$\frac{d}{dt}(X) = \frac{d}{dt}(\sin 2p)$

$\frac{dX}{dt} = 2\cos 2p \frac{dp}{dt}$

Given $\frac{dp}{dt} = 2$

$\frac{dX}{dt} = 4\cos 2p$

When $p = \frac{\pi}{6}$, $\frac{dX}{dt} = 2$.

Question 4

a

$T = \frac{2\pi}{3}\sqrt{L}$

$\frac{d}{dt}(T) = \frac{d}{dt}\left(\frac{2\pi}{3}\sqrt{L}\right)$

$\frac{dT}{dt} = \frac{\pi}{3\sqrt{L}}\frac{dL}{dt}$

Given $\frac{dL}{dt} = \frac{15}{\pi}$

$\frac{dT}{dt} = \frac{5}{\sqrt{L}}$

When $L = 100$

$\frac{dT}{dt} = \frac{1}{2}$

b Rearranging the formula from part **a**

$\frac{dT}{dt} = \frac{\pi}{3\sqrt{L}}\frac{dL}{dt}$

$\frac{dL}{dt} = \frac{3\sqrt{L}}{\pi}\frac{dT}{dt}$

Given $\frac{dT}{dt} = 6\pi$ and $L = 4$

$\frac{dL}{dt} = \frac{3\sqrt{4}}{\pi} \times 6\pi = 36.$

Question 5

$A=\sin^2(3x)$

$\frac{d}{dt}(A)=\frac{d}{dt}(\sin^2(3x))$

$\frac{dA}{dt}=6(\sin 3x)(\cos 3x)\frac{dx}{dt}$

Given $\frac{dx}{dt}=0.1$

$\frac{dA}{dt}=\frac{6}{10}(\sin 3x)(\cos 3x)$

When $x=\frac{\pi}{36}$

$$\frac{dA}{dt}=\frac{6}{10}\left(\sin\frac{\pi}{12}\right)\left(\cos\frac{\pi}{12}\right)$$
$$=\frac{6}{10}\times\frac{\sqrt{2}(\sqrt{3}-1)}{4}\times\frac{\sqrt{2}(\sqrt{3}+1)}{4}$$
$$=\frac{6}{10}\times\frac{2(3-1)}{16}$$
$$=0.15$$

Question 6

$P=4r^2+3$

$\frac{d}{dt}(P)=\frac{d}{dt}(4r^2+3)$

$\frac{dP}{dt}=8r\frac{dr}{dt}$

Given $\frac{dP}{dt}=14,$ rearrange the equation to get:

$\frac{dr}{dt}=\frac{14}{8r}$

When $r=7$

$\frac{dr}{dt}=0.25$

Question 7

$$y^2 = 3x^3 + 1$$

$$\frac{d}{dt}(y^2) = \frac{d}{dt}(3x^3 + 1)$$

$$2y\frac{dy}{dt} = 9x^2\frac{dx}{dt}$$

Given $\frac{dx}{dt} = 0.1$

$$\frac{dy}{dt} = \frac{9x^2}{2y} \times 0.1$$

When $y = 5, x = 2$

$$\frac{dy}{dt} = \frac{9 \times 4}{2 \times 5} \times 0.1$$

$$= 0.36$$

Question 8

$$x^2 + y^2 = 400, x \geq 0$$

$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(400)$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Given $\frac{dx}{dt} = 6$

$$12x + 2y\frac{dy}{dt} = 0$$

When $y = 12,\ x = 16$ (as $x \geq 0$)

$$192 + 24\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -8$$

Question 9

Using the formula for the area of a non-right triangle $A = \frac{1}{2}ab\sin c$

$$A = 50\sin x$$

$$\frac{d}{dt}(A) = \frac{d}{dt}(50\sin x)$$

$$\frac{dA}{dt} = 50\cos x\frac{dx}{dt}$$

Given $\frac{dx}{dt} = 0.01$

$$\frac{dA}{dt} = 0.5\cos x$$

When $x = \frac{\pi}{3}$

$$\frac{dA}{dt} = 0.25\,\text{cm}^2/\text{s}$$

Question 10

$$A = \frac{1}{2}x^2\sin 45° = \frac{1}{2}x^2 \times \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}x^2$$

$$\frac{d}{dt}(A) = \frac{d}{dt}\left(\frac{\sqrt{2}}{4}x^2\right)$$

$$\frac{dA}{dt} = \frac{\sqrt{2}}{2}x\frac{dx}{dt}$$

Given $\frac{dx}{dt} = 0.1$ cm/s

When $AC = AB = 10$ cm ($x = 10\,\text{cm}$)

$$\frac{dA}{dt} = \frac{\sqrt{2}}{2} \times 10 \times 0.1 = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}\,\text{cm}^2/\text{s}$$

Question 11

From the triangle, using Pythagoras' theorem

$$x^2 + y^2 = 10^2$$
$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(100)$$
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Given $\frac{dx}{dt} = 0.1,$

$$0.2x + 2y\frac{dy}{dt} = 0$$

20 seconds after the increase in length of AB commenced $x = 4 + 0.1 \times 20 = 6\,\text{cm}$

Using Pythagoras' theorem to find y, $y = \sqrt{100 - 36} = 8\,\text{cm}$

$$1.2 + 16\frac{dy}{dt} = 0$$
$$\frac{dy}{dt} = -0.075$$

The length of BC is decreasing at a rate of 0.075 cm/s.

