SADLER UNIT 4 MATHEMATICS SPECIALIST

WORKED SOLUTIONS

Chapter 9: Integration techniques and applications

Exercise 9A

Question 1

If $u = x^2 - 3$, then $\frac{du}{dx} = 2x$

Thus

$$\int 60x(x^2-3)^5\,dx = \int 60x(x^2-3)^5 \frac{dx}{du}\,du = \int 60\sqrt{u+3}\,u^5 \frac{1}{2\sqrt{u+3}}\,du$$
$$= \int 30u^5\,du = 5u^6 + c = 5(x^2-3)^6 + c$$

Question 2

If $u = 1-2x$, then $\frac{du}{dx} = -2$.

Thus

$$\int 80x(1-2x)^3\,dx = \int 80x(1-2x)^3 \frac{dx}{du}\,du = \int 80\frac{1-u}{2}u^3\left(-\frac{1}{2}\right)du$$
$$= 20\int (u^4 - 20u^3)\,du = 20\left(\frac{u^5}{5} - \frac{u^4}{4}\right) + c$$
$$= 4u^5 - 5u^4 + c = u^4(4u-5) + c$$
$$= (1-2x)^4(4(1-2x)-5) + c = (1-2x)^4(-8x-1) + c$$
$$= -(1-2x)^4(8x+1) + c$$

Question 3

If $u = 3x+1$, then $\dfrac{du}{dx} = 3$

$$\int 12x(3x+1)^5\,dx = \int 12x(3x+1)^5 \frac{dx}{du}\,du = \int 12\left(\frac{u-1}{3}\right)u^5 \frac{1}{3}\,du$$
$$= \frac{4}{3}\int (u^6 - u^5)\,du = \frac{4}{3}\left(\frac{u^7}{7} - \frac{u^6}{6}\right) + c$$
$$= \frac{4}{126}u^6\left(6u-7\right) + c$$
$$= \frac{2}{63}(3x+1)^6\left[6(3x+1)-7\right] + c$$
$$= \frac{2}{63}\left(3x+1\right)^6(18x-1) + c$$

Question 4

If $u = 2x^2 - 1$, then $\dfrac{du}{dx} = 4x$

$$\int 6x(2x^2-1)^5\,dx = \int 6x(2x^2-1)^5 \frac{dx}{du}\,du$$
$$= \int 6xu^5 \frac{1}{4x}\,du = \frac{3}{2}\int u^5 du$$
$$= \frac{3}{2}\times\frac{u^6}{6} + c = \frac{1}{4}(2x^2-1)^6 + c$$

Question 5

If $u = 3x^2 + 1$, then $\dfrac{du}{dx} = 6x$

$$\int 12x(3x^2+1)^5\,dx = \int 12x(3x^2+1)^5 \frac{dx}{du}\,du$$
$$= \int 12x \times u^5 \times \frac{1}{6x}\,du$$
$$= \int 2u^5 du = \frac{2u^6}{6} + c$$
$$= \frac{1}{3}(3x^2+1)^6 + c$$

Question 6

If $u = x - 2$, then $\frac{du}{dx} = 1$.

$$\int 3x(x-2)^5\,dx = \int 3x(x-2)^5 \frac{dx}{du}\,du = \int 3(u+2)u^5\,du = 3\int (u^6 + 2u^5)\,du$$
$$= 3\left(\frac{u^7}{7} + 2\frac{u^6}{6}\right) + c = 3\left(\frac{u^7}{7} + \frac{u^6}{3}\right) + c = \frac{3}{21}(3u^7 + 7u^6) + c$$
$$= \frac{u^6}{7}(3u+7) + c = \frac{u^6}{7}\left[3(x-2)+7\right] + c$$
$$= \frac{1}{7}(x-2)^6(3x+1) + c$$

Question 7

If $u = 3 - x$, then $\frac{du}{dx} = -1$

$$\int 20x(3-x)^3\,dx = \int 20x(3-x)^3 \frac{dx}{du}\,du = \int 20(3-u)u^3(-1)\,du$$
$$= -20\int (3u^3 - u^4)\,du = -20\left(\frac{3u^4}{4} - \frac{u^5}{5}\right) + c$$
$$= \frac{-20}{20}(15u^4 - 4u^5) + c = -u^4(15-4u) + c$$
$$= -(3-x)^4\left[15 - 4(3-x)\right] + c$$
$$= -(3-x)^4(4x+3) + c = -(4x+3)(3-x)^4 + c$$

Question 8

If $u = 5 - 2x$, then $\frac{du}{dx} = -2$.

$$\int 4x(5-2x)^5\,dx = \int 4x(5-2x)^5 \frac{dx}{du}\,du = \int 4\left(\frac{5-u}{2}\right)u^5 \frac{1}{-2}\,du$$
$$= -\int (5u^5 - u^6)\,du = \int (u^6 - 5u^5)\,du = \frac{u^7}{7} - \frac{5u^6}{6} + c$$
$$= \frac{u^6}{42}(6u - 35) + c = \frac{1}{42}(5-2x)^6\left[6(5-2x) - 35\right] + c$$
$$= \frac{1}{42}(5-2x)^6(-5-12x) + c$$
$$= -\frac{1}{42}(5-2x)^6(12x+5) + c$$

Question 9

If $u=2x+3$, then $\dfrac{du}{dx}=2$

$$\int 20x(2x+3)^3\,dx=\int 20x(2x+3)^3\frac{dx}{du}\,du=\int 20\frac{(u-3)}{2}u^3\frac{1}{2}\,du$$

$$=5\int(u^4-3u^3)\,du=5\left(\frac{u^5}{5}-\frac{3u^4}{4}\right)+c$$

$$=\frac{5u^4}{20}(4u-15)+c=\frac{1}{4}(2x+3)^4\left[4(2x+3)-15\right]+c$$

$$=\frac{1}{4}(2x+3)^4(8x-3)+c$$

Question 10

If $u=3x+1$, then $\dfrac{du}{dx}=3$.

$$\int 18x\sqrt{3x+1}\;dx=\int 18x\sqrt{3x+1}\;\frac{dx}{du}\,du=\int 18\left(\frac{u-1}{3}\right)u^{\frac{1}{2}}\;\frac{1}{3}\,du$$

$$=2\int(u^{\frac{3}{2}}-u^{\frac{1}{2}})\,du=2\left(\frac{2}{5}u^{\frac{5}{2}}-\frac{2}{3}u^{\frac{3}{2}}\right)+c$$

$$=\frac{4}{15}u^{\frac{3}{2}}\left(3u-5\right)+c=\frac{4}{15}(3x+1)^{\frac{3}{2}}(9x-2)+c$$

Question 11

If $u=3x^2+5$, then $\dfrac{du}{dx}=6x$.

$$\int\frac{6x}{\sqrt{3x^2+5}}\,dx=\int\frac{6x}{\sqrt{3x^2+5}}\frac{dx}{du}\,du=\int\frac{6x}{\sqrt{3x^2+5}}\frac{dx}{du}\,du$$

$$=\int\frac{6x}{u^{\frac{1}{2}}}\frac{1}{6x}\,du=\int u^{-\frac{1}{2}}\,du=\frac{u^{\frac{1}{2}}}{\frac{1}{2}}+c$$

$$=2\sqrt{3x^2+5}+c$$

Question 12

If $u = 1-2x$, then $\frac{du}{dx} = -2$.

$$\int \frac{3x}{\sqrt{1-2x}}\,dx = \int \frac{3x}{\sqrt{1-2x}}\frac{dx}{du}\,du = \int \frac{3\left(\frac{1-u}{2}\right)}{\sqrt{u}}\frac{1}{-2}\,du = -\frac{3}{4}\int (1-u)u^{-\frac{1}{2}}\,du$$

$$= -\frac{3}{4}\int (u^{-\frac{1}{2}} - u^{\frac{1}{2}})du = -\frac{3}{4}\left(2u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right) + c = -\frac{1}{2}u^{\frac{1}{2}}(3-u) + c$$

$$= -\frac{1}{2}(3-1+2x)\sqrt{1-2x} + c = -\frac{1}{2}(2x+2)\sqrt{1-2x} + c$$

$$= -(x+1)\sqrt{1-2x} + c$$

Question 13

If $u = \sin 2x$, then $\frac{du}{dx} = 2\cos 2x$.

$$\int 8\sin^5 2x\cos 2x\,dx = \int 8\sin^5 2x\cos 2x\frac{dx}{du}\,du = \int 8u^5\cos 2x\frac{1}{2\cos 2x}\,du$$

$$= 4\int u^5 du = 4\frac{u^6}{6} + c = \frac{2}{3}\sin^6 2x + c$$

Question 14

If $u = \cos 3x$, then $\frac{du}{dx} = -3\sin 3x$.

$$\int 27\cos^7 3x\sin 3x\,dx = \int 27\cos^7 3x\sin 3x\frac{dx}{du}\,du = 27\int u^7\sin 3x\frac{1}{-3\sin 3x}\,du$$

$$= -9\int u^7 du = -9\frac{u^8}{8} + c = -\frac{9}{8}\cos^8 3x + c$$

Question 15

If $u = x^2 + 4$, then $\frac{du}{dx} = 2x$.

$$\int 6x\sin(x^2+4)dx = \int 6x\sin(x^2+4)\frac{dx}{du}\,du = \int 6x\sin u\frac{1}{2x}\,du$$

$$= 3\int \sin u\;du = -3\cos u + c = -3\cos(x^2+4) + c$$

Question 16

If $u = 2x+1$, then $\frac{du}{dx} = 2$.

$$\int (4x+3)(2x+1)^5 \, dx = \int (4x+3)(2x+1)^5 \frac{dx}{du} du = \int \left[4\left(\frac{u-1}{2}\right) + 3 \right] u^5 \, \frac{1}{2} du$$

$$= \frac{1}{2}\int (2u+1)u^5 du = \frac{1}{2}\int (2u^6 + u^5) du = \frac{1}{2}\left(\frac{2u^7}{7} + \frac{u^6}{6}\right) + c$$

$$= \frac{1}{84}u^6(12u+7) + c = \frac{1}{84}(2x+1)^6(24x+19) + c$$

Exercise 9B

Question 1

$$\int (x+\sin 3x)dx = \frac{x^2}{2} - \frac{\cos 3x}{3} + c = \frac{1}{2}x^2 - \frac{1}{3}\cos 3x + c$$

Question 2

$$\int 2\,dx = 2x + c$$

Question 3

$$\int \sin 8x\ dx = -\frac{1}{8}\cos 8x + c$$

Question 4

$$\int (\cos x+\sin x)(\cos x-\sin x)dx = \int (\cos^2 x - \sin^2 x)dx = \int \cos 2x\ dx$$

$$= \frac{1}{2}\sin 2x + c \ \left(\text{or } \frac{1}{2}(\cos x + \sin x)^2 + c\right)$$