Question 12

Using the formula for the area of a square.

$$A = l^2$$
$$\frac{d}{dt}(A) = \frac{d}{dt}(l^2)$$
$$\frac{dA}{dt} = 2l\frac{dl}{dt}$$

Given $\frac{dl}{dt} = 0.01\,\text{cm/s}$

$$\frac{dA}{dt} = 0.02l$$

When $l = 8\,\text{cm}$

$$\frac{dA}{dt} = 0.16\,\text{cm}^2/\text{s}$$

The area of the square is increasing at a rate of 0.16 cm^2 per second.

Question 13

For a particular rectangle $l = 3w$.

$$A = lw = 3w \times w = 3w^2$$

$$\frac{d}{dt}(A) = \frac{d}{dt}(3w^2)$$
$$\frac{dA}{dt} = 6w\frac{dw}{dt}$$

Given $\frac{dw}{dt} = 1\,\text{mm} = 0.1\,\text{cm}$
$$\frac{dA}{dt} = 0.6w$$
When $w = 10\,\text{cm}$
$$\frac{dA}{dt} = 6\,\text{cm}^2/\text{s}$$

The area of the rectangle is increasing at a rate of 6 square centimetres per second.

Question 14

For a regular hexagon the area is $\frac{3\sqrt{3}}{2}a^2$, where a is the length of one side of the hexagon.

$$A = \frac{3\sqrt{3}}{2}a^2$$
$$\frac{d}{dt}(A) = \frac{d}{dt}\left(\frac{3\sqrt{3}}{2}a^2\right)$$
$$\frac{dA}{dt} = 3\sqrt{3}a\frac{da}{dt}$$
Given $\frac{da}{dt} = 1$
$$\frac{dA}{dt} = 3\sqrt{3}a$$
When $a = 20\,\text{cm}$
$$\frac{dA}{dt} = 60\sqrt{3}\,\text{cm}^2/\text{min}$$

The area of the hexagon is increasing at the rate of $60\sqrt{3}\,\text{cm}^2/\text{min}$ when the length of each side is 20 cm.

Question 15

a

$V = \frac{4}{3}\pi r^3$

$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

Given $\frac{dr}{dt} = 0.1$ cm/s

$\frac{dV}{dt} = 0.4\pi r^2 \text{ cm}^3/\text{s}$

When $r = 5$

$\frac{dV}{dt} = 10\pi \text{ cm}^3/\text{s}$

b

When $\frac{dV}{dt} = 40\pi \text{ cm}^3/\text{s}$

$0.4\pi r^2 = 40\pi$

$r^2 = 100$

$r = 10 \text{ cm}$

Question 16

a

The surface area of a cube has formula $A = 6l^2$

$\frac{d}{dt}(A) = \frac{d}{dt}(6l^2)$

$\frac{dA}{dt} = 12l \frac{dl}{dt}$

Given $\frac{dl}{dt} = 0.1 \text{ cm/s}$

$\frac{dA}{dt} = 1.2l \text{ cm}^2/\text{s}$

When $l = 10 \text{ cm}$

$\frac{dA}{dt} = 12 \text{ cm}^2/\text{s}$

$12 \text{ cm}^2/\text{s}$ is the rate that the surface area of the cube is increasing when the side length is 10 cm.

b

The volume of a cube has formula $V = l^3 \text{ cm}^3$

$\frac{d}{dt}(V) = \frac{d}{dt}(l^3)$

$\frac{dV}{dt} = 3l^2 \frac{dl}{dt}$

Given $\frac{dl}{dt} = 0.1$

$\frac{dV}{dt} = 0.3l^2$

When $l = 10 \text{ cm}$

$\frac{dV}{dt} = 30 \text{ cm}^3/\text{s}$

$30 \text{ cm}^3/\text{s}$ is the rate that the volume of the cube is increasing when the side length is 10 cm.

Question 17

The formula for the volume of a cylinder (which is the shape of the oil slick) is $V = \pi r^2 h$.

$$\frac{d}{dt}(V) = \frac{d}{dt}(\pi r^2 h)$$

We know that the height of the cylinder is 5 cm $= 0.05\,\text{m}$ (thickness of the oil slick).

$$\frac{dV}{dt} = \frac{d}{dt}(0.05\pi r^2)$$

$$\frac{dV}{dt} = 0.1\pi r\frac{dr}{dt}$$

Given $\frac{dV}{dt} = 5\,\text{m}^3/\text{s}$

$$5 = 0.1\pi r\frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{50}{\pi r}$$

a When $r = 20\,\text{m}$

$$\frac{dr}{dt} = \frac{50}{20\pi}$$
$$\approx 0.80\,\text{m/min}$$
$$\approx 80\,\text{cm/min}$$

b When $r = 40\,\text{m}$

$$\frac{dr}{dt} = \frac{50}{40\pi}$$
$$\approx 0.40\,\text{m/min}$$
$$\approx 40\,\text{cm/min}$$

c When $r = 100\,\text{m}$

$$\frac{dr}{dt} = \frac{50}{100\pi}$$
$$\approx 0.16\,\text{m/min}$$
$$\approx 16\,\text{cm/min}$$

Question 18

The formula for the volume of a cylinder is $V = \pi r^2 h$.