Question 5

$$\int \frac{x^2+x}{\sqrt{x}}dx = \int (x^{\frac{3}{2}} + x^{\frac{1}{2}})dx = \frac{2}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}} + c$$

Question 6

If $u = x^2$, then $\frac{du}{dx} = 2x$.

$$\int 4x\sin(x^2)\,dx = \int 4x\sin(x^2)\frac{dx}{du}du = \int 4x\sin u \frac{1}{2x}du$$

$$= 2\int \sin u\ du = -2\cos u + c = -2\cos(x^2) + c$$

Question 7

If $u = x^2 - 3$, then $\frac{du}{dx} = 2x$.

$$\int 8x\sin(x^2-3)dx = \int 8x\sin(x^2-3)\frac{dx}{du}du = \int 8x\sin u\frac{1}{2x}du$$
$$= 4\int \sin u\ du = -4\cos u + c = -4\cos(x^2-3) + c$$

Question 8

If $u = 1 + 3x$, then $\frac{du}{dx} = 3$.

$$\int 24\sqrt{1+3x}\,dx = \int 24\sqrt{1+3x}\frac{dx}{du}du = \int 24\sqrt{u}\frac{1}{3}du$$
$$= 8\int u^{\frac{1}{2}}du = 8\times\frac{2}{3}u^{\frac{3}{2}} + c = \frac{16}{3}(1+3x)^{\frac{3}{2}} + c$$

Question 9

If $u = 1 + 3x$, then $\frac{du}{dx} = 3$.

$$\int 15x\sqrt{1+3x}\,dx = \int 15x\sqrt{1+3x}\frac{dx}{du}du = \int 15\left(\frac{u-1}{3}\right)\sqrt{u}\frac{1}{3}du$$
$$= \frac{5}{3}\int (u^{\frac{3}{2}} - u^{\frac{1}{2}})du = \frac{5}{3}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right) + c = \frac{10}{45}u^{\frac{3}{2}}(3u-5) + c$$
$$= \frac{2}{9}(1+3x)^{\frac{3}{2}}(9x-2) + c$$

Question 10

If $u = \sin 2x$, then $\frac{du}{dx} = 2\cos 2x$.

$$\int \sin^4 2x\ \cos 2x\,dx = \int \sin^4 2x\ \cos 2x\frac{dx}{du}du$$
$$= \int u^4\ \cos 2x\frac{1}{2\cos 2x}du = \frac{1}{2}\int u^4 du$$
$$= \frac{1}{2}\frac{u^5}{5} + c = \frac{1}{10}\sin^5 2x + c$$

Question 11

If $u = 2x + 7$, then $\frac{du}{dx} = 2$.

$$\int 6x(2x+7)^5\,dx = \int 6x(2x+7)^5 \frac{dx}{du}\,du = \int 6\left(\frac{u-7}{2}\right)u^5 \frac{1}{2}\,du$$
$$= \frac{3}{2}\int (u^6 - 7u^5)\,du = \frac{3}{2}\left(\frac{u^7}{7} - \frac{7u^6}{6}\right) + c$$
$$= \frac{3}{84}u^6(6u - 49) + c = \frac{1}{28}(2x+7)^6\left[6(2x+7) - 49\right] + c$$
$$= \frac{1}{28}(2x+7)^6(12x - 7) + c$$

Question 12

If $u = 2x + 7$, then $\frac{du}{dx} = 2$.

$$\int 6(2x+7)^5\,dx = \int 6(2x+7)^5 \frac{dx}{du}\,du = \int 6u^5 \frac{1}{2}\,du$$
$$= 3\int u^5\,du = \frac{3u^6}{6} + c = \frac{1}{2}(2x+7)^6 + c$$

Question 13

$$\int (3x^2 - 2)\,dx = x^3 - 2x + c$$

Question 14

If $u = 3x^2 - 2$, then $\frac{du}{dx} = 6x$.

$$\int 4x(3x^2-2)^7\,dx = \int 4x(3x^2-2)^7 \frac{dx}{du}\,du = \int 4xu^7 \frac{1}{6x}\,du$$
$$= \int \frac{2}{3}u^7\,du = \frac{2}{3}\frac{u^8}{8} + c = \frac{1}{12}(3x^2-2)^8 + c$$

Question 15

$$\int(\cos x+\sin 2x)dx=\sin x-\frac{1}{2}\cos 2x+c$$

Question 16

If $u=3x-2$, then $\frac{du}{dx}=3$.

$$\int 6x(3x-2)^7\,dx=\int 6x(3x-2)^7\,\frac{dx}{du}du=\int 6\frac{(u+2)}{3}u^7\,\frac{1}{3}du$$

$$=\frac{2}{3}\int(u^8+2u^7)du=\frac{2}{3}\left(\frac{u^9}{9}+\frac{2u^8}{8}\right)+c=\frac{2}{3\times 72}(8u^9+18u^8)+c$$

$$=\frac{1}{54}u^8(4u+9)+c=\frac{1}{54}(3x-2)^8(12x+1)+c$$

Question 17

$$\int x\,dx=\frac{1}{2}x^2+c$$

Question 18

If $u=1+2x$, then $\frac{du}{dx}=2$.

$$\int\frac{6}{\sqrt{1+2x}}dx=\int\frac{6}{\sqrt{1+2x}}\frac{dx}{du}du=\int\frac{6}{u^{\frac{1}{2}}}\times\frac{1}{2}du$$

$$=3\int u^{-\frac{1}{2}}du=3\frac{u^{\frac{1}{2}}}{\frac{1}{2}}+c=6\sqrt{1+2x}+c$$

Question 19

If $u = 1 + 2x$, then $\frac{du}{dx} = 2$.

$$\int \frac{6x}{\sqrt{1+2x}}\,dx = \int \frac{6x}{\sqrt{1+2x}}\frac{dx}{du}\,du = \int \frac{6 \times \frac{1}{2}(u-1)}{u^{\frac{1}{2}}}\frac{1}{2}\,du$$

$$= \frac{3}{2}\int (u^{\frac{1}{2}} - u^{-\frac{1}{2}})\,du = \frac{3}{2}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + c$$

$$= \frac{3}{2}u^{\frac{1}{2}}\left(\frac{2}{3}u - 2\right) + c = \frac{3}{2}\sqrt{1+2x}\left(\frac{2}{3}(1+2x) - 2\right) + c$$

$$= \sqrt{1+2x}(1+2x-3) + c = 2(x-1)\sqrt{1+2x} + c$$

Question 20

If $u = x^2 + x + 1$, then $\frac{du}{dx} = 2x + 1$.

$$\int (x^2+x+1)^8(2x+1)\,dx = \int (x^2+x+1)^8(2x+1)\frac{dx}{du}\,du$$

$$= \int u^8(2x+1)\frac{1}{2x+1}\,du$$

$$= \frac{u^9}{9} + c = \frac{1}{9}(x^2+x+1)^9 + c$$

Question 21

If $u = x^2 + 3$, then $\frac{du}{dx} = 2x$.

$$\int 24x\sin(x^2+3)\,dx = \int 24x\sin(x^2+3)\frac{dx}{du}\,du = \int 24x\sin u\frac{1}{2x}\,du$$

$$= 12\int \sin u\,du = -12\cos u + c$$

$$= -12\cos(x^2+3) + c$$

Question 22

If $u = x - 5$, then $\dfrac{du}{dx} = 1$.

$$\int (2x+1)\sqrt[3]{x-5}\,dx = \int (2x+1)\sqrt[3]{x-5}\,\frac{dx}{du}\,du = \int \left[2(u+5)+1\right]u^{\frac{1}{3}}\,du$$

$$= \int (2u+11)u^{\frac{1}{3}}du = \int (2u^{\frac{4}{3}} + 11u^{\frac{1}{3}})du$$

$$= 2\frac{u^{\frac{7}{3}}}{\frac{7}{3}} + 11\frac{u^{\frac{4}{3}}}{\frac{4}{3}} + c = \frac{6}{7}u^{\frac{7}{3}} + \frac{33}{4}u^{\frac{4}{3}} + c$$

$$= 3u^{\frac{4}{3}}\left(\frac{2}{7}u + \frac{11}{4}\right) + c = 3(x-5)^{\frac{4}{3}}\left[\frac{2}{7}(x-5) + \frac{11}{4}\right] + c$$

$$= \frac{3}{28}(x-5)^{\frac{4}{3}}\left[8(x-5)+77\right] + c$$

$$= \frac{3}{28}(x-5)^{\frac{4}{3}}(8x+37) + c$$

Question 23

If $u = \sqrt{x} + 5$, then $\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$.

$$\int \frac{(\sqrt{x}+5)^5}{\sqrt{x}}\,dx = \int \frac{(\sqrt{x}+5)^5}{\sqrt{x}}\frac{dx}{du}\,du = \int \frac{u^5}{\sqrt{x}}\frac{2\sqrt{x}}{1}\,du$$

$$= 2\int u^5 du = 2\frac{u^6}{6} + c = \frac{1}{3}(\sqrt{x}+5)^6 + c$$

Question 24

If $u = 2x - 1$, then $\dfrac{du}{dx} = 2$.

$$\int 4(2x-1)^5\,dx = \int 4(2x-1)^5\frac{dx}{du}\,du = \int 4u^5\,\frac{1}{2}\,du$$

$$= 2\frac{u^6}{6} + c = \frac{(2x-1)^6}{3} + c$$

Question 25

If $u = 2x-1$, then $\frac{du}{dx} = 2$.

$$\int 4x(2x-1)^5\,dx = \int 4x(2x-1)^5 \frac{dx}{du}\,du = \int 4\frac{(u+1)}{2}u^5 \frac{1}{2}\,du$$

$$= \int (u^6 + u^5)\,du = \frac{u^7}{7} + \frac{u^6}{6} + c = \frac{1}{42}u^6(6u+7) + c$$

$$= \frac{1}{42}(2x-1)^6(12x+1) + c$$

Question 26

If $u = \cos 6x$, then $\frac{du}{dx} = -6\sin 6x$.

$$\int \cos^3 6x\ \sin 6x\,dx = \int \cos^3 6x\ \sin 6x \frac{dx}{du}\,du = \int u^3\ \sin 6x \frac{1}{-6\sin 6x}\,du$$

$$= -\frac{1}{6}\int u^3\,du = -\frac{1}{6} \times \frac{u^4}{4} + c = -\frac{\cos^4 6x}{24} + c$$

Question 27

If $u = x^2 - 3$, then $\frac{du}{dx} = 2x$.