a

$$\frac{d}{dt}(V) = \frac{d}{dt}(\pi r^2 h)$$

We know that the height of the cylinder is $5r$ cm

$$\frac{dV}{dt} = \frac{d}{dt}((5r)\pi r^2) = \frac{d}{dt}(5\pi r^3)$$

$$\frac{dV}{dt} = 15\pi r^2\frac{dr}{dt} = 15\pi r^2\frac{2}{10\pi}$$
$$= 3r^2\ \text{cm}^3/\text{s}$$

b

$$\frac{d}{dt}(SA) = \frac{d}{dt}(2\pi r^2 + \pi r^2 h)$$

We know that the height of the cylinder is $5r$ cm

$$\frac{dSA}{dt} = \frac{d}{dt}(2\pi r^2 + 2\pi r \times 5r)$$
$$= \frac{d}{dt}(2\pi r^2 + 10\pi r^2)$$
$$= \frac{d}{dt}(12\pi r^2) = 24\pi r\frac{dr}{dt}$$

$$\frac{dSA}{dt} = (24\pi r)\frac{2}{10\pi} = 4.8r = 4.8r\ \text{cm}^2/\text{s}$$

Question 19

The formula for the volume of a cylinder (which is the shape of the oil film) is $V = \pi r^2 h$.

$V = \pi r^2 h$

$\frac{d}{dt}(V) = \frac{d}{dt}(\pi r^2 h)$

We know that the height of the cylinder is 0.02cm (thickness of the oil film).

$\frac{dV}{dt} = \frac{d}{dt}(0.02\pi r^2) = 0.04\pi r \frac{dr}{dt}$

Given $\frac{dV}{dt} = 1\,\text{cm}^3/\text{s}$

$1 = 0.04\pi r \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{25}{\pi r}$

a When $r = 5\,\text{cm}$

$\frac{dr}{dt} = \frac{25}{5\pi} \approx 1.6\,\text{cm/s}$

b When $r = 10\,\text{cm}$

$\frac{dr}{dt} = \frac{25}{\pi r} \approx 0.8\,\text{cm/s}$

Question 20

$v = 2x^2 - 3$

a $\frac{dv}{dx} = 4x$

$\frac{dv}{dt} = \frac{d}{dt}\left(2x^2 - 3\right)$

$= 4x\frac{dx}{dt}$

$= 4x \times v$

$= 4x(2x^2 - 3)\text{m/s}^2$

b When $x = 2$

$v = 5\,\text{m / s}$

When $x = 2$

$a = (8 \times 5) = 40\,\text{m/s}^2$

Question 21

$$A = 0.5\times(20+0.2t)^2\sin 60°$$
$$= \frac{\sqrt{3}}{4}(20+0.2t)^2$$
$$\frac{dA}{dt} = \frac{\sqrt{3}}{10}(20+0.2t)$$

When $t = 0$, side length is 20cm

$$\frac{dA}{dt} = 2\sqrt{3}\ \text{cm}^2/\text{s}$$

Question 22

$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$
$$\frac{dV}{dt} = 0.5\,\text{m}^3/\text{s}$$
$$4\pi r^2\frac{dr}{dt} = 0.5$$
$$\frac{dr}{dt} = \frac{1}{8\pi r^2}$$

a **i** when $r = 1\,\text{m}$, $\frac{dr}{dt} = \frac{1}{8\pi}\,\text{m/s} \approx 4\,\text{cm/s}$

ii when $r = 2\,\text{m}$, $\frac{dr}{dt} = \frac{1}{32\pi}\,\text{m/s} \approx 1\,\text{cm/s}$

b 20 seconds after inflation commences,

$$V = 20\times 0.5 = 10\text{m}^3$$

$$r = \sqrt[3]{\frac{10\times 3}{4\pi}} \approx 1.3365\,\text{m}$$

$$\frac{dr}{dt} = \frac{1}{8\pi\sqrt[3]{\frac{10\times 3}{4\pi}}^{\,2}} \approx 22\,\text{mm/s}$$

Question 23

$V = \frac{1}{3}\pi r^2 h$

Since $h \approx 2r,\ \ V \approx \frac{2}{3}\pi r^3$

$\frac{dV}{dt} \approx 2\pi r^2 \frac{dr}{dt}$

$0.25 \approx 2\pi r^2 \frac{dr}{dt}$

$\frac{dr}{dt} \approx \frac{1}{8\pi r^2}$

a When $r = 2\,\text{m}$, $\frac{dr}{dt} \approx \frac{1}{8\pi(2^2)} \approx \frac{1}{32\pi}\,\text{m/min}$

b When $h = 2\,\text{m},\ r = 1\,\text{m}$.

$\frac{dr}{dt} \approx \frac{1}{8\pi}$

$\frac{dh}{dt} \approx \frac{1}{4\pi}\,\text{m/min}$

Question 24

As the cross section is an equilateral triangle we can find the relationship between the radius and the perpendicular height of the cone.