$$\int \frac{6x}{\sqrt{x^2-3}}\,dx = \int \frac{6x}{\sqrt{x^2-3}} \frac{dx}{du}\,du = \int \frac{6x}{u^{\frac{1}{2}}} \frac{1}{2x}\,du$$

$$= 3\int u^{-\frac{1}{2}}\,du = 3\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c = 6\sqrt{x^2-3} + c$$

Question 28

If $u = \sin 2x$, then $\frac{du}{dx} = 2\cos 2x$.

$$\int \sin 2x\ \cos 2x\,dx = \int \sin 2x\ \cos 2x \frac{dx}{du}\,du = \int u\cos 2x \frac{1}{2\cos 2x}\,du$$

$$= \frac{1}{2}\int u\,du = \frac{1}{2}\frac{u^2}{2} + c = \frac{\sin^2 2x}{4} + c \ \text{ or } \ -\frac{\cos 4x}{8} + c$$

Question 29

If $u = 2x + 5$, then $\frac{du}{dx} = 2$.

$$\int 8x^2(2x-1)^5\,dx = \int 8x^2(2x-1)^5 \frac{dx}{du}\,du = \int 8\left(\frac{u+1}{2}\right)^2 u^5 \frac{1}{2}\,du$$

$$= \int 8\left(\frac{u+1}{2}\right)^2 u^5 \frac{1}{2}\,du = \int (u^2 + 2u + 1)u^5\,du = \int (u^7 + 2u^6 + u^5)\,du$$

$$= \frac{u^8}{8} + \frac{2u^7}{7} + \frac{u^6}{6} + c = \frac{1}{168}u^6(21u^2 + 48u + 28) + c$$

$$= \frac{1}{168}(2x-1)^6(84x^2 + 12x + 1) + c$$

Exercise 9C

Question 1

If $u = 2x+1$, then $\frac{du}{dx} = 2$.

$$\int_0^1 16(2x+1)^3\,dx = \int_{x=0}^{x=1} 16(2x+1)^3 \frac{dx}{du}\,du = \int_{u=1}^{u=3} 16u^3 \frac{1}{2}\,du$$

$$= 8\int_1^3 u^3 du = 8\left[\frac{u^4}{4}\right]_1^3 = 8\left(\frac{81}{4} - \frac{1}{4}\right) = 160$$

Question 2

If $u = 2x+1$, then $\frac{du}{dx} = 2$.

$$\int_0^1 16x(2x+1)^3\,dx = \int_{x=0}^{x=1} 16x(2x+1)^3 \frac{dx}{du}\,du = \int_{u=1}^{u=3} 16\left(\frac{u-1}{2}\right)u^3 \frac{1}{2}\,du$$

$$= 4\int_1^3 (u^4 - u^3)du = 4\left[\frac{u^5}{5} - \frac{u^4}{4}\right]_1^3$$

$$= 4\left(\frac{243}{5} - \frac{81}{4} - \left(\frac{1}{5} - \frac{1}{4}\right)\right) = 113.6$$

Question 3

If $u = x+5$, then $\frac{du}{dx} = 1$.

$$\int_0^1 \frac{6x}{25}(x+5)^4\,dx = \int_{x=0}^{x=1} \frac{6x}{25}(x+5)^4 \frac{dx}{du}\,du = \int_{u=5}^{u=6} \frac{6(u-5)}{25}u^4\,du$$

$$= \frac{6}{25}\int_5^6 (u^5 - 5u^4)du = \frac{6}{25}\left[\frac{u^6}{6} - 5\frac{u^5}{5}\right]_5^6$$

$$= \frac{6}{25}\left(\frac{6^6}{6} - 6^5 - \left(\frac{5^6}{6} - 5^5\right)\right) = 125$$

Question 4

If $u = \sin x$, then $\dfrac{du}{dx} = \cos x$.

$$\int_0^{\frac{\pi}{2}} 12\sin^5 x\cos x\,dx = \int_{x=0}^{x=\frac{\pi}{2}} 12\sin^5 x\cos x\frac{dx}{du}\,du = \int_{u=0}^{u=1} 12u^5\cos x\frac{1}{\cos x}\,du$$

$$= 12\int_{u=0}^{u=1} u^5\,du = 12\left[\frac{u^6}{6}\right]_0^1 = 12\left(\frac{1}{6} - 0\right) = 2$$

Question 5

If $u = 5x + 6$, then $\dfrac{du}{dx} = 5$.

$$\int_2^6 \frac{3x}{\sqrt{x+6}}\,dx = \int_{x=2}^{x=6} \frac{3x}{\sqrt{5x+6}}\frac{dx}{du}\,du = \int_{u=16}^{u=36} \frac{3\frac{u-6}{5}}{\sqrt{u}}\frac{1}{5}\,du$$

$$= \frac{3}{25}\int_{u=16}^{u=36} \frac{u-6}{\sqrt{u}}\,du = \frac{3}{25}\int_{u=16}^{u=36} (u^{\frac{1}{2}} - 6u^{-\frac{1}{2}})\,du = \frac{3}{25}\left[\frac{2}{3}u^{\frac{3}{2}} - 12u^{\frac{1}{2}}\right]_{16}^{36}$$

$$= \frac{3}{25}\left(144 - 72 - \left(\frac{128}{3} - 48\right)\right) = 9.28$$

Question 6

If $u = x - 1$, then $\dfrac{du}{dx} = 1$.

$$\int_2^5 \frac{x+3}{\sqrt{x-1}}\,dx = \int_{x=2}^{x=5} \frac{x+3}{\sqrt{x-1}}\frac{dx}{du}\,du = \int_{u=1}^{u=4} \frac{u+1+3}{\sqrt{u}}\,du$$

$$= \int_{u=1}^{u=4} (u^{\frac{1}{2}} + 4u^{-\frac{1}{2}})\,du = \left[\frac{2}{3}u^{\frac{3}{2}} + 8u^{\frac{1}{2}}\right]_1^4$$

$$= \frac{16}{3} + 16 - \left(\frac{2}{3} + 8\right) = 12\frac{2}{3}$$

Question 7

If $u = 2x + 1$, then $\dfrac{du}{dx} = 2$.

$$\int_0^4 \frac{4}{\sqrt{2x+1}}\,dx = \int_{x=0}^{x=4} \frac{4}{\sqrt{2x+1}} \frac{dx}{du}\,du = \int_{u=1}^{u=9} \frac{4}{u^{\frac{1}{2}}} \frac{1}{2}\,du$$

$$= 2\int_1^9 u^{-\frac{1}{2}}\,du = 2\left[2u^{\frac{1}{2}}\right]_1^9 = 2(6-2)$$

$$= 8 \text{ square units}$$

Question 8

The curve intersects the x-axis at 0 and 3.

If $u = x - 3$, then $\dfrac{du}{dx} = 1$

$$\int_0^3 6x(x-3)^3\,dx = \int_{x=0}^{x=3} 6x(x-3)^3 \frac{dx}{du}\,du = \int_{u=-3}^{u=0} 6(u+3)u^3\,du$$

$$= 6\int_{-3}^0 (u^4 + 3u^3)\,du = 6\left[\frac{u^5}{5} + \frac{3u^4}{4}\right]_{-3}^0$$

$$= 6\left(0 + 0 - \frac{-243}{5} - \frac{243}{4}\right)$$

$$= -72.9$$

This is a negative value because the curve lies below the x-axis for $0 \le x \le 3$. However, area cannot be negative so the required area is 72.9 square units.

Exercise 9D

Question 1

$$\int \cos 5x \cos 4x\,dx = \int \frac{1}{2}[\cos 9x + \cos x]\,dx = \int \left(\frac{1}{2}\cos 9x + \frac{1}{2}\cos x\right)dx$$

$$= \frac{1}{18}\sin 9x + \frac{1}{2}\sin x + c$$

Question 2

$$\int \sin 7x \sin x\,dx = \int \frac{1}{2}[\cos 6x - \cos 8x]\,dx = \frac{1}{12}\sin 6x - \frac{1}{16}\sin 8x + c$$

Question 3

As $\cos x$ is the derivative of $\sin x$, $\int \sin^4 x \cos x\,dx = \frac{1}{5}\sin^5 x + c$.

Question 4

As $\cos x$ is the derivative of $\sin x$, $\int 6\sin^3 x \cos x\,dx = 6\times\frac{1}{4}\sin^4 x + c = \frac{3}{2}\sin^4 x + c$

Question 5

$$\int \sin^3 x\,dx = \int \sin x \sin^2 x\,dx = \int \sin x(1-\cos^2 x)\,dx$$

$$= \int (\sin x - \sin x \cos^2 x)dx = -\cos x + \frac{1}{3}\cos^3 x + c$$

Question 6

$$\int \cos^3 x\,dx = \int \cos x(1-\sin^2 x)\,dx = \int (\cos x - \cos x \sin^2 x)\,dx$$

$$= \sin x - \frac{1}{3}\sin^3 x + c$$

Question 7

$$\int \cos^5 x\,dx = \int \cos x(1-\sin^2 x)^2\,dx = \int \cos x(1-2\sin^2 x+\sin^4 x)dx$$
$$= \int (\cos x - 2\cos x\sin^2 x + \cos x\sin^4 x)dx$$
$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c$$

Question 8

$$\int \cos^2 x\,dx = \int \left(\frac{1+\cos 2x}{2}\right)dx = \frac{1}{2}\int (1+\cos 2x)dx$$
$$= \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + c = \frac{1}{2}x + \frac{1}{4}\sin 2x + c$$

Question 9

$$\int \sin^2 x\,dx = \int \frac{1}{2}(1-\cos 2x)dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + c$$
$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

Question 10

$$\int 8\sin^4 x\,dx = \int 2(4\sin^4 x)dx$$
$$= \int 2(-2\sin^2 x)^2\,dx \quad \text{(from trig identities } \cos 2x - 1 = -2\sin^2 x\text{)}$$
$$= \int 2(\cos 2x - 1)^2\,dx = \int 2(\cos^2 2x - 2\cos 2x + 1)dx$$
$$= \int (2\cos^2 2x - 1 + 1 - 4\cos 2x + 2)dx = \int (\cos 4x - 4\cos 2x + 3)dx$$
$$= \frac{1}{4}\sin 4x - 2\sin 2x + 3x + c$$

Question 11

$$\int (\cos^2 x + \sin^2 x)dx = \int 1\,dx = x + c$$

Question 12

$$\int (\cos^2 x - \sin^2 x)\,dx = \int \cos 2x\,dx = \frac{1}{2}\sin 2x + c$$

Question 13

$$\begin{aligned}\int(\sin^3 x+\cos^2 x)\,dx &= \int(\sin x\sin^2 x+\cos^2 x)\,dx\\ &= \int\left[\sin x(1-\cos^2 x)+\frac{1}{2}(\cos 2x+1)\right]dx\\ &= \int\left(\sin x-\sin x\cos^2 x+\frac{1}{2}\cos 2x+\frac{1}{2}\right)dx\\ &= -\cos x+\frac{1}{3}\cos^3 x+\frac{1}{4}\sin 2x+\frac{1}{2}x+c\end{aligned}$$

Question 14

$$\int 2\sin x\,\cos x\,dx=\int\sin 2x\,dx=-\frac{\cos 2x}{2}+c\ \ (\text{or }\sin^2 x+c\text{ or }-\cos^2 x+c)$$

Question 15

$$\begin{aligned}\int\sin^3 x\cos^2 x\,dx &= \int\sin x\sin^2 x\cos^2 x\,dx=\int\sin x(1-\cos^2 x)\cos^2 x\,dx\\ &= \int(\sin x\cos^2 x-\sin x\cos^4 x)\,dx=-\frac{1}{3}\cos^3 x+\frac{1}{5}\cos^5 x+c\end{aligned}$$

Question 16

$$\begin{aligned}\int\cos^3 x\sin^2 x\,dx &= \int\cos x\cos^2 x\sin^2 x\,dx=\int\cos x(1-\sin^2 x)\sin^2 x\,dx\\ &= \int(\cos x\sin^2 x-\cos x\sin^4 x)dx=\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+c\end{aligned}$$

Question 17

$$\begin{aligned}\int\tan^2 3x\,dx &= \int\left(\frac{\sin^2 3x}{\cos^2 3x}\right)dx=\int\left(\frac{1-\cos^2 3x}{\cos^2 3x}\right)dx\\ &= \int(\sec^2 3x-1)dx=\frac{1}{3}\tan 3x-x+c\end{aligned}$$

Question 18

$$\int (1+\tan^2 x)dx = \int \left(1+\frac{\sin^2 x}{\cos^2 x}\right)dx$$
$$= \int \left(1+\frac{1-\cos^2 x}{\cos^2 x}\right)dx$$
$$= \int \left(1+\sec^2 x - 1\right)dx$$
$$= \tan x + c$$

Question 19

$$\int \left(\frac{\sin x}{1-\sin x}\times\frac{\sin x}{1+\sin x}\right)dx = \int \frac{\sin^2 x}{1-\sin^2 x}dx$$
$$= \int \frac{1-\cos^2 x}{\cos^2 x}dx$$
$$= \int (\sec^2 x - 1)dx$$
$$= \tan x - x + c$$

Question 20

$$\int \sec^2 x \tan^4 x \,\mathrm{d}x = \frac{1}{5}\tan^5 x + c$$

Question 21

$$\int_0^{2\pi} (x+\cos^2 x - \sin^2 x)dx = \int_{x=0}^{x=2\pi} (x+\cos 2x)\frac{dx}{du}du$$
$$= \int_{x=0}^{x=2\pi} (x+\cos 2x)dx$$
$$= \left[\frac{1}{2}x^2 + \frac{1}{2}\sin 2x\right]_0^{2\pi}$$
$$= \frac{1}{2}\times 4\pi^2 + \frac{1}{2}\sin 4\pi - 0 - \frac{1}{2}\sin 0$$
$$= 2\pi^2 \text{ square units}$$

The area under the curve from $x = 0$ to $x = 2\pi$ is 2π square units.

Question 22

a $\mathbf{v} = 4\sin^2 t\mathbf{i} + \tan^2 t\mathbf{j} \quad (0 \le t \le \frac{\pi}{2})$

$$\mathbf{r} = \int 4\sin^2 t\,dt\,\mathbf{i} + \int \tan^2 t\,dt\,\mathbf{j}$$

$$= \int 4\left[-\frac{1}{2}(\cos 2t - 1)\right]dt\,\mathbf{i} + \int\left(\frac{\sin^2 t}{\cos^2 t}\right)dt\mathbf{j}$$

$$= \int (-2\cos 2t + 2)dt\,\mathbf{i} + \int\left(\frac{1-\cos^2 t}{\cos^2 t}\right)dt\mathbf{j}$$

$$= (-\sin 2t + 2t)\mathbf{i} + \mathbf{c} + \int\left(\sec^2 t - 1\right)dt\mathbf{j}$$

$$= (-\sin 2t + 2t)\mathbf{i} + (\tan t - t)\mathbf{j} + \mathbf{c}$$

When $t = 0$, $\mathbf{r} = 3\mathbf{i} + \mathbf{j}$, hence

$$\mathbf{r} = (-\sin 2t + 2t)\mathbf{i} + (\tan t - t)\mathbf{j} + 3\mathbf{i} + \mathbf{j}$$

$$= (-\sin 2t + 2t + 3)\mathbf{i} + (\tan t - t + 1)\mathbf{j}$$

$$= (3 + 2t - \sin 2t)\mathbf{i} + (1 - t + \tan t)\mathbf{j}$$

b $$\mathbf{r}\left(\frac{\pi}{4}\right) = \left[3 + 2\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{2}\right)\right]\mathbf{i} + \left[1 - \frac{\pi}{4} + \tan\left(\frac{\pi}{4}\right)\right]\mathbf{j}$$

$$= \left(3 + \frac{\pi}{2} - 1\right)\mathbf{i} + \left(1 - \frac{\pi}{4} + 1\right)\mathbf{j}$$

$$= \left(2 + \frac{\pi}{2}\right)\mathbf{i} + \left(2 - \frac{\pi}{4}\right)\mathbf{j}$$

Exercise 9E

Question 1

$$\int \frac{7}{x}\,dx = 7\ln|x| + c$$

Question 2

$$\int \left(3x^2 - \frac{4}{x}\right) dx = x^3 - 4\ln|x| + c$$

Question 3

$$\int \frac{8x}{x^2+6}\,dx = 4\int \frac{2x}{x^2+6}\,dx \quad \text{(and now the numerator is the derivative of the denominator)}$$

$$= 4\ln(x^2+6) + c$$

Question 4

If $u = 2x$, then $\frac{du}{dx} = 2$.

$$\int \tan 2x\,dx = \int \tan u \frac{dx}{du}\,du$$

$$\int \frac{\sin u}{\cos u}\frac{1}{2}\,du = \frac{1}{2}\int \frac{\sin u}{\cos u}\,du$$

If $w = \cos u$, then $\frac{dw}{du} = -\sin u$.

$$\frac{1}{2}\int \frac{\sin u}{\cos u}\,du = \frac{1}{2}\int \frac{\sin u}{w}\frac{du}{dw}\,dw = \frac{1}{2}\int \frac{\sin u}{w}\frac{1}{-\sin u}\,dw$$

$$= -\frac{1}{2}\ln|w| + c$$

Substitute $w = \cos u$ and $u = 2x$ back into the equation to get:

$$\int \tan 2x\,dx = -\frac{1}{2}\ln|\cos 2x| + c$$

Question 5

$$\int \frac{x+2}{x}\,dx = \int \left(1+\frac{2}{x}\right)dx = x + 2\ln|x| + c$$

Question 6

If $u = x+2$, then $\frac{du}{dx} = 1$.

$$\int \frac{x}{x+2}\,dx = \int \frac{x}{x+2}\frac{dx}{du}\,du = \int \frac{u-2}{u}\,du = \int \left(1-\frac{2}{u}\right)du$$
$$= u - 2\ln|u| + c = x + 2 - 2\ln|x+2| + c$$
$$= x - 2\ln|x+2| + c$$

(as 2 is a constant, it can be part of c)

Question 7

$$\int \frac{2x-3}{x}\,dx = \int \left(2-\frac{3}{x}\right)dx = 2x - 3\ln|x| + c$$

Question 8

If $u = 2x-3$, then $\frac{du}{dx} = 2$.

$$\int \frac{x}{2x-3}\,dx = \int \frac{x}{2x-3}\frac{dx}{du}\,du = \int \frac{(u+3)}{2u}\frac{1}{2}\,du$$
$$= \frac{1}{4}\int \left(1+\frac{3}{u}\right)du = \frac{1}{4}(u + 3\ln|u|) + c$$
$$= \frac{x}{2} + \frac{3}{4}\ln|2x-3| + c$$

Question 9

$$\int \frac{x^2+4x+1}{x+3}\,dx = \int \left(\frac{x(x+3)}{x+3} + \frac{x+3}{x+3} - \frac{2}{x+3}\right)dx$$
$$= \int \left(x + 1 - \frac{2}{x+3}\right)dx$$
$$= \frac{x^2}{2} + x - 2\ln|x+3| + c$$

Question 10

Let $\dfrac{5x+3}{x(x+1)}=\dfrac{A}{x}+\dfrac{B}{x+1}=\dfrac{A(x+1)+Bx}{x(x+1)}$

$A+B=5$ and $A=3$, so $B=2$.

$$\int\frac{5x+3}{x(x+1)}\,dx=\int\left(\frac{3}{x}+\frac{2}{x+1}\right)dx=3\ln|x|+2\ln|x+1|+c$$

Question 11

Let $\dfrac{4x-7}{(x+2)(x-3)}=\dfrac{A}{x+2}+\dfrac{B}{x-3}=\dfrac{A(x-3)+B(x+2)}{(x+2)(x-3)}$

$A+B=4$ and $-3A+2B=-7$, solving gives $A=3$ and $B=-1$.

$$\int\frac{4x-7}{(x+2)(x-3)}\,dx=\int\left(\frac{3}{x+2}+\frac{1}{x-3}\right)dx=3\ln|x+2|+\ln|x-3|+c$$

Question 12

$$\frac{5x^2-2x+18}{(x-1)(x^2+6)}=\frac{A}{x-1}+\frac{B}{x^2+6}=\frac{A(x^2+6)+(Bx+C)(x-1)}{(x-1)(x^2+6)}$$

$A+B=5$, $C-B=-2$ and $6A-C=18$

Solving gives $A=3$, $B=2$ and $C=0$.

$$\int\frac{5x^2-2x+18}{(x-1)(x^2+6)}\,dx=\int\left(\frac{3}{x-1}+\frac{2x}{x^2+6}\right)dx=3\ln|x-1|+\ln(x^2+6)+c$$

Question 13

$$\frac{7x^2+8x-4}{(x+1)(x^2+x-1)}=\frac{A}{x+1}+\frac{Bx+C}{(x^2+x-1)}=\frac{A(x^2+x-1)+(Bx+C)(x+1)}{(x+1)(x^2+x-1)}$$