$h = \sqrt{(2r)^2 - (r)^2} = \sqrt{3}r$

$V = \frac{1}{3}\pi r^2 h$

Since $h \approx \sqrt{3}r,\quad V \approx \frac{\sqrt{3}}{3}\pi r^3$

$\frac{dV}{dt} \approx \sqrt{3}\pi r^2 \frac{dr}{dt}$

When $r = 20, \frac{dr}{dt} = 0.5\,\text{cm/s}$

$\frac{dV}{dt} \approx \sqrt{3}\pi(20^2)(0.5) \approx 1088.28$

$V \approx 1090\,\text{cm}^3$

Question 25

a

$$S = 2\pi(5^2) + 2\pi(5)h = 50\pi + 10\pi h$$

$$\frac{dS}{dt} = 10\pi\frac{dh}{dt}$$

$$\frac{dh}{dt} = 0.1\,\text{cm/s}$$

$$\frac{dS}{dt} = \pi\,\text{cm}^2/\text{s}$$

b

$$V = \pi(5^2)h = 25\pi h$$

$$\frac{dV}{dt} = 25\pi\frac{dh}{dt}$$

$$\frac{dh}{dt} = 0.1\,\text{cm/s}$$

$$\frac{dV}{dt} = \frac{5\pi}{2}\,\text{cm}^3/\text{s}$$

Question 26

a

$$\text{S} = 2\pi r^2 + 2\pi r(10) = 2\pi r^2 + 20\pi r$$

$$\frac{dS}{dt} = (4\pi r + 20\pi)\frac{dr}{dt}$$

$$\frac{dr}{dt} = 0.1\,\text{cm/s}$$

$$r = 5 + 20 \times 0.1 = 7\,\text{cm}$$

$$\frac{dS}{dt} = (0.4\pi(7) + 2\pi)\,\text{cm}^2/\text{s} = 4.8\pi\text{cm}^2/\text{s}$$

b

$$V = \pi r^2(10) = 10\pi r^2$$

$$\frac{dV}{dt} = 20\pi r\frac{dr}{dt}$$

$$\frac{dr}{dt} = 0.1\,\text{cm/s}$$

$$\frac{dV}{dt} = 2\pi r\,\text{cm}^3/\text{s} = 14\pi\,\text{cm}^3/\text{s}$$

Question 27

Let b be the distance from the base of the ladder to the wall and a be the distance from the base of the wall to the top of the ladder.

$$a^2 + b^2 = 5.2^2$$

$$2a + 2b\frac{db}{da} = 0$$

$$\frac{db}{da} = -\frac{a}{b}$$

$$\frac{da}{dt} = \frac{da}{db}\frac{db}{dt}$$

$$= -0.1\frac{b}{a}\,\text{m/s}$$

When $a = 4.8$

$$b = \sqrt{5.2^2 - 4.8^2} = 2\,\text{m}$$

$$\frac{da}{dt} = -0.1\frac{2}{4.8}\,\text{m/s} = -\frac{1}{24}\,\text{m/s} = -\frac{25}{6}\,\text{cm/s}$$

Question 28

$$V = \frac{\pi h^2}{3}(6-h)$$

$$\frac{dV}{dt} = \left(4\pi h - \pi h^2\right)\frac{dh}{dt}\,\text{m}^3/\text{min}$$

$$\frac{1}{12\pi} = \frac{dh}{dt}\,\text{m}^3/\text{min}$$

$$\frac{dh}{dt} \approx 27\text{mm/min}$$

Question 29

a Let s be the length of the shadow and d the distance from the person to the lamp-post.

$$\frac{s+d}{s} = \frac{6}{1.8}$$

$$s+d = \frac{10}{3}s$$

$$s = \frac{3}{7}d$$

$$\frac{ds}{dt} = \frac{3}{7}\frac{dd}{dt}$$

$$\frac{ds}{dt} = \frac{3}{7}(-1.4)\,\text{m/s}$$

$$= -0.6\,\text{m/s}$$

The shadow is shortening at a rate of 0.6 m/s.

b The tip of the shadow is moving with the combined speed of the person and the shortening length of the shadow, $-0.6-1.4=-2$, so the tip of the shadow is moving towards the wall at a rate of 2 m/s.

Question 30

a Let s be the length of the shadow and d the distance from the person to the lamp-post.

$$\frac{s+d}{s}=\frac{4.5}{1.5}$$
$$s+d=3s$$
$$s=\frac{1}{2}d$$
$$\frac{ds}{dt}=\frac{1}{2}\frac{dd}{dt}=\frac{1}{2}(2)\,\text{m/s}=1\,\text{m/s}$$

The shadow is lengthening at a rate of 1 m/s.

b The tip of the shadow is moving with the combined speed of the person and the lengthening of the shadow, $1+2=3$, so the tip of the shadow is moving away from the wall at a rate of 3 m/s.

Question 31

$$r^2+(2-h)^2=2^2$$
$$2r\frac{dr}{dt}-2(2-h)\frac{dh}{dt}=0$$
$$2r\frac{dr}{dt}=2(2-h)\frac{dh}{dt}$$
$$\frac{dr}{dt}=\frac{2-h}{r}\frac{dh}{dt}$$

When $h=1,\ r=\sqrt{2^2-1^2}=\sqrt{3}$

$$\frac{dr}{dt}=\frac{2-1}{\sqrt{3}}\times(-0.005)=\frac{-1}{200\sqrt{3}}\,\text{m/s}=-\frac{1}{2\sqrt{3}}\,\text{cm/s}$$

Question 32

If the distance from B to C is a, and the distance from A to B is c.

$$c^2=a^2+20^2$$
$$2c\frac{dc}{dt}=2a\frac{da}{dt}$$
$$\frac{dc}{dt}=\frac{a}{c}\frac{da}{dt}$$

When $a=48,\ c=\sqrt{48^2+20^2}=52\text{m}$

$$\frac{dc}{dt}=\frac{48}{52}\times15=\frac{180}{13}\,\text{m/s}\approx13.8\,\text{m/s}$$

Question 33

Let d be the distance from A to the balloon and a be the height of the balloon.

$$d^2 = a^2 + 60^2$$

$$2d\frac{dd}{dt} = 2a\frac{da}{dt}$$

$$\frac{dd}{dt} = \frac{a}{d}\frac{da}{dt}$$

When $a = 80\,\text{m}$, $d = \sqrt{80^2 + 60^2} = 100\,\text{m}$

$$\frac{dd}{dt} = \frac{80}{100}5 = 4\,\text{m/s}$$

Question 34

$$\tan\theta = \frac{x}{8}$$

$$x = 8\tan\theta$$

$$\frac{dx}{dt} = 8\sec^2\theta\frac{d\theta}{dt}$$

$$\frac{dx}{dt} = \frac{8}{\cos^2\theta}4\pi$$

$$\cos^2\theta = \left(\frac{8}{\sqrt{8^2+5^2}}\right)^2 = \frac{64}{89}$$

$$\frac{dx}{dt} = \frac{8}{\frac{64}{89}}4\pi = \frac{89}{2}\pi\text{ m/s} \approx 139.8\text{ m/s}$$

Exercise 8D

Question 1

$f(x) = x^3 - 5x$

If $y = x^3 - 5x$, then $\frac{dy}{dx} = 3x^2 - 5$ and so $\frac{\delta y}{\delta x} \approx 3x^2 - 5$

$\delta y \approx (3x^2 - 5)\delta x$

In this case $x = 5$ and $\delta x = 0.01$, thus $\delta y \approx (3(5^2) - 5)(0.01) = 0.70$

When x changes from 5 to 5.01, the change in $f(x)$ is approximately 0.70.

(Comparing this to $f(5.01) - f(5) = \left[(5.01)^3 - 5(5.01)\right] - \left[(5)^3 - 5(5)\right] \approx 0.70$, which shows the approximation is reasonable).

Question 2

$f(x) = \sin 3x$

If $y = \sin 3x$, then $\frac{dy}{dx} = 3\cos 3x$ and so $\frac{\delta y}{\delta x} \approx 3\cos 3x$

$\delta y \approx (3\cos 3x)\delta x$

In this case $x = \frac{\pi}{9}$ and $\delta x = 0.01$, thus $\delta y \approx 3\cos\frac{\pi}{3} \times 0.01 = 0.015$

When x changes from $\frac{\pi}{9}$ to $\frac{\pi}{9} + 0.01$, the change in $f(x)$ is approximately 0.015.

(Comparing this to $f\left(\frac{\pi}{9} + 0.01\right) - f\left(\frac{\pi}{9}\right) = 0.0146\,(\text{to } 4\,\text{d.p.})$, which shows the approximation is reasonable).

Question 3

$f(x) = 2\sin^3 5x$

If $y = 2\sin^3 5x$, then $\frac{dy}{dx} = 30\sin^2 5x\cos 5x$ and so $\frac{\delta y}{\delta x} \approx 30\sin^2 5x\cos 5x$

$\delta y \approx 30\sin^2 5x\cos 5x\delta x$

In this case $x = \frac{\pi}{3}$ and $\delta x = 0.001$, thus $\delta y \approx \left[30\sin^2 5\left(\frac{\pi}{3}\right)\cos 5\left(\frac{\pi}{3}\right)\right]0.001 = 0.01125$

When x changes from $\frac{\pi}{3}$ to $\frac{\pi}{3} + 0.001$, the change in $f(x)$ is approximately 0.01125.

(comparing this to $f\left(\frac{\pi}{3} + 0.001\right) - f\left(\frac{\pi}{3}\right) = 0.011\ 265\ 9$ (to 7 d.p.), which shows the approximation is reasonable).

Question 4

$C = 5000 + 20\sqrt{x}$

$\frac{dC}{dx} = \frac{10}{\sqrt{x}}$

a When $x = 25$, $\frac{dC}{dx} = \frac{10}{\sqrt{25}} = \$2$ per unit.

b When $x = 100$, $\frac{dC}{dx} = \frac{10}{\sqrt{100}} = \$1$ per unit.

c When $x = 400$, $\frac{dC}{dx} = \frac{10}{\sqrt{400}} = \$0.50$ per unit.

Question 5

$C = 15\,000 + 750x - 15x^2 + \frac{x^3}{10}$

$\frac{dC}{dx} = 750 - 30x + \frac{3x^2}{10}$

a When $x = 30$, $\frac{dC}{dx} = 750 - 30(30) + \frac{3(30)^2}{10} = \$120$ per tonne.

b When $x = 60$, $\frac{dC}{dx} = 750 - 30(60) + \frac{3(60)^2}{10} = \$30$ per tonne.

c When $x = 100$, $\frac{dC}{dx} = 750 - 30(100) + \frac{3(100)^2}{10} = \$750$ per tonne.

Question 6

$C = 450 + 0.5x^2$

$\frac{dC}{dx} = x$

When $x = 10$, $\frac{dC}{dx} = 10$

This means that it will cost approximately \$10 to produce the 11th unit.