$A+B=7$, $A+B+C=8$ and $-A+C=-4$

$A=5, B=2$ and $C=1$

$$\int\frac{7x^2+8x-4}{(x+1)(x^2+x-1)}\,dx=\int\left(\frac{5}{x+1}+\frac{2x+1}{x^2+x-1}\right)dx$$

$$=5\ln|x+1|+\ln|x^2+x-1|+c$$

Question 14

$$\frac{5x^2-10x-3}{(x+1)(x-1)^2}=\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

$$=\frac{A(x^2-2x+1)+B(x^2-1)+C(x+1)}{(x-1)^2}$$

$A+B=5,\ -2A+C=-10$ and $A-B+C=-3$

$A=3,\ B=2$ and $C=-4$

$$\int\frac{5x^2-10x-3}{(x+1)(x-1)^2}\,dx=\int\left(\frac{3}{x+1}+\frac{2}{x-1}-\frac{4}{(x-1)^2}\right)dx$$

$$=3\ln|x+1|+2\ln|x-1|+\frac{4}{x-1}+c$$

Question 15

$$\frac{8x^2-44x+25}{(2x+1)(x-3)^2}=\frac{A}{2x+1}+\frac{B}{x-3}+\frac{C}{(x-3)^2}$$

$$=\frac{A(x^2-6x+9)+B(2x^2-5x-3)+C(2x+1)}{(x-3)^2}$$

$A+2B=8,\ -6A-5B+2C=-44$ and $9A-3B+C=25$

$A=4$, $B=2$ and $C=-5$.

$$\int\left(\frac{8x^2-44x+25}{(2x+1)(x-3)^2}\right)dx=\int\left(\frac{4}{2x+1}+\frac{2}{x-3}-\frac{5}{(x-3)^2}\right)dx$$

$$=2\ln|2x+1|+2\ln|x-3|+\frac{5}{x-3}+c$$

Question 16

The two graphs intersect when:

$$\frac{x}{x-2}=\frac{11x}{x^2+2}$$
$$x(x^2+2)=11x(x-2)$$
$$x^3+2x=11x^2-22x$$
$$x^3-11x^2+24x=0$$
$$x(x-3)(x-8)=0$$
$$x=0,3,8$$

The region enclosed by the curves occurs between $x=3$ and $x=8$.

$$\int_3^8\left(\frac{11x}{x^2+2}\right)dx=\int_3^8\left(\frac{11x}{x^2+2}\right)dx=\left[\frac{11}{2}\ln\left|x^2+2\right|\right]_3^8=\frac{11}{2}\ln|66|-\frac{11}{2}\ln|11|$$

$$\int_3^8\frac{x}{x-2}$$

If $u=x-2$, then $\frac{du}{dx}=1$.

$$\int_3^8\frac{x}{x-2}dx=\int_{x=3}^{x=8}\left(\frac{x}{x-2}\right)\frac{dx}{du}du=\int_{u=1}^{u=6}\left(\frac{u+2}{u}\right)du$$
$$=\int_1^6\left(1+\frac{2}{u}\right)du=\left[u+2\ln|u|\right]_1^6=6+2\ln|6|-1-2\ln|1|$$
$$=5+2\ln|6|-2\times 0=5+2\ln|6|$$

Area enclosed by the curves is:

$$\frac{11}{2}\ln|66|-\frac{11}{2}\ln|11|-\left(5+2\ln|6|-2\ln|1|\right)=\frac{11}{2}(\ln|66|-\ln|11|)-5-2\ln|6|$$
$$=\frac{11}{2}\ln|6|-5-2\ln|6|$$
$$=\left(-5+\frac{7}{2}\ln 6\right)\text{square units}$$

Exercise 9F

Question 1

$$\int_0^2 \pi y^2 dx = \int_0^2 \pi(x^2)^2 dx = \int_0^2 \pi(x^2)^2 dx$$

$$= \pi\int_0^2 x^4 dx = \pi\left[\frac{x^5}{5}\right]_0^2 = \frac{32\pi}{5}\text{units}^3$$

Question 2

$$\int_0^1 \pi y^2 dx = \int_0^1 \pi(3x^2)^2 dx = 9\pi\int_0^1 x^4 dx$$

$$= 9\pi\left[\frac{x^5}{5}\right]_0^1 = \frac{9\pi}{5}\text{units}^3$$

Question 3

$$\int_0^1 \pi y^2 dx = \int_0^1 \pi(\sqrt{x})^2 dx = \pi\int_1^4 x\,dx$$

$$= \pi\left[\frac{x^2}{2}\right]_1^4 = \pi\left[8-\frac{1}{2}\right] = \frac{15\pi}{2}\text{units}^3$$

Question 4

$$\int_2^3 \pi y^2 dx = \int_2^3 \pi(2x+1)^2 dx = \int_2^3 \pi(2x+1)^2 dx$$

$$= \pi\int_2^3 (4x^2+4x+1)\,dx = \pi\left[\frac{4x^3}{3}+\frac{4x^2}{2}+x\right]_2^3$$

$$= \pi\left[36+18+3-\frac{32}{3}-8-2\right] = \frac{109\pi}{3}\text{units}^3$$

Question 5

a

$$\int_1^2 \pi y^2 dx = \int_1^2 \pi\left(\frac{1}{x}\right)^2 dx = \int_1^2 \pi\left(\frac{1}{x}\right)^2 dx$$

$$= \pi\int_1^2 x^{-2} dx = \pi\left[-x^{-1}\right]_1^2$$

$$= \pi\left[-\frac{1}{2}+1\right] = \frac{\pi}{2}\text{units}^3$$

b

$$\int_2^3 \pi y^2 dx = \int_2^3 \pi\left(\frac{1}{x}\right)^2 dx = \int_2^3 \pi\left(\frac{1}{x}\right)^2 dx$$

$$= \pi\int_2^3 x^{-2} dx = \pi\left[-x^{-1}\right]_2^3 = \pi\left[-\frac{1}{3}+\left(\frac{1}{2}\right)\right]$$

$$= \frac{\pi}{6}\text{units}^3$$

Question 6

$$\int_{-1}^2 \pi y^2 dx = \int_{-1}^2 \pi\left(x^2+1\right)^2 dx = \pi\int_{-1}^2\left(x^4+2x^2+1\right)dx$$

$$= \pi\left[\frac{x^5}{5}+\frac{2x^3}{3}+x\right]_{-1}^2 = \pi\left(\frac{32}{5}+\frac{16}{3}+2-\frac{-1}{5}-\frac{-2}{3}-(-1)\right)$$

$$= \frac{78\pi}{5}\text{units}^3$$

Question 7

Using integration to determine the volume of the cone:

$$\int_0^6 \pi y^2 dx = \int_0^6 \pi(0.5x)^2 dx = \frac{\pi}{4}\int_0^6 x^2 dx$$

$$= \frac{\pi}{4}\left[\frac{x^3}{3}\right]_0^6 = \frac{\pi}{4}[72] = 18\pi\,\text{units}^3$$

Using the formula for the volume of a right cone:

$$V = \frac{\pi r^2 h}{3}$$ (the line intersect at (6, 3) so radius = 3, $h = 6$)

$$= \frac{\pi\times 3^2\times 6}{3}$$

$$= 18\pi\,\text{units}^3$$

Both methods give the same answer.

Question 8

$$\int_0^{\pi} \pi y^2 dx = \int_0^{\pi} \pi\left(\sqrt{\sin x}\right)^2 dx = \pi\int_0^{\pi}\left(\sqrt{\sin x}\right)^2 dx$$
$$= \pi\int_0^{\pi}(\sin x)\,dx = \pi\left[-\cos x\right]_0^{\pi}$$
$$= \pi\left[-\cos\pi-(-\cos 0)\right] = \pi(1+1)$$
$$= 2\pi\,\text{units}^3$$

Question 9

$$\int_0^{\pi} \pi y^2 dx = \int_0^{\pi} \pi(\sin x)^2 dx = \pi\int_0^{\pi}\sin^2 x\,dx$$
$$= \pi\int_0^{\pi}\frac{1}{2}(1-\cos 2x)\,dx = \frac{\pi}{2}\left[x-\frac{\sin 2x}{2}\right]_0^{\pi}$$
$$= \frac{\pi}{2}\left(\pi-0-(0-0)\right) = \frac{\pi^2}{2}\,\text{units}^3$$

Question 10

First find the points where $y=x$ and $y=x^2$ intersect.

$$x^2 = x$$
$$x^2 - x = 0$$
$$x(x-1) = 0$$
$$x = 0, 1$$

Now find the volume of the solid formed when the area enclosed by $y=x$, the x-axis and the line $x=1$ is rotated through one revolution about the x-axis.

$$\int_0^1 \pi y^2 dx = \int_0^1 \pi(x)^2 dx = \pi\int_0^1 x^2 dx = \pi\left[\frac{x^3}{3}\right]_0^1 = \frac{\pi}{3}\ \text{units}^3$$

And then find the volume of the solid formed when the area enclosed by $y=x^2$, the x-axis and the line $x=1$ is rotated through one revolution of the x-axis.

$$\int_0^1 \pi y^2 dx = \int_0^1 \pi\left(x^2\right)^2 dx = \pi\int_0^1 x^4 dx = \pi\left[\frac{x^5}{5}\right]_0^1 = \frac{\pi}{5}\text{units}^3$$

The volume of the solid formed by rotating the area enclosed between $y=x^2$ and $y=x$ through one revolution about the x-axis is $\frac{\pi}{3}-\frac{\pi}{5}=\frac{2\pi}{15}\text{units}^3$.

Question 11

First find the points where $y = 0.125x^2$ and $y = \sqrt{x}$ intersect.

$$0.125x^2 = \sqrt{x}$$
$$\frac{1}{64}x^4 = x$$
$$x^4 = 64x$$
$$x^4 - 64x = 0$$
$$x(x^3 - 64) = 0$$
$$x = 0 \text{ or } x^3 = 64 \quad \Rightarrow \quad x = 0,\ 4$$

Now find the volume of the solid formed when the area enclosed by $y = \sqrt{x}$, the x-axis and the line $x = 4$ is rotated through one revolution about the x-axis.

$$\int_0^4 \pi y^2 dx = \int_0^4 \pi\left(\sqrt{x}\right)^2 dx = \pi\int_0^4 x\,dx = \pi\left[\frac{x^2}{2}\right]_0^4 = 8\pi \text{ units}^3$$

And then find the volume of the solid formed when the area enclosed by $y = 0.125x^2$, the x-axis and the line $x = 4$ is rotated through one revolution of the x-axis.

$$\int_0^4 \pi y^2 dx = \int_0^4 \pi\left(0.125x^2\right)^2 dx = \frac{\pi}{64}\int_0^4 x^4 dx = \frac{\pi}{64}\left[\frac{x^5}{5}\right]_0^4 = \frac{16\pi}{5}\text{units}^3$$

The volume of the solid formed by rotating the area enclosed between $y = 0.125x^2$ and $y = \sqrt{x}$ through one revolution about the x-axis is $8\pi - \frac{16\pi}{5} = \frac{24\pi}{5}\text{units}^3$.