Question 7

a SA(cube) $= 6l^3$

If $y = 6l^2$, then $\frac{dy}{dx} = 12l$ and so $\frac{\delta y}{\delta x} \approx 12l$

$\delta y \approx 12l\delta x$

In this case $l = 5$ and $\delta x = 0.2$

$\delta y \approx 12 \times 5 \times 0.2 = 12\,\text{cm}^2$

b V(cube) = l^3

If $y = l^3$, then $\frac{dy}{dx} = 3l^2$ and so $\frac{\delta y}{\delta x} \approx 3l^2$

$\delta y \approx 3l^2\delta x$

In this case $l = 5$ and $\delta x = 0.2$

$\delta y \approx 3 \times 25 \times 0.2 = 15\,\text{cm}^3$

Exercise 8E

Question 1

Using logarithmic differentiation

$$y = x^3(2x+1)^5$$

$$\ln y = \ln\left[x^3(2x+1)^5\right] = \ln x^3 + \ln(2x+1)^5 = 3\ln x + 5\ln(2x+1)$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left[3\ln x + 5\ln(2x+1)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x} + \frac{10}{2x+1}$$

$$\frac{dy}{dx} = \left(\frac{3}{x} + \frac{10}{2x+1}\right)x^3(2x+1)^5 = 3x^2(2x+1)^5 + 10x^3(2x+1)^4$$

Using the product rule

$$\frac{dy}{dx} = x^3 \times 5 \times 2 \times (2x+1)^4 + (2x+1)^5 \times 3x^2 = 10x^3(2x+1)^4 + 3x^2(2x+1)^5$$

Which is the same answer as was found using logarithmic differentiation

Question 2

$$y = \frac{x^3}{x^2+1}$$

$$\ln y = \ln\left(\frac{x^3}{x^2+1}\right) = \ln x^3 - \ln(x^2+1) = 3\ln x - \ln(x^2+1)$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(3\ln x - \ln(x^2+1))$$

$$\frac{d}{dy}(\ln y)\frac{dy}{dx} = \frac{3}{x} - \frac{2x}{x^2+1}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x} - \frac{2x}{x^2+1}$$

$$\frac{dy}{dx} = \frac{3}{x} \times \frac{x^3}{x^2+1} - \frac{2x}{x^2+1} \times \frac{x^3}{x^2+1} = 3 \times \frac{x^2}{x^2+1} - \frac{2x}{x^2+1} \times \frac{x^3}{x^2+1}$$

$$= \frac{3x^2}{x^2+1} - \frac{2x^4}{(x^2+1)^2} = \frac{3x^2(x^2+1) - 2x^4}{(x^2+1)^2} = \frac{x^4+3x^2}{(x^2+1)^2}$$

Using the quotient rule

$$y = \frac{x^3}{x^2+1}$$

$$\frac{dy}{dx} = \frac{(x^2+1) \times 3x^2 - x^3 \times 2x}{(x^2+1)^2} = \frac{3x^4 + 3x^2 - 2x^4}{(x^2+1)^2} = \frac{x^4+3x^2}{(x^2+1)^2}$$

Which is the same answer as was found using logarithmic differentiation.

Question 3

a To differentiate x^x

If $y = x^x$

$$\ln y = \ln x^x = x \ln x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x)$$

$$\frac{d}{dy}(\ln y)\frac{dy}{dx} = x \times \frac{1}{x} + \ln x \times 1$$

$$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$$

$$\frac{dy}{dx} = y(1 + \ln x) = x^x(1 + \ln x)$$

So the derivative of x^x is $x^x(1 + \ln x)$

b To differentiate x^{2x}

If $y = x^{2x}$

$$\ln y = \ln x^{2x} = 2x \ln x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(2x \ln x)$$

$$\frac{d}{dy}(\ln y)\frac{dy}{dx} = 2x \times \frac{1}{x} + \ln x \times 2$$

$$\frac{1}{y}\frac{dy}{dx} = 2 + 2\ln x$$

$$\frac{dy}{dx} = y \times 2(1 + \ln x) = 2x^{2x}(1 + \ln x)$$

So the derivative of x^{2x} is $2x^{2x}(1 + \ln x)$

c To differentiate $x^{\cos x}$

If $y = x^{\cos x}$

$$\ln y = \ln x^{\cos x}$$

$$\ln y = \cos x \ln x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\cos x \ln x)$$

$$\frac{d}{dy}(\ln y)\frac{dy}{dx} = \cos x \times \frac{1}{x} + \ln x(-\sin x)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{\cos x}{x} - \sin x \ln x$$

$$\frac{dy}{dx} = y\left(\frac{\cos x}{x} - \sin x \ln x\right)$$

$$= x^{\cos x}\left(\frac{\cos x - x\sin x \ln x}{x}\right)$$

$$= \frac{x^{\cos x}(\cos x - x\sin x \ln x)}{x}$$

So the derivative of $x^{\cos x}$ is

$$\frac{x^{\cos x}(\cos x - x\sin x \ln x)}{x}$$

d To differentiate $\sqrt{\dfrac{3x+1}{3x-1}}$

If $y=\sqrt{\dfrac{3x+1}{3x-1}}$

$$\ln y=\ln\sqrt{\frac{3x+1}{3x-1}}$$

$$\frac{d}{dx}(\ln y)=\frac{d}{dx}\left(\ln\sqrt{\frac{3x+1}{3x-1}}\right)$$

$$\frac{d}{dy}(y)\frac{dy}{dx}=\frac{d}{dx}\left(\ln(3x+1)^{\frac{1}{2}}-\ln(3x-1)^{\frac{1}{2}}\right)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}\left(\frac{1}{2}\ln(3x+1)-\frac{1}{2}\ln(3x-1)\right)$$

$$\frac{dy}{dx}=y\left(\frac{3}{2(3x+1)}-\frac{3}{2(3x-1)}\right)$$

$$=\sqrt{\frac{3x+1}{3x-1}}\left(\frac{3}{2(3x+1)}-\frac{3}{2(3x-1)}\right)$$

$$=\sqrt{\frac{3x+1}{3x-1}}\left(\frac{3(3x-1)-3(3x+1)}{2(3x+1)(3x-1)}\right)$$

$$=\sqrt{\frac{3x+1}{3x-1}}\left(\frac{9x-3-9x-3)}{2(3x+1)(3x-1)}\right)$$

$$=\left(\frac{-6(3x+1)^{\frac{1}{2}}}{2(3x-1)^{\frac{1}{2}}(3x+1)(3x-1)}\right)$$

$$=-\frac{3}{(3x+1)^{\frac{1}{2}}(3x-1)^{\frac{3}{2}}}$$

$$=-\frac{3}{\sqrt{(3x+1)(3x-1)^3}}$$

So the derivative of $\sqrt{\dfrac{3x+1}{3x-1}}$ is $-\dfrac{3}{\sqrt{(3x+1)(3x-1)^3}}$.

Miscellaneous Exercise 8

Question 1

a $y=\dfrac{2x+1}{3-2x}$

$$\frac{dy}{dx}=\frac{(3-2x)\times 2-(2x+1)\times(-2)}{(3-2x)^2}=\frac{6-4x+4x+2}{(3-2x)^2}=\frac{8}{(3-2x)^2}$$

b $y=\sin^3(2x+1)$

$$\frac{dy}{dx}=3\sin^2(2x+1)\times 2\cos(2x+1)=6\sin^2(2x+1)\cos(2x+1)$$

c Given $3x^2y+y^3=5x+7$

Differentiate with respect to x:

$$\frac{d}{dx}(3x^2y)+\frac{d}{dx}(y^3)=\frac{d}{dx}(5x+7)$$

$$3x^2\times\frac{d}{dy}(y)\frac{dy}{dx}+y\times 6x+\frac{d}{dy}(y^3)\frac{dy}{dx}=5$$

$$3x^2\times\frac{dy}{dx}+y\times 6x+3y^2\frac{dy}{dx}=5$$

$$\frac{dy}{dx}\left(3x^2+3y^2\right)+6xy=5$$

$$\frac{dy}{dx}(3x^2+3y^2)=5-6xy$$

$$\frac{dy}{dx}=\frac{(5-6xy)}{3(x^2+y^2)}$$

d $x=t^2+3t-6,\ y=t^4+1$

$$x=t^2+3t-6$$

$$\frac{dx}{dt}=2t+3$$

$$\frac{dy}{dt}=4t^3$$

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{4t^3}{2t+3}$$

Question 2

Given $x^2 + y^2 = 25$

$$\frac{d}{dx}(x^2) + \frac{d}{dy}(y^2)\frac{dy}{dx} = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

At the point (3, 4), $\frac{dy}{dx} = -\frac{3}{4}$

The equation of the tangent line at (3, 4) is $y = -\frac{3}{4}x + c$.

$$4 = -\frac{3}{4} \times 3 + c$$

$$c = \frac{25}{4}$$

So the equation of the tangent line at (3, 4) is $4y = -3x + 25$.

Question 3

a Given $y + 1 = xy \Rightarrow x = \frac{y+1}{y}$

Differentiate with respect to x:

$$\frac{dy}{dx} + 0 = x \times \frac{dy}{dx} + y \times 1$$

$$\frac{dy}{dx}(1-x) = y$$

$$\frac{dy}{dx} = \frac{y}{1-x}$$

$$\frac{d^2y}{dx^2} = \frac{(1-x)\frac{dy}{dx} + y}{(1-x)^2} = \frac{(1-x)\frac{y}{(1-x)} + y}{(1-x)^2} = \frac{2y}{\left(1 - \frac{y+1}{y}\right)^2} = \frac{2y}{\left(\frac{-1}{y}\right)^2} = 2y^3$$

b Given $y^3 - 5 = xy \Rightarrow x = \dfrac{y^3 - 5}{y}$

$$\frac{d}{dx}(y^3) - \frac{d}{dx}(5) = \frac{d}{dx}(xy)$$

$$3y^2\frac{dy}{dx} - 0 = x\frac{d}{dx}(y) + y \times 1$$

$$3y^2\frac{dy}{dx} = x\frac{dy}{dx} + y$$

$$\frac{dy}{dx}\left(3y^2 - x\right) = y$$

$$\frac{dy}{dx} = \frac{y}{3y^2 - x}$$

$$\frac{d^2y}{dx^2} = \frac{(3y^2 - x)\dfrac{dy}{dx} - y \times \left(6y\dfrac{dy}{dx} - 1\right)}{(3y^2 - x)^2}$$