Question 12

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi y^2\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi(3\cos x)^2\,dx = 9\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x\,dx = 9\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(\cos 2x + 1)\,dx$$
$$= \frac{9\pi}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos 2x + 1)\,dx = \frac{9\pi}{2}\left[\frac{\sin 2x}{2} + x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{9\pi}{2}\left\{0 + \frac{\pi}{2} - \left[0 + \left(-\frac{\pi}{2}\right)\right]\right\}$$
$$= \frac{9\pi^2}{2}\text{units}^3$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi y^2\,dx = \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x\,dx = \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(\cos 2x + 1)\,dx = \frac{\pi}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos 2x + 1)\,dx$$
$$= \frac{\pi}{2}\left[\frac{\sin 2x}{2} + x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{2}\left\{0 + \frac{\pi}{2} - \left[0 + \left(-\frac{\pi}{2}\right)\right]\right\} = \frac{\pi^2}{2}\text{units}^3$$

The volume of the solid formed by rotating the area enclosed between $y = 3\cos x$ and $y = \cos x$, from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$ through one revolution about the x-axis is $\frac{9\pi^2}{2} - \frac{\pi^2}{2} = 4\pi^2 \text{ units}^3$.

Question 13

$$\int_{-r}^{r} \pi y^2\,dx = \int_{-r}^{r} \pi \sqrt{r^2 - x^2}^{\,2}\,dx = \int_{-r}^{r} \pi \sqrt{r^2 - x^2}^{\,2}\,dx$$

$$= \pi \int_{-r}^{r} (r^2 - x^2)\,dx = \pi\left[r^2 x - \frac{1}{3}x^3\right]_{-r}^{r} = \pi\left\{r^3 - \frac{1}{3}r^3 - \left[-r^3 - \left(-\frac{1}{3}r^3\right)\right]\right\}$$

$$= \pi\left[\frac{2}{3}r^3 - \left(-\frac{2}{3}r^3\right)\right] = \frac{4}{3}\pi r^3 \text{ units}^3$$

Question 14

$$\int_0^h \pi y^2\,dx = \pi\int_0^h \left(\frac{r}{h}x\right)^2 dx = \frac{\pi r^2}{h^2}\int_0^h x^2\,dx = \frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h = \frac{\pi r^2}{h^2}\times\frac{h^3}{3} = \frac{1}{3}\pi r^2 h \text{ units}^3$$

Question 15

$$\int_0^2 \pi x^2\,dy = \pi\int_0^2 \left(\sqrt{y}\right)^2 dy = \pi\int_0^2 y\,dy = \pi\left[\frac{y^2}{2}\right]_0^2 = 2\pi \text{ units}^3$$

Question 16

$$\int_1^2 \pi x^2\,dy = \pi\int_1^2 \left(\frac{y}{\sqrt{5}}\right)^2 dy = \pi\int_1^2 \frac{y^2}{5}\,dy = \frac{\pi}{5}\int_1^2 y^2\,dy$$

$$= \frac{\pi}{5}\left[\frac{y^3}{3}\right]_1^2 = \frac{\pi}{5}\left(\frac{8}{3} - \frac{1}{3}\right) = \frac{7\pi}{15} \text{ units}^3$$

Question 17

$$\int_0^{12} \pi x^2\,dy = \pi\int_0^{12} \left(\sqrt{y+3}\right)^2 dy = \pi\int_0^{12} (y+3)\,dy$$

$$= \pi\left[\frac{y^2}{2} + 3y\right]_0^{12} = \pi\left(72 + 36\right) = 108\pi \text{ cm}^3$$

Question 18

$$\int_0^4 \pi y^2\,dx = \pi\int_0^4 \left(\sqrt{x}\right)^2 dx = \pi\int_0^4 x\,dx = \pi\left[\frac{1}{2}x^2\right]_0^4 = 8\pi\,\text{units}^3$$

$$\int_1^4 \pi y^2\,dx = \pi\int_1^4 \left(\sqrt{x-1}\right)^2 dx = \pi\int_1^4 (x-1)\,dx = \pi\left[\frac{1}{2}x^2 - x\right]_1^4 = \frac{9}{2}\pi\,\text{units}^3$$

The volume of the solid formed is $8\pi - \frac{9}{2}\pi = \frac{7}{2}\pi\,\text{units}^3$.

Question 19

Possibility 1:

$$\int_0^{\frac{\pi}{2}} \pi y^2\,dx = \pi\int_0^{\frac{\pi}{2}} \left(\frac{\sin x}{2}\right)^2 dx = \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{\pi}{4}\int_0^{\frac{\pi}{2}}\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \frac{\pi}{8}\int_0^{\frac{\pi}{2}} (1-\cos 2x)\,dx = \frac{\pi}{8}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{8}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi^2}{16}\,\text{m}^3$$

Possibility 2:

$$\int_0^{\frac{\pi}{2}} \pi y^2\,dx = \pi\int_0^{\frac{\pi}{2}} \left(\sqrt{\frac{x}{2\pi}}\right)^2 dx = \pi\int_0^{\frac{\pi}{2}} \frac{x}{2\pi}dx = \frac{\pi}{2\pi}\int_0^{\frac{\pi}{2}} x\,dx$$

$$= \frac{1}{2}\left[\frac{1}{2}x^2\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\times\frac{\pi^2}{8} = \frac{\pi^2}{16}\,\text{m}^3$$

Question 20

Given that $y = kx^2$ and one point on the curve is (4, 20), $k = \frac{5}{4}$.

$$\int_0^{20} \pi x^2\,dy = \pi\int_0^{20}\left(\frac{4y}{5}\right)dy = \frac{4\pi}{5}\int_0^{20} y\,dy = \frac{4\pi}{5}\left[\frac{y^2}{2}\right]_0^{20} = 160\pi\,\text{units}^3$$

Question 21

Volume of "shell" formed when rectangle is rotated about y-axis is large cylinder minus smaller cylinder.

$$\pi\left(x+\frac{\delta x}{2}\right)^2 y-\pi\left(x-\frac{\delta x}{2}\right)^2 y=\pi y\left(x^2+x\delta x+\frac{(\delta x)^2}{4}-\left(x^2-x\delta x+\frac{(\delta x)^2}{4}\right)\right)$$
$$=\pi y(2x\delta x)$$
$$=2\pi xy\delta x$$

So a formula involving a definite integral for the volume of the solid formed by rotating the shaded area shown in the diagram one revolution about the y-axis is $2\pi\int_a^b xy\,dx$.

a

$$2\pi\int_1^2 xy\,dx=2\pi\int_1^2\left(x\times x^2\right)dx$$
$$=2\pi\int_1^2 x^3dx=2\pi\left[\frac{x^4}{4}\right]_1^2$$
$$=2\pi\left(\frac{16}{4}-\frac{1}{4}\right)=\frac{15\pi}{2}\text{units}^3$$

b

$$2\pi\int_1^4 xy\,dx=2\pi\int_1^4 x\left(1+\sqrt{x}\right)dx$$
$$=2\pi\int_1^4\left(x+x^{\frac{3}{2}}\right)dx=2\pi\left[\frac{x^2}{2}+\frac{2x^{\frac{5}{2}}}{5}\right]_1^4$$
$$=2\pi\left(8+\frac{64}{5}-\left(\frac{1}{2}+\frac{2}{5}\right)\right)=\frac{199\pi}{5}\text{units}^3$$

Question 22

Volume of "shell" formed when rectangle is rotated about x-axis is large cylinder minus smaller cylinder.

$$\pi\left(y+\frac{\delta y}{2}\right)^2 x-\pi\left(y-\frac{\delta y}{2}\right)^2 x=\pi x\left[y^2+y\delta y+\frac{(\delta y)^2}{4}-\left(y^2-y\delta y+\frac{(\delta y)^2}{4}\right)\right]$$
$$=\pi x(2y\delta y)$$
$$=2\pi xy\delta y$$

So a formula involving a definite integral for the volume of the solid formed by rotating the shaded area shown in the diagram one revolution about the y-axis is $2\pi\int_a^b xy\,dy$.

a

$$2\pi\int_1^2 xy\,dy=2\pi\int_1^2\left(\frac{1}{y}\times y\right)dy$$
$$=2\pi\int_1^2 1\,dy=2\pi\left[x\right]_1^2$$
$$=2\pi\text{ units}^3$$

b

$$2\pi\int_1^2 xy\,dy=2\pi\int_1^2\left(\frac{y}{2}\times y\right)dy=2\pi\int_1^2\left(\frac{y^2}{2}\right)dy$$
$$=2\pi\left[\frac{y^3}{6}\right]_1^2=2\pi\left(\frac{7}{6}\right)=\frac{7\pi}{3}\text{units}^3$$

Question 1

Let $u = x$ and $\frac{dv}{dx} = \sin x$

then $\frac{du}{dx} = 1$ and $v = -\cos x$

$$\int x\sin x\,dx = -x\cos x - \int(-\cos x)dx$$
$$= -x\cos x - (-\sin x) + c$$
$$= \sin x - x\cos x + c$$

Question 2

Let $u = x$ and $\frac{dv}{dx} = \cos x$

then $\frac{du}{dx} = 1$ and $v = \sin x$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$$
$$= x\sin x - (-\cos x) + c$$
$$= x\sin x + \cos x + c$$

Question 3

Let $u = 3x$ and $\frac{dv}{dx} = \sin 2x$

then $\frac{du}{dx} = 3$ and $v = -\frac{\cos 2x}{2}$

$$\int 3x\sin 2x\,dx = -\frac{3}{2}x\cos 2x - \int\left(\frac{-\cos 2x}{2}\right)3dx$$
$$= -\frac{3}{2}x\cos 2x + \frac{3}{4}\sin 2x + c$$

Question 4

Let $u = x$ and $\frac{dv}{dx} = e^{2x}$

then $\frac{du}{dx} = 1$ and $v = \frac{e^{2x}}{2}$

$$\int x e^{2x} dx = \frac{1}{2} x e^{2x} - \int \left(\frac{e^{2x}}{2} \right) dx$$
$$= \frac{1}{2} x e^{2x} - \frac{e^{2x}}{4} + c$$