$$= \frac{(3y^2 - x) \times \dfrac{y}{(3y^2 - x)} - y\left(\dfrac{6y \times y}{3y^2 - x} - 1\right)}{(3y^2 - x)^2}$$

$$= \frac{\dfrac{y(3y^2 - x) - 6y^3 + y(3y^2 - x)}{(3y^2 - x)}}{(3y^2 - x)^2}$$

$$= \frac{-2xy}{(3y^2 - x)^3} = \frac{-2y\left(\dfrac{y^3 - 5}{y}\right)}{\left(3y^2 - \left(\dfrac{y^3 - 5}{y}\right)\right)^3}$$

$$= \frac{-2y^3 + 10}{\left(3y^2 - y^2 + \dfrac{5}{y}\right)^3} = \frac{-2\left(y^3 - 5\right)}{\left(2y^2 + \dfrac{5}{y}\right)^3}$$

$$= \frac{2\left(5 - y^3\right)}{\left(2y^2 + \dfrac{5}{y}\right)^3} \times \frac{y^3}{y^3}$$

$$= \frac{2y^3\left(5 - y^3\right)}{\left(2y^3 + 5\right)^3}$$

Question 4

Using trigonometric ratios to find the relationship between the angle θ and the height of the rocket.

$$\tan\theta = \frac{h}{200}$$

$$\frac{1}{\cos^2\theta} \times \frac{d\theta}{dt} = \frac{1}{200} \times \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{200}{\cos^2\theta} \times \frac{1}{20} = \frac{10}{\cos^2\theta}$$

$$\frac{d^2h}{dt^2} = \frac{-20(-\sin\theta)}{\cos^3\theta}\frac{d\theta}{dt} = \frac{20\sin\theta}{\cos^3\theta} \times \frac{1}{20} = \frac{\sin\theta}{\cos^3\theta}$$

a When $\theta = \frac{\pi}{6}$

$$v = \frac{10}{\cos^2\frac{\pi}{6}} = \frac{40}{3}\text{ m/s}$$

$$a = \frac{\frac{1}{2}}{\left(\frac{\sqrt{3}}{2}\right)^3} = \frac{1}{2} \times \frac{8}{3\sqrt{3}} = \frac{4\sqrt{3}}{9}\text{ m/s}^2$$

b When $\theta = \frac{\pi}{3}$

$$v = \frac{10}{\cos^2\frac{\pi}{3}} = 40\text{ m/s}$$

$$a = \frac{\frac{\sqrt{3}}{2}}{\left(\frac{1}{2}\right)^3} = \frac{\sqrt{3}}{2} \times \frac{8}{1} = 4\sqrt{3}\text{ m/s}^2$$

Question 5

a Given $y=(2x+3)^3$

$$\frac{dy}{dx}=3(2x+3)^2\times 2=6(2x+3)^2$$

$$\frac{d^2y}{dx^2}=12(2x+3)\times 2=24(2x+3)=48x+72$$

Given $x=\dfrac{\sqrt[3]{y}-3}{2}$

$$\frac{dx}{dy}=\frac{\frac{1}{3}y^{-\frac{2}{3}}}{2}=\frac{1}{6}y^{-\frac{2}{3}}$$

$$\frac{d^2x}{dy^2}=-\frac{1}{9}y^{-\frac{5}{3}}$$

$$-\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3=\frac{1}{9}y^{-\frac{5}{3}}\times\left[6(2x+3)^2\right]^3=\frac{1}{9}\left[(2x+3)^3\right]^{-\frac{5}{3}}\times 6^3(2x+3)^6$$

$$=\frac{1}{9}(2x+3)^{-5}\times 6^3(2x+3)^6=\frac{6^3}{9}(2x+3)=24(2x+3)=48x+72$$

$$=\frac{d^2y}{dx^2}$$

b To prove that $\dfrac{d^2y}{dx^2}=-\left(\dfrac{dy}{dx}\right)^3\times\dfrac{d^2x}{dy^2}$

Given $y=f(x)$, and provided that the necessary derivatives exist.

Differentiate with respect to y.

$$1=\frac{dy}{dx}\times\frac{dx}{dy}\text{ (chain rule)}$$

Differentiate with respect to y again.

$$0=\left(\frac{d^2y}{dx^2}\times\frac{dx}{dy}\right)\times\frac{dx}{dy}+\frac{dy}{dx}\times\frac{d^2x}{dy^2}=\frac{d^2y}{dx^2}\left(\frac{dx}{dy}\right)^2+\frac{dy}{dx}\times\frac{d^2x}{dy^2}$$

$$\frac{d^2y}{dx^2}\left(\frac{dx}{dy}\right)^2=-\frac{dy}{dx}\times\frac{d^2x}{dy^2}$$

$$\frac{d^2y}{dx^2}=\frac{-\frac{dy}{dx}\times\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^2}=\frac{-\frac{dy}{dx}\times\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^2}\times\frac{\left(\frac{dy}{dx}\right)^2}{\left(\frac{dy}{dx}\right)^2}=\frac{-\left(\frac{dy}{dx}\right)^3\times\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\times\frac{dy}{dx}\right)^2}=\frac{-\left(\frac{dy}{dx}\right)^3\times\frac{d^2x}{dy^2}}{1^2}$$

$$=-\left(\frac{dy}{dx}\right)^3\times\frac{d^2x}{dy^2}$$