Question 5

Let $u = \ln x$ and $\frac{dv}{dx} = x^2$

then $\frac{du}{dx} = \frac{1}{x}$ and $v = \frac{x^3}{3}$

$$\int x^2 \ln x \, dx = x^2 \ln x - \int \left(\frac{x^3}{3} \times \frac{1}{x} \right) dx$$
$$= x^2 \ln x - \int \left(\frac{x^2}{3} \right) dx$$
$$= x^2 \ln x - \frac{x^3}{9} + c$$

Question 6

Let $u = x$ and $\frac{dv}{dx} = (x+2)^5$

then $\frac{du}{dx} = 1$ and $v = \frac{1}{6}(x+2)^6$

$$\int x(x+2)^5 \, dx = \frac{x(x+2)^6}{6} - \int \frac{(x+2)^6}{6} dx$$
$$= \frac{x(x+2)^6}{6} - \frac{(x+2)^7}{42} + c$$

Question 7

Let $u = x$ and $\frac{dv}{dx} = \sqrt{2x+1}$

then $\frac{du}{dx} = 1$ and $v = \frac{1}{3}(2x+1)^{\frac{3}{2}}$

$$\int x\sqrt{2x+1}\,dx = \frac{1}{3}x(2x+1)^{\frac{3}{2}} - \int \frac{1}{3}(2x+1)^{\frac{3}{2}}\,dx = \frac{1}{3}x(2x+1)^{\frac{3}{2}} - \frac{1}{15}(2x+1)^{\frac{5}{2}} + c$$

Question 8

Let $u = x^2$ and $\frac{dv}{dx} = e^x$

then $\frac{du}{dx} = 2x$ and $v = e^x$

$\int x^2 e^x\,dx = x^2 e^x - \int 2xe^x dx$ (see below for the $\int 2xe^x dx$)

$$= x^2 e^x - 2xe^x + 2e^x + c$$

Let $u = 2x$ and $\frac{dv}{dx} = e^x$

then $\frac{du}{dx} = 2$ and $v = e^x$

$$\int 2xe^x dx = 2xe^x - \int 2e^x\,dx = 2xe^x - 2e^x + c$$

Question 9

Let $u = x^2$ and $\frac{dv}{dx} = \sin x$

then $\frac{du}{dx} = 2x$ and $v = -\cos x$

$$\int x^2 \sin x\,dx = -x^2\cos x - \int 2x(-\cos x)\,dx$$

$= -x^2\cos x + \int 2x(\cos x)\,dx$ (see below for $\int 2x\cos x\,dx$)

$$= -x^2\cos x + 2x\sin x + 2\cos x + c$$

Let $u = 2x$ and $\frac{dv}{dx} = \cos x$

then $\frac{du}{dx} = 2$ and $v = \sin x$

$$\int 2x\cos x\,dx = 2x\sin x - \int 2\sin x\,dx = 2x\sin x + 2\cos x + c$$

Question 10

Write $\int 2x^3e^{x^2}\,dx$ as $\int x^2 \times 2x \times e^{x^2}\,dx$. Let $u = x^2$. $\dfrac{du}{dx} = 2x, dx = \dfrac{du}{2x}$

$\int x^2 \times 2x \times e^{x^2}\,dx$ becomes $\int u \times 2x \times e^u \dfrac{du}{2x} = \int ue^u du$.

Integration by parts: $\int vw'\,du = vw - \int v'w\,du$

Let $v = u$ and $\dfrac{dw}{du} = e^u$

then $\dfrac{dv}{du} = 1$ and $w = e^u$

$\int ue^u du = ue^u - \int e^u du = ue^u - e^u + c.$

Hence $\int 2x^3e^{x^2}\,dx = x^2e^{x^2} - e^{x^2} + c = e^{x^2}\left(x^2 - 1\right) + c$

Question 11

Let $u = \ln x$ and $\dfrac{dv}{dx} = 1$

then $\dfrac{du}{dx} = \dfrac{1}{x}$ and $v = x$

$$\begin{aligned}\int \ln x\,dx &= x\ln x - \int x \times \frac{1}{x}dx \\ &= x\ln x - x + c\end{aligned}$$

Question 12

Let $u = e^x$ and $\frac{dv}{dx} = \sin x$

then $\frac{du}{dx} = e^x$ and $v = -\cos x$

$$\int e^x \sin x\,dx = -e^x \cos x - \int -\cos x \times e^x dx$$

$$= -e^x \cos x + \int \cos x \times e^x dx \quad \text{(see below for } \int \cos x \times e^x dx\text{)}$$

$$\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx$$

$$2\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x$$

$$\int e^x \sin x\,dx = \frac{-e^x \cos x + e^x \sin x}{2} + c$$

Let $u = e^x$ and $\frac{dv}{dx} = \cos x$

then $\frac{du}{dx} = e^x$ and $v = \sin x$

$$\int \cos x \times e^x dx = e^x \sin x - \int e^x \sin x dx$$

Question 13

Let $u = e^x$ and $\frac{dv}{dx} = \cos 2x$

then $\frac{du}{dx} = e^x$ and $v = \frac{\sin 2x}{2}$

$$\int e^x \cos 2x\,dx = e^x \frac{\sin 2x}{2} - \int e^x \frac{\sin x}{2} dx \quad \text{(see below for } \int e^x \frac{\sin x}{2} dx\text{)}$$

$$\int e^x \cos 2x\,dx = e^x \frac{\sin 2x}{2} - \left(-e^x \frac{\cos 2x}{4} + \frac{1}{4}\int e^x \cos 2x\,dx\right)$$

$$\frac{5}{4}\int e^x \cos 2x\,dx = e^x \frac{\sin 2x}{2} + \frac{e^x \cos 2x}{4} + c$$

$$\int e^x \cos 2x\,dx = \frac{2e^x \sin 2x}{5} + \frac{e^x \cos 2x}{5} + c$$

Let $u = \frac{e^x}{2}$ and $\frac{dv}{dx} = \sin 2x$

then $\frac{du}{dx} = \frac{e^x}{2}$ and $v = -\frac{\cos 2x}{2}$

$$\int \frac{e^x}{2} \sin 2x\,dx = \frac{-e^x \cos 2x}{4} - \int -\frac{\cos 2x}{2}\frac{e^x}{2} dx = -e^x \frac{\cos 2x}{4} + \frac{1}{4}\int e^x \cos 2x\,dx$$

Miscellaneous Exercise 9

Question 1

$$y = (2x+1)^3$$

$$\frac{dy}{dx} = 3(2x+1)^2 \times 2 = 6(2x+1)^2$$

Question 2

$$y = 4\cos 3x + 3\sin 4x$$

$$\frac{dy}{dx} = -12\sin 3x + 12\cos 4x$$

Question 3

$$y = \frac{\sin^4 x}{x}$$

$$\frac{dy}{dx} = \frac{4x\sin^3 x\cos x - \sin^4 x}{x^2} = \frac{\sin^3 x(4x\cos x - \sin x)}{x^2}$$

Question 4

$$y = \frac{1+2\sin x}{1+\cos x}$$

$$\frac{dy}{dx} = \frac{(1+\cos x)2\cos x + (1+2\sin x)\sin x}{(1+\cos x)^2}$$

$$= \frac{2\cos x + 2\cos^2 x + \sin x + 2\sin^2 x}{(1+\cos x)^2}$$

$$= \frac{2\cos x + \sin x + 2(\sin^2 x + \cos^2 x)}{(1+\cos x)^2}$$

$$= \frac{2\cos x + \sin x + 2}{(1+\cos x)^2}$$

Question 5

$$y = \frac{\sin 2x}{1+\sin 2x}$$

$$\frac{dy}{dx} = \frac{(1+\sin 2x)2\cos 2x - \sin 2x(2\cos 2x)}{(1+\sin 2x)^2} = \frac{2\cos 2x}{(1+\sin 2x)^2}$$

Question 6

$$\frac{d}{dx}(5xy+2y^3)=\frac{d}{dx}\left(3x^2-7\right)$$

$$5x\frac{dy}{dx}+5y+6y^2\frac{dy}{dx}=6x$$

$$\frac{dy}{dx}(5x+6y^2)=6x-5y$$

$$\frac{dy}{dx}=\frac{6x-5y}{5x+6y^2}$$

Question 7

$$x=3t^2-5t \quad \Rightarrow \quad \frac{dx}{dt}=6t-5$$

$$y=3-4t^3 \quad \Rightarrow \quad \frac{dy}{dt}=-12t^2$$

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{-12t^2}{6t-5}$$

Question 8

$$x\cos y=y\sin x$$

$$-x\sin y\times\frac{dy}{dx}+\cos y=y\cos x+\sin x\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(\sin x+x\sin y\right)=\cos y-y\cos x$$

$$\frac{dy}{dx}=\frac{\cos y-y\cos x}{\sin x+x\sin y}$$

Question 9

Given $\dfrac{a}{x-1}+\dfrac{b}{x+1}=\dfrac{7x-5}{x^2-1}$

$$\frac{a(x+1)+b(x-1)}{(x-1)(x+1)}=\frac{7x-5}{x^2-1}$$

$$\frac{ax+a+bx-b}{x^2-1}=\frac{7x-5}{x^2-1}$$

From this, equating the co-efficients of the numeratorr gives:

$a+b=7$

$a-b=-5$

$2a=2$ so $a=1$

$a+b=7$ so $b=6$

Then $\displaystyle\int\frac{7x-5}{x^2-1}dx=\int\left(\frac{1}{x-1}+\frac{6}{x+1}\right)dx=\ln\left|x-1\right|+6\ln\left|x+1\right|+c$

Question 10

a $\displaystyle\int 4\cos 8x\,dx=\frac{4\sin 8x}{8}+c=\frac{1}{2}\sin 8x+c$

b Let $u=3+x^2$, $\dfrac{du}{dx}=2x$

$$\int 2x(3+x^2)^5\,dx=\int 2xu^5\frac{dx}{du}du=\int 2xu^5\frac{1}{2x}du$$

$$=\int u^5du=\frac{u^6}{6}+c=\frac{1}{6}(3+x^2)^6+c$$

c Let $u=x+3$, $\dfrac{du}{dx}=1$

$$\int(2-3x)\sqrt[3]{x+3}\,dx=\int(2-3(u-3))\sqrt[3]{u}\,\frac{dx}{du}du=\int(11-3u)u^{\frac{1}{3}}du$$

$$=\int(11u^{\frac{1}{3}}-3u^{\frac{4}{3}})du=\frac{3\times11}{4}u^{\frac{4}{3}}-\frac{3\times3}{7}u^{\frac{7}{3}}+c=\frac{1}{28}(231u^{\frac{4}{3}}-36u^{\frac{7}{3}})+c$$

$$=\frac{1}{28}u^{\frac{4}{3}}(231-36u)+c=\frac{1}{28}(x+3)^{\frac{4}{3}}(231-36(x+3))+c$$

$$=\frac{1}{28}(x+3)^{\frac{4}{3}}(123-36x)+c=\frac{3}{28}(x+3)^{\frac{4}{3}}(41-12x)+c$$

d Let $u = \sin 2x$, $\dfrac{du}{dx} = 2\cos 2x$

$$\int \sin^5 2x \cos 2x \, dx = \int u^5 \cos 2x \frac{dx}{du} du$$

$$= \int u^5 \cos 2x \frac{1}{2\cos 2x} du = \frac{1}{2}\int u^5 du$$

$$= \frac{1}{2} \times \frac{u^6}{6} + c = \frac{1}{12}\sin^6 2x + c$$

e Let $u = \dfrac{x}{2}$, $\dfrac{du}{dx} = \dfrac{1}{2}$

$$\int \sin^2 \frac{x}{2} dx = \int \sin^2 u \frac{dx}{du} du = 2\int \sin^2 u \, du$$

$$= 2\int \left(\frac{1-\cos 2u}{2}\right) du = \int (1-\cos 2u) du$$

$$= \frac{x}{2} - \frac{\sin 2\left(\frac{x}{2}\right)}{2} + c = \frac{1}{2}x - \frac{1}{2}\sin x + c$$

f Let $u = \sin \dfrac{x}{2}$, $\dfrac{du}{dx} = \dfrac{1}{2}\cos\dfrac{x}{2}$

$$\int \cos^3 \frac{x}{2} dx = \int \cos^2 \frac{x}{2} \cos\frac{x}{2} \, dx = \int \left(1 - \sin^2 \frac{x}{2}\right)\cos\frac{x}{2} \, dx$$

$$= \int (1-u^2)\cos\frac{x}{2}\frac{dx}{du} du = \int (1-u^2)\cos\frac{x}{2}\frac{2}{\cos\frac{x}{2}} du$$

$$= 2\int (1-u^2) du = 2\left(u - \frac{u^3}{3}\right) + c = \frac{2}{3}\left(3u - u^3\right) + c$$

$$= \frac{2}{3}\left(3\sin\frac{x}{2} - \sin^3\frac{x}{2}\right) + c = 2\sin\frac{x}{2} - \frac{2}{3}\sin^3\frac{x}{2} + c$$

g Let $u = \cos 2x$, $\dfrac{du}{dx} = -2\sin 2x$

$$\int \sin^3 2x \, dx = \int \sin^2 2x \, \sin 2x \, dx = \int (1-\cos^2 2x) \, \sin 2x \, dx$$

$$= \int (1-u^2)\sin 2x \frac{dx}{du} du = \int (1-u^2)\sin 2x \frac{1}{-2\sin 2x} du$$

$$= -\frac{1}{2}\int (1-u^2) du = -\frac{1}{2}\left(u - \frac{u^3}{3}\right) + c = \frac{1}{6}\left(u^3 - 3u\right) + c$$

$$= \frac{1}{6}\left(\cos^3 2x - 3\cos 2x\right) + c = \frac{1}{6}\cos^3 2x - \frac{1}{2}\cos 2x + c$$

h $\int 6\sin 2x\cos x\,dx = \int 6(2\sin x\cos x)\cos x\,dx = 12\int \sin x\cos^2 x\,dx$

Let $u = \cos x,\ \dfrac{du}{dx} = -\sin x$

$$12\int \sin x\, u^2 \frac{dx}{du}du = 12\int \sin x\, u^2 \frac{1}{-\sin x}du = -12\int u^2 du = \frac{-12u^3}{3} + c$$

$$= -4\cos^3 x + c$$

i $\int 6\cos 2x\sin x\,dx = \int 6(2\cos^2 x - 1)\sin x\,dx$

Let $u = \cos x,\ \dfrac{du}{dx} = -\sin x$

$$\int 6(2\cos^2 x - 1)\sin x\,dx = 6\int (2u^2 - 1)\sin x\frac{dx}{du}du = 6\int (2u^2 - 1)\sin x\frac{1}{-\sin x}du$$

$$= -6\int (2u^2 - 1)du = -6\left(\frac{2u^3}{3} - u\right) + c$$

$$= -4\cos^3 x + 6\cos x + c$$

Question 11

a $\dfrac{d}{dx}(x^2 + xy) = \dfrac{d}{dx}\left(1 + y^2\right)$

$$2x + x\frac{dy}{dx} + y = 2y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x + y}{2y - x}$$

At (2, 3), $\dfrac{dy}{dx} = \dfrac{4+3}{6-2} = \dfrac{7}{4}$

The equation of the tangent at the point (2, 3) is $y = \dfrac{7}{4}x - \dfrac{1}{2}$

b $\dfrac{d}{dx}\left(x^3 + y^3\right) = \dfrac{d}{dx}(35)$

$$3x^2 + 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x^2}{y^2}$$

At (2, 3), $\dfrac{dy}{dx} = -\dfrac{4}{9}$

The equation of the tangent at the point (2, 3) is $y = -\dfrac{4}{9}x + \dfrac{35}{9}$,

which can be rearranged to $9y + 4x = 35$

Question 12

$$\int_1^4 \pi y^2\,dx = \pi\int_1^4 \frac{1}{x}\,dx = \pi\int_1^4 \frac{1}{x}\,dx = \pi\Big[\ln|x|\Big]_1^4$$
$$= \pi(\ln 4 - \ln 1) = \pi \ln 4\,\text{units}^3$$

Question 13

$A = l \times w = 4w \times w = 4w^2$

$\frac{dw}{dt} = 2\text{mm/s}, \quad \frac{dA}{dw} = 8w$

$\frac{dA}{dt} = \frac{dA}{dw} \times \frac{dw}{dt} = 2 \times 8w = 16w$ mm/s

When $w = 150,\ dA = 2400\,\text{mm}^2/\text{s} = 24\,\text{cm}^2/\text{s}$

Question 14

Let $x = 5\sin u, \frac{dx}{du} = 5\cos u$

$$\int_0^5 \sqrt{25 - x^2}\,dx = \int_0^5 \sqrt{25 - 25\sin^2 u}\,\frac{dx}{du}\,du = \int_{x=0}^{x=5} \sqrt{25(1-\sin^2 u)}\,\frac{dx}{du}\,du$$
$$= \int_{u=0}^{u=\frac{\pi}{2}} \sqrt{25\cos^2 u}\,\frac{1}{5\cos u}\,du = \int_0^{\frac{\pi}{2}} 1\,du = [u]_0^{\frac{\pi}{2}} = \frac{\pi}{2}\,\text{units}^2$$

Question 15

$$\int_1^2 \frac{3x^2 + 5x - 1}{(x+2)(x+1)^2}\,dx = \int \frac{A}{x+2}\,dx + \int \frac{B}{x+1}\,dx + \int \frac{C}{(x+1)^2}\,dx$$

$A(x+1)^2 + B(x+2)(x+1) + C(x+2) = 3x^2 + 5x - 1$

$A + B = 3$

$2A + 3B + C = 5$

$A + 2B + 2C = -1$

Solving gives $A = 1,\ B = 2$ and $C = -3$.

$$\int \frac{1}{x+2}\,dx + \int \frac{2}{x+1}\,dx - \int \frac{3}{(x+1)^2}\,dx = \left[\ln|x+2| + 2\ln|x+1| + \frac{3}{x+1}\right]_1^2$$
$$= \ln|4| + 2\ln|3| + \frac{3}{3} - \left(\ln|3| + 2\ln|2| + \frac{3}{2}\right)$$
$$= -\frac{1}{2} + \ln 3$$

Question 16

$\frac{dV}{dt} = -5\,\text{cm}^3/\text{s}$

$\frac{r}{h} = \frac{5}{20}$

$r = \frac{h}{4}$

$V = \frac{\pi r^2 h}{3} = \frac{\pi h^3}{48}$

$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{\pi h^3}{48}\right)$

$\frac{dV}{dt} = \frac{\pi h^2}{16}\frac{dh}{dt}$

$-5 = \frac{\pi h^2}{16}\frac{dh}{dt}$

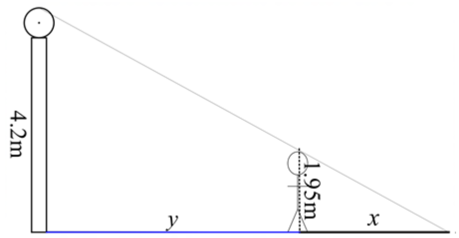
When $h = 10\,\text{cm}$,

$\frac{dh}{dt} = \frac{-5\times 16}{100\pi} = \frac{4}{5\pi}\,\text{cm/s}$

So the height is falling at a rate of $\frac{4}{5\pi}\,\text{cm/s}$

Question 17

Let x be the length of the shadow and y be the distance of the person from the lamp-post.

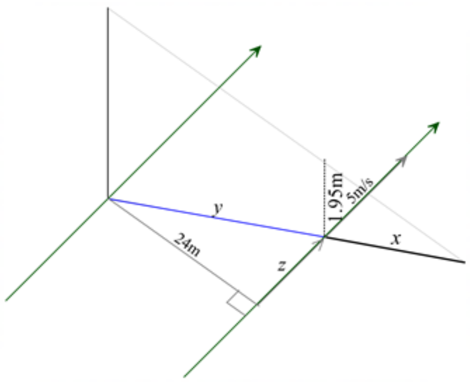


a Using similar triangles

$$\frac{4.2}{1.95} = \frac{x+y}{x}$$
$$4.2x = 1.95x + 1.95y$$
$$2.25x = 1.95y$$
$$x = \frac{1.95}{2.25}y$$
$$\frac{dx}{dt} = \frac{13}{15}\frac{dy}{dt}$$
$$\frac{dx}{dt} = \frac{13}{15}\times 5$$
$$= \frac{13}{3}\,\text{m/s}$$

b Whether the person is travelling East or West the shadow is changing at the same rate, $\frac{13}{3}$ m/s.

c



$$2.25x = 1.95y$$

$$\frac{dx}{dy} = \frac{13}{15}$$

$$y^2 = 24^2 + z^2$$

$$2y\frac{dy}{dz} = 2z$$

$$\frac{dy}{dz} = \frac{z}{y}$$

$$\frac{dx}{dt} = \frac{dx}{dy} \times \frac{dy}{dz} \times \frac{dz}{dt}$$

$$= \frac{13}{15} \times \frac{z}{y} \times 5$$

After 2 seconds, $z = 10$ m and $y = 26$ m.

$$\frac{dx}{dt} = \frac{13}{15} \times \frac{10}{26} \times 5 = \frac{5}{3} \text{ m/s}$$